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ENCYCLOPÆDIA METROPOLITANA;

OR,

UNIVERSAL DICTIONARY OF KNOWLEDGE,

On an Original Plan:

COMPRISING THE TWOFOLD ADVANTAGE OF

A PHILOSOPHICAL AND AN ALPHABETICAL ARRANGEMENT,

WITH APPROPRIATE ENGRAVINGS.

EDITED BY

THE REV. EDWARD SMEDLEY, M.A.,
LATE FELLOW OF SIDNEY COLLEGE, CAMBRIDGE;

THE REV. HUGH JAMES ROSE, B.D.,
PRINCIPAL OF KING'S COLLEGE, LONDON;
AND
THE REV. HENRY JOHN ROSE, B.D.,
LATE FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE.

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ENCYCLOPÆDIA METROPOLITANA;

OR, THE

UNIVERSAL DICTIONARY OF KNOWLEDGE,

ON AN ORIGINAL PLAN.

Second Division.

MECHANICS.

chanics.

MECHANICS (Μηχανὴ) is that science which treats of the laws of equilibrium and motion; and as it relates to solid bodies, it is divided into STATICS and DYNAMICS; the former regarding the theory of their equilibrium, and the latter that of their motion.

Mechanics also will strictly embrace the equilibrium and motion of fluids; in which sense we may consider both HYDROSTATICS and HYDRODYNAMICS as branches of the science. But the applications of mechanical principles to solids and to fluids are so evidently distinct, and each of them, considered separately, of such extreme importance to human life, that we shall not hesitate to follow the usual acceptation of the term, MECHANICS, as confined to the motions and equilibrium of *solid* bodies, and treat of HYDROSTATICS and HYDRODYNAMICS separately; prefixing to this article a brief historical sketch of the rise, progress, and present state of the science in its application to solids; and to the treatise on Hydrostatics, which will immediately succeed it, a history of the application of mechanical principles to the equilibrium and motion of fluids.

§ 1. *Historical sketch of the rise and progress of Mechanics, as relating to solid bodies.*

story of chanic.

It is not our intention, in the present article, to enter upon the history of practical Mechanics, but to confine ourselves exclusively to the theory of the science; we shall not therefore have to travel into those dark ages in which historical facts and fables are so blended, that it is nearly impossible to distinguish the one from the other. We learn at once from the writings of Aristotle, what the state of mechanical theory was in his time: for we find him maintaining that if one body have ten times the density of another, it will move with ten times the velocity, and that both being let fall from the same height, the one will fall through ten times the space that the other will in the same time; that the velocity of the same body, in different mediums, is reciprocally as their densities; and other equally absurd and inconsistent notions: and the difficulty which Galileo experienced in eradicating these false hypotheses, is a proof that in the long interval between his time and that of the Stagirite, no theory of *motion* of a more intelligible and satisfactory description had appeared; although the doctrine of *equilibrium* had already begun to assume a scientific form in the hands of Archimedes and Pappus.

Mechanics. Aristotle. B. C. 330.

In the writings of Archimedes that are still extant, we find the earliest attempt to reduce the laws of equilibrium to order and consistency. His work "De Æquiponderantibus," first unites and assimilates them with the pure principles of geometry. With this view, he began by considering the case of a lever or balance, supported on a fulcrum, and loaded with a weight at each extremity; and assuming it as an axiom, that when the two arms of the balance are equal, the two weights supposed in equilibrio are also necessarily equal, he demonstrated that if one of the arms of this lever be augmented in length, the weight applied to it in order to preserve the equilibrium, must be reduced in the same ratio; and hence he concluded, that generally, when two weights suspended from the unequal arms of a lever, are in equilibrio, these weights ought to be reciprocally proportional to the distance of their respective points of application from the centre of motion. Archimedes, having also observed, that each of the two weights

Archimedes. B. C. 289.

produced the same pressure on the fulcrum or point of support as they would do if they were immediately applied at that point, proceeded to make this substitution mentally, and to combine the sum of the two weights with a third; thus attaining the same conclusion for an assemblage of the three weights as for the first two; and so on for any greater number. Hence he demonstrated, step by step, that there exists in every system of bodies, as well as in every single body, regarded as a system, a general centre of effort, which we denominate the *centre of gravity*. He then applied this theory to certain examples, and determined the centres of gravity in the parallelogram, the triangle, the trapezium, the area of the parabola, &c. &c.

This deduction, as we have above observed, was the first step towards establishing a rational theory of Mechanics, and the surprize expressed by Hiero at the famous assertion of our philosopher, "Give me a place to stand on, and I will move the earth," shows at once the novelty of the doctrine, and the wretched state of mechanical knowledge prior to this discovery. To the same author has also been attributed the theory of the inclined plane, the pulley, and the screw. Much doubt, however, remains upon this subject, and he is too rich in honours to render it desirable to increase them by any of uncertain authority. The machines which he constructed for the annoyance of the Roman army, during the siege of Syracuse, astonish even our present proficients in the science; but as no writings descriptive of them have come down to us, we are in a great measure unacquainted with the nature of their powers: while much of what is invaluable has been lost, much may have been exaggerated by succeeding writers, and little of scientific detail can be relied upon respecting them. No theory of Mechanics, with the exception of what little is found in the collections of Pappus, and which is chiefly a repetition of the doctrine of Archimedes, appeared from the time of the latter philosopher till near the end of the sixteenth century; but at this period (1577) Stevinus, a Flemish mathematician of no considerable notoriety, succeeded in demonstrating the laws of the equilibrium of a body on an inclined plane, without the intervention of the lever. He also illustrated many questions in statics, and determined the conditions of equilibrium between several forces concurring in a common point, in a manner equivalent to the famous proposition of the parallelogram of forces; though it does not appear that he was at all aware of its consequences.

Stevinus.
A.D.1577.

Galileo.
A.D.1592.

To Galileo, the great father of physical science, we are indebted for the next important step in the theory of Mechanics; so important indeed, that he may be regarded as having laid the foundation for most of the subsequent discoveries in this branch of mathematics. This author, in 1592, composed a small treatise on statics, which he reduced to one simple principle; viz. that it requires an equal power to raise two different bodies to heights that are reciprocally proportional to the weights; so that whatever force will raise a body of two pounds to the height of one foot, will raise a body of one pound to the height of two feet; a principle which has since been adopted and generalized by modern mathematicians, and denominated the principle of *virtual velocities*. From this it was easy to conclude, that in all machines in equilibrio, the powers which are opposed to each other, are reciprocally proportional to the spaces that they tend to describe in the same time. The only question then is, to determine with accuracy their spaces from a due consideration of the disposition of the several parts of a machine; of which we may give an example in the simple mechanical power, the screw; where it is obvious, that while the weight is raised only to a distance equal to that of the threads of the screw, the power itself must have passed through the entire circumference described by the extremity of the lever which works it; and consequently, the power is to the weight as the distance of the threads is to that circumference.

Long after this, Descartes, in a small work, entitled "Explication des Machines et Engins," employed the same principle to determine the equilibrium in all machines: though, with a narrowness of mind disgraceful to a promoter of science, he neglected to ascribe it to Galileo, to whom he knew that we owed the discovery of the principle.

Accele
motion

This is, however, but a small part of the obligation which the Florentine philosopher conferred upon the mechanical sciences: the theory of variable motion, wholly unknown to the ancients, received its birth at his hands. He discovered the law of acceleration of bodies falling by the action of gravity, either vertically or down inclined planes, and thus established the general properties of uniformly accelerated motion. The conformity of his theory with the actual phenomena of bodies falling near the surface of the earth, cannot but be regarded as one of the most pleasing results ever attained by an investigator; and as supplying the first link in that chain which ultimately connected the laws of falling bodies with the system of universal gravitation.

Meas
time b
pendu

To Galileo we owe also another memorable discovery, which, though not precisely of the same kind, has its foundation in the same principles; viz. that we may measure time by the oscillations of a pendulum; or, in other words, by the number of vibrations made by a small body attached to a string of a given length. The foundation of this theory is, that the length of the thread of suspension is reciprocally proportional to the square of the number of vibrations made in a given time; whence, for example, by knowing the length of a pendulum which vibrates seconds, a simple proportion will show what must be the length of another string that the pendulum may vibrate once in any other given time; or, the length of the pendulum being given, the time may be determined.

Proje

The theory of projectiles in a vacuum is, again, a branch of mechanical science which we owe to the inventive genius of Galileo. When a body is projected into void space, either in an oblique or in a horizontal direction, it will describe a curve, that depends both upon the initial velocity of projection, and upon the law of acceleration which a body would observe, in its fall, from the action of gravity; and as the latter was not known till the discovery of it by Galileo, it follows, *a fortiori*, that the real path of the projectile was also unknown prior to the time of this distinguished but persecuted philosopher. He not only determined the true nature of the curve, showing it to be the common parabola, but he also investigated and discovered various circumstances relative to its motion. He pointed out the method of ascertaining the impetus for any range and elevation; in what manner the range was to be found when the impetus and elevation were given; and how the elevation may be deduced from the range and impetus. He demonstrated also that the velocity of the projectile in any point of the curve, is equal to that which would be generated by a heavy body in falling through one-fourth of the parameter to the diameter at that point; and that the greatest range, the impetus or velocity being the same, is when the angle of elevation is 45°. He likewise proved that the ranges are the same for all angles equally above and below that which gives the maximum. His enquiries were limited, however, in all these cases, to the horizontal plane. But Torricelli, the excellent pupil of Galileo, pursued this subject to a greater extent, and added several curious propositions to those already discovered by his master.

gens' cid.

1658. Huygens, next in order of time, and equal, perhaps, in talent to Galileo, amongst various other pursuits attached himself particularly to the theory and practice of Mechanics. At this time the motion of bodies falling freely and along inclined planes was all that had been attempted; but Huygens extended his investigations to a subject of a much more difficult nature, viz. the motion of bodies in curves; and first demonstrated that the velocity of a heavy body, which descends along any curve, is the same at every instant in the direction of the tangent, as would have been acquired had the body fallen freely from a height equal to the corresponding vertical absciss. Carrying the same principle to the inverted cycloid, he found that a body from whatever part of the cycloidal arc it falls, always arrives at the lowest part of that arc in the same time. This singular property of the cycloid is what is commonly understood by its *isochronism*: a discovery which, alone, would have been sufficient to elevate its author to the highest reputation.

cussion.

But we are indebted to him for various other important propositions, of which we shall have occasion to speak in treating of the doctrine of central forces; and shall, therefore, at present merely refer to his investigation of the laws of percussion. In what we have stated above, the bodies whose motion have been considered were perfectly isolated; and nothing as yet had been attempted, at least successfully, relative to the action of bodies upon each other. This problem, therefore, next presented itself to the contemplation of mathematicians; and in 1661, Huygens, Wallis, and Sir C. Wren, each succeeded, separately and independently of each other, in establishing the correct laws of percussion, which Descartes had previously attempted in vain. For, being led astray by his metaphysical principles, which induced him to suppose that there was always the same absolute quantity of motion in the universe, he concluded, that the sum of the motions was the same both before and after impact; which was only true in one particular case: but the mathematicians above named found that, in the mutual percussion of many bodies, it was simply the absolute motion of the centre of gravity in the system that was the same; and moreover that when the bodies are elastic, the relative velocities are the same both before and after the stroke.

Huygens, Wallis, Wren.
A.D.1661.

This property has since been generalized, and considered as an inherent principle; and is frequently spoken of under the appellation of the conservation of the centre of gravity.

Mersenne.
A.D.1635.

Two other celebrated problems, both more difficult than any which had been yet investigated relative to the communication of motion, were proposed by Mersenne; the one requiring the determination of what we now call the centre of oscillation in compound pendulums, and the other to find the centre of percussion in a system of bodies revolving about a fixed axis.

Centre of oscillation.

In the first of these problems, we may suppose several heavy bodies connected together at invariable distances, by threads considered as without weight, and oscillating about a horizontal fixed axis, which by counteracting and modifying each other's motion, will not vibrate with the same velocity as if each of them oscillated separately: the bodies nearest the axis will necessarily lose a part of their natural motion, and transmit it to the bodies more distant; whence results a compensation between the motions lost and the motions gained. Now in whatever manner this equilibrium is established, there is necessarily in the system a point at the same distance from the axis of suspension, to which if we apply a small body or indivisible particle, by a thread without a weight, it will oscillate in the same time as the compound pendulum; and the question proposed by Mersenne was to determine the length of that thread; which was no other than to find the centre of oscillation of a compound pendulum.

Centre of percussion.

The second problem, viz. the determination of the centre of percussion (although it appear, generally, to be the same point), is of a different kind; the characteristic property of this centre being that it is found in the direction of the resultant of all the motions of the bodies in a system which turns about a fixed axis, and occupying in this system a situation analogous to that of the centre of gravity in a heavy body. Consequently, if the motion of this point or centre be counteracted by the intervention of a permanent obstacle, the whole system is reduced to a state of momentary equilibrium; therefore, although the centres of oscillation and percussion may be in the same point, and the solution of both problems depends upon the same mechanical principles, the application of these principles is not the same in both cases, but is much more direct and simple in the latter than in the former: to which we may also add that, *a priori*, it would be impossible in many cases to foresee the coincidence. But Descartes and Roberval, having found these two centres to be the same *in some cases*, and finding the latter problem to be by far the most simple, considered both under the same point of view. In a few instances they succeeded in obtaining correct solutions, but in many others their results were erroneous; for their methods, founded upon inaccurate and uncertain suppositions, were precarious and insufficient for the general solution.

Conservatio virium vivarum.

Huygens, again, was here the first who resolved in a complete and general manner the former, and, as we have said above, the most difficult and important of these problems: assuming as a principle, that after the centre of gravity of a compound pendulum has descended to its lowest point, if the several bodies of which it is composed are imagined to be detached from one or other, and that each re-ascends separately with the velocity acquired in their descent; the centre of gravity of the system in this state will re-ascend to the same point as if the connection of the bodies with each other still subsisted.

This principle was much objected to by many of the mathematicians of that time, who very severely criticised the solutions founded upon it. Indeed it must be admitted, that as the foundation (we might almost say) of a new science, it was too general to be acceded to without a previous investigation. At that period this had not been attempted; but it has since received the most luminous and incontestible demonstration; and is, in the present day, generally admitted and adopted under the name of *conservatio virium vivarum*, or the *principle of the conservation of living forces*.

Parallelogram of forces.

We have already had occasion to mention, that the celebrated problem of the composition of forces was in some measure discovered and resolved by Stevinus, though he was not aware of the importance and the extensive application of the principle which he had so nearly developed. Indeed, with the exception of the slight application which Galileo had made of it, in the determination of the path of a projectile, this problem remained unapplied, till Varignon brought it forward again in some sketches given in 1687, in his "Project for a new System of Mechanics;" the success of which encouraged the author to enter into a farther development of his ideas: and the result appeared, after a labour of thirty years, in his "General Mechanics:" a work not published till 1725, three years after the author's death. In both these publications, the author rests his investigation of the equilibrium of machines upon the principle of the composition of forces; which is, that if two forces, the directions of which meet in the same point, be such that each separately would make a body pass in the same time over the sides of a parallelogram constructed upon their direction, their joint action would cause the same body to pass over the diagonal of that parallelogram also in the same time: whence it follows, that if in a state of equilibrium we may represent any two forces by their virtual effects; that is to say, by the sides of a parallelogram which are proportional to these forces, and coinciding with their direction, their resultant will necessarily be denoted and exhibited, both in magnitude and direction, by the diagonal of the same; and, consequently, if we oppose to the two primitive forces another force equal in magnitude and direction to that denoted by the diagonal, the equilibrium will be established.

Demonstration by D. Bernoulli.
A.D. 1726.

Notwithstanding the extreme simplicity of this idea, it is difficult to demonstrate in a precise and satisfactory manner the principle which it involves. Dissatisfied with what had hitherto been given as a demonstration of this proposition, which blended into one point of view the composition of motion with that of the composition of forces, or as it is now more commonly expressed, the *parallelogram of forces*, Daniel Bernoulli undertook the investigation of the latter, entirely independent of any consideration of motion. And many other mathematicians, among whom are D'Alembert and Laplace, have demonstrated this same proposition by different methods. As the principle ought to be established before any farther advances can be made in Mechanics; and as we can, in fact, refer to no elementary treatise that exhibits it with the conciseness and the perspicuity with which it is desirable that it should be treated, we have in this article endeavoured to furnish an illustration of this difficult subject, on principles which we trust will be found to possess, in a certain degree, both the above characteristics.

To return again to the subject of variable motion, which involves the most interesting and delicate considerations of any of the branches of mechanical science; we may remark, that notwithstanding the important discoveries of Galileo and Huygens, little, comparatively speaking, had yet been achieved with regard to the general theory. Galileo had discovered, as we have stated, the properties of rectilinear motion uniformly accelerated; and Huygens had extended his investigations to curvilinear motion, which finally led to his theory of *central forces* in a circle, equally applicable to motion in any curve, by considering it as a series of infinitely small arcs of circles.

Fluxions applied to Mechanics.
A.D. 1704.

The laws of the communication of motion suggested by Descartes, and followed out and improved by Huygens, Wallis, and Wren, tended farther to the advancement of the science, by the solution that Huygens had given of the problems of the centres of percussion and oscillation. These discoveries being combined at length into a few general formulæ by means of the subsequent investigation of the fluxional or differential calculus, the science of Mechanics acquired a new degree of vigour, and was pursued with almost unbounded success. The phænomena of motion were reduced to two classes; containing the general problems of the motion of a single isolated body acted upon by any given force, and in any given direction; and the motion which results from the action and re-action that several bodies exert upon each other, according to any proposed law.

In isolated motion, it was obvious to remark, that matter being of itself indifferent either to motion or to rest, a body once put in motion would continue to move uniformly; and its velocity would suffer neither increase nor diminution, except by the instantaneous action of some constant or variable force: and hence resulted the two principles, viz. 1st. the force of inertia, and 2ndly, compound motion, upon which depends the entire theory of rectilinear and curvilinear motions, both constant and varying according to any given law. In consequence of the force of inertia, the motion at each instant is essentially rectilinear and uniform; abstraction being made of resistance and of every obstacle; and by the nature of compound motion, a body submitted to the action of any number of forces which tend at the same time to change the quantity and direction of its motion, will take such a direction in space that at the last instant it will arrive at the very place, which it would have reached, if it had yielded successively to each of the proposed forces.

In applying the former of these principles to rectilinear motion uniformly accelerated, we see, first, that in this kind of motion the velocity increasing by equal degrees, or being proportionate to the time, the accelerating force must be constant, and consequently the final velocity will be as the product of the accelerating force into the time; and secondly, that each elementary space passed over, being as the product of the corresponding velocity by the element of the time, the total space run over is as the square of the time.

Now these same two principles equally obtain for each element of any variable motion; for nothing need prevent our supposing generally, that the motion, although variable from one instant to another, is constant during each instant, or only receives its variations at the commencement of each element of time. Thus in every rectilinear motion, variable according to any law, the increment of the velocity is as the product of the accelerating force into the element of the time; and the second fluxion or differential is as the product of the accelerating force by the square of the element of the time.

If, now, with this principle we combine that of the composition of motion, we shall arrive at a knowledge of every species of curvilinear motion. In fact, whatever may be the forces applied to a body which describes any curve, we may at each instant reduce all these forces into two only, the one tangential, and the other perpendicular to the element of the curve. To the first of these, which is essentially rectilinear for every instant, we may apply the principle of the force of inertia; and to the second, which is expressed by the square of the actual velocity of the body, divided by the radius of the osculatory circle, the theory of central forces in a circle; which revert again to the same principle, *i. e.* the motion in the direction of the radius of the osculatory circle.

wton. Newton did not write fully upon the doctrine of Mechanics, considering it as a distinct topic: yet he laid down its first principles, as well as investigated several interesting mechanical problems, which naturally fell in his way, when establishing the nature and investigating the truths of general physics, in his "Principia," published in 1687. Newton also exhibited three celebrated axioms, or laws of motion; which now find their way, either explicitly or implicitely, into every treatise of Mechanics. They are these: ncipia. D.1687.

1. Every body continues in its state, whether it be of rest, or of uniform motion in a right line; unless it be compelled to change that state by extraneous forces.

2. The alteration of motion is always proportional to the motive force impressed, and is in the direction of the right line in which that force acts.

3. To every action there is always opposed an equal and opposite re-action.

The mechanical and other momentous verities developed in the Principia, were there given synthetically. Herman also proceeded after the synthetical manner in his "Phoronomia," a valuable work, though containing but few discoveries, published in 1716. erman. .D.1716.

Euler. Euler, in his Mechanics, employs the same principles for putting his problems into equations; but he follows throughout the analytical mode of investigation; which, by reducing all the branches of this doctrine to uniformity, greatly facilitates the connection of the different parts with each other. This work was published in 1736. .D.1736.

But although the method hitherto adopted of laying the foundation of the calculus was convenient, it was not the most perspicuous that might have been used. It is obvious, that instead of the processes employed by most of the foregoing philosophers, the forces and motions at every instant may be resolved into other forces and motions parallel to fixed lines, having determined positions in space. In which case it is only necessary to apply to these motions and forces the equations that involve the principle of the force of inertia, or, as it is commonly termed, the *vis inertia*, and there will then be no longer any occasion to have recourse to the theorem of Huygens.

aclaurin. Maclaurin, in his "Treatise on Fluxions," first suggested this happy idea, which threw a new light on the entire theory of Mechanics. But unfortunately this simple principle has been neglected by later English authors, and much of what our mathematicians at present know and practice of this method, we owe chiefly to the re-importation of it through the medium of modern French works; and many, perhaps, who are admiring the facility which is thus thrown into mechanical investigations of the greatest difficulty, are unconscious that this thought had its origin in our own land. The same fate has attended several other brilliant discoveries in the mathematical sciences. The doctrine of increments, due to Dr. Brooke Taylor; the limitation in the irregularities of the orbits of the planets, suggested by Simpson; various important analytical deductions discovered by Dr. Waring, and afterwards re-modelled into the form of the calculus of derivations; and many curious properties of rotatory motion of bodies about fixed axes, resulting from the profound investigations of Landen: are a few of the many instances that might be enumerated of the unpardonable neglect on the part of modern English mathematicians, in yielding to foreign authors the honour of discoveries which are legitimately due to ourselves; and which, for the sake of the scientific character of England, and in pure justice to the memory of these distinguished philosophers, it is our duty to assert and to defend. In stating this, however, we by no means wish to detract from the well-earned fame of the French mathematicians; we only regret that none of our countrymen should have been found to pursue the ideas thus suggested by Englishmen; and that the fruit should not have been matured in the same soil where the seed was first sown. .D.1742.

Returning from this digression, and referring still to the method of resolution proposed by Maclaurin; it is obvious, that when the body moves constantly in one plane, two fixed axes only will be necessary; and these for the sake of simplicity we may suppose at right angles to each other: but if by the nature of the forces the path of the body be continually changing, so as to describe a curve of double curvature, three axes must be employed perpendicular to each other, or forming the edges of a right angled parallelopipedon. This method of reducing a problem into equations is now commonly adopted by foreign mathematicians; and the generality which is thus given to mechanical investigations has been the means of obtaining the solution of numerous difficult questions which would in all probability have resisted every other mode of research.

Soon after Huygens first gave his solution of the problem of the centre of oscillation, to which we have above referred, his principle was objected to by many mathematicians; but it was defended by James Bernoulli, in the Leipsic acts for 1686, who there undertook to give a direct demonstration of it by means of the lever. He began by considering only two weights attached to an inflexible rod void of gravity, which was supposed to be put in motion on a horizontal fixed axis. Observing then that the velocity of the weight, nearest to the axis of rotation, must necessarily be less, and that of the other greater, than if each acted on the rod separately, he came to this conclusion; that the force lost and the force gained balance each other, and that consequently the product of the quantity of matter in the one into the velocity lost, and that of the other into the velocity gained, must be inversely proportional to the arms of the lever. This reasoning, though accurate in the main, involved a defect which was first pointed out by de l'Hopital, viz. that James Bernoulli began by considering the velocities of the two bodies as finite; instead of which he ought only to have considered the elementary velocities, and have compared them with James Bernoulli. A.D.1686. l'Hopital.

the similar velocities produced every instant by the action of gravitation. The latter author, adopting this method, without departing in any other respect from Bernoulli's principle, arrived at the same conclusion for *two* bodies; and in order thence to proceed to a third he united the two former at their centre of oscillation, and again proceeded as with two weights, which he afterwards combined in the same manner, and so on. This induced Bernoulli to revive his former solution, in order to extend it generally to any number of bodies; a problem which he finally accomplished, by resolving the motion of each body at every instant into two other motions; the one, that which the body actually takes, and the other that which is destroyed; and thus forming equations which express the condition of equilibrium between the motions lost: by these means the problem is brought within the range of the ordinary laws of statics. The author applied the same principle to several examples, and demonstrated strictly, and in the most evident manner, the proposition which Huygens employed as the basis of his solution, and which, since that time, has been commonly denominated the *principle conservatio virium vivarum, i. e. the conservation of living forces;* which may be expressed shortly in the following terms: "In the motion of any system of bodies, the sum of the products of the masses by the squares of the velocities at every instant is the same, whether the bodies have descended conjointly in any manner whatever, or have each descended freely through the same vertical heights." And as Bernoulli, after the example of Leibnitz, had assumed the mass into the square of the velocity, as the measure of a force, we see, in the above, the derivation and origin of the denomination "conservatio virium vivarum" given to this principle; though it is not strictly correct with reference to the measure of force now generally adopted. The propriety of the expression however is of but little importance; while the principle which it involves is of great utility, as it affords us the means of immediately forming an equation between the velocities of the bodies and the variable quantities which determine their position in space; so that when, by the nature of the problem, these variable quantities are reduced to one, this equation, is alone sufficient for its solution.

Virtual velocities.

Besides the principle of the conservation of the *vis viva*, other general properties or principles have been proposed, both prior and subsequently to this, the origin of some of which we have had occasion to notice in our preceding remarks; viz. the principle of virtual velocities; the conservation of the motion of the centre of gravity; the conservation of equal areas; and the principle of least action.

The former, *i. e.* the principle of virtual velocities, we have already alluded to, in speaking of the discoveries of Galileo; but it was not treated by that author in all the generality which it has since attained. According to the present signification of this expression, it asserts, that "If a system, composed of any number of bodies or points, which are drawn in any direction, by any forces, be in equilibrium; and a small motion be given to the system, by virtue of which every point describes an infinitely small space, which will express its virtual velocity; then the sum of the forces, multiplied each by the space described by the point to which it is applied, in the direction of the force, will be equal to zero; estimating as positive the small spaces described in the direction of the forces; and as negative, those described in a contrary direction."

When the forces on a point or a body, or a system of bodies, are not so proportioned as to maintain the system in equilibrio, motion necessarily takes place; and the laws of this motion may be deduced by extending the principles which are employed in the investigation of the state of equilibrium. This is the method pursued by Lagrange, in his "Mecanique Analytique," and subsequently by Laplace. Lagrange, with the principle of virtual velocities, combines what is commonly called the principle of D'Alembert, which is extremely simple, and may indeed be considered as an axiom; or we would rather say, a practical modification of Newton's 3d law.

D'Alembert.

A.D.174

It is this: "If several bodies have a tendency to motion, with velocities and in directions which they are constrained to change, in consequence of their reciprocal action on each other; then these motions may be considered as compounded of two others: *i. e.* one which the bodies actually take, and the other such, that had they been acted upon by these alone, they would have remained in equilibrio." Here, since the force by which the second motion is destroyed, arises merely from the connection between the bodies, whatever such force may be, the body effected by it will, by virtue of the 3d law of motion, exert an equal re-action on the other body or bodies of the system, and destroy an equal quantity of motion. D'Alembert's proposition simply equates these equal quantities of motion. This observation we regard as due to the impartiality of historical deduction.

Conserva tion of th centre of gravity.

The principle of the conservation of the motion of the centre of gravity, to which we have already made some reference, is that which Newton has adopted in his Principia. He there demonstrates that the state of repose or of motion of the centre of gravity of several bodies is not altered by the reciprocal action of the bodies on each other in any manner whatever; so that the centre of gravity of bodies which act upon each other, either by cords or levers, or by the laws of attraction, independently of any exterior action or obstacle, remains always in repose, or moves uniformly in a right line.

D'Alembert has also extended this theorem, by showing, that if every body of the system is solicited by a constant accelerating force, acting either in parallel lines or directed towards a fixed point, and varying with the distance; the centre of gravity will describe the same curve as if the bodies were free. It is evident that this principle serves to determine the motion of the centre of gravity independently of the respective motions of the bodies.

Conserv tion of equal a

A D.17

As to the third of the above principles, viz. the conservation of equal areas, it is of much more recent date than those to which we have already referred, and appears to have been discovered simultaneously by Euler, Bernoulli, and the chevalier D'Arcy, but under different forms. According to the two former, the principle consists in this: that in the motion of several bodies about a fixed axis, the sum of the products of the mass of each body, by the velocity of rotation round the centre, and by its distance from the same centre, is always independent of any mutual action which the bodies may exert on each other, and preserves itself the same as long as there is no exterior action or obstacle. Daniel Bernoulli gave this principle in the first volume of the memoirs of the academy of Berlin in 1746, and D'Alembert the same year in his "Opuscula." The principle of D'Arcy, as given in the Memoires de l'Academie des Sciences for 1746, but not published till 1752, is this: that the sum of the products of the mass of each body, by the area traced by its radius vector about a fixed point, is always proportional to the time. This is obviously only a generalization of one of the laws of Kepler, or the beautiful theorem of Newton of the equalities of areas described by centripetal forces: and to perceive its analogy, or rather its identity with that of Euler and Bernoulli, it is sufficient to recollect that the velocity of rotation is expressed by the element of the circular arc, divided by the element of the time, and that the first of these elements, multiplied by the distance from the centre, gives the element of the area described about this centre; so that this last principle is only the differential expression of that of M. D'Arcy. The author afterwards gave this principle another form, which renders it still more analagous to the preceding: *i.e.* "that the sum of the products of the masses into the velocities, and into the perpendiculars drawn from the centre to the direction of the forces, is always a constant quantity."

Conservation of action.

Under this point of view, the author established a species of metaphysical principle, which he calls the conservation of action, as a convenient substitute for the principle of least action. This principle, viz. of the conservation of action, whether we choose to consider it as an ultimate cause, or as a necessary consequence of some other ultimate cause, takes place in every system of bodies which act on each other in any manner whatever, whether by cords, inflexible lines, attractions, &c. or whether solicited by forces directed to a centre; and the same obtains, whether the system be entirely free or constrained to move about the same centre: *i. e.* "the sum of the product of the masses into the areas described about the centre, and projected on any plane whatever, is always proportional to the time; so that referring these areas to three rectangular planes, three differential equations are obtained of the first order, between the time and the co-ordinates of the curve described by the bodies;" and it is properly in these equations, that the nature of the principle exists.

Principle of least action.
A.D.1744.

The fourth of the principles above enumerated, viz. the principle of least action, so called by Maupertius, and which the writings of several eminent authors have since rendered celebrated, considered analytically, is as follows: "that in the motion of bodies which act on one another, the sum of the products of the masses into the velocities, and into the spaces described, is a minimum." From this principle, the author has deduced the laws of reflection and refraction of light; and Laplace has lately shown its application in investigations relative to the properties of doubly refracting crystals.

It ought to be observed, however, that these applications are too partial to establish the truth of a general principle, besides that they are too vague and arbitrary, which renders the consequences uncertain with regard to the principle itself. It ought not therefore to be classed with the other principles above discussed; although there is one point of view in which it may be considered more general than any of them, and which therefore merits the attention of philosophers. Euler gave the first indication of this in a treatise on Isoperimetrical problems printed at Lausanne in 1744; showing that in trajectories described by central forces, the integral of the velocity, multiplied into the element of the curve, is always a maximum or a minimum.

Lagrange.
A.D.1788.

This principle was only known to Euler as appertaining to isolated bodies: but Lagrange afterwards extended it to the motion of bodies which act on each other in any manner whatever; from which there results this new general principle, viz. the sum of the products of the masses into the integrals of the velocities, multiplied into the elements of the spaces described, is constantly a maximum or a minimum. By combining this with the principle of *vis viva*, a solution of many difficult problems in dynamics may be obtained.

Such is a general outline of the chronological developement of the principles of Mechanics, at least so far as the subject can be treated without involving in our sketch much that relates to the differential calculus: for these two subjects are so intimately connected in many respects, or at least all investigations of the higher kind in dynamics, are so essentially dependent upon the principles of the modern analysis, that it is impossible to enter into a more minute illustration of the former, without embracing more of the latter than properly belongs to an article which professes only to give a concise view of the rise, progress, and present state of the mechanical sciences.

Principal works.

The preceding sketch of the history of Mechanics relates rather to the successive improvements that have been introduced into that science, than to an account of the several works which have been written on the subject: we propose, however, in conclusion, to enumerate some of the most important of these; those to which a reader may with confidence refer for the most accurate information.

Among English works of this kind, are chiefly to be noticed Emerson's Principles of Mechanics, 4to.: Gregory's Treatise of Mechanics, 3 vols. 8vo.: Bridge's Treatise of Mechanics, 8vo.: Parkinson's System of Mechanics, 4to.: and Atwood on the Rectilinear Motion and Rotation of Bodies, 8vo. Emerson's work, besides the theory, contains a variety of information relative to the construction of many useful machines; but Gregory's Mechanics has in a great measure superseded it, by extending both the theory and practice of the science to a length unknown in the days of the former author. For acquiring the first principles of the science, the student might probably consult to advantage Mr. Bridge's book, particularly on account of the numerous practical examples with which it abounds. Parkinson's Mechanics, and Atwood on the Rectilinear and Rotatory Motion of Bodies, are each highly respectable performances, particularly the latter; although the number of press errors with which it is crowded will be found a great inconvenience to the student.

The above are the principal English works to which we think it requisite to refer: and of foreign authors the following are, we believe, all that it is necessary to enumerate, viz. Euler, "Tractatus de Motu:" Lagrange, "Mecanique Analytique;"—a work of great merit and originality, but which can only be advantageously consulted by such as are already well read on the subject: Prony "Mecanique Analytique," and "Architecture Hydraulique;" to which we ought also to add the treatises of Bossut, Poisson and Boucharlat: Poisson's work in particular, which is contained in two octavo volumes, we consider to be a very valuable performance, exhibiting in a concise and accurate manner, a complete view of the present improved state of the mechanical sciences.

§ II. STATICS.

Definitions, &c.

Definitions, &c.

1. *Statics*, as we have already stated, is that branch of Mechanics which treats of the equilibrium of solid bodies.

2. *Force*, or *power*, is used to denote any cause or agency, which has a tendency to put, or which actually puts a body at rest into motion, or which has a tendency to change or destroy, or which actually changes or destroys that motion which a body may already possess.

Forces may be divided into three distinct classes, viz. impulsive, uniform, and variable.

3. *Impulsive forces*, are those which act but for an instant, after which they cease to have effect.

4. *Uniform forces*, are those which act constantly, or incessantly, with the same degree of energy; and these are further denominated *accelerating* or *retarding* forces, according as they are employed in the production or in the destruction of motion.

5. *Variable forces* are those which act incessantly, but every instant with different degrees of energy.

It is, however, only the first and most general of the above definitions which appertains to the theory of Statics.

6. A force acting on a material point may either attract or draw towards it, or repel or push from it; the former of these is called an *attracting*, and the latter a *repelling* force; but in the following pages, unless the contrary be expressed, an attractive force is to be understood.

7. The effect of a force depends, 1st, upon its intensity; 2d, upon its point of application; 3d, upon the direction in which it is estimated; and 4th, upon its own direction with reference to that in which the estimation is made.

8. The *intensity* of a force is its greater or less faculty of producing or destroying motion.

9. A material point being urged by any single force, ought necessarily to move in a right line; for no reason can be assigned for its deviation, either to the one side or the other. Or whatever reason we may imagine for a change of direction on the one side, will apply equally to a symmetrical change on the other; whence, as the body cannot move in both these directions at the same time, we conclude, that it will follow neither, and that its entire direction will be a right line.

10. And the right line in which a force acts, is called its *direction*.

11. When two forces act upon a material point, or an inflexible right line in equilibrio, and these forces are directly opposed to each other, we may assume either of them arbitrarily, provided we give to the other the same intensity; and the same remark will apply to any number of forces. Whence it appears, that it is the ratio of forces, and not their absolute quantity, that is necessary to be considered, in order to establish the condition of equilibrium.

12. Having assumed any force for *unity*, another force is said to be equal to this, when, being directly opposed to it, it retains a material point upon which they both act in equilibrio.

13. Two equal forces applied to the same material point, and acting in the same direction and in the same manner, constitute a double force; and three such forces, a triple force; and so on.

14. The unit of force being arbitrary, we may estimate it by any part of the right line which expresses its direction.

15. When a force is applied to a body, all the points or particles of which are immoveably connected with each other, it is obvious that one point cannot move without a corresponding change in all the others; and consequently a force applied to any point, will have the same effect as if it were applied at any other point taken in the direction of that force, viz. A B C D (fig. 1, plate 1, STATICS), representing any body, and EF the direction of any force acting upon it, the effect of this force will be the same, whether we suppose it attached to the point E, or to any other point, M or N, in the same line of direction.*

It follows from the preceding articles, that if in the line of direction we place an immoveable obstacle, the force can have no effect.

§ III. *On the method of determining the direction of a force in space.*

Directi forces.

16. We know, from the principles of geometry, that the position of a point situated in a given plane, or any point in a plane curve, may always be determined by means of two rectangular co-ordinates, which we commonly call the absciss and ordinate; but that to fix the position of a point in space, or to determine any point in a curve of double curvature, three such axes become necessary, and three only are sufficient in all cases.

In the same manner, the direction of any force acting in a given plane, will be determined by referring it to any two rectangular axes passing through its point of application; and when the plane in which it acts is not given, its direction may still be determined by means of three such rectangular axes.

Let MD (fig. 2) denote the direction of any force, M being its point of application; through which draw the three axes M x, M y, M z, parallel to the co-ordinates which determine the position of the point D, and which we will suppose to be rectangular; then if the angles α, β, γ be known, which the line of direction MD makes with each of those axes, its position will be determined; for there is a necessary relation between these angles, such that whatever may be the value of two of them, the third always depends upon the equation,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$

radius being assumed equal to unity.

That is, the angles α and β being given, the angle γ is determined from the equation

$$\cos\gamma = \pm \sqrt{\{1 - \cos^2\alpha - \cos^2\beta.\}}$$

The preceding relation may be demonstrated as follows. Let x, y, z (fig. 3) be taken to denote the three co-ordinates MP, PQ, QD, of any point D, taking in the direction MD of the given force: then we shall have in the triangle MDQ, which is right angled at Q,

$$MD^2 + QD^2 = MD^2:$$

Again, in the triangle MQP, right angled at P,

$$MQ^2 = MP^2 + QP^2;$$

and substituting this value in the preceding equation, we obtain

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* The reader, however, must here divest himself of any idea of resistance or weight in the body; for this being supposed, it will have a motion of rotation, a consideration which belongs to the doctrine of Dynamics.

$$MP^2 + QP^2 + QD^2 = MD^2, \text{ or}$$
$$X^2 + Y^2 + Z^2 = MD^2.$$

But the triangle MDP, being right angled at P, we have

$$x = MD \cos \alpha;\ y = MD \cos \beta,\ z = MD \cos \gamma;$$

consequently, substituting these values for MP, QP, and QD, we have

$$MD^2 \cos^2\alpha + MD^2 \cos^2\beta + MD^2 \cos^2\gamma = MD^2;$$

or, dividing by the common factor MD², we obtain

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

We have seen that from this equation we immediately deduce the following, viz.

$$\cos\gamma = \pm \surd \{1 - \cos^2\alpha - \cos^2\beta,\}$$

where the double sign with which the radical is affected, indicates that the cosine of the third angle may be arbitrarily assumed positive or negative, which indetermination ought therefore to be obviated by the conditions of the problem; viz. we ought thence to be able to determine whether the third angle is acute or obtuse; for then, in the first case we know that the sign must be taken positive, and in the second negative.

n the signs f the co-ines.

17. As this method of fixing the direction of a force (although, as we have stated in our historical sketch, it is due to Maclaurin) is but little practised by English mathematicians, and as the determination of the signs of the cosines is of great importance, we shall enter a little minutely into this question.

If the line of direction AB (fig. 4) be in the plane of $x\,y$, it is obvious that in order to determine its position, it would be sufficient to increase the angle A from 0 to 360°, in the course of which the revolving line must pass through the direction, and the sine and cosine of this angle only would be sufficient to determine the position of AB; but if, instead of imagining only one axis A x, we refer AB to two axes A x, A y, and denoting the angles which they respectively form with it by α, β, it will only require each of those angles to be increased from 0 to 180°, in order that one of the lines shall pass through AB; viz. the angles by which we refer the direction AB to each of these axes will be less than 180°, or at least cannot exceed that measure; and farther, if we give to AB all the positions it can have with regard to the four right angles formed about the point A, (as in figs. 5, 6, 7, 8) we shall have—in the first case, the angles α and β both acute, and consequently $\cos\alpha$ and $\cos\beta$ both positive: in the second case, the angle α is obtuse and β acute, and therefore $\cos\alpha$ negative and $\cos\beta$ positive: in the third case, α and β being both obtuse, the $\cos\alpha$ and $\cos\beta$ will be both negative: and in the fourth case, α being acute and β obtuse, $\cos\alpha$ will be positive and $\cos\beta$ negative. Now it must be observed, that the signs of these cosines are those also of the co-ordinates x and y of the point B; for example, when the point B is situated in the angle x A y, (fig. 6) x is negative and y positive, and we have also at the same time $\cos\alpha$ negative and $\cos\beta$ positive.

18. When the line of direction is not in the plane of $x\,y$, the angle γ, which it forms with the axis A z, will be acute if it fall above this plane, and obtuse if it fall below, and consequently $\cos\gamma$ will be positive in the first case, and negative in the second. With regard to the signs of the cosines of the other angles, as there are four quarters above and four below the plane of $x\,y$, the signs of the cosines of the angles α and β will be the same as those of the co-ordinates x and y of the point of application of the force; as follows immediately from what is stated above.

It may be farther observed, that the variation in the signs only takes place in the cosines of the angles; for as we have shown that neither of these angles ever exceeds 180°, or two right angles, the sines are necessarily positive.

§ IV. *Of the composition and resolution of forces applied to a material point.*

Of the composition of forces.

19. If two forces act upon a material point in directions diametrically opposite, and these forces are equal, they will destroy each other's effects, and the point will be in a state of equilibrium; but if the forces are unequal, the greatest will preponderate, and the point will be acted upon by a force, the intensity of which is equal to the difference between the two former; and the same will apply to any number of forces.

This follows immediately from our preceding definitions.

20. When two forces, acting upon a material point, form between their lines of direction any angle, an equilibrium cannot subsist, whatever may be the ratio of their intensity: for let us imagine P and Q (fig. 9) to be two forces acting upon the point M, and forming between them the angle PMQ; and let us suppose that they are if possible such as to produce an equilibrium at the point M: now since, by the supposition, the point M is in equilibrio, it is obvious that if we introduce into the system a third force, P′, equal and directly opposite to the force P, motion will ensue in the direction of this third force P′; but since P and P′ are equal and opposite forces, these will alone produce an equilibrium in the point M, and consequently motion will ensue in the direction of the force Q; viz. under the same circumstance, the point M will have a motion in two directions at the same time, which is absurd. Consequently two forces acting upon a material point, and forming between them any angle, cannot produce an equilibrium.

21. Since an equilibrium cannot subsist between two forces forming between them any angle, motion will ensue, and there must be some force whose intensity and direction is exactly such as would produce that motion; and this latter force is said to be the *resultant** of the two former, and the former the *components* of the latter; and the determination of the relation between these quantities is the proposition commonly called the *Parallelogram of Forces*.

Of the parallelogram of forces.

22. In the first place, it may be observed, with reference to this determination, that if any two forces act upon a material point, their resultant will be in the plane of those forces: for whatever reason can be assigned to show that the resultant will pass through

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* *Resultant* is not strictly speaking an English word, nor is it properly a correct substitute for the French word *Resultante*, which is merely an adjective, or rather the active participle, used substantively, which the principles of the French language will frequently admit of: the meaning in French is *force resultante*, which in English therefore requires *resulting force*; but as the constant repetition of the word force would be irksome, it has been the practice of English writers to convert the expression into one single substantive, *resultant*. A similar practice is now commonly adopted in speaking of variable, constant, and arbitrary quantities; where, instead of saying the variable quantity, the constant quantity, &c. we drop the substantive, and simply write the *variable*, the *constant*, the *constant arbitrary*, &c.

a point above that plane, may be equally applied to prove that it will pass through a point symmetrically situated below the same plane; these forces would therefore have two resultants, that is, the body would move in two directions at the same time, which is absurd; the resultant is therefore neither above nor below, but in the plane itself, of the two forces.

Two equal forces.

23. If two *equal* forces PQ, (fig. 10,) form between them any angle, the resultant of these forces will be in the direction of the right line MR, which bisects the angle PMQ of this direction.

For whatever reason can be conceived for the resultant passing in any other direction, as for example through the point *r*, situated on one side of the bisecting line MR, the same reason may be assigned to show that it will pass through the point *s* symmetrically situated on the other side; the resultant therefore can be in neither of these directions. And the same may be demonstrated for every line but MR; this line is therefore in the direction of the resultant of the two equal forces P and Q: and since the diagonal of a rhombus bisects its angle, if we construct upon the two equal lines representing these forces, the rhombus or equilateral parallelogram PMQR, the resultant will be in the direction of that diagonal.

This being established, it only remains now to determine its magnitude or intensity, in which we shall adopt the method followed by M. Poisson, in the first volume of his "Traité de Mecanique."

24. Let mA, mB, (fig. 11) be the direction of any two equal forces, or components, the common value of which let be denoted by P, and let $2x$ denote angle AMB, and m C the direction of the resultant whose intensity is required; which as it bisects the angle A m B, will give the angle $A\,m\,C = B\,m\,C = x$.

Now the intensity of this force can depend only upon the quantities P and x, of which it is therefore some unknown function; if therefore the resultant be denoted by R, we shall have R equal to some function of P and x, that is $R = f, (P, x.)$

In this equation, R and P are the only quantities of which the numerical expression varies with the unit of force; and their ratio therefore is wholly independent of that unit; whence we may conclude that $\frac{R}{P}$ is a simple function of x; and consequently that the function $R = f\ (P, x)$ is of the form $R = P\ \phi\ x$; and the question is therefore reduced to the determination of the form of this function.

In order to effect this, let us draw through the point m, the four lines m A′, m A″, m B′, m B″, such, that the four angles A′ m A, A″ m A; B′ m B, and B″ m B, may be equal to each other, and let this angle be represented by z; this done, let us suppose the force P, acting in the direction m A, to be the resultant of two equal forces acting in the direction m A′, m A″, but of which the value is unknown; then, if we represent each of these two unknown and equal forces by Q, we shall have, on the same principles as above,

$$P = Q.\ \phi\, z;$$

for there must exist the same relation between the three quantities P, Q, and z, as between the three R, P, and x. In the same manner, the equal force P, acting in the direction m B, may be supposed to be the resultant of two equal forces acting in the direction m B′, and m B″; and thus the two original equal forces P, P, are replaced, or are equivalent to the four equal forces Q; and con- consequently the resultants in both cases are the same.

Now if we call Q′ the resultant of the two equal forces Q, acting in the direction m A′, m B′, and observing that the angles $A'\,m\,C = B'\,m\,C = x - z$, we shall have again

$$Q' = Q.\ \phi\ (x - z.)$$

In the same manner, the resultant of the two forces Q, which act in the direction m A″ and m B″, being denoted by Q″, we shall also have

$$Q'' = Q.\ \phi\ (x + z.)$$

Here the two forces Q′ and Q″, acting in the same direction m C, their resultant, which is that of the four equal forces Q, and of the two equal forces P, is equal to their sum; that is, the resultant $R = Q' + Q''$;

$$\text{or } R = Q.\ \phi\ (x - z) + Q.\ \phi\ (x + z.)$$

But we have seen that $R = P.\ \phi\ x$, and $P = Q.\ \phi\ z$; consequently $R = Q.\ \phi\ z.\ \phi\ x$;

and therefore,

$$Q.\ \phi\, z.\ \phi\ x = Q.\ \phi\ (x - z) + Q.\ \phi\ (x + z), \text{ or}$$

$$\phi\ z.\ \phi\ x = \phi\ (x - z) + \phi\ (x + z.)$$

And it therefore now only remains to determine what functions of x and z possess this property; viz. that will establish the above equalities.

Now we know from the principles of trigonometry, that

$$\cos x.\ \cos z = \tfrac{1}{2}\cos (x - z) + \tfrac{1}{2}\cos (x + z,) \text{ and}$$

multiplying this equation by 4, we have

$$2 \cos x.\ 2 \cos z. = 2 \cos (x - z) + 2 \cos (x + z;)$$

and comparing this with our preceding equation,

$$\phi\, x.\ \phi\, z = \phi\ (x - z) + \phi\ (x + z,)$$

it is obvious that we obtain

$$\phi\, x = 2 \cos x,\ \ \phi\, z = 2 \cos z.$$

Whence the equation

$$R = P.\ \phi\, x,$$

becomes

$$R = P.\ 2 \cos x = 2\,P \cos x;$$

which is obviously equal to the diagonal of the parallelogram whose four sides are each equal to P, (as in fig. 10) and whose angle is bisected by that diagonal; viz. $PMQ = 2\,x$.

Consequently the resultant of any two equal forces is represented, both in magnitude and direction, by the diagonal of the rhombus, or equilateral parallelogram, constructed upon the lines denoting the magnitude and direction of these two forces.

Two unequal for

25. We may now easily demonstrate that the resultant of any two unequal forces is exhibited, both in magnitude and direction, by the diagonal of the parallelogram described upon the two lines by which their intensity and directions are represented.

Let AB, AD, (fig. 12) denote any two unequal forces, and let AC be the diagonal of the parallelogram upon these lines; then will AC be their resultant as well in magnitude as direction.

Bisect AC by the perpendicular OP, and join AP; bisect AP by the perpendicular O′P′, and join PP′; again bisect P′P by the perpendicular O″P″; and so on: then it is obvious, since each successive point P″, falls between the two preceding ones P′, P′, there will be a continual approximation of these points to the angular point D; and consequently, by continuing the operation, we shall have ultimately one of those points which either falls into the angle D, or which approaches to it indefinitely near. Let P″ be this point; then because P′P″

$= PP''$, $P'P$ will be resultant of two forces expressed by $P'P''$ (art. 24); and because $AP' = P'P$, AP will, in like manner, be the resultant of the two forces AP' and $P'P$, or of the two forces AP'' and $P''P$. In the same manner, because $AP = PC$, AC will be the resultant of the two forces AP, PC, or of the three forces AP'', $P''P$, and PC, which are obviously equal to the two forces AP'', $P''C$, or to the two AD and AC, because the point P'' coincides with D, and $DC = AC$ being opposite sides of a parallelogram.

Consequently the resultant of any two forces applied at the same point, and represented by lines taken in their direction, and both departing from the point of application, is represented, both in magnitude and direction, by the diagonal of the parallelogram constructed on those forces.

26. *Remark.* The preceding proposition, which is generally spoken of under the term *parallelogram of forces*, is justly considered one of the most important in the theory of statics, and the demonstration of which has engaged the attention of all the most celebrated geometricians since it was first attempted by Daniel Bernoulli, in the Acta Petro. 1726; but many of these are so extremely long and abstruse, that it is impossible, whatever may be their accuracy, to introduce them into a popular theory of mechanics such as we have here in view: we have endeavoured, therefore, by blending together in some measure the principles of different writers, to obtain a concise and satisfactory demonstration of this celebrated proposition. The determination of the *direction* of the resultant of two equal forces, we conceive to be wholly unobjectionable; and in determining the magnitude of the same, in which we have in part followed M. Poisson, we can see nothing which ought not to be perfectly intelligible to any reader; for although it involves, in some measure, considerations relative to the calculus of functions, they are so simple and obvious as to be readily comprehended by any one who has merely a clear idea of the signification of that expression; for we have followed this author no further than where he brings out the equation

$$\phi x . \phi z = \phi (x - z) + \phi (x + z),$$

which we then show is equivalent to the expression

$$2 \cos x . 2 \cos z = 2 \cos (x - z) + 2 \cos (x + z,)$$

by a simple reference to the well-known trigonometrical formula involving this equation; and by this means we have avoided the developement of the above functions, and the successive differenceations, which were the most likely to impede the progress of a student. As to the latter part of the demonstration, by which, from knowing the resultant of two equal forces, we pass to the determination of the same for any two unequal forces, it is, we believe, new; and whatever may be its defects, they belong exclusively to ourselves.

Trigonometrical relation between the components and resultant.

27. One of the first consequences that we may draw from the preceding proposition, is the trigonometrical relation between the two forces P and Q and their resultant R. In order to which, let us take, on the direction of these forces, (fig. 13) the lines AB, AC, proportional to P and Q respectively; construct the parallelogram ABCD, and draw the diagonal AD, and we shall have

$$P : Q : R :: AB : AC : AD :$$

but the side BD being equal to AC, we may, instead of the three lines AB, AC, AD, consider the relation of the three sides of the triangle ABD; whence we shall have

$$P : Q : R :: AB : BD : AD :$$

and as the sides of triangles have the same ratio as the sines of their opposite angles, this latter analogy may be transformed into

$$P : Q : R : \sin BDA : \sin BAD : \sin ABD,$$

where it will be observed that each of these three forces is proportional to the sine of the angle included between the other two.

28. It may also be observed, that the three forces P, Q, R, are not only proportional to the three sides of the triangle ABD, but that they are also proportional to the sides of any triangle formed by lines drawn, so as to form each respectively the same angle with the three sides AB, BD, DA, of the triangle BDA; because it may be demonstrated geometrically, that every such triangle will be similar to BDA, and will therefore have its sides in the same proportion; that is, they will be proportional to the three forces P, Q, R.

29. It results from the above, that three of the six parts which constitute the *triangle of forces* ABD being given, the rest may be found, provided only that one side is included amongst the data.

For example, suppose the two components AC and BC, and the angle of direction BAC, were given to find the resultant DA: here BAC is the supplement of ABD; therefore in the triangle ABD, we know the two sides AB, BD, and the contained angle B, to find the third side, or resultant, DA; and we know from the principles of trigonometry, that

$$\cos B = \frac{BA^2 + BD^2 - DA^2}{2 BD . BA :}$$

whence

$$DA^2 = BA^2 + BD^2 - 2 \cos B . \ BD . \ BA ;$$

that is,

$$R^2 = P^2 + Q^2 - 2 PQ . \cos B ;$$

or,

$$R = \surd \{P^2 + Q^2 - 2 PQ . \ \cos B.\}$$

And on the same principles we have

$$P = \surd \{R^2 - Q^2 + 2 RQ . \cos B.\}$$
$$Q = \surd \{R^2 - P^2 + 2 RP . \cos B.\}$$

These last formula apply to what is commonly called the resolution of forces; viz. to the finding of two forces equivalent to any one given force; in which case it is obvious that either of the required forces may be assumed at pleasure, provided only that it is less than the one proposed.

When the angle B is a right angle, then $\cos B = 0$, and the above becomes simply

$$P = \surd \{R^2 - Q^2 \}$$
$$Q = \surd \{R^2 - P^2 ;\}$$

which is again the same, as

$$P = R \cos BAD,$$
$$Q = R \cos BDA.$$

We shall have frequent occasion for these formulæ in our subsequent enquiries.

The composition of many forces.

30. It is obvious that we may also find the resultant of any number of forces all meeting in a common point: we have for this purpose merely to find, first, the resultant of any two of these forces, and to substitute this one force instead of these two; we must then find the resultant of this new force, and one of the remaining ones of the others, and substitute this; then find the resultant of this last and another, and so

on till the forces are all reduced to one force only, which will be the resultant sought.

Geometrical construction.

31. A geometrical construction of this proposition may be given as follows:

Let P, P', P'', &c. (fig. 14) represent any number of forces, all meeting in the common point A; and let AP, AP', AP'', be taken proportional to these forces respectively: through P draw PR' parallel and equal to AP'; through R', draw $R'R''$ parallel and equal to AP''; and through R'', draw $R''R'''$ equal and parallel to AP''', and join AR'''; this last line is the resultant of the four given forces AP, AP', AP'', and AP''', as is obvious: for by art. 25, AR' is the resultant of the two forces AP, AP'; AR'' is the resultant of the two AR' and AP'', or of the three forces AP, AP', and AP''; and in the same manner AR''' is the resultant of the four forces AP, AP', AP''; and AP''', and so on for any number of forces.

32. It follows from this, that if the last point $R^{(n)}$, in any number of forces, fall in the point A, the resultant will be zero; which shows that in such a case the several forces mutually balance and destroy each other, and that the point A is in a state of equilibrium.

This figure is frequently called the *polygon of forces.*

Forces not in the same plane.

33. We may employ a similar construction for ascertaining the resultant of three forces not situated in the same plane, but which concur in one common point. Let AP, AP', AP'', (fig. 15) represent three such forces; and on any two of them, as AP, AP', construct the parallelogram $APQP'$, and join AQ, so shall AQ be the resultant of these two forces; now draw QR equal and parallel to AP'', and join AR, which will be the resultant sought.

It is obvious from this construction, that the resultant AR is the diagonal of the parallelopiped constructed upon the three forces; on which account this is sometimes denominated the *parallelopipedon of forces.*

When the three forces are at right angles to each other, the parallelopipedon will also be rectangular, as in fig. 16, where the three forces are represented by the three contiguous edges of the solid, viz. AC, AB, AD, and their resultant by AF, which is obviously the diagonal of the parallelopipedon ADEF.

It is evident from the above figure, that

$$AE^2 = AB^2 + AC^2, \text{ and that}$$
$$AF^2 = AE^2 + AD^2, \text{ whence}$$
$$AF^2 = AB^2 + AC^2 + AD^2.$$

Also, if we denote angle FAB, by a, FAC, by b, and FAD by c, that is to say, the angle which the resultant makes with the three original forces, we shall have

$$AB = AF \cos a,\ AC = AF \cos b,\ AD = AF \cos c;$$

whence, by substitution,

$$AF^2 = AF^2 \cos^2 a + AF^2 \cos^2 b + AF^2 \cos^2 c;$$

or, dividing by AF^2, we have

$$\cos^2 a + \cos^2 b + \cos^2 c = 1.$$

We shall have occasion to refer to this equation in some of our subsequent investigations.

§ V. *Of the estimation of forces applied to the same point and situated in the same plane, by means of rectangular co-ordinates.*

Of the method of rectangular co-ordinates.

34. Our preceding investigations are well adapted for conveying an idea of the composition and resolution of forces, because they speak to the eye and to the mind in the most obvious and convincing manner; but still they are not well suited to the purposes of computation. We have seen that every resultant of two forces is found from the formula

$$R = \sqrt{\{P^2 + Q^2 - 2\,PQ \cos B\}} \qquad \text{(art. 29)}$$

But this merely gives the magnitude or intensity of it, and we have still to find its direction; and the same must be repeated for every pair of forces, including the last found force as one of them; consequently any number of forces whose numerical ratios, and the angles which they subtend with each other being given, to find the numerical value, and the angle of direction of the resultant, involve, according to the principles above explained, a very long and laborious calculation. In order to avoid this, the method of rectangular co-ordinates is now generally adopted, which, in the case of two forces, consists in drawing, in the same plane with the forces, two right lines, forming between them a right angle; and instead of denoting the angles which the forces make with each other, we only consider the angles which they each make with these lines; each force is then resolved into two forces in the direction of those axes, by which means the computation and determination of the resultant, as well as its direction, is much facilitated; and the method of effecting this is what we now propose to illustrate.

35. Let P, P', P'', P''', &c. (fig. 17) represent any number of forces, all situated in the same plane, and meeting in one common point A; let us draw through this point the rectangular axes Ax, Ay; then representing the several forces P, P', P'', &c. by proportional parts of their direction AP, AP', AP'', &c. we may decompose each of these forces into two others in the direction of the two axes Ax, and Ay. In order to effect this, let α, α', α'', &c. denote the angles which the forces P, P', P'', &c. make with the axis Ax, and β, β', β'', &c. the angles which they respectively make with the axis Ay.

Now from the principles of trigonometry we know that in any right angled triangle, as ABC, (fig. 18) the side adjacent to either acute angle, as AC is denoted by AB cos A, and the side opposite the same angle by AB sin A, we shall therefore find it very easy to compute the components of P, P', P'', &c. as referred to our two axes; for the force P, being represented by the hypothenuse AB, it makes with the axis Ax, an angle α, and with the axis Ay, an angle β; the components of P will therefore be denoted by

$$AC = P \cos \alpha,\ BC = P \cos \beta.$$

In the same manner the component of the forces P', P'', P''', &c. as referred to the axis Ax, will be found

$$P' \cos \alpha',\ P'' \cos \alpha'',\ P''' \cos \alpha''', \text{ \&c.};$$

and as referred to Ay, they will be

$$P' \cos \beta',\ P'' \cos \beta'',\ P''' \cos \beta''', \text{ \&c.}$$

If, therefore, we now add together all the former components, and denote their sum by X, and all the latter, and call the sum of these Y, we shall have

$$P \cos \alpha + P' \cos \alpha' + P'' \cos \alpha'' + \text{\&c.} = X$$
$$P \cos \beta + P' \cos \beta' + P'' \cos'' \beta + \text{\&c.} = Y:$$

and thus all the original forces are reduced to two forces; the one X, acting in the direction Ax, and the other acting in the direction Ay; calling therefore the resultant of these two forces R, we shall have

$$R^2 = X^2 + Y^2.$$

36. In the preceding article we have supposed all the forces to be situated in the angle yAx, and have therefore given to each of the cosines the positive sign,

the angles being in this case necessarily acute; but if any of these forces had fallen in any other of the angles formed about the point A, we must then have employed the rules indicated in art. 17, in order to determine the proper sign to affix to each.

For example, let it be proposed to ascertain the resultant of the forces P, P′, P″, P‴, &c. (fig. 19) giving the positive sign to the cosines of the acute angles, and the negative to those which are obtuse, the components will be as follow, viz.

$$\begin{array}{llll} \text{of } P \ldots\ldots & + P \cos \alpha & + P \cos \beta \\ \text{of } P' \ldots\ldots & + P' \cos \alpha' & - P' \cos \beta' \\ \text{of } P'' \ldots\ldots & + P'' \cos \alpha'' & - P'' \cos \beta'' \\ \text{of } P''' \ldots\ldots & - P''' \cos \alpha''' & - P''' \cos \beta''' \\ \text{of } P^{\text{iv}} \ldots\ldots & - P^{\text{iv}} \cos \alpha^{\text{iv}} & + P^{\text{iv}} \cos \beta^{\text{iv}}. \end{array}$$

Whence, adding together the components acting in the same direction, and denoting these as before by X and Y, we shall

$$P \cos \alpha + P' \cos \alpha' + P'' \cos \alpha'' - P''' \cos \alpha''' - P^{\text{iv}} \cos \alpha^{\text{iv}} = X,$$
$$P \cos \beta - P' \cos \beta' - P'' \cos \beta'' - P''' \cos \beta''' + P^{\text{iv}} \cos \beta^{\text{iv}} = Y.$$

But generally we may defer the consideration of the signs of the cosines, till we come to the practical solution of the problem, and therefore generally our results may be expressed by the equations

$$P \cos \alpha + P' \cos \alpha' + P'' \cos \alpha'' + \&c. = X, \quad (2)$$
$$P \cos \beta + P' \cos \beta' + P'' \cos \beta'' + \&c. = Y. \quad (3)$$

37. Now the resultant R being the diagonal of a right angled parallelogram, of which the sides are X and Y, we shall have

$$R = \sqrt{(X^2 + Y^2)}. \quad (4)$$

38. At present we have only considered the intensity or magnitude of the resultant; but it is necessary also to know its direction. In order to effect this, let us call a and b the angles which this resultant makes with the rectangular co-ordinates A x, A y, and we shall have

$$X = R \cos a, \text{ and } Y = R \cos b.$$

Whence,

$$\cos a = \frac{X}{R}, \text{ and } \cos b = \frac{Y}{R}. \quad (5)$$

But it is obvious, since the only data of the problem are P, P′, P″, &c. cos α, cos α', cos α'', &c. and cos β, cos β', cos β'', &c. that we cannot find immediately from these the value of cos a and cos b; but that we must first find X and Y from equations (2) and (3), then R from equation (4), and ultimately cos a and cos b from equation (5).

39. The line of direction passing through the point A, which we have assumed as the origin of the co-ordinates, the equation of it will be

$$y = x \tan a, \text{ or } y = x \frac{\sin a}{\cos a};$$

and observing farther, that the angles a and b, formed by the resultant R, with the axes A x, A y, are supplements of each other, we shall have

$$\sin a = \cos b;$$

and therefore the above equation becomes

$$y = x \frac{\cos b}{\cos a};$$

and substituting in this equation the values of cos a and cos b, given by equation 5, we have

$$y = \frac{Y}{X} x.$$

40. In the case of equilibrium, the intensity of the resultant R being zero, formula 4 in this case becomes

$$\sqrt{(X^2 + Y^2)} = 0, \text{ or } X^2 + Y^2 = 0.$$

But as every square is essentially positive, and as the sum of two positive quantities cannot be equal to zero, unless each of the quantities are separately equal to 0, we must have in this case

$$X = 0, \text{ and } Y = 0.$$

Such are the equations of equilibrium of many forces; which, being situated in the same plane, are applied immediately to any one point. If X only were equal to 0, we should have

$$R = Y, \cos a = 0, \text{ and } \cos b = 1;$$

which shows that in this case the resultant would have its direction coinciding with A y, which is otherwise obvious; and in the same manner it will appear, that if Y = 0, the resultant would coincide in its direction with A x.

Example.

41. In order to illustrate by example what we have endeavoured to explain generally in the preceding articles, let us suppose four forces, P, P′, P″, P‴, acting at one common point A, and whose intensities are denoted respectively by the numbers 2, 3, 4, 5; let also the angle PAP′ = 30°, P′AP″ = 15°, and P″AP‴ = 75°; to find their resultant both in intensity and direction.

First assume two rectangular co-ordinates, x, y, passing through A; and as the position of either of these is arbitrary, let us suppose the ordinate A x to coincide with AP; then the angles which the several forces form with A x, will be

$$\left.\begin{array}{l} P\ A x = 0^\circ = \alpha \\ P'\ A x = 30^\circ = \alpha' \\ P''\ A x = 45^\circ = \alpha'' \\ P'''\ A x = 120^\circ = \alpha''' \end{array}\right\}, \quad \begin{array}{l} \cos 0^\circ = 1 \\ \cos 30^\circ = \frac{1}{2}\sqrt{3} \\ \cos 45^\circ = \frac{1}{2}\sqrt{2} \\ \cos 120^\circ = -\frac{1}{2}\sqrt{3}; \end{array}$$

and the angles formed with A y will be

$$\left.\begin{array}{l} P\ A y = 90^\circ = \beta \\ P'\ A y = 60^\circ = \beta' \\ P''\ A y = 45^\circ = \beta'' \\ P'''\ A y = -30^\circ = \beta''' \end{array}\right\}, \quad \begin{array}{l} \cos 90^\circ = 0 \\ \cos 60^\circ = \frac{1}{2} \\ \cos 45^\circ = \frac{1}{2}\sqrt{2} \\ \cos 30^\circ = -\frac{1}{2}\sqrt{3}. \end{array}$$

Our equations, therefore,

$$P \cos \alpha + P' \cos \alpha' + P'' \cos \alpha'' = \&c. = X,$$
$$P \cos \beta + P' \cos \beta' + P'' \cos \beta'' + \&c. = Y,$$

become

$$2 + \tfrac{3}{2}\sqrt{3} + 2\sqrt{2} - \tfrac{5}{2}\sqrt{3} = X,$$
$$0 + \tfrac{3}{2} \qquad + \tfrac{3}{2}\sqrt{2} - \tfrac{4}{2}\sqrt{3} = Y.$$

Whence the numerical values of X and Y are determined, and consequently that of

$$R = \sqrt{\{X^2 + Y^2,\}}$$

which denotes the intensity of the resultant; and hence again,

$$\cos a = \frac{X}{R}, \text{ and } \cos b = \frac{Y}{R},$$

the angles of the directions which the resultant makes with two axes, are also numerically determined.

§ VI. *Of forces any where situated in space, applied to a given point, as referred to three rectangular co-ordinates.*

Composition of forces in different planes.

42. Let P, P′, P″, &c. be different forces soliciting any point A, and let us draw through this point any three rectangular co-ordinates A x, A y, A z, (fig. 21); and denoting respectively the angles which the co-ordinates form with

$$\begin{array}{ll} P \text{ by } & \alpha, \beta, \gamma, \\ P' \text{ by } & \alpha', \beta', \gamma', \\ P'' \text{ by } & \alpha'', \beta'', \gamma'', \\ \quad \&c. & \&c. \end{array}$$

and decomposing each of these forces into others in the direction of these co-ordinates, we shall find,

$P \cos \alpha$, $P \cos \beta$, $P \cos \gamma$, for the components of P,
$P' \cos \alpha'$, $P' \cos \beta'$, $P' \cos \gamma'$, for the components of P′,
$P'' \cos \alpha''$, $P'' \cos \beta''$, $P'' \cos \gamma''$, for the components of P″,
&c. &c. &c.

And if we reserve the determination of the signs of the cosines, till we pass to the practical solution of the problem; and if also we represent by X, Y, and Z, the components of the resultant R sought, as in the preceding articles; we shall have,

$$P \cos \alpha + P' \cos \alpha' + P'' \cos \alpha'' + \&c. = X \qquad (6)$$
$$P \cos \beta + P' \cos \beta' + P'' \cos \beta'' + \&c. = Y \qquad (7)$$
$$P \cos \gamma + P' \cos \gamma' + P'' \cos \gamma'' + \&c. = Z \qquad (8)$$

43. Now X, Y, Z, being the three projections AB, BC, CD, of the right line AD, which represents the resultant, we shall have

$$AB^2 + BC^2 + CD^2 = AD^2;$$

and consequently,

$$X^2 + Y^2 + Z^2 = R^2.$$

And hence we determine immediately the intensity or magnitude of the resultant R; viz.

$$R = \sqrt{(X^2 + Y^2 + Z^2)}. \qquad (9)$$

Again, in order to determine its direction, if we call a, b, c, the angle which this resultant forms with each of the co-ordinates; the components of R, as referred to each of these axes respectively, will be

$$R \cos a,\ R \cos b,\ R \cos c.$$

And as these components have been otherwise represented by X, Y, Z, we shall have

$$X = R \cos a,\ Y = R \cos b,\ Z = R \cos c;$$

whence we draw

$$\cos a = \frac{X}{R},\ \cos b = \frac{Y}{R},\ \cos c = \frac{Z}{R}; \qquad (10)$$

where we may again observe that X, Y, Z, are first to be determined by equations (6), (7), (8), then R from equation (9), and ultimately $\cos a$, $\cos b$, $\cos c$, from equation (10); by which means both the magnitude and direction of R will be determined.

Equilibrium of forces in different planes.

44. In the case of equilibrium, the resultant being equal to zero, we shall have from equation (9)

$$X^2 + Y^2 + Z^2 = 0;$$

which expression being the sum of three squares essentially positive, they must be each respectively equal to 0, as it is otherwise impossible that their sum should have that value; hence, therefore,

$$X = 0,\ Y = 0,\ Z = 0,$$

which values reduce equations (6), (7), (8), to

$$\left.\begin{aligned} P \cos \alpha + P' \cos \alpha' + P'' \cos \alpha'' + \&c. &= 0 \\ P \cos \beta + P' \cos \beta' + P'' \cos \beta'' + \&c. &= 0 \\ P \cos \gamma + P' \cos \gamma' + P'' \cos \gamma'' + \&c. &= 0 \end{aligned}\right\} \quad (11)$$

which are the equations of equilibrium of any system of forces situated in any manner in space, and applied to the same point.

45. The above conditions being fulfilled, we may demonstrate, that any one of these forces is equal and directly opposed to the resultant of all the others.

Let R′ be the resultant of all the forces except $P \cos a$, X′, Y′, Z′ its components, and a', b', c', the angles which it makes with the three axes; then we have

$$P' \cos \alpha' + P'' \cos \alpha'' + P''' \cos \alpha''' + \&c. = X'$$
$$P' \cos \beta' + P'' \cos \beta'' + P''' \cos \beta''' + \&c. = Y'$$
$$P' \cos \gamma' + P''' \cos \gamma'' + P''' \cos \gamma''' + \&c. = Z';$$

and by means of these values, we may reduce equation (11) to

$$P \cos \alpha + X' = 0,$$
$$P \cos \beta + Y' = 0,$$
$$P \cos \gamma + Z' = 0;$$

and eliminating X′, Y′, Z′, by means of the equations

$$X' = R' \cos a',\ Y' = R' \cos b',\ Z = R' \cos c',$$

these become

$$\left.\begin{aligned} P \cos \alpha &= -R' \cos a', \\ P \cos \beta &= -R' \cos b', \\ P \cos \gamma &= -R' \cos c': \end{aligned}\right\} \qquad (12)$$

now squaring each of these, and taking the sum on each side, we have

$$P^2 (\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = R'^2 (\cos^2 a' + \cos^2 b' + \cos^2 c')$$

And as the sum of the squares of the cosines of the angles, which any force makes with the three rectangular co-ordinates, is equal to 1, (art. 33) this last equation gives

$$P^2 = R^2, \text{ or } P = R;$$

where it will be observed that we take P here only with a positive sign, because the particular sign with which it is effected, is to be determined by the rule for the signs of the cosines indicated in art. 18.

Substituting now the value of P in the equations of No. (12), and suppressing afterwards the common factor R′, these equations become

$$\cos \alpha = -\cos a', \qquad (13)$$
$$\cos \beta = -\cos b', \qquad (14)$$
$$\cos \gamma = -\cos c'; \qquad (15)$$

and if we make $\cos a' = m$, equation (13) gives us

$$\cos a = -m.$$

These values of $\cos a'$, and $\cos a$, indicate that the angles a' and a are supplements of each other; and in the same manner it will appear that b' and β, as also c and γ, are respectively the supplements of each other; that is,

b' is the supplement of β, and
c' is the supplement of γ.

It results also from the above, that R′ and P are directly opposite to each other; for if, for example, P was situated below the plane of x, y, in the quarter where x and y were positive, P would be situated in the other part or quarter where x and y were negative.

After having reduced all the forces to three rectangular forces X, Y, Z, we have seen (art. 33) that the resultant R, is the diagonal of a parallelopipedon, of which the contiguous sides AB, BC, CD, (fig. 21) are denoted by X, Y, and Z; consequently, in order to determine the equation of the resultant R, represented by AD, it will be necessary to find that of a right line passing through the origin A, having its co-ordinates equal to zero, and through the point D, having for its co-ordinates X, Y, and Z.

46. We may give still greater generality to this problem, by supposing the point of application A to have for its co-ordinates x', y', z', then the co-ordinates of the point D, (fig. 22) will be

$$x' + X,\ y' + Y,\ z' + Z.$$

This supposed, the equations of the resultant being those of a right line in space, are of the form

$$z = ax + b,\ z = a'y + b'; \qquad (16)$$

and putting in these equations, the co-ordinates of the point D, (fig. 22) instead of x, y, and z, we shall find

$$z' + Z = ax' + aX + b;\ z' + Z = a'y' + a'Y + b'; \qquad (17)$$

and as the co-ordinates x', y', z', of the point A, ought also to satisfy equation (16), we shall have,

$$z' = ax' + b;\ z' = a'y' + b'; \qquad (18)$$

and subtracting these equations from those of No. 17, we shall obtain

$$Z = aX,\ Z = a'Y;$$

whence also, $a = \frac{Z}{X}$, $a' = \frac{Z}{Y}$.

On the other hand, eliminating b and b' by means of the equations (16) and (18), we shall have

$$z - z' = a(x - x');\ z - z' = a'(y - y');$$

putting in these equations, the values of a and a', we shall have finally, for the equation of the resultant,

$$z - z' = \frac{Z}{X}(x - x');\ z - z' = \frac{Z}{Y}(y - y').$$

§ VII. *Of the equilibrium of solid bodies.*

47. At present we have considered all our forces as acting at the same point, and have therefore only determined the conditions necessary for establishing an equilibrium of a material point; but we must now extend our enquiries to the equilibrium of solid bodies, or to the equilibrium of a system of forces acting upon different points which are supposed to be connected together in an invariable manner. This problem, in all its extent, involves the greatest generality; but we shall at present consider only two particular cases, viz. first, when the forces are all parallel to each other; and, second, when they all act in the same plane; to which all the other cases are readily referred, as will be seen in the subsequent articles.

Of parallel forces.

Parallel forces applied to the extremities of a line.

48. Let us first consider two parallel forces P and Q, (fig. 23) applied to the extremities of a right line AB, cutting their directions at right angles; and let us represent these forces by the right lines AP, BQ, proportional to them. Now we may add to the system two other forces AM, BN, equal, and directly opposed to each other, and substitute for the four forces AP, AM, BQ, and BN, the diagonals AD and BI; and as these diagonals meet in the point C, we may conceive the two forces BQ, BN, as transferred to this point (art. 15); and taking CE = AD, and CF = BI, to represent these two latter forces; if we decompose, or resolve the two forces CE, CF, into two rectangular forces CG, CL, and CH, CK, these will be obviously respectively equal to the four former forces AM, AP, BN and BQ, of which GC and CH are equal and directly opposed, being the equivalents of AM and BN, which are so by hypothesis; abstracting therefore from these, which destroy each other, there remain at the point C, only the two forces CL, CK, equal to P and Q; which acting both in the same direction, are equivalent to one force equal to their sum; denoting therefore the resultant by R, we obtain

$$R = P + Q.$$

And as a force may be applied to any point in its direction, we may suppose this resultant to be applied at O, and it will then only be necessary to ascertain the distance of this point from A and B.

Now the triangles CAO, CEL, being similar, give

$$CO : AO :: CL : EL;$$

and in the same manner, the similar triangles COB, CKF, give

$$OB : OC :: KF : CK.$$

Multiplying these two analogies, term for term, and suppressing the common factor OC of the two first products, we have

$$OB : OA :: CL \times KF : EL \times CK.$$

But the EL = KF, being respectively equal to AM and BN, which are equal by hypothesis, these factors in the two latter terms may therefore also be suppressed, and then we shall have

$$OB : OA :: CL : CK;$$

but CL and CK are equal to AP and BQ, or they are equivalent to the forces P and Q; consequently,

$$OB : OA :: P : Q; \qquad (19)$$

that is, the point of application O of the resultant of the two forces P and Q, divides the right line AB into two parts OB, OA, which are reciprocally proportional to the intensity of these forces.

Remark. It may be objected to the above demonstration, that we are reasoning upon a point of application C, which is out of the line AB, of which we have been endeavouring to establish the laws of equilibrium: but it is obvious that the process would have been just the same, if our object had been to determine the equilibrium of a plane of which the line AB was the base, or which passed through any part of that plane; and it is evident, if this had been our enquiry, that the plane itself would have been in equilibrio, and consequently the line AB, both being supposed without weight.

49. The preceding analogy furnishes us with the following, viz.

$$\text{by composition} \left\{ \begin{array}{l} OB + OA : AO :: P + Q : Q, \\ OB + OA : OB :: P + Q : P; \end{array} \right.$$

$$\text{that is} \left\{ \begin{array}{l} AB : AO :: R : Q \\ AB : OB :: R : P \end{array} \right\} \qquad (20)$$

whence we readily deduce

$$Q : P : R :: OA : OB : AB;$$

from which we draw immediately the following conclusion; viz.

Theorem.

The three forces Q, P, R, are respectively to each other as the distances included between the other two.

50. By means of the above theorem, it is obvious that any three of the six terms being given, the others may be found; thus if Q, P, and OA, are given, we have

$$Q : P :: OA : OB = \frac{P \times OA}{Q}:$$

If Q, P, and OA are given, we find

$$P : Q : OB : OA = \frac{Q \times OB}{P}:$$

If AB, BO, and R be given, then

$$AB : BO :: R : P = \frac{BO \times R}{AB}:$$

If AB, AO, and R be given, then

$$AB : AO :: R : Q = \frac{AO \times R}{AB}:$$

and so on of others.

In the preceding demonstration, we have supposed the directions AP, BQ, of the forces P, Q, to have been perpendicular to the right line AB; but if they were oblique to it, as in fig. 24, we might through the point of application O of the resultant, draw CD perpendicular to the direction of these forces: then the force P applied at A, would have the same effect as if it were applied at C, and it would be the same with regard to the force Q applied at the point D; and as we have by the preceding

$$P : Q :: OD : OC,$$

and the triangles CAO and OBD being similar, we obtain also

$$P : Q :: OB : OA.$$

Parallel forces in opposite directions.

51. When two forces P and Q act in opposite directions, as in fig. 25, the resultant is equal to the difference of these forces.

For let S be the resultant of the two forces P and R, which act in the same direction, we shall have, as above,

$$S = P + R.$$

Now replacing S, by a force Q, which we must suppose equal and acting in a contrary direction, an equilibrium will be established between the three forces P, Q, R; consequently we may consider R as the resultant of P and Q, and we have therefore, in order to obtain R,

$$Q = P + R, \text{ or}$$
$$R = Q - P.$$

Whence the point O, where the resultant R ought to be applied, will be determined from the following proportion; viz.

$$AB : BO :: R : P.$$

Whence

$$BO = \frac{AB \times P}{R};$$

or, writing for R its equal Q — P, we have

$$BO = \frac{AB \times P}{Q - P}.$$

Here it is evident that the less the difference is between Q and P, the greater will be the distance BO; and consequently when P and Q are equal, BO is infinitely great, and R infinitely small; that is, in this case no equilibrium can be practically established.

Composition of several parallel forces.

52. We may apply to any number of parallel forces, the same theory as that given above: for let P, P', P'', P''', P'''', &c. (fig. 26) be parallel forces applied to the different points A, B, C, D, E, &c. connected together by inflexible right lines, and let it be required to determine their resultant and its point of application:

Here we first find the point of application M of the resultant of the forces P and P', by the proportion

$$AB : AM :: P + P' : P';$$

which gives

$$AM = \frac{AB \times P'}{P + P'}.$$

Having then drawn the right line MC, we find the point of application N, of the resultant of the forces P, P' applied in M, and the force P'' applied at C, by the proportion

$$MC : MN :: P + P' + P'' : P'';$$

whence we have

$$MN = \frac{MC \times P''}{P + P' + P''}:$$

drawing then the line ND, we determine in the same manner, the point of application of the resultant of these three forces applied at N, and the force P''' applied at D; this gives us we will suppose the point O: finally draw OE, and by a similar operation determine the point K, which will be the resultant of all the proposed forces.

53. If any of the forces have a contrary direction, let them be denoted by Q, Q', Q''; and we must then find, first, the point of application of the resultant of the forces P, P', P'', &c. which let be K; then of the forces Q, Q', Q'', &c. which let be L; and the system will thus be reduced to two opposite parallel forces, the one applied

in the point $K = P + P' + P'' + P''' +$ &c. and the other at the point $L = Q + Q' + Q'' + Q''' +$ &c.;

of which the resultant and its point of application may be found according to the principles explained in art. 51.

54. If the forces P, P', P'', &c. (fig. 27) remain always parallel and applied to the same points, but taking the direction AQ, BQ', CQ'', DQ''', &c. the resultant will neither be changed in intensity nor in its point of application, but will merely change its direction by becoming parallel to the general direction of the forces; that is, instead of being parallel to the directions AP, BP', CP'', &c. it will now be parallel to the new directions AQ, BQ', CQ'', &c.

For the operation by which we determined the point of application of the resultant, depends only upon the intensities of the forces, and the respective distances of the several points of application; consequently the data will remain the same, whatever may be the direction of the forces.

And the point through which the resultant of all the forces pass, whatever may be their inclination, is called the *centre of the parallel forces.*

Equatio the cent the forc

55. Now in order to find the co-ordinates of the centre of any parallel forces, let M, M', M'', &c. be the points of application of the forces P, P', P'', &c and let

x, y, z, be the co-ordinates of the point M,
x', y', z', the co-ordinates of the point M',
x'', y'', z'', those of the point M'',
and $x_{,}$, $y_{,}$, $z_{,}$
those of the centre of the parallel forces.

Let us denote by N, (fig. 28) the point of application of the resultant of the parallel forces P, P', we shall have

$$MM' : M'N :: P' + P : P;$$

and because of the similar triangles ML'M', NLM', we in like manner obtain

$$MM' : M'N :: ML' : NL;$$

whence we readily draw the following proportions,

$$ML' : NL :: P + P' : P;$$

therefore

$$(P + P')\,NL = P + ML';$$

and adding $(P + P')\,LK$ to both sides

$$(P + P')(NL + LK) = P\,(ML' + KL) + P.KL:$$

now observing that

$$NL + LK = NK$$
$$ML' + KL = M'H',$$

the preceding equation reduces to

$$(P + P')\,NK = P.MH + P'.M'H';$$

and if we call Q the resultant of the forces P, P' and Z, the ordinate NK of its point of application, this equation becomes

$$QZ = Pz + P'z'.$$

Calling again Q' the resultant of the forces Q and P'', and Z' the ordinate of the point of application of Q', we shall have,

$$Q'Z' = QZ + P''z'';$$

or substituting for QZ its value found above, we obtain

$$Q'Z' = Pz + P'z' + P''z'' +, \text{ \&c.};$$

and by continuing the same operation, it is obvious that we shall have ultimately

$$R z_, = P z + P'z' + P''z'' + P'''z''' +, \&c.: \quad (21)$$

R being the resultant of all the forces P, P′, P″, &c. and $z_,$, the ordinate of its point of application in the direction of z.

efinition f moments.

56. *The moment of a force, with reference to any plane*, is the product of its intensity by the distance of its point of application from this plane: the preceding equation indicates therefore that the moment of the resultant of any forces P, P′, P″, &c. with reference to the plane of x, y, is equal to the sum of the moments of each of these forces as referred to the same plane; and taking the moments with reference to two other co-ordinate planes, we shall have again

$$R y_, = P y + P'y' + P''y'' + P'''y''' +, \&c. \quad (22)$$

$$R x_, = P x + P'x' + P''x'' + P'''x''' +, \&c. \quad (23)$$

Therefore when we know the co-ordinates, x, y, z, x', y', z', &c. of the points of application, and the intensities P, P′, P″, &c. of the forces, we know also the resultant R, which is equal to the sum of these intensities; and consequently we may calculate the co-ordinates $x_,, y_,, z_,$, of the centre of these parallel forces.

With regard to the signs, we consider the forces acting in one direction as affected with the positive sign, and those which act in an opposite direction as affected with the negative; and since the co-ordinates are positive or negative, according as they fall on one side or on the other of the origin, we must give the positive sign to a moment in which the force and that of its co-ordinates have the same sign, and the negative to those which have contrary signs.

Points of application in the same plane.

57. If the points of application M, M′, M″, M‴, &c. are in the same plane MM″, (fig. 29,) we may so dispose the co-ordinate planes, that the plane of x, y, may be parallel to MM″; in which case all the co-ordinates z, z', z'', z''', &c. are equal to each other, or

$$z = z' = z'' = z''', \&c.$$

and if we represent by $z_,$, the ordinate of the centre of the parallel forces, this ordinate will also be equal to z, z', &c.; consequently, in this case, the preceding equation

$$R z_, = P z + P'z' + P''z'' + \&c.$$

will become simply

$$R = P + P' + P'' + P''' +, \&c.$$

If the points of application of the forces, beside being all found in one plane, were also all in one right line AB (fig. 30), we might suppose this parallel to one of the axes, x for example, and then we should have at the same time

$$z_, = z = z' = z'' = z''' =, \&c. \text{ and}$$

$$y_, = y = y' = y'' = y''' =, \&c.$$

and consequently in this case, equations 21 and 22 both reduce to

$$R = P + P' + P'' + P''' +, \&c. \quad (24)$$

and there will only remain that of

$$R x_, = P x + P'x' + P''x'' +, \&c. \quad (25)$$

we may therefore here dispense with employing three rectangular co-ordinates, by merely referring x, x', x'', &c. to the line AB.

For example, let $x = 9, x' = 3, x'' = -3, x''' = -4$;

also, $P = -\frac{1}{3} P', P'' = -\frac{2}{3} P, P''' = 2 P$:

then putting these values into equations 24 and 25, we should have

$$R = P - \tfrac{1}{3} P - \tfrac{2}{3} P + 2 P = 2 P,$$

$$R x_, = 9P - P + 2P - 8P = 2P:$$

whence we obtain $x_, = 1$.

58. Let us now endeavour to investigate the conditions of equilibrium in such a system of parallel forces as that we have been considering.

Conditions of equilibrium.

In the first place, as the position of the co-ordinate planes is perfectly arbitrary, we may always suppose the axis of z parallel to the direction of the forces. This, therefore, being supposed, and having reduced all the forces which act in one direction to one resultant $R_,$ (fig. 31), and all those that act in a contrary direction to another resultant $R_{,,}$, we shall have an equilibrium in the system if these two resultants are equal and directly opposed to each other.

Now, in order that the first of these conditions may be fulfilled, the distance C′C″ must be zero; in order to which, it is necessary that the co-ordinates $x_,$ and $y_,$, from the centre C′, may be the same as the co-ordinates $x_{,,}, y_{,,}$, from the centre C″: consequently we shall have

$$x_, = x_{,,}, \; y_, = y_{,,};$$

and the second will be fulfilled if

$$R_, = - R_{,,}. \quad (26)$$

Now, multiplying the two first equations by the third, we have

$$R_, x_, = - R_{,,} x_{,,}, \quad (27)$$

$$R_, y_, = - R_{,,} y_{,,}. \quad (28)$$

And by the property of moments, (art. 56) we shall have, by calling P, P′, P″, &c. the components of $R_,$, and P‴, P$^{\text{iv}}$, those of $R_{,,}$, &c.

$$R_, x_, = P x + P'x' + P''x'' +, \&c.$$

$$R_{,,} x_{,,} = P'''x''' + P^{\text{iv}}x^{\text{iv}} + P^{\text{v}}x^{\text{v}} +, \&c.$$

And substituting these values in equation (27), we reduce it to

$$P x + P'x' + P''x'' + P'''x''' + P^{\text{iv}}x^{\text{iv}} + \&c. = 0; \quad (29)$$

and by a similar proceeding

$$P y + P'y' + P''y'' + P'''y''' + P^{\text{iv}}y^{\text{iv}} +, \&c. = 0. \quad (30)$$

Finally, if in equation (26) we substitute the values of R′, R″, we obtain

$$P + P' + P'' + P''' + P^{\text{iv}} +, \&c. = 0; \quad (31)$$

59. When these latter equations are satisfied, the parallel forces are in equilibrio.

Whence, then, we may draw the following general conclusion, viz.

There will be an equilibrium in the system, if the sum of the moments of the forces, taken with reference to two rectangular planes, parallel to the common direction, is equal to zero; and if, at the same time, the sum of the forces is also equal to zero.

60. The equilibrium will also subsist, if the resultant of the parallel forces pass through a fixed point; for it will then be destroyed by the resistance of this point.

§ VIII. *Of forces situated in a plane, and applied to different points connected together in an invariable manner.*

Forces situated in the same plane.

61. Let P, P′, P″, &c. (fig. 32) represent several forces acting in the same plane, which we will suppose to be that of x, y, and which are applied to the points A, B, C, D, situated in this plane, and invariably connected together.

First, having taken the parts A *a*, B *b*, C *c*, D *d*, proportional to the intensity of these forces, we prolong the right lines A *a*, B *b*, to their point of meeting G, and having transferred into this point the two forces A *a*, B *b*, construct the parallelogram GG′, and its diagonal GG′ will represent the intensity and direction of the resultant of these two forces; prolong again GG′ and C *c* to their point of intersection H; then having transferred to this point the two forces GG' and C *c*, construct the parallelogram HH', of which the diagonal HH', will represent the resultant of the forces GG' and C *c*, and consequently those of A *a*, B *b*, C *c*; prolong now HH' and D *d*, till they intersect in the point I, and having constructed the parallelogram I I', the diagonal I I' will be the resultant of the entire system.

62. It should be observed, that if amongst the proposed forces there should be two or more parallel to each other, these must be reduced or compounded into one force by the method explained in the preceding section; and this compounded force or resultant will be equal to their sum or difference, according as they act in the same or in opposite directions, as explained in articles 49 and 51.

But if two of the forces are equal, and not directly opposed to each other, then, as in this case, we cannot assign to them any resultant (art. 51); we must compound each of these with other forces of the system, and continue the operations in the same manner as explained above; but if these two forces, or if two such resultants should be the resultants of the whole system, then we must conclude, as in the article above referred to, that it is impossible to reduce the forces to one resultant, or, which is the same, that it is impossible under those circumstances, to reduce the system to an equilibrium. We may also farther observe, that if in this construction, the last resultant is zero, the system will be already found in a state of equilibrium.

Analytical principle of these forces.

63. Such is the geometrical construction of this problem, but it is not well suited to the purposes of computation; in order to accommodate it to this, we may observe, that it is precisely the same thing as if we had transferred to the point I, each of the original forces parallel to themselves, and at that point compounded them into one only as II'.

Let us, for example, first take only three forces, P,Q,S (fig. 33), the resultant DC of the forces P and Q, being transferred to D'C', we may resolve D'C' into D'P' and D'Q', where it is obvious that D'P' and D'Q', are parallel to DP and DQ, the parallelograms D'C' and DC being equal.

This then being premised, let us endeavour to investigate the analytical conditions of equilibrium; and with this view, we shall consider, at first, only three forces P, P', P'', applied to different fixed points of any system; now one of the necessary conditions, in order that these forces may be in equilibrio, is that they all meet in one point, or, which is the same, that their resultant be equal to zero.

As the forces must therefore meet in one point, we may assume this as their common point of application, by supposing them each transferred to this point parallel to themselves; then the conditions of equilibrium will be precisely the same as those of forces acting upon a material point, viz.

$$\left.\begin{array}{l} P\cos\alpha + P'\cos\alpha' + P''\cos\alpha'' = 0 \\ P\cos\beta + P'\cos\beta' + P''\cos\beta'' = 0 \end{array}\right\} \quad \text{(art. 44)}$$

To which equations we must, however, in our present case, unite that which we shall now proceed to determine, namely the condition of the concurrence of the forces.

64. Let P, P', and R, be three forces which meet in the point A, (fig. 34). If through any point C, assumed at pleasure, we draw also through A the right line CA, and from the point C, demit the perpendiculars CI, CI', CI'', the right angled triangles ACI', ACI'', A'CI, will have the same common hypothenuse AC; and it is this which constitutes the condition of the concurrence of the forces; for this hypothenuse is common to the three triangles only in consequence of their having the same vertex A.

Now if from the point A, we draw AB, at right angles to AC, and from the extremities of the right lines AP, AP', and AR, which represent the intensities of these forces, we let fall the perpendiculars PD, P'D', RD'', on AB, the right angled triangles ACI, APD, will be similar, because the angle formed by AP with the parallel AC and PD, are equal; whence the following proportion, viz.

$$AC : CI :: AP : AD;$$

and if we make AC = c, and CI = p, we have

$$c : p :: P : AD;$$

whence

$$AD = \frac{p'P.}{c}$$

In the same manner, if we take p' and r to represent the perpendicular CI' and CI'', we shall find

$$AD' = \frac{P'p'}{c} \text{ and } AD'' = \frac{Rr}{c}:$$

now if R be the resultant of P and P', the component of R, in the direction of AB, will be equal to the sum of the components of AP and AP'', in the direction of the same right line.

The truth of this is involved in our equations of equilibrium (art. 40), viz.

$$X = 0 \text{ and } Y = 0;$$

but it may be demonstrated as follows;

Construct the parallelogram APRP' (fig. 35), and draw PE parallel to AB; then the triangles P'ER.ADP are equal, and give

$$AD = P'E = D'D'';$$

substituting D'D'' for AD in the identical equation

$$AD'' = AD' + D'D'',$$

we have

$$AD'' = AD' + AD;$$

that is, the component AD'' of the force R, is equal to the two components AD', AD of the forces P, P'; and if here we substitute for AD'', AD' and AD, their values, as above determined, we shall have

$$\frac{Rr}{c} = \frac{Pp}{c} + \frac{P'p'}{c},$$

or

$$Rr = Pp + P'p'. \qquad (32)$$

But if the point C were situated in the angle formed by the two forces P, P', or in the opposite vertical angle R, then in this case we should have

$$Rr = Pp - P'p'; \qquad (33)$$

as will appear obvious by referring to fig. 36.

Moment a force r ferred to point.

65. We have defined (art. 56) the moment of a force, *with reference to any plane*, to be the product of the intensity of that force, by the perpendicular let fall from its point of application to that plane. And, by analogy, we call the *moment of a force, with reference to any point*, the product of the perpendicular let fall from this point upon the direction of the force into the force. Moments therefore, with reference to a point, differ essentially from moments referred to a plane. The latter depend upon the point of application of the force, and are independent of its direction; while moments, with reference to a point, depend entirely on the direction, and are independent of the point of applica-

tion. The first are only employed in the theory of parallel forces, and may be positive or negative; their signs being determined from those of the force and the ordinate of the point to which they are applied, since they are the product of these two quantities. On the contrary, in moments with reference to a point, the two factors, that is, the force and the perpendicular let fall upon its direction, are quantities which we always consider as positive, and consequently the moments themselves are positive also.

ations quili- m.

Equations (32) and (33) indicate therefore, that the moment of the resultant of two forces is equal to the sum or to the difference of the moments of the components, according as the point C, which is called the *centre of the moments*, is situated without or within the angles formed by the two components, or the vertical opposite angle formed by producing the lines of their direction.

Or, if we suppose the positive or negative sign as belonging to the moments themselves, which must be determined by the conditions of the question, then we may reduce the two cases of the preceding theorem to one only, and say,

That the moment of any resultant of two forces is equal to the sum of the moments of the two components.

But when three forces retain each other in equilibrio, the third force must be equal and directly opposed to the resultant of the other two; our third force P''' must therefore be equal to $-$ R, and denoting also the perpendicular demited from it by $p'' = r$, we shall have

$$\text{P}p + \text{P}'p' + \text{P}''p'' = 0.$$

Thus the equations of equilibrium of three forces, situated in a plane, and applied to different points, will be

$$\text{P} \cos \alpha + \text{P}' \cos \alpha' + \text{P}'' \cos \alpha'' = 0 \quad (34)$$
$$\text{P} \cos \beta + \text{P}' \cos \beta' + \text{P}'' \cos \beta'' = 0 \quad (35)$$
$$\text{P}p + \text{P}'p' + \text{P}''p'' = 0. \quad (36)$$

hen the ces ex- ed three.

66. In order to proceed now to the case in which the number of forces exceed three, we may consider P, as the resultant of the two forces P''' and P$^{\text{iv}}$; whence we have, as before,

$$\text{P}p = \text{P}'''p''' + \text{P}^{\text{iv}}p^{\text{iv}};$$

and substituting this value of Pp, in equation (36), we change it into

$$\text{P}'p' + \text{P}''p'' + \text{P}'''p''' + \text{P}^{\text{iv}}p^{\text{iv}} = 0;$$

and by a similar proceeding, equations 34 and 35, become

$$\text{P}' \cos \alpha' + \text{P}'' \cos \alpha'' + \text{P}''' \cos \alpha''' + \text{P}^{\text{iv}} \cos \alpha^{\text{iv}} = 0,$$
$$\text{P}' \cos \beta' + \text{P}'' \cos \beta'' + \text{P}''' \cos \beta''' + \text{P}^{\text{iv}} \cos \beta^{\text{iv}} = 0.$$

And the same method may be extended to any greater number of forces; and, consequently, we shall have for the general equations of equilibrium of any number of forces applied to different points, but acting in the same plane,

$$\text{P} \cos \alpha + \text{P}' \cos \alpha' + \text{P}'' \cos \alpha'' +, \&\text{c.} = 0, \quad (37)$$
$$\text{P} \cos \beta + \text{P}' \cos \beta' + \text{P}'' \cos \beta'' +, \&\text{c.} = 0, \quad (38)$$
$$\text{P}p + \text{P}'p' + \text{P}''p'' +, \&\text{c.} = 0. \quad (39)$$

Particular notation.

67. It is very convenient, and we shall frequently avail ourselves of it, to denote the sum of such quantities as those above, by inserting the first of them in parentheses, and prefixing the Greek Σ; thus—

$$\Sigma\,(\text{P} \cos \alpha),\ \Sigma\,(\text{P} \cos \beta),\ \Sigma\,(\text{P}p);$$

under which notation, the above equations will therefore be written,

$$\Sigma\,(\text{P} \cos \alpha) = 0,\ \Sigma\,(\text{P} \cos \beta) = 0, \Sigma\,(\text{P}\,p) = 0, \&\text{c.}$$

Properties of moments

68. Let figs. (37) and (38) represent such a system of forces as we have been considering; the former having C, the centre of the moments, situated beyond the angle of the two extreme forces, and of their vertical opposite angle; and the latter having that point situated within the angle formed by any two of those forces; it is obvious, that if we suppose the several forces P, P', P'', &c. to act by impulsions, and to remain invariably fixed to their respective perpendiculars, denoted by p, p', p'', &c. then, in the first of the above figures, these forces will have a tendency to make the perpendiculars all revolve about the point C, in one and the same direction; while in fig. (38), the forces P, P', P'', will have a tendency to cause the rotation to take place in one direction, and the forces P''', P$^{\text{iv}}$, in an opposite one.

Now the expressions $\frac{\text{P}p}{c}$, $\frac{\text{P}'p'}{c}$, &c. represented by AD, AD', &c. being affected with different signs from AD''', AD$^{\text{iv}}$, &c. it follows that the moments that have the same sign, have a tendency to cause the system to revolve in one direction; and those with a different sign, to make it turn in the contrary one; and therefore, in the case of an equilibrium, the moment of the resultant will be equal to the sum of the moments which have a tendency to make the system revolve in the same direction as this resultant, *minus* the sum of the moments which tend to produce a revolution in the opposite direction.

Or, if we consider the resultant as another force similar to the others, as we have done in establishing our equations of equilibrium in the preceding cases, it follows from the equation $\Sigma\,(\text{P}\,p) = 0$, that, in the case of an equilibrium, the sum of the moments of the forces which tend to produce motion in one direction, is equal to the sum of the moments of the forces which tend to produce a motion in the contrary direction.

Direction of the resultant.

69. If, in a system supposed to be in equilibrio, we suppress one of the forces, as for example P, the other forces will have a resultant, which we may denote by R, and the angles which it makes with the axes by a and b; and as this resultant ought to act in the opposite direction to the suppressed force P, which retained the other forces in equilibrio, instead of the equations (37), (38), (39), we shall have the following; viz.

$$\text{R} \cos a = \text{P}' \cos \alpha' + \text{P}'' \cos \alpha'' + \text{P}''' \cos \alpha''', +, \&\text{c.}$$
$$\text{R} \cos b = \text{P}' \cos \beta' + \text{P}'' \cos \beta'' + \text{P}''' \cos \beta''', +, \&\text{c.}$$

or,

$$\text{R} \cos a = \Sigma\,(\text{P} \cos \alpha) = \text{X},$$
$$\text{R} \cos b = \Sigma\,(\text{P} \cos \beta) = \text{Y},$$
$$\text{R}\,r = \Sigma\,(\text{P}\,p);$$

and, by means of these equations, we readily obtain every condition relative to the resultant R.

For example, the first two equations give

$$\text{R}^2\,(\cos^2 a + \cos^2 b) = \text{X}^2 + \text{Y}^2:$$

or, because a and b are the complements to each other, $\cos^2 a + \cos^2 b = 1$; therefore

$$\text{R}^2 = \text{X}^2 + \text{Y}^2:$$

or

$$\text{R} = \surd\,\{\text{X}^2 + \text{Y}^2.\}$$

The same equations give also the angle of the direction of the resultant; for we have

$$\cos a = \frac{\text{X}}{\text{R}},$$

$$\cos b = \frac{Y}{R}.$$

70. Now, in order to introduce the resultant into the system, we must begin by determining the position of the right line AB (fig. 39), which shall pass through the origin A, and be parallel to the resultant; and with this view, it will be observed that the sign of the cosine b, will indicate whether the line AB is situated above or below the axis of x; for when the $\cos b$ is positive, the right line AB must form, with the axis of y, an acute angle; and it ought, on the contrary, to form an obtuse angle when that sign is negative.

Similar considerations may be applied to the sign of the $\cos a$, and the consequent position of AB, with regard to the axis of x; when the combination of two cases will give the direction sought.

Having thus ascertained the direction of AB, we draw through its origin A, a perpendicular $r = \frac{\Sigma(Pp)}{R}$, which perpendicular is denoted by OA, or O′A, its particular direction being ascertained by the preceding rule.

Equation of the resultant.

71. And now, in order to obtain the equation of this resultant, we may observe that in the most general case, as it cuts the axis y in a point B (fig. 40), its equation ought to be of the form

$$y = x \tan D + AB; \qquad (40)$$

and as the angle, which the direction of the resultant makes with the axis x, is represented by a, we have $D = a$, and consequently

$$\tan D = \frac{\sin a}{\cos a} = \frac{\cos b}{\cos a} = \frac{P \cos b}{P \cos a} = \frac{Y}{X}.$$

And with regard to the magnitude or value of AB, it it is given by the equation

$$OA = AB \cos OAB.$$

The angle OAB, which enters into this equation, is equal to D, because these angles are each complements of the angle OAD; we may therefore, for OAB, write D or a; and as OA is the perpendicular let fall from the centre of the moments upon the direction of the resultant, which we have in the preceding article denoted by r, we have, by making these substitutions,

$$r = AB \cos a,$$

and consequently

$$AB = \frac{r}{\cos a}.$$

Now writing this value of AB, and that for tan D, in equation (40), we obtain

$$y = x\frac{Y}{X} + \frac{r}{\cos a}, \text{ or}$$

$$y = x\frac{Y}{X} + \frac{R r}{R \cos a}, \text{ or}$$

$$y = x\frac{Y}{X} + \frac{R r}{X};$$

or, multiplying by X, we have

$$x y = Y + R r, \text{ or}$$

$$y x - Y = R r.$$

Again, replacing $R r$ by its value $\Sigma(Pp)$, we shall have for the resulting equation,

$$x y - Y = \Sigma(P p).$$

And, as in the case of equilibrium, $X = 0$, and $Y = 0$, the equation then reduces to $\Sigma(Pp) = 0$, which is agreeable to the preceding deduction.

72. Since the data which enable us to determine the direction of the resultant are, first, the intensity of the forces; secondly, their directions; and thirdly, the co-ordinates of their points of application; it will be convenient to substitute for equation (39), another, in which, instead of p, p', p'', &c. we introduce the co-ordinates above referred to. In order to this, let us place the origin at A (fig. 41), and let x and y be the co-ordinates of the point of application M, of a force P represented in intensity by the right line MP; then the components of P, parallel to the axes A x, and A y, will be

$$MN = P \cos a,$$

$$MQ = P \cos \beta.$$

Draw from the origin A, the perpendiculars AO, AF, and AE, on the prolongation of MP, and of its components, and we shall find

OA × MP = moment of the resultant P,
AF × MN = moment of the component P cos a,
AE × MQ = moment of the component P cos β.

Now regarding these forces as acting by impulsions, the resultant P, and the component P cos a, will tend to make OA, AF, revolve about the point A in the same direction: let us therefore assign to each of these moments the positive sign, and to the component P cos β, which has a tendency to produce a contrary motion, the negative sign; and thus we shall obtain the equation

$$P p = y P \cos a - x P \cos \beta.$$

For the same reason,

$$P'p' = y' P' \cos a' - x' P' \cos \beta',$$

$$P''p'' = y'' P'' \cos a'' - x'' P'' \cos \beta'',$$

$$P'''p''' = y''' P''' \cos a''' - x''' P''' \cos \beta''',$$

&c. &c.

and substituting these values in equation (39) of moments, it becomes

$$P(y \cos a - x \cos \beta) + P'(y' \cos a' - x' \cos \beta') + \&c. = 0; \ldots\ldots\ldots (41)$$

and consequently we have for that of the resultant

$$y X - x Y = \Sigma \{ P(y \cos a - x \cos \beta). \}$$

Signs mome deteri analy

73. We have seen that in order to determine the signs of the moments, we must refer to the rule given under article 68, which is somewhat foreign to the spirit of analysis; but when by a farther transformation this equation is converted into the form above given, it will be sufficient, in order to determine the signs of the moments, to make use of the rules given in art. 17, only observing to change the signs of the co-ordinates when necessary.

For example, let a force be situated as represented in fig. 42, with regard to the axes A x, A y; the moment of the force being generally denoted by

$$P(y \cos a - x \cos \beta);$$

in order to modify it, so as to answer to this particular case, we shall have x *negative*, and y *positive*; cos a *negative*, and cos β *negative*; therefore the moment of this force, having regard to the signs, will become

$$P(-y \cos a - x \cos \beta).$$

Parti dedu

74. The equations of equilibrium (34), 35), (36), express the condition, that all the forces of the system are reduced to two equal and directly opposite forces. In fact, if we call P cos a, P′ cos a', &c. the components parallel to the axes of x, which act in one direction; and P″ cos a'', P‴ cos a''', &c. those which act in the opposite one, the equation will become

$$P \cos a + P' \cos a' + \&c. = P'' \cos a'' + P''' \cos a''' + \&c.$$

The forces P cos a, P′ cos a', &c. being parallel, we may, by the composition of forces, reduce them to one only, X', equal to their sum and parallel to their direction; and we may proceed in the same manner with $P'' \cos a''$, $P''' \cos a'''$, &c. which we may denote by X''; then the system of all the forces, parallel to the axis x, is reduced to the two equal and opposite forces X' and X''; and in the same manner we may convert all the forces parallel to y into two equal and opposite forces Y' and Y''.

Now transferring the forces X' and Y' to their point of intersection M (fig. 43), and the forces X'' and Y'' to their point of meeting N, we may construct the rectangles MA, NB, of which the sides MC, MD, NE, NF, shall represent the forces X', Y', X'', Y''; and as the homologous sides of these rectangles are equal, the diagonals MA, NB, will also be equal, and their directions will be parallel on account of the equality of the triangles AMD, BNE.

The equations $X = 0$ and $Y = 0$, express therefore the following condition; viz. that all the forces situated in a plane, may be reduced to two forces MA and NB, equal, parallel, and having opposite directions; but they do not indicate that they both act in the same right line. In order that this may have place, it is necessary that the equation $\Sigma (Pp) = 0$ be fulfilled; that is, assuming R' and R'' to represent the two forces, and r', r'', to denote the perpendiculars OP, OQ, let fall from the point O, taken beyond the direction of the prolongation of these forces; then as R' and R'' act in a contrary direction, the moments of these forces will have different signs, and the equation $\Sigma (Pp) = 0$, will become

$$R' r' - R'' r'' = 0.$$

Now by hypothesis $R' = R''$; and therefore suppressing this common factor, we have

$$r' - r'' = 0.$$

Consequently $r' = r''$, or $OP = OQ$; that is, the points O and Q coincide, and consequently AM and NB are in the same right line.

And from this it follows, that when the equation $\Sigma (Pp) = 0$, is not satisfied, as well as the two $X = 0$, and $Y = 0$, the system is reduced to equal forces parallel to each other, and acting in opposite directions, but not directly opposed; in which case no equilibrium can be obtained, as is shown in art. 51.

On the other hand, if the equation $\Sigma (Pp) = 0$ were fulfilled, but not the equations $X = 0$, and $Y = 0$, there would still be no equilibrium; because in this case, instead of having $R = 0$, we should have

$$R = \surd (X^2 + Y^2).$$

And in this case the equation $\Sigma (Pp) = 0$, or rather $Rr = 0$, could only have place on the supposition of $r = 0$; now r being a perpendicular, let fall from the centre of the moments upon the resultant, it follows thence that the centre of the moments will be in the resultant.

Thus, when of the three equations

$$\Sigma (P \cos a) = 0,\ \Sigma (P \cos \beta) = 0, \text{ and } \Sigma (Pp) = 0,$$

the last only is satisfied, it is necessary, in order that an equilibrium may be sustained in the system, that there be a fixed point in the resultant: for example, if the forces P, P', P'', P''', &c. are applied to different points of a lever, and the point C, through which the resultant passes, be a fixed and immoveable obstacle which destroys the effect of this resultant; then the conditions of the equation $\Sigma (Pp) = 0$ being fulfilled, will be sufficient to put the system into a state of equilibrium; and it will be seen that the intensity of this resultant is the pressure exerted upon the point C.

When the system is reduced to two forces, equal and parallel, but not directly opposed; it will only be necessary to introduce any new arbitrary force S, in order to render the system capable of a resultant. In fact, there can happen only the two following cases; viz. either that the new force S will be parallel to P and Q, or it will not. In the first case, we may resolve S into two forces P', Q', and the system of the three forces P, Q, S, will be replaced by the two unequal forces $P + P'$ and $Q + Q'$, and consequently a resultant may in this case be obtained.

And if the new force S be not parallel to the two others, it is obvious from all that has been before said, that a resultant is in this case readily obtained, by transferring to the point of intersection of any two of the forces, that of the third force; consequently, under either supposition, the introduction of any third force will be sufficient to render the system of two equal forces, acting in opposite directions but not directly opposed, susceptible of a resultant, and consequently of an equilibrium.

§ IX. *Of forces acting in any manner in space.*

Of forces in space.

75. This is far more general than any of the cases which we have yet considered, involving, in fact, the whole of them in one comprehensive proposition. We began first by treating of the equilibrium of a system of forces, all directed to the same point, and all acting in the same plane; secondly, the forces, instead of all acting in the same plane, were supposed to be situated any where in space, but still all directed to one and the same point; thirdly, we have treated of the equilibrium of forces acting parallel to each other, and directed to different points; and fourthly, we have considered forces not necessarily parallel, but all acting in one plane. All these are obviously only so many particular cases of the proposition to which we are now arrived: the determination of the conditions of equilibrium of forces any way situated in space, and acting on any number of points connected together in an invariable manner, under every possible arrangement and position, is here required.

76. Let P', P'', P''', &c. denote the several forces any where situated in space, and let

x', y', z', be the co-ordinates of the point of application of P'
x'', y'', z'', ... of P''
x''', y''', z''', ... of P'''
&c. &c. &c.
α', β', γ', the angles formed by P' with the axes;
α'', β'', γ'', ... by P'' with the axes;
α''', β''', γ''', ... by P''' with the axes;
&c. &c. &c.

We have to investigate the conditions of equilibrium of such a system of forces, which we shall endeavour to do by showing that these may always be resolved into two distinct classes of forces; the one class having all the forces situated in one plane, and the others all parallel to each other: also, that the first shall coincide with the plane of any two of the co-ordinates, as for example of x and y, and the others be parallel to the third axis z.

Forces not parallel to the plane of x, y.

76. If amongst the forces of the system there are found none which are parallel to the plane of x, y, it will be very easy to obtain the decomposition proposed. For let P′ (fig. 44) be one of the forces which we shall suppose applied at the point M′, we may prolong its direction till it meets the plane of x, y, in any point C′; and then transferring to C′ its point of application, we may there resolve it into two others; the one C′L, parallel to the axis z; and the other C′N, situated in the plane of x, y. And it is obvious, that the same may be done with any number of forces, provided they are not situated in a direction parallel to the proposed plane.

Forces parallel to the plane of $x\,y$.

77. But when the force P′ is parallel to the plane of x, y, this mode of decomposition cannot be adopted; another, therefore, is to be sought, in which this inconvenience may be avoided.

In order to this, let us draw through the point M′ a line parallel to the axis z, and take on this parallel two equal parts or lines M′O, M′O′ (fig. 45), which may be taken to denote any two equal and opposite forces g' and $-g'$. Now the introduction of these two forces, as they are equal and diametrically opposed, will not disturb the equilibrium, and by this means, instead of the single force P′, we shall have an equivalent in the three forces, P′, g', and $-g'$; we may now therefore compound P′ with $-g$, and calling R the resultant, we shall find the force P′ replaced by two forces, R and g', both applied at the point M, of which the force g' will be parallel to the axis of z, and the force R being oblique to the plane of x, y, may be resolved in the manner explained above.

Co-ordinates of the points of application.

78. The co-ordinates of the points of application being necessary when we express analytically the conditions of the equilibrium of the forces, it remains for us to determine the co-ordinates of the point C′; and this is readily done by means of the equations of the resultant R, which passes through a point of which the co-ordinates are x', y', z'. Now we have seen (art. 46) that these equations are,

$$\left.\begin{aligned} z - z' &= \frac{Z}{X}(x - x',) \\ z - z' &= \frac{Z}{Y}(y - y'.) \end{aligned}\right\} \qquad (42)$$

In these equations, X, Y, Z, represent the projections of the right line R, on the co-ordinate axes; and as these projections are equal to the components of R′ parallel to the axes, it will only be necessary to replace x, y, z, by these components. Now R′, being the resultant of P′, and of $-g'$, we may substitute for P′ its three components, $P' \cos \alpha'$, $P' \cos \beta'$, $P' \cos \gamma'$, and R′ will be the resultant of the four forces

$$P' \cos \alpha',\ P' \cos \beta',\ P' \cos \gamma',\ \text{and } -g.$$

But these forces acting parallel to the co-ordinates, we shall have

$$\left.\begin{aligned} X = P' \cos \alpha',\ Y = P' \cos \beta, \\ Z = P' \cos \gamma', -g'. \end{aligned}\right\} \qquad (43)$$

and introducing these values into equation (42), we obtain for the equation of R′,

$$\left.\begin{aligned} z - z' &= \frac{P' \cos \gamma' - g'}{P' \cos \alpha'}(x - x',) \\ z - z' &= \frac{P' \cos \gamma' - g'}{P' \cos \beta'.}(y - y'.) \end{aligned}\right\} \qquad (44)$$

Now, in order to have the co-ordinates of the point C′, where the right line R′ cuts the plane of $x\,y$, it may be observed that in this point $z = 0$; and if we denote by $a_{,}$, and $b_{,}$, the other co-ordinates, equations (43) will, in this case, become (since

$$x = a_{,},\ y = b_{,},\ \text{and } z = 0)$$

$$-z' = \frac{P' \cos \gamma' - g'}{P' \cos \alpha'}(a_{,} - x',)$$

$$-z' = \frac{P' \cos \gamma' - g'}{P' \cos \beta'}(b_{,} - y'.)$$

Whence we have

$$a_{,} = x' - \frac{z' P' \cos \alpha'}{P' \cos \gamma' - g'}$$

$$b_{,} = y' - \frac{z' P' \cos \beta'}{P' \cos \gamma' - g'}.$$

Such therefore are the co-ordinates $a_{,}$ and $b_{,}$ of the point $C_{,}$, where the resultant R′ cuts the plane of x, y.

79. The force R′ (fig. 46) being represented by the part M′R′ of its direction, we may transfer it to the point C′, by taking C′D′ = M′R′; then resolving C′D′ into three rectangular forces applied at C′, these forces will be the same as the components of M′R′; and consequently we may consider the point C′ as solicited by the three forces

$$P' \cos \alpha',\ P' \cos \beta',\ \text{and } P' \cos \gamma' - g';$$

the two first being situated in the plane of x, y, and the third out of that plane and parallel to the axis of z: thus, instead of the force P′ applied at M′, we shall have

in M′, the	force g',	parallel to the axis z;
in C′,	$P' \cos \gamma - g'$,	parallel to the axis z;
in C′,	$P' \cos \alpha'$,	in the plane of x, y;
in C′,	$P' \cos \beta$,	in the plane of x, y:

and proceeding in the same manner with any other forces P″, P‴, P$^{\text{iv}}$, &c. by means of the forces g'', $-g''$; g''', $-g'''$; g^{iv}, $-g^{\text{iv}}$, &c. we shall be able to resolve any number of such forces into two distinct classes; the one acting in the plane of x, y, and the other parallel to the axis z.

Of the latter, g', g'', g''', &c. will be the forces acting at the points of application M′, M″, M‴, &c.; and $P' \cos \gamma' - g'$; $P'' \cos \gamma'' - g''$; $P''' \cos \gamma''' - g'''$, &c. will be those applied at the points C′, C″, C‴, &c.; and the former, that is the forces acting in the plane of x, y, will be

$$\left\{\begin{aligned} &P' \cos \alpha',\ P'' \cos \alpha'',\ P''' \cos \alpha''', \&c. \\ &P' \cos \beta',\ P'' \cos \beta'',\ P''' \cos \beta''', \&c. \end{aligned}\right.$$

We have, therefore, to consider the conditions of equilibrium in these two classes of forces.

Condit of equi brium.

80. With regard to those which are parallel to the axis of z, the conditions of equilibrium will be the same as we have already found for parallel forces, article 59; which require that we make equal to zero,

the sum of the forces parallel to the axis of z;

the sum of the moments with reference to the plane x, y;

the sum of the moments with reference to the plane x, z.

The first of these conditions gives us

$$\{P' \cos \gamma' - g' + g'\} + \{P'' \cos \gamma'' - g'' + g''\} + \&c. = 0;$$

which reduces to

$$P' \cos \gamma' + P'' \cos \gamma'' + P''' \cos \gamma''' + \&c. = 0. \qquad (45)$$

In order, now, to fulfil the second condition, we have two sets of moments to take; viz.

1st. Those of the forces g', g'', g''', &c. applied to the points M′, M″, M‴, &c. and

2d. Those of the forces $P' \cos \gamma' - g'$, $P'' \cos \gamma'' - g''$, &c. applied to the points C′, C″, C‴, &c.

In the first place, the moment of the force g', (which acts at the point M′ (fig. 48), with reference to the plane of y, z, is $g' \times M'N'$. But $M'N' = B'D' = AG' = x'$; whence the moment sought is $g'x'$.

Again, with regard to the moment of the force $P' \cos \gamma' - g'$, which acts at C', it is obvious that as referred to the plane of y, z, it will be

$$(P' \cos \gamma' - g') \times E'C', \text{ or}$$
$$(P' \cos \gamma' - g') \times a_{,};$$

therefore the sum of the moments of the forces g' and $(P' \cos \gamma' - g'_{,})$ with reference to the plane of y, z, will be represented by

$$g'x' + (P' \cos \gamma' - g')\, a_{,}:$$

or, introducing here the value of a' found in article 78, we shall have

$$g'x' + (P' \cos \gamma' - g')\left(x' - \frac{z' P' \cos a'}{P' \cos \gamma' - g'}\right);$$

or, by performing the multiplication indicated in the formula, this reduces to

$$x' P' \cos \gamma' - z' P' \cos a'.$$

The same form also appertains to each of the forces P'', P''', &c. applied at the points M'', M''', &c. C'', C''', &c.; and the sum therefore, which according to the conditions of equilibrium, is equal to zero, will be expressed by the following equation:

$$P'(x' \cos \gamma' - z' \cos a'_{,})$$
$$+ P''(x'' \cos \gamma'' - z'' \cos a'') + \&c. = 0. \qquad (46)$$

And now, in order to obtain the third equation of equilibrium for the parallel forces, the moment of the force g' applied at M', with regard to the plane of x, z, will be

$$g' \times M'L' = g' \times B'G' = g'y';$$

and that of the force $P' \cos \gamma' - g'$ applied at C', will be

$$(P' \cos \gamma' - g')\, b_{,}:$$

hence we shall have, for the sum of these two moments,

$$g'y' + (P' \cos \gamma' - g')\, b_{,};$$

and substituting for $b_{,}$, from art. 78, this becomes, when reduced,

$$y' P' \cos \gamma' - z' P' \cos \beta':$$

determining again in the same manner the moments of the other forces with reference to the plane of $x\ z$, we shall have, for the third equation of equilibrium,

$$P'(y' \cos \gamma' - z' \cos \beta').$$
$$+ P''(y'' \cos \gamma'' - z \cos \beta) + \&c. = 0. \qquad (47)$$

Whence, collecting the three equations under one point of view, they will be

$$\begin{cases} P' \cos \gamma' + P'' \cos \gamma'' + P''' \cos \gamma''' + \&c. = 0, \\ P'(x' \cos \gamma' - z' \cos a') + P''(x'' \cos \gamma'' - z'' \cos a'') + \&c. = 0, \\ P'(y' \cos \gamma' - z' \cos \beta') + P''(y'' \cos \gamma'' - z'' \cos \beta'') + \&c. = 0. \end{cases}$$

librium the forces the plane z, y.

81. Now in order to find the conditions of equilibrium of the forces situated in x, y, it will be seen that they are the same as those we have already determined for this particular case, art. 66, et seq.; viz.

1. The sum of the forces parallel to the axis of x, ought to be equal to zero.

2. The sum of the same, with reference to the axis of y, equal to zero.

3. The sum of the moments of the forces, with reference to the origin, also equal to zero.

The two first of these conditions are expressed by the equations

$$P' \cos a' + P'' \cos a'' + P''' \cos a''' + \&c. = 0, \qquad (48)$$
$$P' \cos \beta' + P'' \cos \beta'' + P''' \cos \beta''' + \&c. = 0. \qquad (49)$$

And with regard to the third condition, considering first only the point C' (fig. 49), we have the two forces $P' \cos a'$, $P' \cos \beta'$, applied to this point; and taking the moments of these forces with regard to the origin A, the moment of the force $P' \cos a'$, will be

$$P' \cos a' \times AE' = P' \cos a' \times C'F' = P' \cos a' \times b_{,}.$$

In the same manner, the moment of the force $P' \cos \beta'$, will be

$$P' \cos \beta' \times C'E' = P' \cos \beta' \times AF' = P' \cos \beta' \times a_{,}.$$

Since these moments ought to have contrary signs, having tendencies to produce the rotation in different directions; we have, by considering the moment in which $P' \cos a' \times b_{,}$ enters as positive,

$$P' \cos a' \times b_{,} - P' \cos \beta \times a_{,};$$

whence introducing into this expression the values of a and b, as before, we find them changed into the following; viz.

$$P' \cos a' \left(y' - \frac{z' P' \cos \beta'}{P' \cos \gamma' - g'}\right) - P' \cos \beta' \left(x' - \frac{z' P' \cos a'}{P' \cos \gamma' - g'}\right)$$

which, by multiplying, reduce to

$$y' P' \cos a' - x' P' \cos \beta':$$

and proceeding in the same manner with the other forces as applied to the points C'', C''', &c. we shall have, for the last equation of equilibrium,

$$P'(y' \cos a' - x' \cos \beta')$$
$$+ P''(y'' \cos a'' - x'' \cos \beta'') + \&c. - 0. \qquad (50)$$

General equations of equilibrium.

Collecting, now, all our six last equations of equilibrium, and adopting the notation explained under article 67, we may write them thus:

$$\Sigma (P \cos a) = 0.$$
$$\Sigma (P \cos \beta) = 0.$$
$$\Sigma (P \cos \gamma) = 0.$$
$$\Sigma P (y \cos a - x \cos \beta) = 0.$$
$$\Sigma P (x \cos \gamma - z \cos a) = 0.$$
$$\Sigma P (y \cos \gamma - z \cos \beta) = 0.$$

Such are the equations of equilibrium of a solid body, or system of bodies, free to move in every direction, and acted upon by forces any where situated in space.

Let us now consider those cases in which the system is not wholly free; as for example, when it is constrained to revolve about a fixed point, a fixed axis, &c.

Equilibrium of a system having a fixed point.

82. When there is a fixed point in the system, all the above equations are not necessary. For example, if we suppose the origin in this point, there will be an equilibrium between the forces situated in the plane of x, y, if the system of these forces cannot turn about the fixed point; and this condition will be fulfilled, if we have

$$\Sigma P (y \cos a - x \cos \beta) = 0.$$

It then only remains to find the conditions of equilibrium of the forces parallel to the axis z. In order to this, let x, y, and 0, be the co-ordinates of the point where the resultant of the parallel forces meet the plane of x, y; then in whatever part it may be situated, it follows, from the property of parallel forces, that the moment of this resultant with reference to one of the planes of x, z, and of y, z, is equal to the sum of the moments of the parallel forces as referred to this plane; consequently we shall have

$$R a_{,} = \Sigma P (x \cos \gamma - z \cos a)$$
$$R b_{,} = \Sigma P (y \cos \gamma - z \cos \beta).$$

Now in order that there may be an equilibrium between the parallel forces, it is necessary that their resultant pass through the fixed point, which is also their origin; we must therefore have $a_{,} = 0$, $b_{,} = 0$, which hypothesis reduces the preceding equations to

$$\Sigma P (x \cos \gamma - z \cos a) = 0,$$
$$\Sigma P (y \cos \gamma - z \cos \beta) = 0.$$

Thus, when there is a fixed point in the system, there will be an equilibrium between all the forces, when the equations Nos. 54, 55, and 56, are made to obtain.

Equilibrium in a system having a fixed axis.

83. If the system is only free to turn about a fixed axis, by supposing this axis to coincide with that of z, all the forces parallel to it will be destroyed, and there will remain only the forces directed according to the plane of x, y. Now, in order that these forces may be in equilibrio, it will be sufficient that their resultant pass through the point A, which is fixed as appertaining to the axis Az. The necessary condition, in order that the resultant should pass through this point is, as we have before seen, that we have

$$\Sigma P(y \cos \alpha - x \cos \beta) = 0;$$

which equation alone, therefore, will be sufficient for establishing an equilibrium in a system having a fixed axis of rotation.

In the above, z is supposed to be the fixed axis; if x be that axis, then we have the symmetrical equation

$$\Sigma P(y \cos \gamma - z \cos \beta) = 0;$$

and if y is the fixed axis, the equation is

$$\Sigma P(x \cos \gamma - z \cos \alpha) = 0.$$

84. We may also suppose a system which is only free to slide along a fixed axis; and in this case, the condition of equilibrium will obviously be fulfilled, if we have

$$\Sigma P(\cos \gamma) = 0.$$

Particular deductions.

85. Comparing the conditions of equilibrium of a system which is moveable about a fixed axis, with those which have place when the system is moveable about a fixed point; we shall find, *that the equilibrium about the fixed point will obtain, provided the same has place about each of the axes of the system.*

Forces acting on a plane.

86. With regard to forces acting on a fixed plane, it is evident, that those which are perpendicular, are destroyed by the resistance of the plane; whence the conditions of equilibrium in this case are reduced to those of forces situated in a plane; and consequently we have here

$$\Sigma P(\cos \alpha) = 0,$$
$$\Sigma P(\cos \beta) = 0,$$
$$\Sigma P(\cos \alpha) - x \cos \beta) = 0.$$

87. But if the body rest upon the plane, so that it may overset, then we must add to the above three conditions, another, viz. that the resultant of the forces perpendicular to the plane, shall pass through a point common both to the plane and to the body, or that it falls within the figure formed by joining the points of contact with each other.

Example.

88. We propose, now, to conclude this section, by the solution of the following problem.

To find the equation of condition, which must have place in order that all the forces situated in space may have only one resultant.

There will be but one resultant in the system, if the resultant of the forces parallel to the axis of z meets the plane of x, y, in a point E (fig. 50), which is in the resultant of the forces situated in that plane. Now, in order to express this condition, it must first be observed that when the equilibrium has place, it must also exist between the forces parallel to the axes of z, and the conditions of equilibrium of these forces are (art. 80)

$$P \cos \gamma + P' \cos \gamma' + P'' \cos \gamma'' + \&c. = 0,$$
$$P(x \cos \gamma - z \cos \alpha) + P'(x' \cos \gamma' - z' \cos \alpha') + \&c. = 0,$$
$$P(y \cos \gamma - z \cos \beta) + P'(y' \cos \gamma' - z' \cos \beta') + \&c. = 0.$$

Now, if we consider the first of these forces as equal and directly opposed to the resultant of all the others, we shall have, in order to determine the resultant of the parallel forces,

$$P \cos \gamma = P' \cos \gamma' + P'' \cos \gamma'' + \&c.$$
$$P(x \cos \gamma - z \cos \alpha) = P'(x' \cos \gamma' - z' \cos \alpha') + \&c.$$
$$P(y \cos \gamma - z \cos \beta) = P'(y' \cos \gamma' - z' \cos \beta) + \&c.$$

If the point of application of this resultant is in the plane of x, y, and the three co-ordinates of this point be $x_{,}$, $y_{,}$, and 0, this will reduce the above equations to

$$P \cos \gamma = P' \cos \gamma' + P'' \cos \gamma'' + \&c.$$
$$P x_{,} \cos \gamma = P'(x' \cos \gamma' - z' \cos \alpha') + \&c.$$
$$P y_{,} \cos \gamma = P'(y' \cos \gamma' - z' \cos \beta') + \&c.:$$

or, making $P \cos \gamma = Z$, $P'(x' \cos \gamma' - z' \cos \alpha') = M$, and $P'(y' \cos \gamma' - z' \cos \beta') = N$; they are still more simply expressed by

$$Z = P' \cos \gamma' +, \&c.$$
$$Z x_{,} = M,$$
$$Z y_{,} = N;$$

whence we immediately deduce

$$x_{,} = \frac{M}{Z}, \text{ and } y_{,} = \frac{N}{Z}.$$

Having thus determined the co-ordinates $x_{,}$ and $y_{,}$, where the resultant of the parallel forces meet the plane of x, y, we must next express the condition necessary, in order that this point may be found in the resultant of the forces situated in the plane of x, y: the equation of this resultant (art. 72) is

$$Xy - xY = \Sigma P(y \cos \alpha - x \cos \beta);$$

or, denoting the latter term by L,

$$Xy - xY = L;$$

and substituting in this equation, instead of x and y, the values of $x_{,}$ and $y_{,}$ above determined; this last equation becomes

$$\frac{XN}{Z} - \frac{MY}{Z} = L;$$

or, which is the same,

$$XN = LX + MY:$$

and therefore, when this equation obtains, the forces will be reduced to one resultant only; excepting, however, the particular instance when $X = 0$, $Y = 0$, and $Z = 0$.

We have seen (art. 74) that the equations $X = 0$, $Y = 0$, indicates that the forces situated in the plane of x, y, may be reduced to two components R', R'', equal and acting in opposite directions; and by a similar proceeding, we may also reduce the forces parallel to the axis z to two forces Z' and Z'', equal and acting in contrary directions. Whence, in the case where we have only $X = 0$, $Y = 0$, the system of all our forces becomes that of the four forces R', R'', Z', Z'', which we may reduce to two equal forces acting in contrary or opposite directions.

§ X. *On the equilibrium of flexible bodies.*

89. Hitherto our investigations have been made with reference only to the equilibrium of forces, as exerted upon certain fixed points connected with each other by inflexible right lines in an invariable manner; and it remains now, in order to complete the doctrine of equilibrium, to consider the same with reference to points connected by flexible lines, which may be considered as a distinct branch of the general theory of statics.

In order to simplify our investigations, we shall here adopt the hypothesis, that these lines or cords are perfectly flexible; that they are void of gravity; inextensible; and that they are reduced to their axes: or that they have no sensible lateral dimensions.

This being supposed, we shall commence by observing, that if a cord be fixed at one end, and have a weight P suspended at the other, the tension of that line is measured by, and is said to be equal to the weight P; and if the fixed point be removed, and another equal force or weight Q be substituted for it, the tension of the cord is still the same as before, for the re-action of the fixed point being exactly equal to the weight P, it is now merely replaced by the action of the equal weight Q; and if Q exceed P, the tension of the line will still have the same measure; for the excess of Q above P, will only be employed in producing motion in the system, in the direction of the greater force.

We may further observe, that the conditions of equilibrium, which exist between three cords united together by a *fixed* knot, are the same as those of any three other forces soliciting a material point; that is, one of these forces must be equal, and directly opposed, to the resultant of the other two; and if there be any greater number of cords united in the same manner, they may be reduced, as we have already shewn (article 31) to that of three forces only: we shall, therefore, in what follows, suppose this reduction to have been effected, and shall only consider the action of three forces at each knot; for the present, also, we shall treat of the conditions of equilibrium in the case of *fixed* knots only, reserving for another place those investigations which have reference to running knots or moveable rings.

90. Let us then suppose a cord perfectly flexible and inextensible, and of any length; of which, K and K′ (fig. 51), are the two extreme points: let m, m', m'', &c. be other points in the same cord; and suppose the cords $m\,h$, $m'h'$, $m''h''$, &c. attached to these points, in the direction of which the forces P, P′, P″, &c. are to act: let us apply also to the point m, a given force A, acting in the direction of the cord Km; and to the last of the points m, m', &c. another given force A′ directed towards the point K.

unicular polygon.

Now, in the case of an equilibrium, the inextensible cord will form itself into a certain polygonal figure, commonly called the *funicular polygon;* and of which the angular points or summits will be K, m, m', m'', &c.: and the question is to determine the figure of this polygon, and the relation and direction of the forces, in order that an equilibrium may be established in the system.

In order to discover these relations, or the conditions of equilibrium, we shall proceed upon this evident principle, viz. that when the equilibrium obtains, each of the cords $m\,m'$, $m'm''$, $m''m'''$, &c. is drawn at its two extremities by equal forces, acting in the direction of its prolongation.

It immediately follows from this, that the resultant of the two forces A and P, applied to the point m, ought to coincide with the prolongation mf, of the cord $m\,m'$. We may therefore transfer the point of application of this force to the point m', situated in its direction; and at that point compound it with the force P′, applied at the same point in the direction $m'h'$; this second resultant will therefore be that of the three forces A, P, P′, and will coincide with the prolongation $m'f'$, of the cord $m'm''$. Whence we may again transfer this resultant to the point m'', which is necessarily in its direction. Here again the resultant of these three forces, and the force P″ being compounded as before, the new resultant will be that of the four forces A, P, P′, P″, its point of application m'', and its direction in the prolongation $m''f''$. Proceeding in a similar manner from the other extremity of the system, it will appear, that the force which draws the same cord at its other extremity m''', and which ought to coincide with the prolongation $m'''f'''$ of this cord, is the resultant of the forces P‴, P^{iv}, P^{v}, and A′. These two resultants are therefore equal and directly opposed to each other, and consequently the resultant of all the forces A, P, P′, P″, &c. .. A′ is equal to zero. And we shall evidently arrive at the same conclusion relative to the forces acting at the extremities of any of the other cords.

Hence then it follows, that the forces applied to the funicular polygon, ought to be such, that in transferring them, parallel to themselves, to one and the same point, they would, in that point, produce an equilibrium. Now the condition of equilibrium in this case is, as we have shewn (article 43), involved in three equations of the form

$$\left.\begin{array}{l} \mathrm{A}\cos a + \mathrm{A}'\cos a' + \mathrm{P}\cos\alpha + \mathrm{P}'\cos\alpha' +, \&c. = 0 \\ \mathrm{A}\cos b + \mathrm{A}'\cos b' + \mathrm{P}\cos\beta + \mathrm{P}'\cos\beta' +, \&c. = 0 \\ \mathrm{A}\cos c + \mathrm{A}'\cos c' + \mathrm{P}\cos\gamma + \mathrm{P}'\cos\gamma' +, \&c. = 0 \end{array}\right\}$$

where a, a', α, α', &c. denote the angles relative to one axis; b, b', β, β', &c. those relative to a second; and c, c', γ, γ', &c those relative to the third.

91. When the given forces A, P, P′, &c... A′, and the directions in which the cords act, will not satisfy these equations, it will be impossible, whatever figure we give to the polygon, to produce an equilibrium; and on the other hand, when these conditions are found to obtain, we may always give to the polygon such a figure that the equilibrium shall be established, and this independent of the lengths of the sides $m\,m'$, $m'm''$, &c. But if the magnitude of the forces, and the lengths of the several cords $m\,m'$, $m'm''$, $m''m'''$, &c. be given, then the figure of the polygon is determinable, and the construction of it may be effected as follows:

Knowing the direction of the cords Km, and $h\,m$, according to which the two given forces A and P act, it is easy to determine the magnitude and direction of the resultant. On the prolongation of this direction, set off from the point m, the given line $m\,m'$; which being done, apply to the point m', the resultant of A and P, on the line $m\,m'$, and the force P′ in the given direction of the cord $m'h'$; take the resultant of these two forces, which will give the direction of the cord $m'm''$; and on this line, from the point m', setting off the given line $m\,m''$, will determine the point m''. Now applying at the point m'', the force P″, acting in the direction $m''h''$, and compounding it with the resultant whose intensity and direction are both given, we may determine in the same manner the point m'''.

Continuing to proceed in the same way, we shall ultimately arrive at the last of the knots m, m', m'', &c. which we will suppose to be m^{v}; so that m^{v}K′ may be the last side of the polygon: now the direction of this line is known, since it is supposed to represent the force A′, which is given; consequently the prolongation of the direction of the resultant of the two forces, applied at the point m^{v}, in the directions $m^{\mathrm{iv}}m^{\mathrm{v}}$, and $h^{\mathrm{v}}m^{\mathrm{v}}$, must coincide with the given direction $m'K'$, of the force A'; and which coincidence must obviously have place when an equilibrium obtains in the system.

If we draw through the point K, the three axes to which we refer the several directions of the forces, viz. the angles a, a', α, α', α'', &c. b, b', β, β', &c. c, c', γ, γ', &c.; and if we consider the co-ordinates of the several points m, m', m'', &c. parallel to these axes, it is obvious that these co-ordinates will be determinate functions of the angles, and the intensities of the forces A, P, P′, &c. and of the lengths of the lines mm', $m'm''$, &c.; and we might therefore calculate, in functions of these quantities, the three co-ordinates of each point; but as the computation involves very considerable intricacy, and the ultimate expressions being extremely complicated, we shall dispense with the operation in this place.

When the given forces fulfil the conditions of the preceding equations, and if we give to the polygon the figure necessary to establish the equilibrium; we know, before applying these forces, that this state will exist; and we may therefore previously ascertain the tension of each of the cords mm', $m'm''$, &c. and proportion their diameters accordingly; for each of these cords will be drawn in the direction of its length, by two equal and contrary forces, applied at its extremities; and one of these equal forces will, as we have seen (art. 89), express the tension which the cord has to sustain.

It appears, from the preceding investigations, that the tensions of the different sides of the polygon, vary according to their respective situations. The tension of the side mm' is, for example, equal to the resultant of the forces A and P, or of the forces P″, P‴, &c. . . A′; that of the side mm', will be equal to the resultant of the forces A, P, P′, or of P″, P‴, . . A′; and so on; whence it will be easy to determine in each particular case, the tension sustained by every side of the polygon, when the magnitude and the direction of the forces are given.

92. If the two extreme points K, K′, of the polygon be fixed, then it is obvious that the equilibrium will always obtain, whatever may be the intensity and direction of the other forces; for, in this case, the two extreme forces which we have denoted by A and A′, and the angles which their directions make with the three axes, will not form any part of the data of the problem, being themselves determinable from the conditions of the equilibrium. The quantities to be found, therefore, are obviously A, A′, and the angles a, a', b, b', c and c'; viz. they are eight in number, and will consequently require eight equations: viz. first, the three equations of equilibrium given in art. 90; secondly, those of art. 33; viz.

$$\left.\begin{array}{l}\cos^2 a + \cos^2 b + \cos^2 c = 1 \\ \cos^2 a' + \cos^2 b' + \cos^2 c' = 1\end{array}\right\}$$

and three others resulting from the given positions of the two fixed points K, K′; that is to say, the co-ordinates of the point K′, as referred to the three axes drawn through the other point K being given, these must be equated with the expressions for the same, calculated upon the principles indicated in the conclusion of art. 91. Consequently, there being as many equations as there are unknown quantities, these latter may always be determined, and the equilibrium established.

Sliding knots.

93. Whether the points K, K′, be fixed or free; if one or many of the fixed knots m, m', m'', &c. be replaced by sliding knots, or moveable rings, this circumstance will give place to new conditions of equilibrium. Let us suppose, for example, that m'' is now a sliding knot, or a moveable ring, which glides freely along the cord m', m'', m''', &c.; it is, in the first place, clear that in this motion the sum of the distances $m'\,m''$, and $m''\,m'''$ of the point m'', from the points m', m''', will be constant. Now, if the equilibrium exist, it is obvious that this state of the system will not be disturbed by considering the last two points as fixed; and in this case it is also evident, that the point m'' will be constrained in its motion to pass over a part of the surface of an ellipsoid of revolution, of which m' and m''' are the two foci, and of which the greater axis is equal to the given length of the cord $m'\,m''\,m'''$. This point cannot therefore be in equilibrio while the force P″ applied to it is resolvable into two components, one of which is in the direction of the element of the curve; but this may always be done, except when the direction of the force is a normal to the curve. Whence it follows that the direction of the force P″, in case of an equilibrium, must be a normal to the ellipse; and hence, by a known property of that curve, it will bisect the angle formed by the *radii vectores* $m'\,m''$ and $m''\,m'''$.

Therefore, when, in executing the construction indicated in art. 91, we meet with a moveable ring, such as we have supposed at m'', and we have taken the resultant of the two forces acting in the direction $m''\,m'$ and $m''\,h''$, which will determine the direction of the cord $m''\,m'''$; if we find the angles $h''\,m''\,m'$, and $h''\,m''\,m'''$ are not equal to each other, it must be concluded that the equilibrium cannot obtain. If the direction of the force attached to the moveable ring be not given, then we may always determine it so that the equilibrium shall subsist, by giving to it such a direction as may fulfil the necessary conditions; viz. of the equality of the angles $h''\,m''\,m'$ and $h''\,m''\,m'''$.

We may also observe, that in a state of equilibrium the tensions of the cords adjacent to the moveable ring will be equal to each other, in consequence of each of them making equal angles with the force applied to this ring; the same is otherwise obvious, since, in fact, the two cords in this case may be considered as forming only one and the same cord, which has therefore necessarily the same tension throughout.

General principl

94. What is stated above, relative to a ring constrained to slide along an inextensible line, may be extended to all the points of a system in equilibrio, whatever may be the connection of these points with each other; for we shall not disturb this equilibrium by supposing all the points of the system fixed except one only. And if the connection of this point with the others be such, that it may still describe a surface, or only a curve line, about these fixed points; it is evident that the moveable point will be situated exactly in the same manner as if the surface or curve really existed; and consequently the force, which is applied to it so as to produce an equilibrium, must necessarily be a normal to that curve.

Hence, then, we may conclude, that in every system in equilibrio, the force applied at each point of the system is perpendicular to the surface, or to the curve on which each point would be constrained to move, if all the other points with which it is connected were for a moment considered as fixed.

When this condition relative to the direction of the forces, and to the connection of the parts of the system, is not fulfilled, we may be certain that an equilibrium will not obtain; although we cannot draw the opposite conclusion when it is, for this condition alone is not sufficient to assure the equilibrium.

ertical ights.

95. Let us consider now the particular case where all the forces P, P′ P″, &c. applied to the funicular polygon, are weights; and where the two extreme points, K and K′, are conceived to be fixed; and for the greater simplicity, we shall suppose that there is no sliding or moveable ring in the system, but that the several points $m\ m'\ m''$, &c. (fig. 52) are all fixed knots.

We have seen, that in this case the equilibrium is always possible; and we have therefore only to investigate the figure of the polygon proper to this state. Now first it may be observed, that all these sides will be in the same plane; that is to say, in the vertical plane passing through the two fixed points K, K′. This is obvious from our construction, article 91; for the three cords which proceed from each of the knots $m\ m'\ m''$ are in the same plane, and the plane of the cords K m, $h\ m$, and $m\,m'$ is vertical, because it contains in it the vertical line of direction $h\ m$, of the weight P. The plane of the cords $m\,m'$, $h'\ m'$, and $m'\ m''$, is also a vertical plane, because it contains in it the line of direction $h'\ m'$ of the weight P′; and as the two vertical planes have in them one common line $m'\,m$, they necessarily coincide. Hence the three sides K m, $m\ m'$, and $m'\ m''$, are in the same vertical plane; and the same may obviously be demonstrated with regard to all the other sides of the polygon.

This being established, draw in this plane and through the point K, the two axes K x, K y, to which we refer the angles a, α', α'', &c.; a', b, β, β', β'', &c. b', (art. 90); the angles c, γ, γ', γ'', and c', being all right angles. And if we take the axes K x horizontal, and K y vertical, the angles a, α', α'', &c. will also be right angles, and the angles β, β', β'', &c. each equal to zero; the equations therefore, above referred to, become

$$\mathrm{A} \cos a + \mathrm{A}' \cos a' = 0$$
$$\mathrm{A} \cos b + \mathrm{A}' \cos b' + \Pi = 0;$$

Π denoting the sum of the weights P + P′ + P″, &c.

Tension of the cords.

Let now T denote the tension of any side of the polygon; then, if we consider this force as applied to that extremity of the side which is nearest to the point K; and if we represent by u the angle which its direction makes with the horizontal axis, and by v that which it makes with the vertical axis; and by p the sum of the weights suspended to the polygon from the point K, to the first extremity, inclusive: the force T will be the resultant of the force A and of the weight p (art. 91); consequently we shall have,

$$\left.\begin{array}{l}\mathrm{T} \cos u = \mathrm{A} \cos a, \\ \mathrm{T} \cos v = \mathrm{A} \cos b + p'.\end{array}\right\} \quad (51)$$

If, on the contrary, we regard the force T as applied to the other extremity of the side of which it expresses the tension, its direction will be opposed to that which it had at first; we must therefore replace the angles u and v by their supplements; in which case it will then be the resultant of the force A′, and of the weights whose sum is $\Pi - p$; and we have therefore

$$\left.\begin{array}{l}-\mathrm{T} \cos u = \mathrm{A}' \cos a', \\ -\mathrm{T} \cos v = \mathrm{A}' \cos b' + (\Pi - p).\end{array}\right\} \quad (52)$$

By means of these values of T cos u, and T cos v, we may determine immediately the tension and the direction of each side of the polygon, after the values of the unknowns A, A′, a, a', b, b', c, c', shall have been calculated according to the method indicated in art. 91.

Catenary.

96. *Of the catenary.*—If a perfectly flexible heavy chain be suspended to any two fixed points at its extremities, it will form itself into a certain curve, which is called the *catenary*, from *catena*, a chain: the properties of which, as being intimately related to our preceding enquiries, we here propose to investigate.

In the first place, it is obvious, that all which has been said in the preceding article being independent of the number and lengths of the sides of the polygon, ought equally to obtain when this number becomes indefinitely great, and each of the sides indefinitely small.

Let us then suppose K n K′ (fig. 53) to be a heavy chain suspended to two fixed points K and K′; and let us suppose this to be decomposed into an infinite number of parts; and let P, P′, P″, &c. be the weights of these elements; p the weight of the part of the chain commencing at the point K, and terminating at any point n. And if we suppose the chain homogeneous, and of an equal thickness through its whole extent, this weight will be proportional to the length of that portion of the chain; so that, denoting the length of the arc K n by s and by h, a constant co-efficient, expressing the weight of any unit of length of the chain; we shall have $p = h\,s$.

After this decomposition, the *catenary* is the same as a polygon of an infinite number of sides, each being indefinitely small; and the side of the polygon which is in contact with the point n, will sustain a tension T, exerted in the direction of the prolongation of the side; which, in the present case, is the tangent of the curve at the same point n. Hence the values T cos u, and T cos v, of the preceding article, will give us the direction of this tangent, and the tension which has place at the point n; that is, at the extremity of the arc s. If, therefore, we take in the plane of the catenary the horizontal axis K x, and the vertical axis K y, for those of the co-ordinates; and if we represent by x and y the absciss and ordinate of the point n, and supposing $n\,t$ the tangent at the point n; we know that the differential co-efficient $\frac{dy}{dx}$, or the fluxional co-efficient $\frac{\dot{y}}{\dot{x}}$, will express the tangent of the angle K $t\,n$. But because the line $n\,t$ represents also the direction of the tension T at the point n, which is determined by the angles u and v; this tangent is also equal to, or is expressed by the ratio $\frac{\cos v}{\cos u}$. But we have seen (art. 95) that

$$\mathrm{T} \cos u = \mathrm{A} \cos a,$$
$$\mathrm{T} \cos v = \mathrm{A} \cos b + p.$$

Whence

$$\frac{\dot{y}}{\dot{x}} = \frac{\cos v}{\cos u} = \frac{\mathrm{A} \cos b + p}{\mathrm{A} \cos a}.$$

Or replacing p by its equivalent $h\,s$, we have

$$\frac{\dot{y}}{\dot{x}} = \frac{\mathrm{A} \cos b + h\,s}{\mathrm{A} \cos a}.$$

Or,

$$\mathrm{A} \cos a\,\dot{y} - \mathrm{A} \cos b\,\dot{x} = h\,s\,\dot{x} \qquad (53)$$

Let now K a be the tangent to the point K, and make the angle n K $a = c$, then y K $a = 90° - c$; and we shall have $\cos a = -\cos c$, $\cos b = -\sin c$. For the direction of the force A is the same as that of the prolongation of the infinitely small side of the polygon coinciding with K; or, which is the same, the prolongation of the tangent to that point; and therefore the angles x K a, and y K a are those we have before denoted by a and b; viz. the angles by which we refer the direction of the force A to the two rectangular coordinates A x and A y. And as the angles x K a and x K a' are supplements of each other, as also the angles y K a and y K a; it follows that cos a and cos b are equal, but have contrary signs to cos c and sin c. Employing therefore the angle c, instead of the angles a and b, the preceding equation will become

$$A \sin c\,\dot{x} - A \cos c\,\dot{y} = h s \dot{x}, \qquad (54)$$

Equation of. which is the equation of the catenary; A and c being unknown constant quantities which still remain to be determined.

Tension. 97. By making the above substitutions in equations (51), these become

$$T \cos u = -A \cos c,$$
$$T \cos v = -A \sin c + h s;$$

and by squaring

$$T^2 \cos^2 u = A^2 \cos^2 c,$$
$$T^2 \cos^2 v = A^2 \sin^2 c - 2\,A\,hs.\,\sin c + h^2 s^2.$$

Adding these, and observing that $\sin^2 c + \cos^2 c = 1$, and $\cos^2 u + \cos^2 v = 1$, we have

$$T^2 = A^2 - 2\,A h s.\,\sin c + h^2 s^2. \qquad (55)$$

From which it is obvious, that when $s = 0$, A will express the tension at the point K, or its action upon the fixed point, as we know already: we see, also, that the tension is variable from one point to another, and that it is a minimum when $-2\,A h s.\,\sin c + h^2 s^2$ is a maximum; that is, when $-2\,A\,h.\,\sin c.\,\dot{s} + 2h^2 s\,\dot{s} = 0$; or when $s = \frac{A.\,\sin c}{h}$.

Substituting, therefore, this value of s in equation (55), we have for the minimum tension

$$T^2 = A^2 (1 - \sin^2 c) \text{ or } T^2 = A^2 \cos^2 c;$$

or $\quad T = A \cos c.$

But we have seen that generally

$$T \cos u = A \cos c.$$

Consequently when the tension is a minimum, $\cos u = 1$, or angle $u = 0$; that is, the tangent is then horizontal, which answers to the lowest point of the curve.

98. Our equation (54) will admit of a more simple form by eliminating the arc s; which may be done by putting that equation into fluxions, and observing that $\dot{s} = \sqrt{(\dot{x}^2 + \dot{y}^2)}$; while $\dot{x}$ is considered constant. This operation gives

$$-A \cos c\,\ddot{y} = h \dot{x} \sqrt{(\dot{x}^2 + \dot{y}^2)};$$

and multiplying both sides by

$$\frac{\dot{y}}{\sqrt{(\dot{x}^2 + \dot{y}^2)}},$$

it becomes

$$\frac{-A \cos c\,\dot{y}\,\ddot{y}}{\sqrt{(\dot{x}^2 + \dot{y}^2)}} = h \dot{y} \dot{x}.$$

But $\frac{\dot{y}\;\ddot{y}}{\sqrt{(\dot{x}^2 + \dot{y}^2)}}$ = the fluxion[*] of $\sqrt{(\dot{x}^2 + \dot{y}^2)}$: denoting therefore the latter by f $\sqrt{(\dot{x}^2 + \dot{y}^2)}$, we have

$$-A \cos c\,\mathrm{f} \sqrt{(\dot{x}^2 + \dot{y}^2)} = h \dot{y} \dot{x};$$

and taking the fluents on both sides, we have

$$-A \cos c \sqrt{(\dot{x}^2 + \dot{y}^2)} = h y \dot{x} + C \dot{x},$$

C being a constant arbitrary quantity.

In order to determine its value, we observe that at the point K, where $y = 0$, this equation reduces to

$$-A \cos c \sqrt{(\dot{x}^2 + \dot{y}^2)} = C \dot{x};$$

but at the same point we have also $\frac{\dot{y}}{\dot{x}} = \tan c$, or $\dot{y} = \tan c\,\dot{x}$; where, by substituting and dividing by $\dot{x}$, we have

$$-A \cos c \sqrt{(1 + \tan^2 c)} = C;$$

that is, $C = -A$; and hence we have for the fluxional equation of the catenary,

$$A \cos c \sqrt{(\dot{x}^2 + \dot{y}^2)} = (A - hy)\,\dot{x}, \qquad (56)$$

at the lowest point of the curve $\dot{y} = 0$; and consequently $\sqrt{(\dot{x}^2 + \dot{y}^2)} = \dot{x}$ at that point; we have therefore, for the corresponding value of y,

$$A \cos c\,\dot{x} = (A - hy)\,\dot{x};$$

whence $\quad y = \frac{A\,(r - \cos c)}{h}.$

Now the general equation of the catenary we have seen equation (54) is

$$A \sin c\,\dot{x} - A \cos c\,\dot{y} = h s \dot{x};$$

and if we substitute the value of $\dot{y}$ from this last equation into equation (56); we shall obtain the following value of s, viz. Length of arc.

$$s = \frac{A \sin c}{h} \mp \frac{\sqrt{\{(A - hy)^2 - A^2 \cos^2 c\}}}{h}. \qquad (57)$$

In which expression, the upper sign *minus* must be used from $y = 0$, to $y = \frac{A}{h}$; that is from the commencement to the lowest point of the curve; and beyond that point we must employ the *plus* or lower sign.

It is clear from the above equation, that the curve of the catenary is rectifiable.

99. Hitherto we have only determined the fluxional equation of the curve; and in order now to exhibit the same in a finite form, we may proceed as follows:

Resolving equation (56), with reference to $\dot{x}$, we find

$$\dot{x} = \frac{\pm A \cos c\,\dot{y}}{\sqrt{\{(A - h y)^2 - A^2 \cos^2 c}};$$

and as the absciss x is always increasing, it is evident that we must take the upper sign, while the fluxion $\dot{y}$ is positive, and the lower sign when $\dot{y}$ is negative, in order to preserve, in all cases, the positive sign to $\dot{x}$; that is, the upper sign must be employed from the origin K, to the lowest point of the curve; after which, as y is then decreasing, and therefore $\dot{y}$ negative, we must take the lower or negative sign.

Taking the fluent of the preceding fluxional equation, we have

$$x = C' + \frac{A \cos c}{h} h \log.\{A - hy \mp \surd)\,(A - hy)^2 - A^2 \cos^2 c)\};$$

C′ being an arbitrary constant quantity, or as we usually term it, the correction.

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[*] We have no fixed notation for indicating the fluxion of a compound quantity, except by inclosing that quantity in a parenthesis, and placing the fluxional mark over it; which is certainly not sufficiently conspicuous. We shall therefore, in the present treatise, where such a notation is necessary, make use of the roman f for fluxion, in exactly the same sense as the French writers employ the d to denote a differential.

Now when $x = 0$, and $y = 0$, that is to say, at the point K, and where the radical is positive this gives

$$0 = C' + \frac{A \cos c}{h} h \log. (A - A \sin c);$$

or,

$$C' = -\frac{A \cos c}{h} h \log. (A - A \sin c).$$

Substituting this for C′ in the above equation, we obtain

$$x = \frac{A \cos c}{h} h \log. \left\{ \frac{A - hy \mp \sqrt{\{(A - hy)^2 - A^2 \cos^2 c\}}}{A(1 - \sin c)} \right\} \quad 58$$

the curve is therefore transcendental, of the same order as the logarithmic curve.

If, in this equation, we substitute the value of the ordinate as before determined for the lowest point; viz.

$$y = \frac{A(1 - \cos c)}{h};$$

we have for the corresponding value of x,

$$x = \frac{A \cos c}{h} h \log. \frac{\cos c}{1 - \sin c}.$$

100. It still remains to determine the values of A and c, which depend on the position of the point K′ with reference to K. If we suppose, as the simplest case, that the two points of suspension are in the same horizontal line; it is obvious that the two branches of the curve, comprised between the lowest point and the two fixed points, will be similar, and each equal to half the length of the chain; and the absciss of the lowest point will be equal to half the given distance K K′. Let this given length and given distance be respectively denoted by l and l'; then we have

$$\tfrac{1}{2} l = \frac{A \sin c}{h},$$

$$\tfrac{1}{2} l' = \frac{A \cos c}{h} h \log. \frac{\cos c}{1 - \sin c};$$

whence, by division,

$$\frac{l'}{l} = \frac{\cos c}{\sin c} h \log. \frac{\cos c}{1 - \sin c}; \qquad (59)$$

from which the value of c may be determined by approximation; and this being known, A may be found from either of the above equations.

101. We shall now, in conclusion, place under one point of view, the principal deductions drawn from the preceding investigations;

Let l denote the length of a heavy chain;

l' the distance of its two points of suspension, supposed in the same horizontal line;

c the angle comprised between the horizontal line or distance and the tangent at the point of suspension;

A the tension of the chain at the same point;

T the tension at any other point;

x any variable absciss;

y its corresponding ordinate, &c.;

s its corresponding arc;

h the weight of one inch, or one foot of the chain;

x, y and s being estimated by the same unit of measure;

and y' the greatest ordinate.

1. Then, first, to find c, we have

$$\frac{l'}{l} = \frac{\cos c}{\sin c} h \log. \frac{\cos c}{1 - \sin c};$$

but as this equation is transcendental, c must be found by approximation, and cannot be expressed in finite terms.

2. To find A, we have

$$\frac{A \sin c}{h} = \tfrac{1}{2} l, \text{ or}$$

$$A = \frac{h\,l}{2 \sin c};$$

the tension at the point of suspension.

3. And for the tension (T) at any other point x, corresponding to a given arc s, we have

$$T = \sqrt{\{A^2 - 2 A h s . \sin c + h^2 s^2\}};$$

which, at the lowest point of the curve, becomes

$$T = A \cos c.$$

4. The general equation of the curve, in terms of x and y, is

$$x = \frac{A \cos c}{h} h \log. \left\{ \frac{A - hy \mp \sqrt{\{(A - hy)^2 - A^2 \cos^2 c\}}}{A(1 - \sin c} \right.;$$

from which, either x or y may be determined, the one from the other.

5. At the lowest point of the curve, we have, for the corresponding ordinate,

$$y' = \frac{A(1 - \cos c)}{h}.$$

6. To which we may add,

$$\tfrac{1}{2} l = \frac{A \sin c}{h}; \text{ and } \tfrac{1}{2} l' = \frac{A \cos c}{h} h \log. \frac{\cos c}{1 - \sin c};$$

from which two equations, our first equation above, is deduced; and,

7. For the length of any arc s, corresponding to the absciss x and ordinate y, we have

$$s = \frac{A \sin c}{h} \mp \frac{\sqrt{\{(A - hy)^2 - A^2 \cos^2 c\}}}{h}.$$

All the principal properties of the catenary, both geometrical and mechanical, are involved in the preceding equations.

Let us propose the following problem by way of illustration.

PROBLEM.

102. A heavy flexible chain, 100 feet in length, and weighing 1,000 lbs. is suspended at its extremities to two fixed points in the same horizontal line, 95 ft. 1 ½ in. asunder. It is required to determine the greatest depth of the curve, the tension at the lowest point, and its tensions at the points of suspension. Example.

Here, first, 95 ft. 1 ½ in. = 95·125; and by the formula $\frac{\cos c}{\sin c} h \log. \frac{\cos c}{1 - \sin c} = \frac{l'}{l} = \frac{95{\cdot}125}{100}$,

we have $\frac{\cos c}{\sin c} h \log. \frac{\cos c}{1 - \sin c} = {\cdot}95125$;

which, by approximation, gives $\cos c = {\cdot}866$ and $\sin c = {\cdot}5$ very nearly.

Again, since 100 feet weighs 1000 lb. $h = 10$; and hence

$$A = \frac{h\,l}{2 \sin c} = \frac{10 \times 100}{\cdot 1} = 1{,}000 \text{ lbs.}$$

for the tension at the point of suspension.

And $T = A \cos c = 1000 \times {\cdot}866 = {\cdot}866$ lbs.

the tension at the lowest point.

Lastly, $y' = \frac{A(1 - \cos c)}{h} = \frac{1000 \times {\cdot}134}{10} = 13{\cdot}4$ ft.

the depth of the lowest point below the horizontal line of suspension.

§ XI. *On the equilibrium of elastic bodies.*

Elastic curve.

103. In the preceding section, we have given the equations of equilibrium of a line supposed perfectly flexible, but entirely devoid of elasticity; we propose now to consider the influence of this quality of matter upon the conditions of equilibrium.

Let us suppose an inextensible lamina of uniform dimensions throughout, and such that, when unsolicited by any force, it assumes the form of a rectangular plane of given dimensions. Conceive this lamina fixed at one of its extremities, and to have a force applied at the other, which changes its figure from that of a rectilinear plane to a curved surface: the problem which we propose to investigate, is the determination of the nature of this curve.

Let K′ K″ (fig. 54) represent the length and original position of the elastic lamina; and K′ K its position after inflexion; the curvature, for the greater generality, we shall suppose to be produced by a force applied at C, KC being an inflexible line attached to the point K, and coinciding with its original position; the direction of this force being in the plane of the curve, and exerted in the direction CD. We have then to consider the equilibrium of this force as opposed to the elasticity of the lamina, which may be regarded as another force, whose tendency is to restore the body to its original form and position.

Conditions of equilibrium.

104. In order to find the conditions of this equilibrium, we shall substitute for the curve K′ m K, a polygon of an indefinite number of sides, of which the first K′ H (fig. 55) is taken on the fixed line K′ K″ given in position. Let $m\,m_{,}$ and $m\,m'$ be two consecutive sides; and $m\,t$ and $m'\,t'$, their prolongations; and let us suppose that the two parts K′ m, K m of the polygon become inflexible, in such a manner, that the lamina may preserve its flexibility and elasticity only at the point m; and let us imagine moreover the part K′m to be fixed, and rendered immovable, which supposition will not destroy the existing equilibrium. The elasticity has now, according to this supposition, only a tendency to cause the side $m\,m'$ to turn about the point m, in order to restore it to the same line of direction as the side $m_{,}m$; and we may therefore consider it as a force acting perpendicularly to the line $m\,t'$, and applied in any point of it as at B, in the perpendicular direction BE. Let F represent this unknown force, and call P the given force acting in the direction CD, and p the perpendicular let fall from the point m to the line CD; the perpendicular m B to the line BE, being consideredas unity. Then, by the theory of moments (art. 65), we have (since the point m is kept in equilibrio by the action of thesetwo forces),

$$\mathrm{F} \times 1 = \mathrm{P}p, \text{ or } \mathrm{F} = \mathrm{P}p.$$

This equation of equilibrium will obviously have place for every side of the polygon, that is in every point of the curve K′ K″; or rather, if this equation obtain for every point of the curve, the equilibrium will be established.

105. The force F varies, or is different, in different points of the curve, or at the several angular points of the polygon which we have supposed substituted for the curve; and depends upon the angle formed between any two consecutive tangents, or the angles formed between the prolongation of two adjacent sides of the polygon; as, for instance, the two lines $m\,t$, $m'\,t'$, which angle is called the *angle of contingence*. What may be the exact relation between the angle of contingence and the force which produces it, is not positively determined; but it is usual to consider them directly proportional: viz. if A be taken to denote that value of F, which answers to any determined point of the curve or angle of the polygon, where the angle of contingence is α, and if $\tilde{\omega}$ be the angle $t\,m\,t'$ at any other point m, we shall have

$$\mathrm{F} : \mathrm{A} :: \tilde{\omega} : \alpha;$$

whence $$\mathrm{F} = \mathrm{A}.\frac{\omega}{\alpha};$$

or, since these angles are inversely as the radii of curvature of these points, if we denote the radius of curvature at the point m, viz. $m\,g$, by r, and the radius corresponding to the angle α by a, we have

$$\mathrm{F} : \mathrm{A} :: a : r;$$

or, $$\mathrm{F} = \mathrm{A}.\frac{a}{r}.$$

Since F varies inversely as r, while the lamina is homogeneous, and of the same uniform thickness, we have A a = a constant quantity; if, therefore, we denote this by b, we shall have

$$\mathrm{F} = \mathrm{P}p = \frac{b}{r}; \qquad (60)$$

which may be considered as the fundamental equation of the curve: but it will be necessary, in order to render it applicable to the purposes of our farther investigations, to introduce into it the co-ordinates of the point m instead of the variables r and p.

Equati the cu

106. Let us assume the point K (fig. 54) for the origin of the co-ordinates; the line K x, perpendicular to the right line KC, for the axis of the abscisses; and the prolongation of this line K y, for that of the ordinates. Let x and y be the absciss and ordinate of the point m; make also KC = c, and the angle DCK = γ. Resolve now the force P, acting in the direction CD, into two forces, in the direction CK and CA, the latter being perpendicular to CK.

These components will be P cos γ, and P sin γ; and their moments, with reference to the point m, will be equal to x P cos γ, and $(y + c)$ P sin γ; whence we have (art. 65)

$$\mathrm{P}p = \mathrm{P}\{y + c) \sin\gamma - x\cos\gamma\} \qquad (61)$$

We know also, that considering $\dot{x}$ as constant, we have the radius of curvature

$$r = \frac{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}{-\dot{x}\ \ddot{y};}$$

whence by means of equations (60) and (61), we have

$$\frac{-b\,\dot{x}\,\ddot{y}}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}} = \mathrm{P}\{(y + c)\sin\gamma - x\cos\gamma.\} \qquad (62)$$

Here, if we make

$$\{(y + c)\sin\gamma - x\cos\gamma\} = z,$$

and

$$\dot{y}\sin\gamma - \dot{x}\cos\gamma = \dot{z},$$

and write

$$\frac{\dot{z}}{\dot{x}} = \frac{\dot{y}\sin\gamma}{\dot{x}} - \cos\gamma = u;$$

we have

$$\sin\gamma\,\dot{y} = u\,\dot{x} + \cos\gamma\,\dot{x}$$
$$\sin\gamma\,\ddot{y} = \dot{u}\,\dot{x}$$
$$(\dot{y}^2 + \dot{x}^2)\sin^2\gamma = (u^2 + 2\,u\cos\gamma + 1)\,\dot{x}^2.$$

Now substituting this in equation (62), we obtain

$$\frac{-b\sin^2\gamma\dfrac{\dot{u}}{\dot{x}}}{(u^2+2u\cos\gamma+1)^{\frac{3}{2}}}=\mathrm{P}\,z;$$

and multiplying the first side by $u\,\dot{x}$, and the second by its equal $\dot{z}$, we have

$$\frac{-b\sin^2\gamma\, u\,\dot{u}}{(u^2+2u\cos\gamma+1)^{\frac{3}{2}}}=\mathrm{P}\,z\,\dot{z}.$$

If here, as the simplest case, we suppose the force P to act perpendicularly to KC, so that $\gamma=90°$; then $\cos\gamma=0$, $\sin\gamma=1$; whence the above becomes

$$\frac{-b\,u\,\dot{u}}{(u^2+1)^{\frac{3}{2}}}=\mathrm{P}\,z\,\dot{z};$$

the fluent of which gives

$$\frac{2b}{\sqrt{(u^2+1)}}=\mathrm{P}\,z^2+\mathrm{C}:$$

C being the correction, or a constant arbitrary quantity, which remains to be determined. If we resolve this equation with reference to u, we have

$$u^2+1=\left(\frac{2b}{\mathrm{P}\,z^2+\mathrm{C};}\right)^2$$

or,

$$u=\sqrt{\left\{\left(\frac{2b}{\mathrm{P}\,z^2+\mathrm{C}}\right)^2-1.\right\}}$$

And since, in the case of $\gamma=90°$, $z=y+c$, and by our former substitution $\frac{\dot{z}}{\dot{x}}=u$, we have by introducing these values, and reducing the equation,

$$\dot{x}=\frac{\{\mathrm{P}(y+c)^2+\mathrm{C}\}\,\dot{y}}{\sqrt{\{4b^2-\mathrm{C}^2-2\,\mathrm{CP}(y+c)^2-\mathrm{P}^2(y+c)^4\}}}.$$

But as we cannot find the fluent of the second side of this equation, in finite terms; the relation of the absciss x, and ordinate y, in the elastic curve, can only be exhibited by means of an infinite series.

In integrating the above equation in any manner, it is obvious that we shall have to introduce into it another correction C′; the value of which, as well as that of C, may however be determined from the following considerations.

At the point K, the origin of the co-ordinates, we have $x=0$, and $y=0$; moreover, as the right line KC, is supposed to be the continuation of the elastic lamina, it follows that it is the tangent at the point K: and as we have taken K y, for the axis of the ordinates, it follows that this tangent will be perpendicular to the axis of the abscisses; whence we shall have at the point K, $\frac{\dot{y}}{\dot{x}}=0$, or $\frac{\dot{x}}{\dot{y}}=0$. These two conditions, viz. $y=0$, and $\frac{\dot{x}}{\dot{y}}=0$, when $x=0$, will be sufficient for determining the two constant arbitrary constant quantities C and C′.

107. When the above equation of the curve K′mK is completely determined, we may deduce the value of any arc Km, in functions of the ordinate y, of the point m; and consequently we shall have likewise the length of the entire curve K′mK, in functions of the ordinate K′L of the point K′; and because this length is given, being equal to the right line K′K″, we may reciprocally from this determine the value of K′L. And if we substitute this value, instead of y in equation (63), it will give that of $\frac{\dot{x}}{\dot{y}}$ which answers to the point K′. This result will contain the quantities c, P, b, and the length of the lamina, which we may denote by l: now at the point K′, the tangent of the curve is the right line K′ K″ given in position; denoting therefore by θ, the angle K′Gy, which this right line makes with the axis K′y, we shall have for the point K′, $\frac{\dot{x}}{\dot{y}}=\tan\theta$; so that putting for $\frac{\dot{x}}{\dot{y}}$ this value, there will result an equation of condition between the five quantities c, P, b, l, and θ; any four of which therefore being given, the fifth may, theoretically, be determined; but, except in one of the most simple cases, that is to say, in which we suppose the total inflection θ very small, the computation, in consequence of the non-integrability of equation (63), becomes so complicated as to render it almost or entirely impracticable.

When the lamina is not fixed.

108. We shall examine the particular condition above alluded to in a subsequent article, after making an observation relative to the case in which the point K′ is not fixed.

In the first place, if the point K′ were entirely free, then it is obvious, that in order to produce the equilibrium, of which we have above given the conditions, a force equal to P must be applied at the point K′; and these two must be directly opposed to each other: but if the point K′ be resisted by a plane, so that the lamina is at liberty to turn about that point, then it will be sufficient, for establishing the equilibrium, that the line of direction of the force P passes through K′; that is, it must be exerted in a direction coinciding with the line K′C, which joins the two points of application; but the figure of the curve in this, and all other cases, is essentially the same, and determinable by means of the general equation (63).

109. Let us now attend to the particular case above referred to, viz. when the inflection of the lamina is very small; and for the sake of greater simplicity, let us suppose the force P to be applied immediately at the point K, so that $c=0$; let us suppose also, that the other extremity K′ is merely supported on a fixed plane which will then require, in case of equilibrio, that the force P be made to act in the direction of the right line KK′, which joins the points of application K and K′, as in fig. 56; and let it be required, to find the particular relation of x and y, upon the above supposition, x being taken on the vertical line KK′; that is, KK′ is the axis of the abscisses, and Ky that of the ordinates.

Referring to equation (62), and assuming $\gamma=90°$, and $c=0$, it will become

$$\frac{-b\,\dot{x}\,\ddot{y}}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}=\mathrm{P}\,y,$$

$$\frac{-b\dfrac{\ddot{y}}{\dot{x}^2}}{\left(1+\dfrac{\dot{y}^2}{\dot{u}^2}\right)^{\frac{3}{2}}}=\mathrm{P}\,y.$$

Now the value of $\frac{\dot{y}}{\dot{x}}$, viz. the tangent of the angle of contingence, is, by the supposition, very small; and we may therefore, in our approximation, neglect all the powers of this quantity higher than the first; whence the above denominator reduces to unity, and the equation itself to

$$-b\frac{\ddot{y}}{\dot{x}^2} = \mathrm{P}y;$$

of which the complete fluent is

$$y = h \sin\left(x\sqrt{\frac{\mathrm{P}}{b}} + f\right);$$

h and f being two constant arbitrary quantities or corrections: but since, at the point K, when $x = 0, y = 0$, we have also $f = 0$; therefore the equation of the curve, in the case we have supposed, is

$$y = h \sin x \sqrt{\frac{\mathrm{P}}{b}};$$

or, writing $\sqrt{\frac{\mathrm{P}}{p}} = \beta$, it becomes

$$y = h \sin x\beta.$$

110. Here, since $\sin x\beta$ can never exceed unity, y can never exceed h; that is, the constant quantity h, expresses the greatest value of y; or the greatest inflection of the curve from the vertical KK′: and in order to determine its value, we may observe, that as at the point K′, we have $y = 0$, we must also have its equal $h \sin x\beta = 0$; and consequently, either $h = 0$, or $\sin x\beta = 0$: now $\sin x\beta = 0$, when $\beta x = 0$, or any multiple of π, the semi-circumference of a circle. If therefore we denote by π, the semi-circumference of a circle whose radius is 1, or $\pi = 3.14159$, &c.; and x by l, to which it becomes equal in this case, l denoting the length of the lamina; also by i, any integral number whatever, we ought to have

$$\beta x \text{ or } \beta l = l\sqrt{\frac{\mathrm{P}}{b}} = i\pi; \text{ or}$$

$$\frac{\mathrm{P}l^2}{b} = i^2\pi^2, \qquad (64)$$

in order to fulfil the conditions of $y = 0$; or of

$$h \sin x\beta = 0;$$

or otherwise we must have $h = 0$: therefore when the equation $\frac{\mathrm{P}l^2}{b} = i^2\pi^2$ cannot be made to obtain, it follows necessarily, that $h = 0$; that is, the curve will then coincide with its axis, and there will be no inflection.

Equation (64) gives by division

$$\mathrm{P} = \frac{b i^2 \pi^2}{l^2};$$

Inflection $= 0$, under a certain limit.

in which b, l, and π^2, are given quantities; and $i = 0$, or some integer. If $i = 0$, then $\mathrm{P} = 0$, and consequently $h = 0$; if $i = 1$, its least integral value, then $\mathrm{P} = \frac{b\pi^2}{l^2}$; therefore when P is less than $\frac{b\pi^2}{l^2}$, this equation will not obtain; that is, we shall not then have $\sin\beta x = 0$; consequently, when $\mathrm{P} < \frac{b\pi^2}{l^2}$, we must have $h = 0$, in order to fulfil the condition $y = 0$; that is, when P is less $\frac{b\pi^2}{l^2}$, there will be no inflection; so that an elastic lamina, placed vertically on a horizontal plane, and loaded with a weight P at its upper extremity, will experience no inflection from that weight till it exceeds the constant quantity $\frac{b\pi^2}{l^2}$.

This singular result, to which we shall have occasion to refer in treating of the vertical resistance of beams and columns, was first obtained by Euler, in a memoir published by him on the latter subject; but as it seemed a contradiction to what we commonly call the *law of continuity*, he examined the subject with much attention, in order to explain away this apparent paradox: it was Lagrange, however, who first saw clearly through the mystery, by observing that while P was less than $\frac{b\pi^2}{l^2}$, the value of h was imaginary; and hence the supposed violation of the above law remained no longer an objection to the conclusion derived from the analytical investigation.

§ XII. *Of gravity, and the centre of gravity. Definitions, &c.*

111. Gravity is a term employed to denote that force or agency, by which every material particle is urged towards the surface of the earth as soon as it is left unsupported; and its action is exerted always in directions perpendicular to this surface. The earth being nearly spherical, these vertical lines necessarily converge towards a point, situated in or near the centre of the earth: but when we contemplate the magnitude of the terrestrial radius, in comparison with that of any body which we have commonly to consider, it is obvious, that we may abstract from the convergency of these lines, and regard, without any sensible error, the several directions as parallel to each other throughout the extent of any isolated mass or material substance.

From experiments which we shall have occasion to refer to, in treating of the motion of heavy bodies, it has been demonstrated, that the intensity of this gravity varies in different parts of the surface of the earth: it is less at the equator than it is as we approach the poles, and its increase is found to be proportional to the square of the sine of the latitude. It is known also, that in the same latitude it diminishes as we ascend vertically from the earth's surface, and that the decrease of energy is inversely proportional to the square of the distance from the centre: if, therefore, the utmost rigour were used, we might say that gravity was not constant for all the particles of the same mass, by reason of their different distances from the equator and from the centre of attraction; but this would obviously be a useless refinement, when we consider the little difference in this respect as compared with the absolute distance of the centre of the earth. We may then, without any appreciable error, regard the action of gravity as constant for every particle of the same body; and assume that it is exerted in right lines parallel to each other. A heavy body therefore, under this point of view, will be the same as an assemblage of material points, to each of which is applied an equal force; these forces being parallel, and all exerted in the same direction; which direction is indicated by that of a plumb line suspended from the point of application.

112. These parallel forces will have a resultant equal

to their sum and parallel to their direction (art. 54), and this resultant is what we denominate the *weight of the body*. Gravity and weight are frequently employed as synonymous expressions; they are however distinguishable the one from the other, differing as cause and effect: the former being the power of transmitting, or a tendency to transmit, into every particle of matter a certain velocity which is independent of their number; and the latter, the effort which must be exerted to prevent a given mass from obeying the action impressed upon it by gravity. Weight therefore is dependent on the mass, but gravity is wholly independent of it.

It follows from the above definition that two homogeneous bodies of equal magnitude, will be of equal weights; and generally in all such bodies the weights are proportional to the magnitudes.

In heterogeneous bodies, this proportionality does not necessarily obtain; and we express the difference by saying, that one body is more or less *dense* than another.

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113. *Density* therefore is a relative expression, by which we indicate the proportionality of the number of material particles contained under equal volumes; that body being said to have the greater density that has the greater mass, or which is the same, the greater number of material particles under the same volume.

The term density is seldom used with reference to particular numbers; whenever we employ specific numbers to express the proportionality above mentioned, it is customary to change the term density to *specific gravity*.

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In order to construct a table of the specific gravities of different bodies, it is necessary to fix upon some one as a standard; and this body, by the universal approbation of philosophers, is distilled water, taken at the temperature of its greatest condensation, viz. 4° of the centigrade, or 39° of Fahrenheit's thermometer: when we say therefore, that the specific gravity of gold is 19, or that gold has nineteen times the density of water; we mean, that of two equal volumes, the one of gold, and the other of distilled water; the former is nineteen times heavier than the latter.

It follows from the above (since in all homogeneous bodies the magnitude is as the weight), that if we write M to denote the magnitude of a body, W its weight, and D its density, D will vary as, $\frac{W}{M}$; that is, we shall have

$$\frac{W}{M} = c\,D,$$

where c is a co-efficient, which depends upon the units of weight and measure that we assume. The French take the centimetre as the unit of measure, and the gramme as the unit of weight; and since a centimetre of water is what constitutes the gramme of weight, it follows that when these are taken as the units of W and M, the co-efficient c becomes $= 1$; and therefore, that in all cases they have,

$$\frac{W}{M} = D.$$

The English have not this advantage, although by a most remarkable coincidence, considering it as wholly accidental, we possess one nearly equivalent, depending upon this, that a cubic foot of water weighs exactly, or very nearly, 1000 ounces,* avoirdupois; therefore, calling the density or specific gravity of water 1000, and assuming the ounce avoirdupois for the unit of W, and 1 foot as the unit of M, we shall have also

$$\frac{W}{M} = D.$$

Removing afterwards the decimal point three places, our tables of densities will correspond with those of the French.

When we transfer a body from one place to another on the earth, its weight W varies proportionally to gravity, although it is impossible to ascertain the difference by means of a balance; because both the weight and the body undergo the same change; we must therefore employ some other means, independent of, or which are not effected by gravity, as a spring, or the elasticity of the air: we shall then find that the weight varies with the gravity: and it is obvious from the above equation, that in the same place

$$W = M\,D;$$

introducing therefore g, to denote the relation which subsists between the gravity at the place where the weight of the body is estimated, and the same at a determined place, which is assumed as *unity*, we shall obtain

$$W = M\,D\,g;$$

which shows that the weight of a body varies as the magnitude, density, and gravity; while the mass varies simply as the magnitude and density, being the same in all places.

Of the centre of gravity.

Characteristic properties of the centre of gravity.

114. We have seen (art. 112) that under the point of view in which we consider a heavy body, it is the same as an assemblage of material points, to each of which an equal force or weight is applied: these forces being all parallel to each other, and all acting in the same direction: and we have seen (art. 54) that in every such system of forces, there is a certain point through which their resultant will pass whatever may be the position of the system; and this point, which in the above article, is defined the *centre of the parallel forces*,

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* In the philosophical transactions for 1798, we find a very interesting memoir, in part connected with this subject, by Sir George Shuckburgh Evelyn, who made numerous accurate experiments with a view to the above determination. He ascertained that a cubic inch of pure distilled water, when the barometer stood at 29·74 inches, and the thermometer at 66°, weighed 252·587 grains troy.

Now if we compute the weight of the same, answering to the greatest condensation of water, viz. 39°; we shall have

$$\cdot 99845 : 1 :: 252\cdot 587 : 252\cdot 969 \text{ grains},$$

the weight of a cubic inch of water, when the thermometer is 39°, or in the greatest state of condensation of that fluid; which gives $252\cdot 969 \times 1728 = 437130\cdot 432$ grains, the weight of the cubic foot. And if we assume the ratio of the pound avoirdupois to the pound troy, as 7000 : 5760, we find

437130·432 grs. troy = 999·6352 ounces avoirdupois for the cubic foot.

But if we take the ratio of the two lbs. as it is commonly given; viz. 13999 : 11520, we have

$$437130\cdot 432 \text{ grs. troy} = 999\cdot 8066 \text{ oz. avoirdupois}:$$

either of which results approaches extremely near to the 1000 ounces commonly assumed; so near indeed, that in any new system of weights and measures, it is much to be desired that it may be converted into a complete coincidence. We should thus, without any perceptible change whatever, have as complete a scientific relation, between our units of weight and measure, as that which the French philosophers have obtained by the entire destruction of their old system, and in direct opposition to the prejudices and long established customs of the people.

receives, in the case at present under consideration, the particular denomination of the *centre of gravity.*

115. The characteristic property therefore of the centre of gravity of a heavy body is, that if this point is supposed fixed, the body to which it appertains will remain in equilibrio in every possible position about that centre; because in all these positions, the resultant of the forces applied to the different points of the body will pass through this fixed point.

Hence, also, it is easy to conceive, that if a solid body be retained by any other fixed point, it will be necessary for the equilibrium, and it will be sufficient for it, that the right line which joins this point and the centre of gravity be vertical, and it is indifferent whether the centre of gravity be above or below the fixed point. In fact, the weight of the body being a vertical force applied to its centre of gravity, its direction will, according to our supposition, coincide with the right line which joins this centre and the fixed point; and consequently that force will be destroyed by the resistance of the latter point, just the same as if it were its immediate point of application.

Another distinguishing characteristic of this centre is, that the product of each particle of a body into its respective distance from any plane, is equal to the product of the whole mass, or resultant, into the distance of the centre of gravity from the same plane. The former of these properties indicates to us the common mechanical method of determining the centre of gravity in bodies; and on the latter depends the analytical principle of solution illustrated in a subsequent article.

To find the centre of gravity of a body mechanically.

Centre of gravity found mechanically.

116. In order to effect this, it is only necessary to dispose the body successively in two positions of equilibrium, by aid of two forces in vertical directions, applied in succession to two different points of the body, not diametrically opposite; then the point of intersection of these two directions will show the centre required. This may be exemplified by particularising a few methods: for example, if the body have plane faces, as a piece of board, hang it up by any point; then a plumb-line suspended from the same point, will pass through the centre of gravity; therefore mark that line upon it; and after suspending the body by another point, not diametrically opposite, apply the plummet to find another such line; then will their intersection give the centre of gravity sought.

Otherwise; hang the body from the same point, by two strings attached to different parts of it; then a plumb-line hung from the same point of suspension, will meet the body in its centre of gravity.

Or a third method is, to lay the body upon the edge of a triangular prism, or on any other similar thing, moving it about till the parts on both sides are in equilibrio; then impress the mark of the edge upon the surface: do the same with the body in any other position, and again impress the line of the edge upon the body; the point of the intersection of the two lines will give the centre of gravity required. A similar method to the above, is to rest the body partly on a plane table, till you find it in equilibrio upon the edge of the table; draw a line along this edge, and repeating the same in another direction, will give the point of intersection as before.

Note. All the preceding methods suppose the body to be merely a plane, and the points which we have found are points upon its surface: they are not therefore, strictly speaking, the centres of gravity, except in the case of mere surfaces, but the extremities of lines in which the centres lie; for this centre is, generally speaking, within the solid matter of the body; although there are some few exceptions; such for example as the centre of gravity of a ring, which is in the axis of the cylinder of which the ring is a part; and other instances of a similar kind might be mentioned.

To determine the centre of gravity of bodies by analysis.

117. The general problem of which we here propose to give the solution, is this: *a body or system of bodies being given, consisting of any number of parts, of which the centres of gravity are supposed known, to find that of the whole body or system:*

In order to resolve this, we must consider the weight of each part as a vertical force applied at its centre of gravity; after which we must find, by means of the formulæ articles 54 & 55, viz. equations (21), (22), and (23), the co-ordinates of the centre of these parallel forces, which will give the centre of gravity demanded.

The several points or weights being supposed invariably connected with each other, let them be denoted by P, P′, P″, &c.; and let x, y, z represent the co-ordinates of the centre of gravity of P; x', y', z', those of P′; x'', y'', z'', those of P″, &c.; let also $x_{,}$, $y_{,}$, $z_{,}$ be the co-ordinates of the centre of gravity of the whole body or system; and $P_{,}$ its weight, or $P_{,} = P + P' + P'' + \&c.$ $P_{,}$ being therefore the resultant, we shall have from the articles above referred to

$$x_{,} = \frac{Px + P'x' + P''x'' + \&c.}{P + P' + P'' + \&c.}$$
$$y_{,} = \frac{Py + P'y' + P''y'' + \&c.}{P + P' + P'' + \&c.}$$
$$z_{,} = \frac{Pz + P'z' + P''z'' + \&c.}{P + P' + P'' + \&c.}$$

Or,

$$P_{,}x_{,} = Px + P'x' + P''x'' + \&c.$$
$$P_{,}y_{,} = Py + P'y' + P''y'' + \&c.$$
$$P_{,}z_{,} = Pz + P'z' + P''z'' + \&c.$$

From which the values of $x_{,}$, $y_{,}$, and $z_{,}$; that is, the co-ordinates of the centre of force, or in this case the centre of gravity, may be determined.

It appears from the above equations, that the situation of the centre of gravity is independent of gravity, that is of the quantity which we have denoted by g, art. 113; for it is obvious, that if we introduced this quantity into one side of the above equations, by writing Pg, $P'g$, $P''g$, &c. for P, P′, P″, &c. we must also introduce it into the other side, by writing $P_{,}g$, for $P_{,}$, and it will therefore be the same if we omit it on both sides; the value of the co-ordinates $x_{,}$, $y_{,}$, $z_{,}$, &c. being the same in either case: whence it follows that the centre of gravity does not change its position with a change of place, that is in removing the body from one point of space to another, although the gravity itself be different in these different places.

Since the weight of a body is as the mass into the gravity; and, as we have seen above, the co-ordinates are independent of gravity; it is obvious that we may use the mass instead of the weight in all homogeneous bodies: employing therefore $Q_{,}$, Q, Q′, Q″, &c. to denote the masses or quantities of matter, instead of

$P_,$, P, P′, P″, &c. which represent their weights; the above equations are readily converted into

$$\left.\begin{aligned} x_, &= \frac{Qx + Q'x' + Q''x'' + \&c.}{Q_,} \\ y_, &= \frac{Qy + Q'y' + Q''y'' + \&c.}{Q_,} \\ z_, &= \frac{Qz + Q'z' + Q''z'' + \&c.}{Q_,} \end{aligned}\right\} \quad (65)$$

These formulæ being wholly independent of the number of parts Q, Q′, Q″, &c. into which the mass $Q_,$ is divided, will be true when the number of the former are infinite, and the mass of each infinitely small; and hence it is that we arrive at that property of the centre of gravity which may be considered its particular characteristic, and which we have already introduced as a definition of it: viz.

"*The mass of a body, multiplied by the distance of its centre of gravity from any plane, is equal to the sum of the products of each of the elements of the body, into its respective distance from the same plane.*"

118. In order to render the above equations subservient to the purposes of computation, it is necessary to introduce considerations depending upon the principles of the fluxional or differential calculus; viz. instead of $Q_,$, we may use generally, *the fluent of* $\dot{Q}$, or $\int \dot{Q} = Q_,$; and instead of Qx, Qx', $Q''x''$, &c. we may employ the general expression *the fluent of* $x'\dot{Q}$, or $\int x'\dot{Q}$, &c.: which abbreviations will be readily understood by those who are conservant with the fluxional analysis; and thus the above equations become

fluxional. differential.

$$\left.\begin{aligned} x_, &= \frac{\int x'\dot{Q}}{\int \dot{Q}}, \text{ or } \frac{\int x'd\,\dot{Q}}{\int dQ} \\ y_, &= \frac{\int y'\dot{Q}}{\int \dot{Q}}, \text{ or } \frac{\int y'd\,Q}{\int d\,Q} \\ z_, &= \frac{\int z'\dot{Q}}{\int \dot{Q}}, \text{ or } \frac{\int z'd\,Q}{\int dQ.} \end{aligned}\right\} \quad (66)$$

These equations therefore will determine the three co-ordinates of the centre of gravity of a body, as referred to three rectangular planes; and consequently the position of that centre is known in all cases.

It may not be improper to observe, that x', y', and z', in the above equations, are not necessarily the same letters as those which in any particular example express the fluxion of Q; being only employed to denote the distances of the centre of gravity of $\dot{Q}$ from the several axes. This however will be better understood from the following illustration, in which also we shall show in what cases these three equations may be reduced to two, and in some cases to one equation only.

In the first place, it is obvious that when it is proposed to find the centre of gravity of any plane curve, as AMB (fig. 57), we may always suppose this plane to coincide with the plane of the two rectangular axes Ax, Ay; whereby $z = 0$, and the third of the above equations then disappear. And if the curve has another branch AB′ symmetrical to AB, since in this case the centre of gravity must necessarily lie in the axis AC, we may then make $y = 0$, and merely determine the distance AG′ upon the axis AC, by means of the first of the above equations.

It is known, from the principles of the fluxional calculus, that if we make $AP = x$, $PM = y$, and the arc $AM = z$, we shall have

$$\dot{z} = \surd\,(\dot{x}^2 + \dot{y}^2),$$

which is the quantity denoted by $\dot{Q}$ in the above equations; and farther, if we place the origin of the co-ordinates at the vertex A, x' and y' will be the same as x and y; because $\dot{z}$ being indefinitely small, the centre of gravity of $\dot{z}$ will be in M: substituting therefore z for Q, and $\surd\,(\dot{x}^2 + \dot{y}^2)$ for $\dot{Q}$, our two equations become

$$\left.\begin{aligned} x_, &= EG = \frac{\int x \surd\,(\dot{x}^2 + \dot{y}^2)}{z} \\ y_, &= G'G = \frac{\int y \surd\,(\dot{x}^2 + \dot{y}^2)}{z.} \end{aligned}\right\} \quad (67)$$

When the curve is symmetrical with regard to any axis, y becomes $= 0$, and we have only the first equation to solve; viz.

$$x_, = \frac{\int x \surd\,(\dot{x}^2 + \dot{y}^2)}{z.} \quad (68)$$

119. If the figure be a plane, the centre of gravity is necessarily in the same plane; and we have therefore also in this case $z = 0$, and our three equations are reduced to the two

$$x_, = \frac{\int x'\dot{Q}}{\int \dot{Q}}, \text{ and } y_, = \frac{\int y'\dot{Q}}{\int \dot{Q}}.$$

Now in order to reduce these to a form ready for the purposes of computation, let ABC (fig. 58) be supposed to represent any area comprised between a plane curve AB, and the absciss and ordinate AC, CB; let Ax, Ay, be the two rectangular co-ordinate axes, having their origin in the vertex A; make $AP = x$ and $PM = y$; then the fluxion of the area Pm or $\dot{Q}$, will be $y\dot{x}$; and since PM pm, are supposed indefinitely near $FG = AP = x = x'$, and $pg = \frac{1}{2}y = y'$; whence the above two equations become

$$\left.\begin{aligned} x_, &= \frac{\int xy\dot{x}}{\int y\dot{x}} = \frac{\int xy\dot{x}}{Q} \\ y_, &= \frac{\frac{1}{2}\int y^2\dot{x}}{\int y\dot{x}} = \frac{\int y^2\dot{x}}{2Q;} \end{aligned}\right\} \quad (69)$$

where Q denotes the area ABC.

Here again, if the plane is symmetrical with reference to the axis AC, we may make $y = 0$, and reduce the whole to the first equation only; viz. to the determination of the distance of the centre of gravity from the vertex, measuring along the axis AC.

Curve superficies.

120. When the figure is the superficies of a body generated by the rotation of any line about an axis, then also we may suppose $z_, = 0$ and $y_, = 0$; and denoting the semi-circumference of a circle whose radius is 1, by π; viz. $\pi = 3{\cdot}14159$, &c. $2\pi y$ will denote the circumference of the generating circle; and consequently $2\pi y\dot{z} = \dot{Q}$ the fluxion of the surface; wherefore

$$x_, = \frac{\int 2\pi yx\dot{z}}{\int 2\pi y\dot{z}} = \frac{\int yx\dot{z}}{\int y\dot{z}}. \quad (70)$$

Of a solid.

121. When the figure is a solid of revolution, the centre of gravity being in its axis, we may take again

$y_{,} = 0$, $z_{,} = 0$; and in this case πy^2 denoting the area of the circle whose radius is y, and $\pi y^2 \dot{x} = \dot{Q}$ the fluxion of the solid, we readily find

$$x_{,} = \frac{\int \pi y^2 x \dot{x}}{\int \pi y^2 \dot{x}} = \frac{\int y^2 x \dot{x}}{\int y^2 \dot{x}}. \qquad (71)$$

Examples.

122. Let us now illustrate the application of these formulæ by the solution of a few examples:

Ex. 1. To find the centre of gravity of a right line AB (fig. 59):

Here, as the centre sought must lie in the proposed line, it will only be necessary to have reference to one axis, which we shall suppose to pass through the extremity A: let AD be any variable distance $= x$; then the fluxion of the body denoted by $\dot{Q}$, in equation 69, will be $\dot{x}$; consequently the equation itself will now be

$$x_{,} = \frac{\int . x\dot{x}}{\int \dot{x}} = \frac{\frac{1}{2} x^2}{x} = \tfrac{1}{2} x:$$

and when $x = a$ (denoting by a, the whole length AB) we have $x_{,} = \frac{1}{2} a$.

That is, the centre of gravity of a right line is in the middle of its length.

Ex. 2. To find the centre of gravity of a circular arc:

Here the plane being symmetrical with reference to axis AC (fig. 60), we need employ only one co-ordinate axis, which we shall suppose to pass through the point A; for since the centre sought must be in the line AP, when the distance AG is found, the centre itself will be determined.

From the middle point of the proposed arc MAM′, conceive the line AC drawn through C, the centre of the circle, on which let AP be denoted by x; the variable ordinate PM by y; the radius AC of the circle by r; and the half arc by z.

Then considering A as the origin of the co-ordinates, since the curve is symmetrical with respect to AC, we only make use of equation (68); in which, substituting $\dot{z}$ for $\dot{Q}$, it becomes

$$x_{,} = \frac{\int x \sqrt{(\dot{x}^2 + \dot{y}^2)}}{\int \dot{z}}, \text{ or } \frac{\int x \sqrt{(\dot{x}^2 + \dot{y}^2)}}{z}.$$

Now the equation expressing the relation between the co-ordinates of a circle being

$$y^2 = 2rx - x^2;$$

we have, by substituting for $\dot{y}^2$,

$$x_{,} = \frac{r}{z} \int \frac{x \dot{x}}{\sqrt{(2rx - x^2)}};$$

that is, taking the fluent,

$$x_{,} = \frac{r}{z}(z - y) = r - \frac{ry}{z},$$

which needs no correction; because when $y = 0, z = 0$.

Assuming therefore $\text{AG} = x_{,}$, G will be the centre of gravity, and consequently

$$\text{AG} = r - \frac{ry}{z} = \text{AC} - \frac{ry}{z};$$

whence $\quad \text{CG} = \text{AC} - \text{AG} = \frac{ry}{z}$; that is,

The distance of the centre of gravity of a circular arc, from its centre, is a fourth proportional to the arc, the radius, and the chord of the arc.

Hence when the arc is a semicircle, the chord being then double the radius, we shall have

$$\text{CG} = \frac{2r}{\pi} = \frac{r}{\frac{1}{2}\pi} = \cdot 63662\, r;$$

π denoting as before the semi-circumference of a circle to radius 1.

When the arc is a complete circle, then $y = 0$; and consequently $\text{CG} = 0$; that is, the centre of gravity of the whole circumference coincides with the centre of the circle, as is otherwise obvious.

Ex. 3. To find the centre of gravity of a circular segment:

Here again, as the figure proposed is symmetrical, the centre of gravity must lie in the axis AC; and we need therefore only find the distance AG.

Let MAM′P (fig. 60) be the proposed segment, and let the several parts be denoted as in the last example: then taking the first of the equations, No. 69, that is

$$x_{,} = \frac{\int x y \dot{x}}{\int y \dot{x}} = \frac{\int x y \dot{x}}{Q};$$

and substituting in this the value of y found from the equation

$$y = \sqrt{(2rx - x^2)};$$

we have

$$x_{,} Q = \int x y \dot{x} = \int x \dot{x} \sqrt{(2rx - x^2)}, \text{ or}$$

$$x_{,} Q = \frac{-(2rx - x^2)^{\frac{3}{2}}}{3} + r Q, \text{ or}$$

$$x = \text{AG} = r - \frac{(\text{PM}')^3}{3\,Q}, \text{ or}$$

$$x = \text{AC} - \frac{(\text{PM}')^3}{3 \text{ area APM}'}.$$

Consequently

$$\text{CG} = \frac{(\text{PM}')^3}{3 \text{ area APM}'} = \frac{(\text{MM})^3}{12 \text{ area of the seg.}}$$

When the area becomes a semicircle, then $\text{MM}' = 2r$, and $\text{CG} = \frac{8r^3}{6 \text{ area circle}} = \frac{r}{2 \cdot 356193} = \cdot 42441\, r.$

Ex. 4. To find the centre of gravity of any parabola:

Here the general equation is $y = \frac{x^n}{a^{n-1}}$. And by substitution in the first equation No. 69, we have

$$x_{,} = \frac{\int x y \dot{x}}{\int y \dot{x}} = \frac{\int x^{n+1} \dot{x}}{\int x^{n} \dot{x}} = \frac{n+1}{n+2} x.$$

When $n = \frac{1}{2}$, as in the common parabola, then

$$x_{,} = \tfrac{3}{5} x, \text{ or AG} = \tfrac{3}{5}\,\text{AC}. \qquad \text{(fig. 61)}.$$

When $n = \frac{1}{2}$, the figure becomes an isosceles triangle, and

$$x_{,} = \tfrac{2}{3} x, \text{ or AG} = \tfrac{2}{3}\,\text{AE}. \qquad \text{(fig. 62)}$$

Ex. 5. To find the centre of gravity of any semi-parabola:

Here the plane not being symmetrical to any axis, we only know that the centre of gravity lies in the plane ADC, fig. 61; and we therefore in this case must have reference to two rectangular co-ordinate axes A x, A y, which we shall suppose to have their origin at the vertex A; and we must thence determine the distance AG, represented by $x_{,}$ of the centre from the axis A x, and then the distance G g of the same from the axis A y; by which means the centre sought will be determined.

Now the distance on the absciss AC, or the value of $x_{,}$ being the same as above, is determined by the foregoing problem; and we have therefore only to find

G g. In order to which we have the second equation, No. 69; viz.

$$y_{,} = \frac{\frac{1}{2}\int y^2 \dot{x}}{\int y \dot{x}} = \frac{\int y^2 \dot{x}}{2\,Q}.$$

By substituting for y^2 in the numerator of this expression, its value given by the equation of the curve, we have

$$2 y_{,} Q = \int \frac{x^{2n} \dot{x}}{a^{2n-2}};$$

which by the known rules for the finding of fluents, is

$$2\, y_{,}\, Q = y^2 \; \frac{x}{2n+1};$$

let $h =$ AC the height of the parabola; then, when $x = h$, this becomes

$$y_{,}\, Q = y^2 \times \frac{h}{4n+2}.$$

Or dividing by the area $Q = \frac{h \times CD}{n+1}$, we obtain

$$y_{,} = Gg = \frac{(n+1)\,CD}{4n+2}.$$

Having thus the value of Gg, and that of AG or A$'g$ being given by the last example, the point g will become known.

If $n = \frac{1}{2}$, as in the common parabola, then G$g = \frac{3}{8}$ CD.

If $n = 1$, as in the isosceles triangle, then G$g = \frac{1}{3}$ CD.

Ex. 6. To find the centre of gravity of any parabolic conoid.

Here the equation of the curve being as before, (Ex. 4)

$$y = \frac{x^n}{a^{n-1}},$$

and the solid being one of revolution, is symmetrical with reference to its axis; if therefore we reckon the vertex as the origin of the co-ordinates, we have only occasion to make use of equation (71); viz.

$$x_{,} = \frac{\int y^2 x \dot{x}}{\int y^2 \dot{x}};$$

which by substituting for y^2, becomes

$$x_{,} = \frac{\int x^{2n+1} \dot{x}}{\int x^{2n} \dot{x}} = \frac{2n+1}{2n+2} x.$$

When $n = \frac{1}{2}$, as in the common parabola, then $x_{,} = \frac{2}{3} x$.

When $n = 1$, the conoid becomes a cone, and $x_{,} = \frac{3}{4} x$.

That is, in the common parabola the centre of gravity is at $\frac{2}{3}$ of the axis from the vertex, and in the cone it is $\frac{3}{4}$ of the axis from the vertex, or $\frac{1}{4}$ from the centre of the base.

Ex. 7. To find the centre of gravity of the segment of a spheroid.

Here the solid being symmetrical to the axis, if we account the vertex A the origin of the co-ordinates, we shall only require the equation for $x_{,}$, given No. 71; viz.

$$x_{,} = \frac{\int y^2 x \dot{x}}{\int y^2 \dot{x}}.$$

Let the fixed axis of the spheroid be called a, the revolving axis c, then the equation to the curve is

$$y^2 = \frac{c^2}{a^2}(a x - x^2) \text{ or } y = \frac{c}{a} \surd (a x - x^2.)$$

Consequently

$$x_{,} = \frac{\int y^2 x \dot{x}}{\int y \dot{x}} = \frac{\int (a x - x^2)\, x \dot{x}}{\int (a x - x^2)\, \dot{x}}, \text{ or}$$

$$x_{,} = \frac{\frac{1}{3} a x^3 - \frac{1}{4} x^4}{\frac{1}{2} a x - \frac{1}{3} x^3} = \frac{4a - 3x}{6a - 4x}\, x = \text{AG, the}$$

distance of the centre of gravity from the vertex.

When the segment becomes a hemisphoid $x = \frac{1}{2} a$, and

$$x_{,} = \frac{4a - 3x}{6a - 4x}\, x = \frac{5}{8} a, \text{ the distance from}$$

the vertex, and consequently $\frac{3}{8} a$ the distance from the centre of its base.

When $c = a$ the spheroid becomes a sphere, and as the above result is independent of c, the same expressions apply to both solids and their corresponding segments.

Since the equation to the hyperbola is,

$$y^2 = \frac{c^2}{a^2}(a x + x^2);$$

which differs from that to the ellipse only in the change of signs from − to +; a similar process to that above, will give us

$$x_{,} = \text{AG} = \frac{4a + 3x}{6a + 4x} x;$$

for the distance of the centre of gravity from the vertex of a hyperboloid.

Ex. 8. To find the centre of gravity of a sector of a sphere.

A spheroidal sector may be considered as a solid of revolution, generated by the rotation of a circular sector; thus (in fig. 60) if the circular sector CMA make a complete revolution upon AC as an axis, it will describe the spheric sector, of which CMAM′ will be a section.

This solid is composed of a spheric segment and a cone; the centre of gravity of the first of these we suppose at G, and that of the second at g. And if AP be denoted by x, and AC by r, we shall have

$$\text{AG} = \frac{8r - 3x}{12r - 4x}\, x, \text{ as in example 7;}$$

and by example 6,

$$Cg = \tfrac{3}{4}\, CP = \tfrac{3}{4}(r - x);$$

consequently,

$$Ag = AC - Cg = \frac{3x + r}{4}.$$

Let now $\pi = 3{\cdot}14169$, &c. as before; then, by the known rules for finding the solidity of these solids, we shall have

$\frac{2}{3} \pi r^2 x =$ content spheric sector;

$\frac{1}{3} \pi x^2 (3r - x) =$ the segment;

$\frac{1}{3} \pi (2rx - x^2)(r - x)$ the cone.

Now, conceiving each solid to be condensed into its respective centre of gravity, we must divide the distance between the two latter centres, so that the product of each solid into the distance of its centre of gravity, from the required point, may be equal to each other; which being done, we shall find

$$Ag = \tfrac{1}{8}(2r + 3x),$$

the distance of the centre of gravity of the spheric sector from A the centre of its base.

Note. The preceding division of the distance between the two centres, in the reciprocal ratio of the mass supposed to be collected at each point, the reader will perceive is founded upon the general property of the centre of gravity of a body or system; viz. that the sum of the products of each mass into its respective distance from any plane, is equal to the product of the whole mass into its distance from the same plane. Now, in the present case, our plane is supposed to pass through the centre of gravity of the system; the latter distance is therefore equal to zero, and consequently the sum of the other two products $= 0$: that is, one being accounted positive and the other negative, they must be equal to each other.

To find the centre of gravity of a body by a geometrical construction.

Centre of gravity by geometrical construction.

123. The analytical method of determining the centre of gravity of any homogeneous body or system of bodies, is the most general that it is possible to give, as no case can arise which does not yield to the processes above illustrated; although it must be acknowledged, that there is frequently some difficulty in solving the equations which they involve: but theoretically, every problem that can be proposed, as to the determination of the centre of gravity of any body or system, has its solution included in the three general equations (No. 66); it may still, however, not be improper to explain here some other methods, which, although less general than the above, are extremely simple in the few cases in which they can be applied.

124. To find the centre of gravity of the perimeter of any right lined figure.

If the figure be a regular polygon, the centre of gravity will be in the centre of its circumscribing or inscribed circle. But if it be irregular, we may conceive the particles of each line to be all placed at their respective centres of gravity; that is, in the middle of each line. Therefore, join the two middle points of any two of the sides, as FB, BC (fig. 63), by the line fb; make

$$fG : Gb :: BC : BF, \text{ or}$$
$$fG : fb :: BC :: BC + BF :$$

And G will be the common centre of gravity of the two sides BC, BF. Then join Gc, c being the middle point of a third side CD, make

$$GG' : cG' :: CD : BF + BC;$$

and G′ will be the centre of gravity of the three sides FB, BC, CD.

In like manner join G′ and the middle point d of a fourth line, and find a new centre of gravity G″, and so on for all the other sides of the figure.

125. To find the centre of gravity of any plane triangle:

Let ABC (fig. 64) be any triangle; draw AE from one of its angles to the middle of the opposite side; then will AE divide every line which can be drawn in the triangle parallel to BC into two equal parts; consequently the surface of the triangle is symmetrical with regard to the axis AE, and therefore the centre of gravity will be found in that line. For the same reason, if from any other angle, as C, we draw CD to the middle of the opposite side, the centre of gravity of the triangle will also be somewhere upon that line; and it must therefore be found in their common point of intersection G. Since the points D and E divide the sides BA, BC of the triangle proportionally, the line DE, which joins them, must be parallel to the side AC: hence the triangles BDE, BAC, are similar, as are also the triangles GDE, GCA; consequently

$$GD : GC :: GE : GA,$$
$$GD : GC :: DE : AC :: BD : BA :: 1 : 2;$$

therefore $AG = \frac{2}{3} AE$; $CG = \frac{2}{3} CD$; $BG = \frac{2}{3} BF$. If AB, BC, CA, be denoted by a, b, c, and AG, BG, CG, by m, n, r, we have, by a well-known theorem in geometry, the three following equations:

$$\begin{cases} AB^2 + AC^2 = 2\,BE^2 + 2\,AE^2, \\ AB^2 + BC^2 = 2\,CF^2 + 2\,BF^2, \\ AC \ + BC^2 = 2\,AD^2 + 2\,CD^2; \end{cases}$$

or,

$$\begin{cases} a^2 + c^2 = \frac{1}{2}\,b^2 + \frac{9}{2}\,m^2, \\ a^2 + b^2 = \frac{1}{2}\,c^2 + \frac{9}{2}\,n^2, \\ c^2 + b^2 = \frac{1}{2}\,a^2 + \frac{9}{2}\,r^2. \end{cases}$$

Whence, by addition, we obtain

$$a^2 + b^2 + c^2 = 3\,(m^2 + n^2 + r^2).$$

That is, in any plane triangle, the sum of the squares of the three sides is equal to three times the sum of the squares of the three lines drawn from each of its angles to the centre of gravity.

126. To find the centre of gravity of a trapezium. Let AB, CD (fig. 65), be any trapezium; divide it into two triangles by the diagonal AC; find their centres of gravity H and I, by the last proposition; then join IH; and make IG : IG as the triangle ABC : ADC and G will be the centre of gravity of the trapezium. Or, divide the trapezium into two other triangles by the diagonal BD; find their centres of gravity E and F, and join EF; then since the centre of gravity must be in the line HI, and since it must also be in the line EF, it must necessarily be in their common point of intersection G.

By a little extension of the methods above described, the centre of gravity of any irregular plane figure may be determined; as also the centre of gravity of some right lined solids: for example, any triangular or square pyramid, &c. &c. It would, however, be useless to give many partial solutions of these and similar cases; which, as we have before stated, may all be made to submit to the general equations (No. 66).

Centre c gravity i various bodies.

127. A knowledge of the position of the centre of gravity of bodies of various forms being frequently required in mechanical investigations, we propose, as ready means of reference, and as they may also be considered as examples for the practice of a student, to give a few results for most of those cases which are more commonly required in the practical solution of mechanical problems.

1. In a circular arc, $CG = \dfrac{r\,y}{z}$;

where C is the centre of the circle, r the radius, z the half arc, y the ordinate, and G the centre of gravity.

2. When the arc is a semicircle, $CG = \cdot 63662\,r$.

3. In a circular segment, $CG = \dfrac{8\,y^3}{12 \text{ area of seg.}}$

4. When the seg. is a semicircle, $CG = \cdot 42441\,r$.

5. In a circular sector, $CG = \dfrac{4\,r\,y}{6\,z}$.

In all these examples r, y, z, C and G, are made to denote the same as in example 1.

6. The centres of gravity of the surface of a cylinder, of a cone, and of a conic frustrum, are respectively at the same distances from the origin, as are the centres of

gravity of the parallelogram, the triangle, and trapezoid, which are vertical sections of the respective solids; viz.

In the parallelogram, $CG = \frac{1}{2} a$.
In the triangle, $CG = \frac{2}{3} a$ from vertex.
In the trapezoid, $CG = \frac{1}{3} a \times \frac{b + 2B}{b + B}$;

from the centre of the less parallel side; a in each of those cases denoting the altitude of the figure; and in the last b and B are two parallel sides, b the less and B the greater.

7. The centre of gravity of the surface of a spheric segment is at the middle of its height or versed sine.

8. In a cone, and in all regular pyramids, the centre of gravity is at $\frac{3}{4}$ of the axis distant from the vertex.

9. In a conic frustrum the distance on the axis from the centre of the less end is

$$\tfrac{1}{4} h \cdot \frac{3R^2 + 2Rr + r^2}{R^2 + Rr + r^2};$$

where h is the height, R and r the radii of the greater and less ends.

10. The same theorem will apply to the frustrum of any regular pyramid, taking R and r in such cases to denote the sides of the polygonal ends.

11. In the paraboloid the centre of gravity is at $\frac{2}{3}$ of the axis from the vertex.

12. In the frustrum of a paraboloid, the distance on the axis from the centre of the less end is

$$\tfrac{1}{3} h \times \frac{2R^2 + r^2}{R^2 + r^2};$$

h, R, and r denoting the same as in No. 9.

§ XIII. *Of Guldin's theorem; or, the centrobaryc method.*

128. Among the many interesting properties appertaining to the centre of gravity, there is one which, for its elegance and simplicity, deserves particularly to be mentioned in this place. It has been commonly called the centrobaryc method; the magnitudes of surfaces and solids may thereby often be determined with great facility. The relation between the centres of gravity of lines or surfaces, and the figures generated by them, in their revolution about a certain fixed point or axis, which is the foundation of this method, was first alluded to by Pappus in the preface to his 7th book; but it was not completely developed till the time of father Guldin; who, in the 2d and 3d books of his treatise on the "Centre of Gravity," treated this method very fully, and shewed its utility in a variety of examples. The theorem itself is comprised in the following proposition.

129. "If any line, right or curved; or any plane figure, whether it be bounded by right lines or curves, revolve about any axis in the plane of the figure; the surface or solid generated will be respectively equal to the surface or solid, whose base is the given line or figure, and its height equal to the arc described by the centre of gravity of the said generating line or figure."

Or the same may be otherwise stated as follows. "The surface or solid, generated by the rotation of any right line or curve, or of any plane figure about a given axis, is equal to the product of the generating line or plane into the circle describing its centre of gravity."

Let AMD (fig. 61) represent any curve, g its centre of gravity, and AC its axis; DAB the curve surface described by AD in revolving about the axis AC. Make $\pi = 3{\cdot}14159$, &c. $AP = x$, $PM = y$, the arc $AM = z$; then we know, from the principles of the fluxional calculus, that

$$\int 2 \pi y \dot{z} = \text{the surface generated by AMD.}$$

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We have seen, also (art. 118), that the distance Gg denoted by y, is equal to $\frac{\int y \dot{Q}}{\int \dot{Q}}$, or writing z for Q, which in this case denotes the arc AM; we have

$$y_{,} = \frac{\int y \dot{z}}{z}, \text{ or } y z = \int y \dot{z}.$$

Multiply both sides by 2π, and we obtain

$$2 \pi y_{,} z = \int 2 \pi y_{,} \dot{z}.$$

Now here $2 \pi y_{,}$ is evidently the circle described with the radius $y_{,}$, or Gg, and z is the arc by hypothesis; also, $\int 2 \pi y \dot{z}$ is the fluxional expression for the generated surface; consequently, *the surface generated by any right line or curve about a fixed axis, is equal to the product of the revolving arc into the circle described by its centre of gravity.*

Again, let AMD (fig. 61) denote any plane curve, and g its centre of gravity; which by its revolution about the axis AC, generates the solid BAD. If we denote the plane surface by Q, we know, from the principles of fluxions, that $\dot{Q} = y \dot{x}$; and that the fluxion of the solid is $\pi y^2 \dot{x}$. We have also (art. 119) for the distance of the centre of gravity g, from the axis AC,

$$y_{,} = \frac{\int y^2 \dot{x}}{2Q}, \text{ or } 2 y_{,} Q = \int y^2 \dot{x};$$

multiplying both sides by π, we obtain

$$2 \pi y_{,} Q = \int \pi y^2 \dot{x}.$$

Now, $2 \pi y$ is obviously the circle described with the extremity of the radius Gg, that is by the centre of gravity of the surface, and therefore
$2 \pi y_{,} \times Q =$ the product of the path of the centre of gravity, into the generating area; which is equal to $\int \pi y^2 \dot{x}$, the solid generated by the same area.

Whence the solid generated by the revolution of a plane surface about any fixed axis, is equal to the product of the revolving area into the path or circle described by its centre of gravity.

Deductions from the centrobaryc method.

130. If, instead of 2π, we introduce any fraction $\frac{\pi}{n} \pi$ into the numerators and denominators of the same formulæ, it would thence be equally obvious, that the same property is applicable to the curve surfaces or the capacities of figures generated by a part of a revolution about a fixed axis. And if many curves comprised in the same plane generate at once as many surfaces and solids; we may apply the same proposition, by taking the common centre of gravity of the system. Hence, therefore, we may draw the following corollaries, which flow immediately from the above general principles.

1. If the distance of the centre of gravity of any line or surface from the axis of rotation, and the magnitude of the line or surface be given; the value of the surface or solid generated, by either a total or partial revo-

lution, may be found; and consequently any two of the three being given, the third may be easily determined.

Deductions from the centrobaryc method.

Thus, if the rectangle, whose sides are m and n, revolve upon m as an axis, it will generate a cylinder, whose altitude is m, and the radius of its base n. Now the centre of gravity of a rectangle will be at the distance of $\frac{1}{2}n$ from the axis, and will therefore describe in a complete rotation the circumference $\frac{1}{2}n \times 2\pi = \pi n$; consequently, by the proposition $mn \times \pi n = \pi m n^2$, is the capacity of the solid; and this is equal to $\pi n^2 \times m$, the base of the cylinder into its altitude, as we know, from other principles, it ought to be.

Again, putting b the base, and a the axis of a semi-parabola, which by its rotation generates a paraboloid; we have $\frac{3}{8}b$, for the distance of the centre of gravity from the axis, and

$\frac{3}{8}b \times 2\pi = \frac{3}{4}b\pi =$ *circumference* described by the centre of gravity. And since the area of the parabola is $\frac{2}{3}ab$, we have

$$\tfrac{3}{4}b\pi \times \tfrac{2}{3}ab = \tfrac{1}{2}ab^2\pi,$$

for the content of the paraboloid, which is manifestly equal to half the circumscribing cylinder.

As another example, let us take a sphere whose radius is r; and capacity therefore, $\frac{4}{3}\pi r^3$, being generated by the rotation of a semicircle, whose area is $\frac{1}{2}\pi r^2$. Here

$$\frac{\frac{4}{3}\pi r^3}{\frac{1}{2}\pi r^2} = \tfrac{8}{3}r,$$

the circumference described by the centre of gravity; which, divided by 2π, gives

$$\tfrac{8}{3}r \div 2\pi = \frac{4r}{3\pi} = \cdot 42441\,r,$$

for the distance of the centre of gravity of a semicircle from its centre, agreeing with our preceding result (art. 122). By a similar process, the centre of gravity may be found in several figures more easily than by the direct method.

2. If the generating or revolving lines or surfaces are equal, but the distances of their centres of gravity from the axes unequal; then the generated surfaces or solids will be directly as the distances of their centres of gravity from their respective axes of motion.

Thus, if two solid rings of iron have both the same thickness of metal, but have different diameters; the capacities of the rings, or their weights, will be as the arithmetical means between their respective inner and outer diameters.

3. If the distances of the centres of gravity of the generating or revolving lines or surfaces from the axis of motion be equal, but the generating lines or surfaces unequal; then the generated surfaces or solids will be as the revolving lines or surfaces.

4. If neither the generating lines or surfaces, nor their distances from the axes of motion be equal; the generated surfaces or solids will be to each other, in the ratio compounded of the ratio of the generating lines or surfaces, and that of the distances of their centres of gravity from the axis of motion.

Thus, for example, the triangle is to its circumscribing parallelogram, as 1 : 2; and the distance of the centre of gravity from the vertex of the triangle is $\frac{2}{3}$ of the axis; but the distance of the centre of gravity of a parallelogram from its side is $\frac{1}{2}$ of the other side, or of the axis; therefore the distance of the centre of gravity of the triangle from the axis of rotation, is to the distance of the centre of gravity of the parallelogram, as 4 to 3. So that if both the spaces are supposed to revolve about a line touching the vertex of the triangle, and parallel to its base; the solid generated by the triangular space will be to that generated by the parallelogram, as 4 to 6, or as 2 to 3.

Or, if we suppose the triangle to be right angled, and the rotation to take place about one of its legs, then the distance of the centre of gravity of the triangle from the axis of motion will be equal to $\frac{1}{3}$ of the base, and the distance of the centre of gravity of the parallelogram from the same, $\frac{1}{2}$ its base. The areas, therefore, being as 1 : 2; and the distances of the centres of gravity from the axis of motion, as 2 to 3; the generated solids will be to each other as 2 to 6, or as 1 : 3; which is the known ratio of a cone to its circumscribing cylinder. Again, the common parabola is to its circumscribing parallelogram as 2 : 3; and the distance of the centre of gravity of the parabola from the vertex, is to the distance of that of the parallelogram, from the same point of suspension, as 6 to 5; wherefore the solid generated by the parabola is to that generated by the parallelogram, each revolving about the same common axis, passing through the vertex of the parabola, and parallel to its base, as 12 to 15, or as 4 : 5. And consequently the solid generated by a triangle, is to that generated by a parabola of the same base and altitude, and each revolving about the vertex in the ratio compounded of the ratios of 5 to 4, and 4 to 6: that is, of 20 : 24, or of 5 : 6.

After a similar manner may the value of any surface or solid be found, and those of all kinds of surfaces and solids be compared with each other. For various other examples connected with this subject, we refer to Dr. Wallis's "Treatise de Calculo Centri, Gravitatis;" Dr. Hutton's "Mensuration;" and Gregory's "Mechanics."

§ XIV. *Of the mechanical powers.*

131. A *machine* is any thing which serves to augment, or to regulate, moving forces or powers; or it is any agent designed to produce motion, so as to save either time or force. Or we may, in general, apply the term *machines* to the material agents, by the aid of which forces operate one upon another; and which when employed to second the effort of certain powers, enable them to overcome others which may be more considerable.

132. Machines are distinguished into *simple* and *compound*; but a difference of opinion exists as to this division, some writers reducing the simple machines to three only; viz. the *lever*, the *inclined plane* and *cords*, or as the latter is sometimes called, the *funicular machine*. Others make seven; adding to the above three, the *wheel and axle*, the *pulley*, the *screw*, and the *wedge*; some again, rejecting the cords, reckon but six simple machines; or, as they are commonly called, *mechanical powers*.

In all these mechanical agents, the advantage arises from distributing the power to be overcome amongst the different parts of the machine, so that the part sustained by the contrary power shall bear but a small ratio to the whole: thus, a power incapable of communicating motion to a body, or of supporting its pressure without mechanical assistance, may effect its designed purpose by transferring a part of the weight upon a fulcrum, distributing it among a system of pulleys, placing it upon an inclined plane, &c.

133. Forces of different kinds may, it is obvious, be made to operate so as to counteract each others effects, by the intervention of machines; as, for example, the muscular force of men and animals may be applied to machines so as to oppose or counteract the force of gravity; but in such cases, since the just and adequate measures of these forces are their simultaneous effects in similar situations, the theoretic processes must be conducted in the same manner, whether the forces exerted are unlike, or the same. Hence, then, whatever is shown to obtain with relation to any two forces whose effects are known, while their nature is undefined, may be safely applied to all which are respectively equal; and operate in a similar manner, whether they are dead weights, living powers, or forces naturally existing.

134. In the theory of simple machines, as well as that of machines in general, they are first considered as in a state of balanced rest; and when the ratio of the forces is ascertained, which by the intervention of any given machine produces such equilibrium; it is manifest that a slight change in either of the forces, or in their points of application, by causing a preponderance on the one side or the other, will produce motion. In actual practice, the change must always be made in favour of the power, which is to produce motion; and this may be easily accomplished, if the conditions of equilibrium be once established.

When forces act upon each other by means of machines, they meet with various obstacles, on account of the roughness of surfaces, the stiffness of cords, &c. which, if introduced into the commencement of our investigations, would render them very embarrassing and intricate; we shall therefore first suppose surfaces perfectly smooth, levers perfectly inflexible, cords flexible and inextensible, &c. and shall afterwards consider the circumstances arising from the actual operation of the impediments here supposed as non-existing.

With regard to the principle upon which we propose to found our theory of the equilibrium of the simple mechanical powers, we shall employ, as the most natural, though perhaps not the most general, that of the composition and resolution of forces already illustrated in the preceding sections: but it may not be amiss, before we proceed to this subject, to offer a slight enunciation and illustration of a very general principle; namely, that of *virtual velocities*, which is now, by most foreign writers, made the foundation of the whole theory of statics. We cannot, however, consider it in any other light than as matter of induction, which it is extremely difficult, if not impossible to demonstrate, independent of results drawn from other principles; and therefore not well or properly suited to become the foundation of a science.

Principle of virtual velocities.

135. In comparing with each other the conditions of equilibrium of the several simple mechanical powers, and looking to what is common to them all, mathematicians have discovered by induction a general law, which is observed by every system of bodies in equilibrio; and it is this law which has received the denomination of the *principle of virtual velocities*, and which may be illustrated as follows.

Let us represent by P, P′, P″, &c. the given forces; by mA, m'A′, m''A″, &c. (fig. 66) their directions; and by m, m', m'', &c. their points of application. Those points may be supposed to be connected with each other in any manner whatever, by inextensible and flexible or inflexible lines, elastic or non-elastic, or by any other physical means that can be imagined; also all or any of those points may be conceived to be resting on plane or curve surfaces, and others wholly immoveable; in short, they may be supposed to have any relation or connection with each other, that does not involve a physical impossibility.

Now let us communicate to this system of points an infinitely small motion, such that the point m be transferred to n, the point m' to n', &c. without violating the conditions which connect the several points of the system with each other; then the several infinitely small lines $m\,n$, $m'\,n'$, $m''\,n''$, &c. are what we call the *virtual velocities* of those points; and if we project one of these right lines on the direction of the force applied at the same point, we shall have the virtual velocity of this point estimated according to the direction of the force. Thus, if we let fall a perpendicular $n\,a$, on the direction mA of the force P, or on the prolongation of the same; and from the point n' a perpendicular $n'\,a'$ on the direction of P′, or on its prolongation, &c. we shall in this manner have the lines $m\,a$, $m'\,a'$, $m''\,a''$, &c. for the virtual velocities of the points m, m', m'', &c. estimated according to the direction of the forces acting on those points. Then giving to the projections which fall on the direction of the forces the sign +, and to those which fall on the prolongations the sign —, the general principle of virtual velocities may be thus enunciated, viz.

"If the forces P, P′, P″, &c. be in equilibrio, the sum of the forces multiplied respectively by the virtual velocities p, p', p'', &c. estimated according to their directions, is equal to zero; that is,

$$\mathrm{P}p + \mathrm{P}'p' + \mathrm{P}''p'' + \&\text{c.} = 0;$$

and reciprocally the forces P, P′, P″, &c. will be in equilibrio, when this equation has place, for every infinitely small motion that we can give to the system of points m, m', m'', &c.

Such is the general enunciation of this important principle; and with respect to its application to the solution of every problem of statics, it only requires that we distinguish in each particular case the different infinitely small motions that the system of points m, m', m'', &c. is susceptible of taking, and determine for each of these motions the virtual velocities $p, p'\,p''$, &c. estimated according to the above principles; which being done, the above equation will give immediately the equations of equilibrium, and these will be equal in number to those of the possible motions. But as, for the reason stated above, we do not intend to employ this principle in establishing our conditions of equilibrium, we shall enter no farther in this place into the application of this theory; but shall refer such of our readers who are desirous of seeing a farther developement of it to the "Mecanique Celeste," of La Place, and the "Mecanique Analytique," by La Grange; see also Poisson "Traité de Mecanique."

Let us now proceed to our investigations relative to the equilibrium of the simple mechanical powers.

§ XV. *Of cords, or the funicular machine.*

Funicular machine.

136. We have already in our tenth section investigated the principles and conditions of equilibrium as

connected with the funicular machine; we shall therefore, in this place, merely collect together the deductions there obtained, and endeavour to illustrate them by the solution of a few examples; and shall adopt here, as in the article above alluded to, the hypothesis of cords being perfectly flexible and inextensible, and as devoid of gravity or weight, unless the contrary be stated; and that they are reduced to their axes, or that they have no sensible lateral dimensions.

Funicular machine.

The deductions to which we have above alluded, may be stated as follows:

1. If a cord be acted upon at its two extremities by two equal forces in opposite directions, its tension will be measured by one of those forces or weights; and if the forces are unequal, the tension will be measured by the least of the two.

2. If a system of forces be made to act upon a material point by means of any number of flexible cords, the conditions of equilibrium are the same as if the forces acted through any other medium; and the tension of each line will, when the system is in equilibrio, be equal to the force applied at its extremity.

3. Consequently any such system of forces may be reduced to that of three forces only; and the tension of the cords by which this new state of equilibrium is sustained, will be equal to the resultant of the forces which each respectively is made to replace.

4. Hence if we have any number of forces acting at different points of a flexible line, we may reduce each of those system of forces to that of three forces only; and therefore, in any investigation connected with a system of this kind, we may always suppose this reduction to have been effected, and confine our researches to the case in which the forces are connected, three and three by fixed knots, as in fig. 67.

5. This polygon will be in equilibrio, when all the forces being transferred to a single point parallel to themselves, will sustain each other in equilibrio in that point: and if this condition have not place, the equilibrium cannot subsist, whatever form we give to the funicular polygon.

6. In order to find the ratio of the extreme tensions P and P^{iv}, referring to the above fig.; let them be denoted by t and t', and the

$$\begin{cases} \text{angle PAP}' \text{ by } a; \quad \text{P}''\text{BA by } a'; \quad \text{BCP}''' \text{ by } a'', \\ \text{angle P}'\text{AB by } b; \quad \text{P}''\text{BC by } b'; \quad \text{P}'''\text{CP}^{iv} \text{ by } b''; \end{cases}$$

then we shall have

$$\begin{aligned} P &: t \;::\; \sin b : \sin a, \\ t &: t' \;::\; \sin b' : \sin a', \\ t' &: P'' \;::\; \sin b'' : \sin a''. \end{aligned}$$

And multiplying these proportions together, and cancelling the common factors; we have

$$P : P^{iv} :: \sin b.\, \sin b'.\, \sin b'' : \sin a.\, \sin a'.\, \sin a'':$$

and in the same manner we may find the ratio of any two other forces or tensions.

7. The above gives only the ratio of the tensions; but we may find the absolute tension of any side of the polygon, as for example BC, by finding the resultant of the two forces P''' and P^{iv}, or of the three P, P', P'', supposed acting at the same point; for these must obviously, when the system is in equilibrio, be equal and directly opposed to each other.

As none of the above cases can involve any practical difficulty, except what may appertain to the tediousness of the computation when the number of forces is considerable, we shall dispense with giving the solution of any particular problem by way of illustration.

8. If instead of the fixed knots which we have hitherto considered, we suppose the ends of the cord to be fixed, and a force to be applied acting by means of a moveable ring or running knot; then we have seen that in order to establish an equilibrium, the direction of the force must be such as to become a normal to the ellipsoid generated by an ellipse, which has the two fixed points for its foci, and its transverse axis equal to the given length of the cord.

As an example, let P and R (fig. 68) be the two fixed points, PCR the given flexible line, AQ′ the direction of the force Q; conceive this force to be connected with the line PR by means of a running knot or moveable ring C; and let it be required to determine the position of the line CQ, when that force is in equilibrio; and the tension of the line PCR.

The distance of the points P and R, as also the length of the line PCR being given, describe with P and R as foci, and with the length PCR as the transverse diameter, the ellipse ACB in the plane of the given direction Q′A; and find the point C in the ellipse ACB, such that the tangent to that point T*t* shall be perpendicular to the given line AQ′; from C draw CQ parallel to AQ′, and it will be the direction and position of the force Q as required.

Again, because by this construction, the angle PCR is bisected by the continuation of the line of direction QC, the tension of PC and CR are equal; and therefore denoting the tension sought by t, and completing the parallelogram CPVR, we shall have

$$CV : PC :: Q : \frac{PC \times Q}{CV} = t \text{ the tension of}$$

the cord PCR.

137. It only now remains to consider the equilibrium and tension of a number of weights connected by fixed knots to a flexible line, the extremities of which are attached to any two given fixed points. This problem we have treated in all its generality in art. 95, where our results are the most general that can be obtained, being applicable to every possible case that can be proposed: but at the same time it must be acknowledged, that the computations which they involve are extremely laborious, and that in many cases we may obtain the solution of particular problems more directly by employing methods which, though less general, are suited to the case under consideration; we shall therefore give an example or two of this kind, and with them conclude this article.

Ex

Let A and B be two fixed pulleys given in position, and w, w', w'' three given weights suspended as represented (fig. 69) to the flexible line w ACB w''; it is required to determine the point C, at which the weight w must be suspended, that is the lengths AC and CB, to ensure the equilibrium of the system.

Let the angle EAC $= a$, the angle DBC $= b$, AC $= x$, and BC $= y$; and call the horizontal distance AE + DB which is given, $= d$, EC − DC $= c$, and the sum of AC + BC $= s$; then we have four following equations, viz. as the weights w and w'' express respectively the tensions of the cords AC, CB; these, with the weight w', are the three forces which keep the point C in equilibrio; we must therefore have

$$\begin{aligned} w \cos a - w'' \cos b &= o, \\ w \sin a + w'' \sin b &= w', \\ x \cos a + y \cos b &= d, \\ x \sin a - y \sin b &= c. \end{aligned}$$

From which the four unknown quantities x, y, $\cos a$, and $\cos b$, may be determined; but as their solution only requires the common operations of algebra, we shall leave it for the amusement of the reader. and proceed to a second example.

Let A and B (fig. 70) denote two pulleys given in position, over which are made to pass the two extremities of the flexible cord ACDEB, of which the several parts AC, CD, DE, &c. are given; as also the three weights w, w', w''; and let it be required to find the weights P and Q which are requisite to preserve the system in equilibrio, and to determine the form of the funicular polygon.

Here we have eight unknown quantities which must be determined in order to arrive at the complete solution of the problem; viz. the weights P and Q, or the tensions of the cords AC, EB; the tensions of CD and DE; and the angles which these four lines make with the horizontal axis A x, or A y, for the angles made by the directions of these lines with one of those axes being known, the other may be immediately determined.

Let the given lengths of the four cords AC, CD, DE, EB, be denoted by a, a', a'', a'''; and their tensions by t, t', t'', t'''; their angles of direction with the vertical axis A y, by a, a', a'', a'''; also let the co-ordinates of the point B, viz. AK and KB be called m and n.

Then the equations of these co-ordinates, and the resolution and comparisons of the tensions, give

$$\begin{aligned} a \cos a + a' \cos a' + a'' \cos a'' + a''' \cos a''' &= n, \\ a \sin a + a' \sin a' + a'' \sin a'' + a''' \sin a''' &= m, \\ t \sin a - t' \sin a' &= o, \\ t \cos a - t' \cos a' &= w, \\ t' \sin a' - t'' \sin a'' &= o, \\ t' \cos a' - t'' \cos a'' &= w', \\ t'' \sin a'' - t''' \sin a''' &= o, \\ t'' \cos a'' - t''' \cos a''' &= w''. \end{aligned}$$

The solution of which eight equations will give the value of the eight unknown quantities sought, but the actual operation is necessarily attended with considerable labour; let us therefore, for the sake of simplification, suppose the points A and B to be in the same horizontal line which makes $n = o$; the four lengths a, a', a'', a''', to be all equal; as also the three weights w, w', w''. By this means the above equations will be reduced to the four following; viz.

$$\begin{aligned} \sin a + \sin a' &= \frac{m}{2a}, \\ t \sin a - t' \sin a' &= 0, \\ t \cos a - t' \cos a' &= w, \\ 2t' \cos a' &= w; \end{aligned}$$

which admit of an easy and ready solution: the actual operation of which we leave as before for the practice of the student.

§ XVI. *Of the lever.*

Lever. Definition.

138. A *lever* is an inflexible bar or rod, which may be subjected to the action of two or more forces, or powers, at different points, while it is supported on a fulcrum or prop about which it is free to move.

There are two kinds of levers which differ essentially from each other; viz. those in which the forces act on *contrary* sides of the prop or fuclrum, and those in which they act on the same side. They are, however, usually distinguished into three kinds according to the respective positions of the prop, with reference to the situation of the power and the resistance. In levers of the *first* kind, the fulcrum is between the power and the resistance, as in fig. 71: in the *second* kind, the resistance is between the fulcrum and power, as in fig. 72; and in the *third*, the power is between the prop and resistance, as in fig. 73.

139. For the greater simplicity, we may, in the first instance, consider the lever as void of gravity, and reduced to a right line coinciding with its axis; and the power and weight as two forces applied perpendicularly to it; in which case the equilibrium of the lever differs in no respect from what we have already considered under articles 48 and 49; viz. the equilibrium of parallel forces, from which we have drawn the following general theorem; viz.

"There will be an equilibrium between three parallel forces, when each of them is directly proportional to the distance of the points of application of the other two."

Properties of the lever.

130. In treating of the lever, it is usual with English writers, to distinguish the forces by the names, power, and weight, or power and resistance; terms that have merely a reference to the common application of the machine, and must not be understood as implying any necessary distinction of the two forces.

Denoting then the power by P, and the weight by W, we shall have from the preceding theorem (and referring to figs. 71, 72, and 73), the following analogies:

P : W :: FB : FA;

or the power and weight, in case of equilibrio, are reciprocally to each other as their respective distances from the fulcrum, or centre of motion. From these also we have

P : P + W :: FB : FB + FA,
W : P + W :: FA : FB + FA.

Which proportions are equally applicable to all the three kinds of levers; as they are also to levers of every form; observing only, that in all cases, FA and FB must be taken to denote the distance of the points of application of the forces from a plane passing through the fulcrum F perpendicularly to the direction of the lever; that is, the bent lever AB (fig. 74) will be in equilibrio when

P : W :: FB : FA, or B′b : A′a:

for it is obvious, that we may suppose the power to be applied at A, and the weight at B, without in any respect affecting the equilibrium; these points being each in the direction of its respective force.

131. When the power and weight act obliquely upon the lever, as in fig. 75, and the lever itself turns on a fixed pivot instead of simply resting on the fulcrum; then we must resolve each of these forces into two forces, the one perpendicular, and the other parallel, to the lever; that is, denoting the power P by DA, and the weight W by BG; also calling the angle DAC $= a$, and GBE $= \beta$; we must resolve the former into the two forces AC $= \mathrm{P} \sin a$, and DE $= \mathrm{P} \cos \alpha$; and the latter into BE $= \mathrm{W} \sin\beta$, and EG $= \mathrm{W} \cos \beta$; and then, in case of equilibrio, we shall have

$$\mathrm{P} \sin \alpha : \mathrm{W} \sin \beta :: \mathrm{FB} : \mathrm{FA}.$$

But if the lever merely rest upon a fulcrum, on which it is free to slide, then, beside the above proportion, we must have an equality in the horizontal forces $P \cos \alpha$, and $W \cos \beta$; that is, we must, in addition to the preceding analogy, have

$$P : W :: \cos\beta : \cos\alpha.$$

132. When several forces, as P, P', P'', and W, W', W'', acting on the contrary sides of the fulcrum F (fig. 76), these will be in equilibrio when the sum of the products of each force into its respective distance from the fulcrum on the one side, is equal to the sum of the like products on the other side; that is, when

$$P.FA + P'.FA' + P''FA'', \&c. = W.FB + W'FB' + W''FB'', \&c.$$

as is evident from what we have already demonstrated in the chapter on parallel forces; the fulcrum in this case answering to what is there defined the centre of the parallel forces.

But if there are many forces acting at different angles of direction, then, denoting these angles by α, α', α'', &c. and β, β', β'', &c. the lever will be in equilibrio, supposing it only free to revolve about a fixed axis; when

$$P \sin\alpha . FA + P' \sin\alpha' . FA' + P'' \sin\alpha'' . FA' + \&c. = W \sin\beta . FB + W' \sin\beta' . FB' + W'' \sin\beta'' . FB'' + \&c.$$

But when the lever merely rests upon a fulcrum on which it is free to slide, then the equilibrium cannot obtain unless there is also an equality in the horizontal forces; viz. unless,

$$P \cos\alpha + P' \cos\alpha' + P'' \cos\alpha'' = W \cos\beta + W' \cos\beta' + W'' \cos\beta'';$$

as we have already shown for two forces in the preceding article.

133. And in all the preceding cases of equilibrium, the same will obtain (when the lever acts on a fixed axis) whatever may be its inclination with the horizon, because the distances of the several lines of direction will still have the same proportionality to each other; but if the lever merely rest upon a fulcrum, it will immediately slide upon it as soon as it is disturbed from its horizontal position; and therefore an equilibrium is in this case impossible.

126. *The compound lever*, is a combination of two or more levers acting upon each other as in fig. 77, where the extremity B of the first lever, presses upon A′ of the second lever, and the extremity B′ of the second, upon the point A″ of the third; and so on for any farther combinations. And it is obvious here, from what has been already stated of the simple lever, that when P and W are in equilibrio, we shall have

$$P \times FA \times F'A' \times F''A'' = W \times FB \times F'B' \times F''B''.$$

134. Hitherto we have considered the lever as without weight; but it is necessary in all practical cases to have reference to this datum, which we must regard as a force acting at its centre of gravity; and in all cases of equilibrium, the moment of this force must be included with those acting on the same side of the fulcrum as itself; that is, denoting the weight of the lever by Q, and its distance FG of the centre of gravity from the fulcrum by q,

The equilibrium will require in

$$\begin{array}{lll} \text{fig. 71,} & P.FA + Qq = W.FB; & (1) \\ \left.\begin{array}{l}\text{fig. 72,}\\ \text{fig. 73,}\\ \text{fig. 74,}\end{array}\right\} & P.FA = W.FB + Qq; & (2) \\ \text{fig. 75,} & P \sin\alpha\, FA = W \sin\beta .\, FB + Qq; & (3) \\ \text{fig. 76,} & \left\{\begin{array}{l} P.FA + P'FA' + \&c. = \\ W.FB + W'.FB' + \&c. + Qq; \end{array}\right\} & (4) \end{array}$$

and to estimate the pressure upon the prop in any of the above cases, it will only be necessary to compute the weight which, acting at that part of the lever in an opposite direction, would produce an equilibrium in the system.

135. Such are the mechanical properties of the lever; which it may not be amiss to illustrate by a few practical examples.

Ex. 1. What power will be gained by a lever 100 effective inches in length, having a fulcrum, about which it turns, at the distance of $7\frac{1}{2}$ inches from the extremity where the resistance is applied. Or, what weight will a man, who presses with a force equal to $1\frac{1}{2}$ cwt. or 168 lb. be able to raise by such a lever.

Here the two arms of the lever are $7\frac{1}{2}$ inches and $92\frac{1}{2}$ inches; whence, denoting the required weight by W, we shall have from equation (1), rejecting Q as inconsiderable,

$$168 \times 92\tfrac{1}{2} = 7\tfrac{1}{2} \times W;$$

$$\text{whence} \qquad W = \frac{168 \times 92\frac{1}{2}}{7\frac{1}{2}} = 2072 \text{ lbs.}$$

If we suppose the weight of the lever to be 20 lb. and that it is a uniform bar, having its centre of gravity 50 inches from each end, or $42\frac{1}{2}$ inches from the fulcrum, then

$$168 \times 92\tfrac{1}{2} + 20 \times 42\tfrac{1}{2} = 7\tfrac{1}{2}\, W,$$

$$\text{and } W = \frac{168 \times 92\frac{1}{2} + 20 \times 42\frac{1}{2}}{7\frac{1}{2}} = 2187\tfrac{1}{3} \text{ lbs.}$$

Ex. 2. The length and weight of the lever, as also the power remaining the same, what weight will the same man be able to raise, supposing the resistance to be placed between the fulcrum and power as in fig. 72.

Here FA = 100, FB = $7\frac{1}{2}$, and FG = 50; whence by equation (2), we have

$$168 \times 100 = W \times 7\tfrac{1}{2} + 50 \times 20;$$

$$\text{or, } W = \frac{168 \times 100 - 50 \times 20}{7\frac{1}{2}} = 2106\tfrac{2}{3} \text{ lbs.}$$

Ex. 3. A tapering piece of timber, 24 feet long, is balanced, when resting on a prop 13 feet from the less end; whereas, when the prop is placed in the middle of its length, a weight of 210 lb. suspended at the least end, exactly produces the equilibrium; required the weight of the tree.

Here, since the piece balances itself when the prop is 13 feet from the less end, the centre of gravity must there be vertically over the prop. Supposing, therefore, the whole weight of the tree to be accumulated in that point, the length of the arm of the lever by which it acts in the second position of the prop will be 1 foot, and the arm by which the power 210 lbs. acts, will be 12 feet; calling, therefore, the required weight W, we have

$$210 \times 12 = 1 \times W, \text{ or}$$
$$W = 2520 \text{ lbs.}$$

Ex. 4. In a false balance; viz. one in the which the arms are of unequal length; a body appears to weigh, when placed in one scale, and the weight in the other, 36 lbs.; but when weighed in the other, 49 lbs.: what is its real weight,

Writing x and y to denote the lengths of the unequal arms, and w the weight of the body, we have

$$\left\{\begin{array}{l} 36x = yw, \\ 49y = xw; \end{array}\right.$$

multiplying these equations together,

$$36 \times 49xy = xyw^2;$$

$$\text{or,} \qquad w^2 = 36 \times 49;$$

and therefore $w = \surd(36 \times 49) = 6 \times 7 = 42$ lbs.; that is, the real weight is a geometrical mean between the two false weights shown by the balance.

Some of the best constructions of the balance are explained in the Alphabetical department of this work.

136. The Roman *statera or steel-yard*, is a particular application of the principles of the lever of the first kind, having unequal arms, and is so contrived, that one weight alone may serve to counterpoise a great variety of others, by sliding it to different distances from the point of suspension upon the longer arm of the lever.

Let W, W′, W″, &c. denote the weights which are suspended successively from B (fig. 78), w the invariable distance FB, and p, p', p'' the variable distances FD, FD′, &c. (at which the constant weight P must be hung) from the fulcrum, in order to keep the beam in equilibrio; let also Q denote the weight of the beam, and q the distance of the centre of gravity from the fulcrum; then our equation (1) gives us the following series of equation:

$$\begin{aligned} W w &= P p + Q q, \\ W' w &= P p' + Q q, \\ W'' w &= P p'' + Q q, \\ \&c. &= \&c.; \end{aligned}$$

whence we have

$$\begin{aligned} p &= \frac{W w - Q q}{P}, \\ p' &= \frac{W' w - Q q}{P}, \\ p'' &= \frac{W'' w - Q q}{P}, \\ \&c. &= \&c.; \end{aligned}$$

consequently, these several distances are readily determined.

Note. We have here supposed the centre of gravity of the beam to be in the longer arm; if it should be found on the other side of the fulcrum, then $Q q$ in our first equations will be negative, and, consequently, in the last positive. And if the fulcrum be found immediately in the centre of gravity, then $Q q = 0$, and we have

$$p = \frac{W w}{P},\ p' = \frac{W' w}{P},\ p'' = \frac{W'' w}{P},\ \&c.$$

As an example, let us suppose the constant weight P to be 1 lb. the length of the beam 36 inches, its weight 2 lb. and the centre of gravity 1 inch from the fulcrum, which latter is 2 inches from the extremity of the beam, where the hook is suspended; to find the greatest weight that the steel-yard will ascertain, and the divisions on the longer arm for all the intermediate lbs. between the greatest and least weight:

Here the longer arm from the fulcrum being 34 inches, and the constant weight 1 lb. the greatest moment will be $1 \times 34 = 34$; also, $2 \times 1 = 2$, will be the moment arising from the centre of gravity; consequently, $\frac{34 + 2}{2} = 18$ lbs. the greatest weight. And supposing the weight P to be removed entirely over the fulcrum, the least weight is obviously 1 lb.; we have therefore to find the several distances answering to 2 lb. 3 lb. &c. 18 lb.; that is, we have to substitute for W W′, W″, &c.; the numbers 2, 3, 4, &c. in which case our equations become

$$\begin{aligned} p &= \frac{W w - Q q}{P} = \frac{4-2}{1} = 2 \text{ for } 2\text{ lb.} \\ p' &= \frac{W' w - Q q}{P} = \frac{6-2}{1} = 4 \text{ for } 3\text{ lb.} \\ p'' &= \frac{W'' w - Q q}{P} = \frac{8-2}{1} = 6 \text{ for } 4\text{ lb.} \\ \&c. \quad & \quad \&c. \qquad \&c. \end{aligned}$$

Steel-yards admit of a variety of constructions, as will be shown in the Miscellaneous department of this work.

§ XVII. *Of the wheel and axle.*

Wheel and axle. Definition.

137. The *wheel and axle;* or as it is often called, the *axis in peritrochio,* is a machine consisting of a cylinder and a wheel having the same axis; at the two extremities of the former are pivots on which both may turn. The power is applied at the extremity of the wheel, generally in the direction of a tangent, and a cord is wrapped about the cylinder, in order to overcome the resistance or to elevate the weight.

This machine is sometimes, without any impropriety, called the perpetual lever; it being, in reality, a lever on whose arms the power and weight may always act perpendicularly, although the lever turns about its fulcrum. It is frequently constructed of the form represented in fig. 79, where CD is the cylinder, at the ends of which are the pivots EF, turning in the solid pieces of timber HF, AE; and the weight W is raised by means of a rope coiled about the cylinder; the power being applied to the wheel SSB, either by the cord II, or by the handles S, S, S. Sometimes, instead of the wheel, we find the machine made up of levers fixed into the cylinder, as spokes into the nave of a wheel; at others, a simple handle serves for the application of the power, as in fig. 80; but the effect is still the same, except that the rotation is less uniform. In some cases, the cylinder is horizontal, as in the figures above referred to; as it is also in some description of cranes; while in others it is vertical, as in the capstan, &c. But whether the cylinder be horizontal or vertical, this machine has a manifest advantage over the simple lever in point of convenience; for by the continual rotation of the wheel, the weight may be raised to any height, or from any depth, while it could be raised only to a very small altitude by the former machine.

General property.

138. In the wheel and axle, if the power acts perpendicularly to the radius of the former, and the weight perpendicular to that of the latter, there will be an equilibrium when the weight and the power are reciprocally as the radii or distances at which they act.

This will be evident by referring to fig. 81, which we may conceive to be a vertical section of the machine perpendicular to the axis; and since the effort of the weight to turn the axle round is the same at whatever point of that axle it be applied, suppose both the power P and the weight W, or the resistance, to have their points of application A and B in the same plane perpendicular to the axis of rotation. Then, whether the power P acts at A or A′, since it acts perpendicularly to the radius CA, CA′, while the weight acts perpendicularly to CB, it is manifest that either ACB, or A′CB may be considered as a lever, whose fulcrum is C, and consequently

$$P : W :: CB : CA.$$

139. The above is the general theoretical property of this very convenient mechanical power: but it may not

Practical properties.

be amiss to add a few remarks as to the practical properties of it under certain and particular csses.

First, we may observe, that if the power act in any other direction than the tangential one, as in AP′ for example, then drawing from C, the line CD, perpendicular to the direction of the power, we have

$$\mathrm{P : W :: CB : CD;}$$

whence, because CD is always less than AC, it is obvious that the tangential is the most advantageous direction in which the power can be applied.

Again, since we have, as shown above,

$$\mathrm{P : W :: CB : CA;} \text{ or}$$
$$\mathrm{CA \times P = CB \times W;}$$

it follows that when the power is variable, if the diameter of the wheel increase in the same proportion as the power diminishes, and *vice versa*, the force with which the wheel will continue to revolve will always be of the same magnitude. An example of the application of this principle is ingeniously introduced into the construction of the fuzee of a watch, and into the main spring on the tumbler of gun locks.

When the moving force is applied by means of handles, as at S, S, S (fig. 79), it frequently happens that many forces act simultaneously, one at each handle; then if they all act in directions perpendicularly to the respective radii, there will be an equilibrium when the sum of all the powers is to the weight as the radius of the wheel to the mean distance at which the powers act; or when the sum of the moments of the powers is equal to the moment of the weight.

When the power acts by means of a rope having any sensible thickness, compared with that of the axle, there will be an equilibrium when the power is to the weight as the sum of the radii of the axle and rope to the sum of the radii of the wheel and rope. For the action of both forces is transmitted by the axis of the rope, and consequently, its radius ought to be added both to that of the axle and to that of the wheel.

It is obvious, from the above, that the greater the rope is, the less will be the power gained by this machine; for the terms of a ratio increased, each by the same constant quantity, *i. e.* in this case, half the thickness of the rope, the quantity of that ratio will be diminished; and consequently, if the rope be so folded on the axle as to cover the surface, and the weight act by a second spire of rope, the power must be augmented to maintain an equilibrium; and so on continually for every increased course of rope upon the axle.

Lastly, we may observe, that the distance which the weight hangs from the axle will make no difference in the result, except that which may be occasioned by the weight of the rope, or that which might be caused by a varying force of gravity, at different distances from the earth's centre; this last, however, must be a mere theoretical consideration, and will never become sensible in any practical application of this machine.

When the weight and power act in parallel directions, and on opposite sides of the horizontal axis of motion, the pressure upon the pivots will arise conjointly from the action of the weight, that of the power, the weight of the wheel, and that of the axle: let us denote the weight of the axle by A, that of the wheel by V, the power and resistance to be overcome, being represented as before by P and W. Then (fig. 81-*a*) the weight A may be considered as acting in the middle of the axle, and consequently the pressure upon each pivot, as arising from this part of the system, will be $\frac{1}{2}$ A. The weight V will act at C, the centre of the wheel; therefore the pressure upon $\mathrm{A} = \frac{\mathrm{V.BC}}{\mathrm{AB}}$, and upon $\mathrm{B} = \frac{\mathrm{V.AC}}{\mathrm{AB}}$, as follows from the principle stated, relatively to the pressure of the lever, art. 134; these being the weights that would be requisite to equilibrate the weight V, if applied at the same points, and in opposite directions, to the pressure on the pivots.

The forces P and W, may be considered as acting together at their common centre; that is, if I be the centre of a section of the axle where W acts, and

$$\mathrm{ID : DC :: P : W,}$$

D will be the point where P + W will act together; hence the third pressure upon A, will be

$$= \mathrm{(P + W)} \frac{\mathrm{BD}}{\mathrm{AB}}, \text{ and upon } \mathrm{B} = \mathrm{(P + W)} \frac{\mathrm{AD}}{\mathrm{AB}}:$$

consequently, in the case of an equilibrium, the entire pressure upon

$$\mathrm{A} = \tfrac{1}{2}\,\mathrm{A} + \frac{\mathrm{V.BC} + \mathrm{(P + W)\,BD}}{\mathrm{AB}},$$
$$\mathrm{B} = \tfrac{1}{2}\,\mathrm{A} + \frac{\mathrm{V.AC} + \mathrm{(P + W)\,AD}}{\mathrm{AB}}.$$

Whee axle c pounc

140. If a series of wheels and axles be so connected by cords as to act upon each other, the powers being applied tangentially to the first wheel, and the weight, in like manner, to the last axle; there will be an equilibrium when the power is to the weight as the continued product of the radii of the axles to the continued product of the radii of the wheel. For let such a system of wheels and axles be represented in fig. 82, and let a force Q, in the direction OF, balance P, acting at G; and a force Q′, in the direction NE, balance Q; then, from what is shown (art. 138), we have

$$\mathrm{P : Q :: OD : DG,}$$
$$\mathrm{Q : Q' :: CN : CF,}$$
$$\mathrm{Q' : W :: AB : AE;}$$

whence, by compounding the ratios, we have

$$\mathrm{P : W :: OD.CN.AB : DG.CF.AE;}$$

that is, the power is to the weight, as the continued product of the radii of the axles to the continued product of the radii of the wheels.

Pract opera

141. We may here remark again, with respect to the practical application of this compound machine, that if the wheels and axles, instead of being at a distance from each other, and connected by cords, are placed as in fig. 83, and the axle of one wheel be made to act on the circumference of another, by means of equal teeth, the proportion between the power and weight will still be the same as above; for each wheel and axle will act as a lever, whose fulcrum is at the centre of the axle, and its arms respectively equal to the radii of the wheel and the axle.

This is commonly called *tooth and pinion work*; the axle, when its surface is indented, being called a *pinion*; and because the number of the teeth in the wheels and axles are to each other as their peripheries, or as their radii, it follows, that the power is to the weight, as the continued product of the teeth in the pinions is to the continued product of the teeth in the wheels. Or, if P act on the wheel A by a cord, and W be suspended from the axle *c* by another, we shall have

$$\mathrm{P : W ::}\ t{\cdot}t'{\cdot}t''{\cdot}t'''{\cdot}\ \mathrm{rad.}\ c\ \mathrm{: T.T'.T''.T'''.\ rad.\ A,}$$

where t, t', t'', &c. T, T', T'', &c. denote, respectively, the number of teeth in the axles a, b, c, &c. and in the wheels B, C, D, &c.; and in whatever other manner the application may be made or the motion communicated, still the ratio of the weight to the power will be obtained on the same principles, and by a similar process.

If the wheels be all equal to each other, and the axles equal to each other; or if each wheel be to the axle on which it is fixed in a constant ratio, as for instance that of $r : s$, and if n be the number of wheels or axles, then we shall have

$$P : W :: r^n : s^n.$$

Let us suppose, for example, that the radius of the wheel is to that of the axle as 10 to 1, and that there are three wheels and three axles, then

$$P : W :: 10^3 : 1^3 :: 1000 : 1;$$

and if there were four such wheels, then

$$P : W :: 10{,}000 : 1, \text{ and so on.}$$

Since, when a toothed pinion works a wheel, the peripheries of both work with the same velocity, and the pinion of the second wheel moves the third wheel with a less velocity than the second moves, being at a less distance from the centre of motion; and that the same obtains throughout the system: it follows, that in a combination of wheels in motion, the number of revolutions of the wheel, where the power acts, is to the number of revolutions of the wheel where the weight is applied, as the product of the radii, or the teeth of the wheels, to the product of those of the axles; and, consequently, the velocity of the power is to that of the weight in the same ratio, which again returns to this: that the product of the power into its velocity is equal to that of the weight into its velocity; a general property belonging to every species of machine, and which, as we have stated (art. 135), is made by Lagrange, and some other modern writers, the fundamental property, or that on which they have founded the whole theory of mechanics, under the denomination of *virtual velocities.*

142. Before we dismiss this part of our subject, it may not be improper to observe, that in forming the teeth, it is of considerable importance to determine their proper curvature, so that the motion may be communicated equally and with as little friction as possible.

Two methods of accomplishing this end have been recommended; of which the first was originally proposed by M. de la Hire, who endeavoured to demonstrate that the pressure would be uniform if the teeth were formed into epicycloids; and Camus, in his Cours des Mathematiques, has adopted the same idea, and applied it to the various cases that are likely to arise in practice. This construction, however, is subject to certain limitations, on which account a second method has been proposed, which secures a perfect uniformity of action without being liable to any limitation. This consists in making both teeth and pinions of the evolutes of circles: thus let IHF, KE b (fig. 84), be the wheels to which the teeth are to be accommodated, the acting face of the tooth a, must have the form of the curve traced by the extremity H, of the flexible line F a H, as it evolves from the circumference; and, in like manner, the acting face of the tooth b must be formed by evolving of a thread from the circumference of the circle KE b. The line FCE, drawn to touch both circles, will cut the surface of the two teeth in C, the point where they touch each other; and the faces of both teeth will always touch each other at a point in the common tangents to both circles; and the force arising from their mutual pressure, will always act in the direction of the circumference of the wheels E and F. This form, by allowing the teeth to act on each other through the whole extent of the line FCE, will admit several teeth to act at the same time; and thus, by dividing the pressure amongst many, will diminish its quantity upon any one of them, and therefore diminish the cause of the indentations they unavoidably make upon each other: and consequently, a considerable number of teeth thus formed, and acting at the same time, will have a tendency to produce a very regular and uniform motion. But more of this, in the Miscellaneous department.

§ XVIII. *Of the pulley.*

Pulley. Definition.

142. The *pulley* is a wheel of wood, brass, or iron, turning on an axis, and enclosed in a kind of frame or case, called a *block*, which admits of the rope to pass freely over the circumference of the pulley; in the latter there is usually a groove, to prevent the rope from slipping off. The pulley is said to be *fixed* or *moveable*, according as its block is fixed, or rises and falls with the weight. An assemblage of several pulleys is called a *system* of pulleys, a *muffle* or *polyspacton*, of which some are in fixed blocks, and others in a moveable one.

The properties and application of the pulley have, by some writers, been explained, by considering a fixed pulley as a lever of the first order, and a moveable one as a lever of the second order; but one of the most simple and natural methods of computing the power, and explaining the effects of the pulley, appears to be, that every moveable pulley hangs by two parts of the same rope equally stretched, and which must therefore sustain equal parts of the weight; and consequently, when one and the same rope passes over several fixed and moveable pulleys, since all its parts are equally stretched, the entire weight must be equally divided amongst all the ropes by which the moveable pulleys are suspended: hence, if the power which acts on one rope, be equal to the weight divided by the number of ropes, the power and weight will mutually balance each other, and establish an equilibrium in the system.

This principle may be applied with facility to most cases which occur in practice, where the directions of the several cords are parallel, or very nearly parallel, to each other, as in the several systems exhibited in fig. 85; but when the ropes are drawn in directions which are not parallel, some modification of the above principle becomes necessary; viz. we must introduce into our computation, considerations founded upon the principles of the composition and resolution of forces, similar to those employed in treating of the funicular polygon: on which account we prefer giving one general proposition, depending upon the pure principles of equilibrium, from which the application to particular cases may be readily deduced.

Fixed pulley.

143. Let DEC (fig. 86) represent a pulley kept in equilibrio by three forces P, W, R; of which the two former act by a rope PEGDW, which touches the pulley throughout the arc EGD; and the latter acting by a rope or bar RG, whose direction passes

through C, the axis of the pulley. Now since the whole is in equilibrio, the force R is equal and opposite to the resultant of the other two: but this resultant is destroyed or counterbalanced by the centre C of the pulley, and therefore necessarily passes through that point; consequently it bisects the angle DRE, whence DC = CE, or P = W. Produce therefore WD, PE, till they meet CR; and setting off, from the point of concourse, the equal distances RA, RB, on the direction of the powers to represent them; and complete the parallelogram RAIB, then we shall have

$$P : W : R :: RB : BI : IR.$$

But the triangles RBI, ECD, having their sides respectively proportional to each other, are similar; consequently,

$$P : W :: R :: CE : CD : DE;$$

that is, *each of two forces in equilibrio about a pulley, is to that which retains the axis, as the radius of the pulley is to the chord of that arc with which the rope is in contact.*

Moveable pulley.

144. When the direction of the power and weight are parallel, viz. when WD is parallel to PE, CD and CE would coincide with the semi-chords FD, FE, and we should have P = W, or the power equal to the weight, as in the fixed pulley with parallel ropes. Or supposing the figure inverted, the rope fixed at one end as W, and the power acting at the other end P, R would then be the weight, and would be equal to 2 P.

And if the angle DRE be denoted by 2 a, then will

$$R = 2\,P \cos a;$$

for since

$$P : R :: CE : DE,$$

we have

$$R = P\,\frac{DE}{CE};$$

and because

$$DE = 2\,FE = 2\,CE \cos a,$$

$$R = P.\frac{2\,CE \cos a}{CE} = 2\,P \cos a.$$

Consequently, if DRE = 120°, then, because cos a = ½ rad. we have P = W = R; that is, the power, the weight, and resistance, are all equal to each other.

System of pulleys.

145. We have at present only considered the properties of a single pulley fixed and moveable: it remains now, therefore, to examine the case of a system of moveable pulleys.

Let the weight W (fig. 87) be kept in equilibrio by the power P, through the medium of the pulleys A, A', A'', &c.; the angles n AA', n A'A'', &c. made at each, being denoted by 2 a, 2 a', 2 a'', &c.; and let t, t', t'', &c. be the tensions of the cords by which the pulleys A, A', A'', &c. are supported; then, by the preceding article, we shall have

$$\begin{aligned} W &= 2\,t' \cos a, \\ t &= 2\,t' \cos a', \\ t' &= 2\,t'' \cos a'', \\ \&c. &= \&c. \\ t^{(n-1)} &= 2\,t^{(n)} \cos a^{(n)}; \end{aligned}$$

multiplying together the first two, the first three, &c. of these equations, and we shall have the tensions of the several cords in succession; also multiplying together the whole system of the equations, the product will exhibit the relation between P the power, and the weight W, viz.

$$W = 2^n\,P\,(\cos a.\cos a'.\cos a''.\cos a'''..\cos a^n):$$

that is, *in a system of moveable pulleys, each of which has a separate rope, the power is to the weight, as radius to the continued product of the cosines of half the angles, made by the rope sustaining each pulley, into that power of 2 whose exponent is equal to the number of pulleys.*

If the cords be parallel, as in fig. 88, then will cos a = cos a' = cos a'', &c. = 1, and the preceding equation becomes

$$W = 2^n P.$$

And if the angles be all equal to each other, but not all parallel with a vertical, then

$$W = 2^n P \cos^n a,$$

as is obvious.

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146. The systems of pulleys, whose powers we have investigated, and of which we have given a representation in the figures above referred to, include all those in most common use, although a variety of other systems might have been exhibited. This machine possesses many very important advantages; such for example, that it is comparatively of small weight, and readily transferable from place to place, and that it can be made to act in any required direction; and is, in many cases, applicable where we should find it difficult to apply any other mechanical power. But it must be acknowledged that its disadvantages are also considerable, arising principally from the friction, which is, in some cases, very great; and from the rigidity of the ropes with which the pulleys are necessarily connected.

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tent

Several ingenious contrivances have been devised for removing, in a great degree, these and other inconveniences; of which, the merit of one of the most important is due to Mr. White, who obtained a patent for his invention, of which he gives the following description:

The machine is exhibited in fig. 89, and consists of two pulleys Q and R; one fixed and the other moveable. Each has six concentric grooves capable of having a cord passed round them, and thus acting like as many different pulleys, having diameters equal to those of the grooves. Now if we suppose each of these grooves to be a distinct pulley, and that all their diameters are equal, it is evident, that if any weight, for instance 144 lbs. were to be raised by pulling at S till the pulleys touch each other, the first pulley must receive the length of the string as many times as there are parts of the string hanging between it and the lower pulley.

In the present case there are twelve strings, b, d, f, &c. between the two principal pulleys, formed by the passing of the cord over the six upper and six lower grooves; whence as much line must pass over the uppermost pulley as is equal to twelve times the distance of the two. But from an inspection of the figure, it is obvious that the second pulley cannot receive the full quantity of cord, by as much as is equal to the distance between it and the first. In like manner, the third pulley receives less cord than the first, by as much as is the distance between the first and third; and so on to the last, which receives only one-twelfth of the whole: for this receives its portion of the line or cord n, from a fixed point in the upper frame, from which it gains nothing; while all the others in the frame receive the line partly by turning to meet it, and partly by its meeting them. Supposing now these pulleys to be equal in size, and to move freely as the lines determine them; then it will be evident, from the nature of the system, that the number of their revolutions, and consequently their velocities, must be in proportion to the number of suspending parts that are between the fixed point above mentioned, and each pulley respectively. Thus the outermost pulley would go twelve times round in the time that the pulley, under which the part *n* of the line, if equal to it, would revolve once; and the intermediate times and velocities, would be a series of arithmetical proportionals, of which, if the first were 1, the last would always be equal to the whole number of terms.

Since, then, the revolutions of equal and distinct pulleys are measured by their velocities, and that it is possible to find any proportion of velocity on a single body running on a centre, viz. by finding proportional distances from that centre; it follows, that if the diameter of certain grooves, in the same substance, be exactly adapted to the above series (the line itself being supposed inextensible, and of no sensible magnitude), the necessity of using several pulleys in one frame will be obviated, and with that some of the inconveniences to which the use of the pulley is liable. In practice, however, the diameter of the rope employed, without which the smaller grooves, to which the said diameter bears a larger proportion than to the larger ones, will tend to rise and fall faster than they, and thus introduce worse defects than those which they are intended to obviate. The principal advantage of this kind of pulley is, that it destroys that lateral friction, and the shaking motion that is so extremely inconvenient in the common pulley.

§ XIX. *Of the inclined plane.*

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147. The *inclined plane*, considered as a mechanical power, is any plane forming an angle with the horizon on which a body may be kept in equilibrio by the action of other forces.

Let M (fig. 90) denote any body, the weight of which, W, let us consider as attached to its centre of gravity by a vertical line MW.

Now, in order that this body may be in equilibrio on the inclined plane by the action of a power P, the first condition necessarily is, that these forces have only one resultant; consequently their directions must meet in the point M; and since M is in the vertical which passes through the centre of gravity, the plane WMP will also be vertical, and contain in it the centre of gravity M. The first condition, therefore, involves this principle: that the power P must have its direction MW in a vertical plane passing through the centre of gravity of the body.

The second condition is, that the resultant MN of the forces W, P, be destroyed by the resistance of the inclined plane; which cannot have place, unless the resultant be perpendicular to the plane, and coincide with some point of the body which is in contact with it. This second condition is, in a certain degree, modified, in the case where the body comes in contact with the plane in several points only; for then, joining those points by right lines, it will be sufficient if the point, in which the resultant meets the plane, falls within the polygon formed by joining the above points.

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148. These conditions being fulfilled, let KL denote any body retained in equilibrio on the inclined plane AC, by a force or power P; and let us take the parts ME, MF, respectively proportional to the weight W and power P, and construct the parallelogram FMER; then the diagonal MR will represent the pressure that the body exerts on the plane AC; ME or FR, the weight W; and MF or ER, the power P: it is obvious, therefore, that we shall have immediately

$$P : W : R :: MF : FR : MR; \text{ or}$$

$$P : W : R :: \sin WMR : \sin PMR : \sin WMP.$$

Also, in consequence of the similar triangles AOW and OMN, the angle WMR and CAB are equal; whence

$$\sin WMR = \sin A = \frac{CB}{CA}.$$

Substituting now this value in the above proportions, we obtain, after multiplying the three last terms by AC,

$$P : W : R :: CB : AC \sin PMR : AC \sin WMP.$$

This ratio obtains, whatever may be the direction of the power MP; but when this becomes parallel to the length of the inclined plane AC, then the triangles MER and ACB are similar, because the angles C and E are then each equal to the angle MOC; whence it follows, that in this case we have

$$ER : ME :: CB : CA;$$

but ER denotes the power P, and ME the weight W; whence again

$$P : W :: CB : CA;$$

that is, when the power acts parallel to the inclined plane, we shall have an equilibrium when *the power is to the weight as the height of the plane is to the length, or as the sine of inclination to radius.*

If the power, or the line MF which denotes it, become parallel to the base of the plane, the triangles MER, CAB, which will then be similar, will give

$$ER : ME :: CB : AB; \text{ or}$$

$$P : W :: CB : AB;$$

that is, when the power acts parallel to the base, there will be an equilibrium *when the power is to the weight as the height of the plane to its base.*

Consequently, if in this last case, the angle A be half a right angle, the power and weight will be equal to each other; but if the angle be less than half a right angle, the power will be less than the weight; and if greater, the weight will be less than the power. It follows, from the above investigations, that the most advantageous direction in which the power can act, is when it is parallel to the face of the plane AC.

Of a body sustained on two planes.

149. When a heavy body is supported upon two planes, as in fig. 91, and it is required to determine the pressure sustained upon each plane, we may draw through G, the centre of gravity of the body, a vertical line F *b*; and at the points of contact of the body with each plane, erect the two perpendiculars EF and CF, which must necessarily intersect each other in the vertical F *b*, for otherwise the equilibrium would not be sustained: consequently, if from *any* point F in that vertical, perpendiculars be demitted to the planes, these will either pass through the points of contact, or be parallel to the lines which pass through them. Let these be FE and CF, and take F *b* to denote the weight of the body; on which, as a diagonal, complete the parallelogram *c* F *b e*, two of the sides of which fall upon FC, FE; then will F *c* or *e b*, represent the pressure P upon the plane AB, and F *e* or *c b*, the pressure P′ upon the plane BD. Hence, if KL be drawn parallel to HI, the sides of the triangle BKL will be respectively perpendicular to the sides of the triangle *c* FB, and we shall have

$$W : P : P' :: Fb : Fe : bc ::$$

$$KL : BL : BK :: \sin ABD : \sin ABH : \sin DBI :$$

that is, *if the weight of the body be represented by the sine of the angle comprehended between the two planes, the pressures upon them are reciprocally proportional to the sines of the inclination of those planes with the horizon.*

When a body is sustained by three planes, we must, in a similar manner, demit perpendiculars from some point in the vertical, passing through its centre of gravity upon those planes; and upon them construct a parallelopipedon whose vertical diagonal shall represent the weight of the body; then will its three contiguous edges denote the pressure upon the planes to which they are perpendicular.

§ XX. *Of the screw.*

Screw. Definition.

150. The *screw* is a mechanical power principally employed in pressing bodies close, and sometimes in raising weights; but its character, as a simple mechanical power, is certainly very questionable, although it is by most writers considered as falling under that denomination. It consists of a spiral thread revolving round a cylinder, and combines in itself the properties of the lever and the inclined plane.

If an isosceles triangle BFG, turn about the axle AZ (fig. 92), there will be generated by that revolution two conic frustrums, united by their greater ends: conceive now, that besides the motion of rotation, this triangle has also a motion of translation in the direction of the axis AZ, so that while the triangle makes a complete revolution, the point B is moved to G, and the whole triangle is found in the position GFG, and so on: the solid thus generated, is called the *interior screw*, and the height GB is called the *distance of the threads*: the *exterior screw* is so adapted to the other as if it were its mould, and is no other than the solid generated by the polygon HGFBC, supposing it to partake of the same motion as the triangle BGF. For the sake of distinction, we may apply the name *spindle* to denote the interior screw, calling the exterior one only the *screw*.

The spindle, then, is a cylinder encompassed with a spiral band of uniform thickness, and of which the inclination, with respect to the axis of the cylinder, is constant: the screw, on the contrary, is a solid, having a corresponding spiral hollow. In some cases the spindle is fixed in a solid block, as AB (fig. 93), while the screw E is moved upon it by means of a lever DC; in other cases the screw is fixed and the spindle is moveable; but this difference of action causes no difference whatever in the theory.

The curve, which any one of the points of the generating polygon, as for example N, describes about AZ, is obviously traced on the surface of a right cylinder whose axis is AZ, and radius of its base EN (fig. 92). If we develop this, then $d\,c$ (fig. 94), being the circumference which has EN for its radius, and taking the perpendicular $b\,c$, equal to the distance between two contiguous threads, the hypothenuse $d\,b$ will be the developement of an entire revolution of the point N. In fact, the helix being throughout of constant inclination, with respect to any position whatever of the generating line of the cylinder, every parallel to AD will make, with the developement of that curve, the same angle; thus the developement will be a right line, as $b\,d$; and in like manner, the right line $a\,f$, will be the developement of a second revolution.

Mecha Mecha proper of the screw.

151. In order now to investigate the power gained by the mechanical properties of this machine, let us suppose the spindle AB (fig. 93) to be fixed, and that the screw is moveable by the aid of a power P, applied to the extremity C, of a lever CE $=$ R, acting horizontally and perpendicular to the lever. Let W be the weight of the screw, or that which the screw supports, or the resistance opposed by the screw to the power P. If the screw press only on one of the points of the spindle, suppose it to be at the distance r from the axis, and that its position on the developement $d\,b$ of the spiral be at n, then will the pressure on the spindle be exactly the same as on the inclined plane $d\,c$; whence (art. 148) the power M, which we have supposed applied horizontally in the direction M n, must, in order to retain the equilibrium, fulfil the conditions of the analogy

$$M : W :: c\,b : c\,d;$$

whence
$$M = W \cdot \frac{cb}{cd} = W \cdot \frac{h}{2\,\pi\,r},$$

where $h = b\,c$, and $\pi = 3{\cdot}14159$ &c.; *i. e.* the semicircumference of a circle whose radius is 1.

Now substituting for this subsidiary power M, the power P, acting at the distance R, we have, from the principles of the lever,

$$PR = M\,r,$$

the lengths of the arms being R and r.

And if for M, in this equation, we substitute its value in the former, there will be obtained the equation

$$2\,\pi\,RP + W\,h.$$

This equation not containing r, is wholly independent of the distance that the point n is supposed to be from the axis, and it will therefore be the same if we suppose that point any where else on the spindle; and hence we deduce the general result; for this equation will obviously still obtain if we suppose the screw, instead of touching in a single point, to touch in any number of points whatever; in which latter case, every point on the thread of the spindle bears a portion of the weight W: these portions being denoted by W', W'', W''', &c. give

$$W' + W'' + W''' +, \text{&c.} = W.$$

But on the other hand, the force P, which supports these weights, may be considered as the sum of as many forces P', P'', P''', &c. as there are points of contact, each of which is employed in supporting its corresponding weight W', W'', W''', &c. And as to each of these the last equation applies, we have

$$W'h = 2\,\pi\,RP',\ W''h = 2\,\pi\,RP'',\ W'''h = 2\,\pi\,RP''', \text{&c.}$$

and since

$$W' + W'' + W''' +, \text{&c.} = W,$$
$$P' + P'' + P''' +, \text{&c.} = P;$$

we evidently obtain

$$W\,h = 2\,\pi\,RP;$$

that is, in case of equilibrium, the product of the weight into the distance of the threads, is equal to the product of the power into the circumference which it describes in one entire revolution.

Or, which is the same, *there will be an equilibrium in the screw when the power is to the resistance as the distance between two contiguous threads, in a direction parallel to the axis to the circumference described by a complete revolution of the power.*

It may not be amiss to observe here, that if the screw had a square or rectangular fillet, instead of a triangular one, the conclusion would still be the same, as this is wholly independent of the figure of the generating polygon.

In the endless or *perpetual screw* BC (fig. 95), which drives the teeth of the wheel FD, we shall, in the case of an equilibrium, have

$$\left\{\begin{array}{l} P \times AB \times \text{rad. of FD} = \\ W \times \text{dist. of threads} \times \text{rad. of the axle.} \end{array}\right.$$

For the perpetual screw is a combination of the axis in peritrochio and the screw.

152. The screw is of very extensive use in mechanics, its great power rendering it more eligible for compressing bodies than most other machines, at least, if we except the hydrostatic press; and the great disparity between the velocity of the handle and that of the threads of the screw, renders it proper for dividing space into an almost infinite number of parts, which is another important application of this engine. Hence, in the construction of many mathematical instruments, such as telescopes, where it is necessary to adjust the focus to the eyes of different people, the screw is always made use of, in order to move the eye-glass, by almost imperceptibly small quantities. In the Philosophical Transactions, vol. 71, a new method of applying the screw, so as to make it act with greater accuracy, is described by Mr. Hunter. These methods depend upon the following general principles, which are applicable to most machines.

1. That the strength of the several parts of the engine be adjusted in such a manner to the force they are intended to exert, that they shall not break under the weight they are intended to counteract; nor yet encumber the motion by a greater quantity of matter than is necessary to give them the requisite degree of strength.

2. That the increase of power, by means of the machine, may be so regulated, that while the force is thereby rendered adequate to the effect, it may not be retarded by procuring it more than is absolutely necessary.

3. That the machine be as simple as consistent with the other conditions.

4. That it be as portable and as little complicated as possible in its application.

5. That the moving power be applied in such a manner as to act to the greatest advantage, and that the motion ultimately produced may have that direction and velocity which is most adapted to the execution of the ultimate design of the machine.

6. Of two machines, in other respects equal, that is to be preferred in which the friction least diminishes the effect proposed to be accomplished.

Hunter's screw.

To attain all these advantages in any one machine is, perhaps, impossible; but Mr. Hunter's method of applying the screw certainly combines a great portion of them. Let AB (fig. 96) be a plate of metal in which the screw CD plays, having a certain number of threads in an inch, suppose 10. Within the screw CD there is an exterior screw, which receives the smaller screw DE of 11 threads in an inch, which is kept from moving about with the former by means of the apparatus AFGB. Now if the handle CKL be turned 10 times round, the screw CD will advance an inch upwards; and if we suppose the screw DE to move along with CD, the point E will advance an inch. And if then we turn the screw DE ten times backward, the point E will move down $\frac{10}{11}$ths of an inch, and the result of both motions will be to lift the point E $\frac{1}{11}$th of an inch upwards. But if, while the screw CD is turned 10 times round, DE be kept from moving, the effect will be the same as if it had moved 10 times round with CD, and then been turned back again 10 times; that is, it will advance $\frac{1}{11}$th of an inch: at one turn, therefore, it will advance $\frac{1}{10}$ of $\frac{1}{11}$, or $\frac{1}{110}$th part of an inch. Consequently, if we suppose the handle to be only 6 inches long, the power to produce an equilibrium must be to the weight, as

$$1 \text{ to } 110 \times 6 \times 2\Pi = 4146{\cdot}912.$$

Hence the force of this machine is found greatly superior to that of the common screw; for in the latter, to gain the same power, we must have 110 threads in an inch, which would render them too weak to resist any considerable violence.

With regard to the second general maxim, both kinds of screws are generally applicable; only that the more complicated structure, and consequently the greater expence of Mr. Hunter's screw, render it convenient to use the common one where only a small degree of power is requisite, and the improved one where a great power is wanted. The handle of this machine being short, makes it accord with the fourth of the above maxims; to answer the fifth, both seem equally proper; but for the sixth, the preference must be given to such as best answer the specific purpose proposed. Thus, if the screw DE be designed to carry an index, which must turn round at the same time that it rises upwards, the common screw appears preferable; though Mr. Hunter proposes a method by which his construction may answer the same purpose. With this view, a still smaller screw ought to play within DE, and be connected with CD, so as to move along with it. It must have, according to the foregoing proportions, 111 threads in an inch, and they must lie in a contrary direction to those of CD; so that when they are both turned together, and CD moves upwards, this other may move downwards. At one turn this will move upwards $\frac{1}{12210}$th of an inch, and at the same time will move in a circular direction.

Similar methods may be applied in many other cases: indeed, they have lately found very frequent application, though few of those who have adopted them have acknowledged by whom they were first proposed: a want of liberality which is inconsistent with the spirit of true genius.

§ XXI. *Of the wedge.*

Wedge. Definition.

153. The *wedge* is a triangular prism, or a solid considered to be generated by the motion of a plane triangle parallel to itself, upon a straight line which passes through one of its angular points; and it is called *isosceles*, *rectangular*, or *scalene*, according as the generating triangle is of the one or other of these descriptions.

The wedge is frequently used in cleaving wood, and sometimes in raising great weights; most cutting instruments, as knives, scissars, the hatchet, &c. are in part a species of the wedge, to which they are commonly referred.

The method of application is, as is very well known, by introducing the edge of the prism or wedge into a small opening in the substance intended to be separated, and which, by repeated blows upon the head of the wedge, is rendered greater and greater, till the end proposed is accomplished. The part of the prism which first enters is called the *edge*, the face opposite to it the *head*, and the two rectangular sides the *faces*.

Mechanical properties.

Let ABC (fig. 97) be a rectangular wedge, whose edge is C, face BC, and back AB; and let this wedge slide freely along the plane LN, the body E being drawn or urged in the direction KE, against the face of the wedge, and let it be kept in that direction by a force acting perpendicularly to KE, as in the line DE. Now there are three forces acting on the body E, viz. the resisting force KE, the sustaining force DE, and the re-action of the wedge in the direction AE, perpendicular to the surface BC.

On ED let fall the perpendicular AG; and since the three forces are in equilibrio, they will be to each other as the sides of the triangle AEG, drawn parallel to their directions. Draw EF perpendicular to AC, and the force AE will be resolved into two, one of which EF presses the wedge perpendicularly against LN, and is balanced by the re-action of the plane; the other FA tends to move the wedge upwards along the plane LN, and is balanced by the power on the back of the wedge. If, therefore, AG represent the force KE, EG will be the sustaining force, and AF the power applied to the back of the wedge when these forces balance each other; therefore making AE radius, AF is the sine of the angle AEF, or ACB; and AG is the sine of the angle AEG, or KEC; these two angles being the complements of AEK: hence,

When a resisting body is sustained against the face of a wedge, by a force acting at right angles to its direction, in the case of equilibrium, the power is to the resistance, as the sine of half the angle of the wedge to the sine of the angle, which the direction of the resistance makes with the face of the wedge; and the sustaining force will be as the cosine of the latter angle.

In the above investigation, we have supposed the wedge BCA to be resisted by the plane AC; but if we suppose the wedge doubled, or to become isosceles as BCH, the only difference will be that the force EF, which in the former case was counteracted by the plane, will now be supplied by the other half of the wedge; and consequently the power, the resistance, and the sustaining force, will be still the same as before.

When EK is parallel to BA, AG becomes equal and parallel to EF, and EG equal and parallel to AF, and the power is to the resistance AF to FE, or AB to AC, and equal to the sustaining force.

If EK be perpendicular to BA, the direction of the resisting force will be parallel to AB; therefore, the resisting and sustaining forces changing denominations, this will be a case corresponding with the former.

When KE is perpendicular to BC, the sine of the angle KEC is radius, and its cosine, which represents the sustaining force, vanishes; therefore, in this case, the power is to the resistance, as the sine of half the angle of the wedge to radius.

When the resistance is made against the face of a wedge by a body which is not sustained, but will adhere to the place to which it is applied without sliding; then, if from any point K′ we draw the line K′E through the middle point of the back, meeting the face of the wedge in E; and if we suppose E′ to be the resisting body, which acts in the direction EK′; and if the magnitude of the force by which it is urged be represented by AE, and from E we let fall the perpendicular EF upon AC; then resolving the force AE into two forces, one of them, as EF, will be balanced by the opposite half of the wedge, and the other AF, will be counteracted by the power; therefore the power, in this case, is to the resistance as AF to AE; that is, making AE radius, as the cosine of the angle EAF to radius.

Here, when K′E is perpendicular to BC, the power is to the resistance as AF to AE; that is, as the sine of half the angle of the wedge to radius. And when K′E is parallel to AB, AF vanishes; that is, the power is indefinitely less than the weight. When K′E is perpendicular to AB, EF vanishes, and AE and AF which represent the power, and the weight become equal.

Imperf tions o above theory.

We might add here a few other properties of the wedge according to the nature of the resistance to which it is opposed; but they are so little applicable to any practical cases, that it seems useless to dwell longer upon this subject.

In calculating the power of the wedge, its edge is supposed of no thickness, and its sides perfectly smooth; also the friction which it experiences is not considered, nor can it be perhaps accurately or even approximately valued: moreover, the power applied to the wedge is generally percussive from the stroke of a mall or mallet, which if it be not absolutely incomparable with pressure, at least involves much difficulty in the comparison,* and the shock which it occasions produces effects in many cases perhaps totally different from what would be caused by any degree of pressure. The theory therefore of the wedge, as above delivered, must not be considered as appertaining to any practical application of this powerful engine, but as pointing out merely, what would be the results, if the several powers and resistances were such as we have supposed them to be.

§ XXII. *General laws of the equilibrium of machines.*

Genera princip

154. By combining together in different ways, the simple mechanical powers we have been considering, various, and innumerable compound machines may be constructed; and with regard to these, the ratio of the power to the resistance necessary for establishing the equilibrium, may be computed by combining together the several principles deduced from the preceding investigations: but there exists a law much more general than those to which we now refer, and which belongs to every combination whatever, whence we may immediately determine the ratio sought. This consists in comparing the spaces the points of application of the power and resistance have a tendency to describe; and therefore may be considered, as in fact it is, a particular application of the principle of *virtual velocities* referred to under art. 135.

According to this theory, if we impress on a machine in equilibrio, a very small motion, and the points of application of the power and resistance describe curves tangents to the direction of these forces; also if we call p the space run over by the point of application of the power, and r that passed over by the point of application of the resistance; and if we denote the power by P, the resistance by R, and the ratio of the two latter be required, we shall have an approximation towards it, by means of the equation

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* The writer of this article has been informed, that from the results of some experiments made in Portsmouth dock-yard, in driving and pressing in large iron and copper bolts, a man of medium strength, striking with a mall weighing 18 lbs. and the handle of which was 44 inches, would *start* a bolt about one-eighth of an inch every blow; and that it required a pressure of 107 tons to press the same bolt down the same quantity, but a small additional weight pressed it completely home.

$$\frac{P}{R}=\frac{r}{p};$$

which approximation will be so much the more exact, in proportion as the motion given to the machine is less considerable; so that in taking the limit of the ratio $\frac{r}{p}$, we shall have exactly that of $\frac{P}{R}$; for it will be observed, that it is not in *all cases* the ratio of the spaces run over, which gives that of the power and resistance, but the spaces that the points of application have a *tendency* to describe.

If the curves described by the points of application be not tangents to their directions, we must take, instead of the spaces run over by these points, the projection of those spaces on the direction of the forces, and the inverse ratio of these projections, will be the same as that of the forces in the case of an equilibrio; moreover, we are at liberty to take, as the point of application of either force, any point taken in its direction, provided only that we employ it as if it were fixedly attached to the machine.

stration.

We shall illustrate the above principles by a few examples applicable to the cases which we have already investigated on other principles, and by this means verify the truth of the theory in these particular instances; and hence by induction infer its application to all others. This, it must be allowed, is not agreeable to the spirit of mathematical investigation; but after all that has been written on the principle in question, we doubt much if its truth depends upon any more secure ground; at the same time, as it never fails of being verified in every case in which the ratio has been otherwise determined, we may with safety, though not with complete satisfaction, consider it as a fundamental principle of Mechanics.

If in the bent lever (fig. 98), we assume as the points of application of the forces P and R, the extremities of the perpendiculars A b, A c, let fall upon their directions from the point A; then, when the lever turns about the latter point, the points b and c will describe arcs of circles, as $b\,b'$, $c\,c'$, tangents to the directions b E, c F of the forces, and each will have the same measure or number of degrees, and the lengths of these arcs are therefore as their radii A b, A c; so that we shall have

$$\frac{c\,c'}{b\,b'}=\frac{r}{p}=\frac{A\,c}{A\,b};$$

and since this ratio will remain the same, however small the motion impressed upon the lever may be, it follows that we shall have, according to the above principle,

$$\frac{P}{R}=\frac{A\,c}{A\,b}:$$

that is, in case of an equilibrium, the two forces P and R will be inversely proportional to the perpendiculars let fall from the point of support upon their respective directions.

If we had assumed the extremities B and C of the lever for the points of application of the forces P and R, the arcs of circles described by these points during the motion of the lever would not have been tangents to the direction of the forces, and we must therefore in this case have projected these arcs on the right lines BE, CF; then taking the ratio of these projections, we must have sought the limit of this ratio, which we should, as above, have found to be equal to the ratio of the perpendiculars A b and A c.

When the wheel AA'C, &c. (fig. 81) turns on its axis, the points of application of the power and weight, or resistance, will describe similar arcs of circles of the same number of degrees; the one on the circumference of the wheel, and the other on the circumference of the axle; and as these arcs are tangents to the direction of the forces, their lengths are as the radii AC, CB, of the wheel and axle; therefore in this machine

$$\frac{p}{r}=\frac{AC}{CB}\text{ and }\frac{P}{R}\text{ or }\frac{P}{W}=\frac{CB}{CA};$$

that is, the power is to the weight as the radius of axle to that of the wheel.

In a system of pulleys, as for instance the first in fig. 85, if the weight W ascend through any space r, each of the four lines passing over the moveable pulleys will be shortened by the same quantity; consequently the cord to which the power is suspended will be lengthened, or the power P will descend through a space equal to $4\,r$; we have, therefore, in this case

$$\frac{p}{r}=\frac{4r}{r}=\frac{W}{P}\text{ or }W=4\,P.$$

Again, in the system fig. 88, if the power P descend through any space p, the pulley A''' will ascend through a space $=\frac{1}{2}\,p$; the pulley A'' through $\frac{1}{4}\,p$; the pulley A' through $\frac{1}{8}\,p$; and lastly, the pulley A, or the weight W, through $\frac{1}{16}\,p$: that is, $r=\frac{1}{16}\,p$; whence

$$\frac{P}{W}=\frac{r}{p}=\tfrac{1}{16},\text{ or }W=16\,P.$$

In the inclined plane, referring for example to fig. 90, it is obvious, that whatever distance the power P descends, the weight or body will ascend along the plane by an equal quantity, which let be $a\,c$; but this latter motion not being made in a tangent to the direction of the force, we must project it upon that direction, that is, upon the perpendicular $c\,b$; and calling this last r, and the descent of the power p, we shall obviously have

$$r:p\;::\;c\,b:a\,c\;::\;c\text{B}:\text{A}\,c;$$

and since this ratio obtains, however small the motion may be, we shall have

$$\frac{r}{p}=\frac{CB}{AC}=\frac{P}{W};$$

that is, the power is to the weight as the height of the plane to the length.

In the screw, it is clear that while the weight or resistance is elevated through a space equal to the distances to two contiguous threads, the power will have described a space equal to the circumference of the circle it passes through; and, moreover, these spaces are so related to each other, that if the power passes through any arc of that circle, the weight will ascend through a vertical space, which is to that arc as the distance of the threads to the entire circumference.

Hence, calling r the space passed over by the weight, and s for that described by the power, we shall have

r : *dist. of threads* :: s : *circum. of the power;*

which would be the final result if the curve described by the power was a complete circle; but as this is a helix, and not a tangent, to the direction of the power, we must, in order to apply the general law of equilibrium to this case, project a small arc of the helix on the direction of the force: since, however, this direction is by the supposition a tangent to the circumference described by the handle, we may take the arc or tangent (these being indefinitely small), the one for the other; consequently, the power P to the resistance of the screw R, will be found by taking the limit of the ratio of $\frac{r}{s}$; and as this is constantly the same, and equal to

dist. of threads, circum. of the power;

it follows that in this machine, when in equilibrio, the power is to the resistance as the distance of the threads to the circumference described in one complete revolution of the power.

§ XXIII. *Experimental data, relating to the theory of Statics.*

Experimental data.

155. In establishing the first principles of the theory of Statics, we are under the necessity of considering lines without weight or thickness, surfaces perfectly polished, cords perfectly flexible, axles and wheels divested of gravity and friction, &c.; circumstances which in no instance will obtain in the practical application of the doctrine of equilibrium to the several useful purposes of life: and it becomes therefore absolutely necessary, when we wish to employ our theory in actually computing the results of certain combinations of the machines whose properties we have been illustrating, and whose powers have been investigated upon suppositions which have no place in nature; to establish a due estimation of the several counteracting causes, in order that we may thence determine, not merely the theoretical, but the actual practical powers and resistances which it is necessary to apply or to overcome. In other cases also, our investigations are intended to show, not the absolute powers and resistances, but their comparative value with regard to each other under various circumstances: as for example, in computing the relative strains upon solid bodies excited by the action of different forces under various conditions, and the proportional resistance which such bodies oppose to these several existing forces.

In these and other similar cases, it is impossible to state the absolute magnitude or intensity of either, without reference to experiment. We propose, therefore, in concluding the article Statics, to enter upon this practical consideration of the subject, and to give a series of the most useful and valuable experiments, as far as they relate to the principles of this doctrine, in order that the reader may, at any time, by introducing them into his theoretical investigations, estimate correctly the practical efficiency of any proposed mechanical combination or construction.

Of friction, and the rigidity of ropes.

Measure of friction.

156. It has been observed above, that in our preceding investigations relative to the power and equilibrium of the several mechanical powers, we have supposed surfaces to be perfectly polished and free from friction, ropes and cords perfectly flexible, the axes of motion as mere points, &c.: no one of which conditions has place in actual practice. It is important therefore to be able to form a due estimation of the several inert causes of resistance; as friction, rigidity, &c. which must be considered as distinct and passive forces, having no power to produce, but a constant tendency to destroy, the effect of the efficient forces of the machine. In the funicular polygon, the principal counteracting power, is the rigidity of the cords; but in the lever, it is chiefly friction; in the wheel and axle, and the pulley, both the above inert forces are combined; in the inclined plane, the screw, and the wedge, friction only enters, but it forms in the case of their actual equilibrium, a most important datum, particularly in the two latter. It is therefore indispensable to have regard to these forces, when we wish to apply the general laws of equilibrium to particular problems.

Frictio

It should be observed, however, that friction may be considered both as it tends to counteract the effort made merely to disturb the equilibrium of a machine, in which respect it has a relation with the doctrine of statics; or as it tends to destroy the motion of a body, in which sense it appertains to the theory of dynamics; but it is only in its former relation that we have to treat of it in the present chapter.

In consequence of the above relation of friction, both to the theory of Statics and Dynamics, we cannot enter fully upon the subject in this place: we shall therefore refer, for a practical investigation of its effects, to the article Friction in our general alphabetical arrangement; and in this place shall merely point out the nature of the experiments that have been made with a view of establishing the numerical and proportional values of this inert force, and the methods by which we may introduce them into our computations.

157. Two different means of estimating the power of friction are at times employed: the one consists in placing a body on a horizontal plane, having a fine cord fixed to it, and passing over a pulley, to which different small weights are attached till some one is found that is just sufficient to give a slight degree of motion to the body; this weight is then said to be equal to, or to be the measure of, the friction: and from several experiments of this kind, it has been found that the friction is always proportional to the weight of the body, and that it is independent of the area of the surface of contact; so that a body, of whatever figure, will experience the same degree or quantity of friction on which ever of its faces it is in contact with the plane. This, however, only obtains as far as friction is considered with reference to the resistance it opposes to the motion of a body at rest, that is, as it relates to the theory of Statics; for it has been found from the results of numerous experiments performed by Vince, that when the body has attained a certain velocity, the area of contact becomes a part of the data of the problem: but this, as we have stated above, will be discussed more at length under the term *friction.*

In the present article we shall assume, that within certain limits, the friction is as the pressure, and that it is independent of the surface of contact; so that if f be made to denote the friction due to the unit of weight in any case, $2f$ will be the friction due to a double weight under similar circumstances; $3f$ the friction due to a treble weight; and Wf the friction due to the weight W.

We have at present only spoken of one method of estimating the numerical value of friction: another, and that which has been very commonly employed, is as follows: place a body MN, on an inclined plane AC (fig. 99), and suppose it to be in the first instance retained upon the plane in consequence of its friction; if then we increase the angle A, which the plane forms with the horizon, till the body is just beginning to slide down the plane, the friction F will be equal to the weight multiplied by the tangent of the angle A.

ion.

For let GD be taken to denote the gravity or weight of the body, the perpendicular to the plane GK will denote the pressure on the plane, and DK will be the force by which the body endeavours to descend; which, at the instant before the motion ensues, is exactly equal to the friction.

Now since F is the friction due to the weight or pressure of the body B, we shall have

$$\mathrm{GD} : \mathrm{GK} :: \mathrm{F} : \frac{\mathrm{GK}}{\mathrm{GD}}\mathrm{F},$$

the friction due to the pressure of the body B in the position of the plane AC; consequently we have the equation

$$\mathrm{DK} = \frac{\mathrm{GK}}{\mathrm{GD}}\mathrm{F}, \text{ or}$$

$$\mathrm{F} = \frac{\mathrm{DK{\cdot}GD}}{\mathrm{GK}} = \frac{\mathrm{DK}}{\mathrm{GK}}\mathrm{W}.$$

By the similar triangles DKG, and ABC,

$$\mathrm{DK} : \mathrm{GK} :: \mathrm{BC} : \mathrm{AB} :: \sin \mathrm{A} : \cos \mathrm{A};$$

whence $\frac{\mathrm{DK}}{\mathrm{GK}} = \frac{\mathrm{BC}}{\mathrm{AB}} = \frac{\sin \mathrm{A}}{\cos \mathrm{A}} = \tan \mathrm{A}$:

consequently the above equation becomes

$$\mathrm{F} = \tan \mathrm{A} \times \mathrm{W};$$

that is, *the friction due to any body* B, *is measured by the weight of that body into the tangent of the angle at which it begins to slide down an inclined plane.*

Or, which is the same, calling f the friction due to the unit of weight, we shall have $f = \tan \mathrm{A}$; and 1 to f, will express the constant ratio between the pressure and friction.

mple. 158. The value of f, due to the unit of weight, for example 1 lb. being given, as also the weight of the body and the inclination of a plane; to determine the power necessary to preserve the body upon it in equilibrio.

Let W denote the weight of the body, and a the inclination of the plane; then by theory (art. 148),

the pressure on the plane $= \cos a \,\mathrm{W}$,
and the sustaining force $= \sin a \,\mathrm{W}$;

which, abstracting from friction, would denote the power necessary to preserve the body in equilibrio.

But from the preceding article, we have

$$1 : \cos a :: f\mathrm{W} : f\mathrm{W}\cos a,$$

the resistance due to the friction at the angle of inclination a, and which we must consider as opposed to the power $\sin a \,\mathrm{W}$; whence we have

$$(\sin a - f\cos a)\,\mathrm{W} = \mathrm{P},$$

the sustaining power sought.

If the weight W, the sustaining power P, and the value of f dueto 1 lb. were given, to find the angle of inclination at which the body would be in equilibrio; then we must proceed as follows: let x be the angle of inclination sought, then

$$1 : \cos x :: f\mathrm{W} : f\mathrm{W}\cos x,$$

the resistance due to friction: and $\sin x =$ sustaining power independent of friction. Whence

Mechanics.

$$(\sin x - f\cos x)\,\mathrm{W} = \mathrm{P};$$

or,

$$\sin x - f\sqrt{(1 - \sin^2 x)} = \frac{\mathrm{P}}{\mathrm{W}};$$

or,

$$f^2 - f^2\sin^2 x = \sin^2 x - \frac{2\,\mathrm{P}}{\mathrm{W}}\sin x + \frac{\mathrm{P}^2}{\mathrm{W}^2},$$

$$(f^2 + 1)\sin^2 x - \frac{2\,\mathrm{P}}{\mathrm{W}}\sin x = f^2 - \frac{\mathrm{P}^2}{\mathrm{W}^2},$$

$$\sin^2 x - \frac{2\,\mathrm{P}}{\mathrm{W}\,(1 + f^2)}\sin x = \frac{\mathrm{W}^2 f^2 - \mathrm{P}^2}{\mathrm{W}^2\,(1 + f^2)};$$

consequently

$$\sin x = \frac{\mathrm{P}}{\mathrm{W}(1+f^2)} \pm \sqrt{\left\{\frac{\mathrm{P}^2}{\mathrm{W}^2(1+f^2)^2} + \frac{\mathrm{W}^2 f^2 - \mathrm{P}^2}{\mathrm{W}^2(1+f^2)}\right\}};$$

or,

$$\sin x - \frac{\mathrm{P} \pm \sqrt{\{\mathrm{P}^2 + (\mathrm{W}^2 f^2 - \mathrm{P}^2)(1 + f^2)\}}}{\mathrm{W}\,(1 + f^2)}.$$

Friction of the lever.

159. Let us now endeavour to estimate the effect of friction upon the equilibrium of the lever: for this purpose let ABCD (fig. 100), represent a part of a heavy lever; *a n b*, a section of the circular hole, through which passes the cylindrical axis *n e d*; which two circles we shall suppose so nearly equal to each other, that the centre *o* of the one may be considered as the centre of the other, without any appreciable error. Now in the first place, if the sum of the several forces acting on the lever on the one side of *o*, or rather the moments of these forces, be equal to the moments of the forces on the other side of *o*, the lever, as we have seen in treating of this mechanical power, will be kept in equilibrio, independent of friction; but in consequence of this inert force, the moments on one side of the axis may be diminished to a certain extent, before any motion will ensue; that is, the friction being a tangential force at the point or line of contact *n*, between the circle of the lever and the cylindrical axis, will still maintain the equilibrium of the system.

Examples.

Let F denote the friction; then, as we have said above, this may be considered as a force acting in the direction of a tangent at the point *n*; and the moment of this force, as exerted to turn the lever about the point *o*, or as repelling a contrary force, will be denoted by F × *h*, where *h* is the radius *o n*. But the friction F will be proportional to the pressure of the lever upon the point *n*; and this pressure will be equal to the resultant of all the forces acting upon the lever, including F as one of them; and, moreover, this resultant must, since the system is in equilibrio, pass through the fixed point *o*; that is, *n o* must be the direction of all the forces of the system. Without at present being able to express the quantity of this resultant, let us denote it by Q, which will, as is obvious, also express the pressure upon the point *n*; and since F is proportional to the pressure, let it be denoted by F = f Q, where f is a constant quantity, which expresses the ratio between the pressure and friction.

We may, therefore, consider the lever as kept in equilibrio by the action of three forces, all acting at the point *n*; viz. the friction F or f Q in the direction *n* F; the resultant of the forces of the system, except the friction in the direction *n* E; and the re-action of the fixed centre *o*, in the direction *o n*.

These forces are, therefore, each the resultant of the other two; consequently R, the resultant of all the efficient forces on the lever, is equal to the resultant of the two forces fQ and Q; and since these are at right angles to each other,

$$R = \sqrt{(f^2Q^2 + Q^2)} = Q\sqrt{(1 + f^2)}.$$

If, now, r be taken to denote the perpendicular let fall from the centre o upon the direction of the resultant R, we shall have Rr to denote the moment of this resultant, or the tendency which it has to cause the lever to turn about the centre o; but this tendency, as we have seen, is counteracted by the friction acting at the perpendicular distance $on = h$; whence

$$Rr = fQh;$$

or, $$rQ\sqrt{(1 + f^2)} = fQh,$$

and $$r\sqrt{(1 + f^2)} = fh;$$

whence $$r = \frac{fh}{\sqrt{(1 + f^2)}}.$$

And here, since f is a known quantity, being supposed determined from experiment, as indicated in art. 157, and h being the given radius of the axle, r is also determinate. Hence, therefore, we conclude that the limit where the equilibrium of a heavy lever begins to be destroyed, or the friction of it overcome, depends only on the distance of the resultant of the given efficient forces of the lever from the centre of motion o.

While this distance is less than

$$\frac{fh}{\sqrt{(1 + f^2)}},$$

the given forces, or their resultant, will be held in equilibrio by the friction; but as soon as this distance exceeds the above quantity, the forces will overcome the friction, and the equilibrium will be destroyed.

Example. As an example, let us suppose that from experiment it has been ascertained, that a body of the same matter as the lever, and with the same degree of polish, being placed upon a plane of similar matter and polish as the axle, begins to slide down when the plane was brought to an angle of inclination of 11° 19′. The tangent of which $= \frac{1}{5}$ or $\cdot 2$; consequently $f = \cdot 2$ (art. 157); and suppose the lever, in the first instance, to be kept in perfect equilibrium by two weights of 10 lbs. each hanging at the distance of 12 inches respectively from the fulcrum, and let it be required to determine how much one of these weights may be moved inward towards the fulcrum, before the lever will begin to turn; the radius of the fulcrum being 1 inch, and the weight of the lever 20 lbs. Here, in the first place, we have $f = \cdot 2$ and $h = 1$; therefore

$$r = \frac{fh}{\sqrt{(1 + f^2)}} = \frac{\cdot 2}{\sqrt{(1 + \cdot 04)}} = \cdot 196.$$

The resultant of the forces acting on the lever must therefore be placed ·196 of an inch from the centre.

Now the resultant of the lever itself will always remain in the centre of the axis, it being supposed perfectly uniform, and the resultant of the two weights will be in the middle point between them; therefore if $2x$ denote the distance which the weight is moved inward, x will be the distance of the resultant from the centre, and $\frac{1}{2}x$ the distance of the resultant of the weight of the lever and the weights of the bodies; whence

$$r = \tfrac{1}{2}x = \cdot 196, \text{ and } 2x = \cdot 784 \text{ of an inch},$$

the distance one of the weights may be removed towards the centre before the equilibrium will be destroyed.

If, instead of requiring how far one of the given weights might be displaced before the equilibrium would be destroyed, it had been required to determine how much one of those weights might be diminished; we should still have to proceed in a similar manner.

In this case, let x represent the required quantity, and we shall have at once $12x$ for the difference of the moments upon the lever.

This, as we have seen, is equal to rR, or

$$\cdot 196\,(40 - x) = 12x;$$

or $$12\cdot 196\,x = \cdot 196 \times 40;$$

whence $$x = \frac{\cdot 196 \times 40}{12\cdot 196} = \cdot 642 \text{ of a lb.}$$

the quantity sought. That is, we may diminish one of the weights ·642 lb. and the equilibrium will still be maintained by the friction; but if we reduce the weight any farther, the lever will begin to revolve.

The quantity f, determined as in article 157, will obviously vary with the matter of the lever and the fixed axis or fulcrum, the degree of polish, and the extent of the surfaces in contact; but it is evident, that however great its value may be, the fraction $\dfrac{f}{\sqrt{(1 + f^2)}}$ will always be less than unity; whence it follows, that while the equilibrium obtains, the distance r of the resultant R from the point o is always less than the radius h of the cylindrical axis. Consequently, this force cannot be held in equilibrio by the friction of the lever against the axis, only while its direction meets this cylinder; and therefore if this condition be not fulfilled, we may conclude, without examining whether the friction is more or less considerable, that the equilibrium is impossible.

The reader, however, will bear constantly in mind, that all the preceding deductions are made upon the hypothesis that the lever and axle, or fulcrum, touch only in a point, or in a line formed by a plane passing through the parts in contact; and therefore that they no longer obtain if the friction has place on different points at the same time.

As another example, let it be proposed to determine the direction in which a force ought to act upon a body resting on an inclined plane, in order to put it in a state of equilibrium with the least effort possible; or, which is the same, to determine the most advantageous direction of a force to move a body along a horizontal plane.

Here it is obvious, that if there were no friction, the horizontal direction would be the most advantageous; but when we consider the friction, it is evident that, by making the force act in a direction above the plane, the pressure of the body will be in part relieved, and the effect of friction also diminished; and our object is to determine the measure of the angle of direction, that the resistance may be a minimum.

Let x denote the sine of the angle of direction, then $(1 - x^2)^{\frac{1}{2}} =$ the cosine of the same angle; let W be the weight of the body, and fW the friction; then the force requisite to move the body must be equal to the horizontal force diminished by the friction. Now, xfW will denote the friction relieved by the direction of the force, and $(1 - x)fW$ the quantity remaining; also $W(1 - x^2)^{\frac{1}{2}} =$ the horizontal force; whence we must determine

$$(1 - x^2)^{\frac{1}{2}}W - (1 - x)fW = \text{a minimum};$$

or, $$(1 - x^2)^{\frac{1}{2}} - (1 - x)f = \text{a minimum};$$

the fluxion of which gives

$$f\dot{x} - \frac{x\dot{x}}{(1-x^2)^{\frac{1}{2}}} = 0,$$

or $$f^2(1-x^2) = x^2,$$

or $$(f^2+1)x^2 = f^2,$$

whence $$x = \frac{f}{\sqrt{f^2+1}}.$$

Whence the numerical value of f being supposed to be determined by experiment, that of x also becomes known, and consequently the angle of direction:

when $f = \frac{1}{3}$, then $x = \frac{1}{\sqrt{10}} = \sin 18° 26'$ nearly.

Rigidity. 160. *On the rigidity of ropes.* Cords and ropes not being perfectly flexible, it becomes necessary, in estimating the advantages of pulleys, to make some allowance for this want of flexibility; and in this case the theory by which we compute these deductions is far more satisfactory than that given above with reference to friction, and accords much better with experiment.

Let AC = CB = r (fig. 85-a) be the radius of the pulley, and P,W two weights in equilibrio. If W should prevail, it is obvious that the cord DP becomes in the upper part bent, so as to fit the groove of the pulley, and in the lower part bent inwards, so as to fall into the vertical b W; if the cord have any considerable degree of flexibility, the curving is pretty regular from B almost down to W; but if it be very rigid, BEW, ADP, are found to be nearly straight lines, but neither of them vertical; the weight P being found to hang vertically below some point a, making C a greater than CA, and the weight W hanging below some point b, where C b is less than CB. So that as the arm of the lever at which one of the forces acts, is become greater, and that of the other less than r, the condition of equilibrium is no longer P = W.

When the cord is moderately rigid, as in most practical cases, the distance B b is always found so extremely small that it may safely be neglected in the discussion; that is, we need not in such cases pay any regard to the want of flexibility in the part BEW corresponding to the weight W, which is supposed to preponderate; but merely enquire into that of the part ADP, by which the other weight is suspended. Hence, therefore, if we put A a = q, the condition of equilibrium will be expressed by the equation

$$W r = P(r+q);$$

from which it results, that if W − P be the quantity by which we should augment the power, that it may be on the point of prevailing; and if we have regard to the thickness of the cord, this quantity will be $W - P = P\frac{q}{r}$. Consequently, *in order to introduce the consideration of the stiffness of cords*, we have only to suppose that the arm of the lever, at which the resistance acts, is greater than it really is by a determinate quantity q.

Now, in order to ascertain the quantity q, it may be observed that a cord resists on two accounts the efforts that are made to bend it. The first is due to the tension of the cord, and is proportional to it; and may, therefore, be denoted by b P; the second is due to its warping or twisting, and the force requisite to overcome it may be denoted by a.

Here it is obvious that a and b are variable co-efficients. For one and the same cord, $a + b$ P, may represent the force required to bend it; but if the cord be changed, the diameter d will be different, and we may conclude that, *cæteris paribus*, the force which must be employed, will be proportional to some certain power of d, as d^n; for the force necessary to bend a cord will increase with its diameter; the power, on the contrary, will decrease with the radius of the pulley; therefore

$$\frac{d^n}{r}(a + b\,P)$$

may be made to represent the force necessary to overcome the stiffness of the cord; n being, independent of experiment, an indeterminate quantity. This value being the augmentation, which must be given to the force or weight W, that it may be on the point of prevailing over the power or resistance P, must, as before shown, be equal to $P\frac{q}{r}$; thus we have

$$d^n(a + b\,P) = P\,q, \text{ or } q = \frac{d^n}{P}(a + b\,P).$$

The above equation, it will be observed, is only furnished from general consideration, and not by a rigorous investigation; it contains moreover the unknown co-efficients a and b, and the unknown index n, which doubtless vary for different cords; but there is a simple method of determining these co-efficients, and of assuring ourselves, that the expression is sufficiently exact in practice.

We propose now to give an enumeration of the principal experimental results that have been made on this subject, after first indicating the methods which have been pursued with a view to this determination, and which may be easily repeated. Choose any cord, and after bending it along the groove of a pulley, as for example the cord PDABEW (fig. a-85), attach to it two equal weights, and augment one of them till it is just upon the point of prevailing, so as to give motion to the system, noting the difference of the weights. Make similar experiments on four different weights, cords, and pulleys, and let the several resulting values of q in the equation

$$q = \frac{d^n}{P}(a + b\,P),$$

be denoted by q', q'', q''', q^{iv}; the four radii by r', r'', r''', r^{iv}; also the weights and powers by W′, W″, W‴, &c. P′, P″, P‴, &c.; whence we shall obtain

$$q' = \frac{d^n}{r'}(a + b\,P') \qquad q'' = \frac{d^n}{r''}(a + b\,P''),$$

$$q''' = \frac{d^n}{r'''}(a + b\,P''') \qquad q^{iv} = \frac{d^n}{r^{iv}}(a + b\,P^{iv}).$$

The first three, or indeed any three, of these equations may be employed for determining the values of a, b, and n; and by means of the fourth we may assure ourselves, whether these formulæ have all the generality we can desire; for the values of a, b, and n, as determined from the former, when substituted in the latter, ought to produce a numerical equality in the results of both sides of the equation, without which no dependence can be placed upon them.

Experiments on rigidity. 161. The principal experiments that have been made on this subject, are those of M. M. Coulumb and Amontons. The apparatus of the former consisted of two tressels six feet in height, and sufficiently solid and firm, on which were laid two pieces of squared wood; and upon these were fixed two rulers of oak, well planed and polished; then two cylinders of lignum vitæ were procured, one of six inches diameter, the other two inches; together with several cylinders of elm, from two to twelve inches in diameter.

These things being prepared, in order first to find the friction of the rollers, they were laid horizontally upon the two rulers of oak, and crossing their directions perpendicularly, the rulers being in a perfectly horizontal position; then suspending on each side of the roller in use, a weight of 50 lbs. with very fine and flexible packthread, or by means of several such threads distributed over the roller, and charged each with 50 lbs. at its extremities, any determinate pressure might be produced upon the rulers. And by the addition of small weights on one side of the roller, it was easy to ascertain what weight was necessary to give to them a motion barely perceptible.

The friction of the rollers being estimated by the preceding method, it is easy to allow for it; when, instead of the flexible packthread, the cords or ropes of which the rigidity is to be determined are substituted. And this determination is made in the same manner as that with respect to the nascent friction, by suspending small weights alternately on each side of the roller, so as to cause the least sensible motion.

It is obvious that this method of estimating the effects of the rigidity of cords, furnishes results directly applicable to the preceding formulæ; for the weights which produce the very small motion in the cylinders will be precisely equal to the augmentation of the resistance arising from the stiffness of the cord; estimated in the direction of that portion of the cord to which the resistance is applied that represents the useful effects of the machine.

We shall first exhibit the results of M. Coulumb's experiments, on the second species of friction produced by the rollers of lignum vitæ of six and of two inches diameter.

Charge of the Rollers, their weight being comprised.	Weights which produced an extremely slow motion, the diameters of the Rollers being	
	6 Inches.	2 Inches.
100 lbs.	0·6 lbs.	1·6 lbs.
500 lbs.	3·0 lbs.	9·4 lbs.
1,000 lbs.	6·0 lbs.	18·00 lbs.

From this table M. Coulumb infers, that the friction of cylinders that roll upon horizontal planes is directly as the pressures, and inversely as the diameters of the rollers. He also found that greasing the surfaces did not cause in these cases any diminution of the friction.

Rollers of elm produced a friction about $\frac{2}{5}$ths greater than lignum vitæ; and under small pressures the friction was rather greater than would result from the law of friction being proportional to the pressure.

Note.—The foot and lb. here referred to are those of the old French system.

We shall next present the results of M. Coulumb's experiments on the rigidity of cords, and different rollers between 2 and 12 inches diameter. The deduction for the friction being stated in the table, and a column exhibiting the comparative rigidity deduced from the experiments of M. Amontons. The cords were of three kinds: No. 1, of *six* threads in a yarn, or *two* in a strand, the circumference 12½ lines, and the weight of a foot in length 4½ drs.; No. 2, of 15 threads in a yarn, or 5 in a strand, circumference 20 lines, and the weight of a foot in length 12½ drs.; No. 3, of 30 threads in a yarn, or 10 in a strand, circumference 28 lines, and the weight of a foot in length 24½ drs.

No. of Experiments.	Cords employed in the Experiment.	Kinds of Wood, diameter and weight of the Rollers.	Weights hung on each side of the Roller, in lbs.	Additional Weight to produce the Motion, in lbs.	Total Charge of the Rulers which support the Roller.	Friction of the Roller.	Stiffness of the Cord.	
							Valued by Coulumb's Apparatus.	Valued by Amonton's Apparatus.
1	Cord No. 3, 30 threads to a yarn.	*Elm.* 12 inches diameter, wt. 110 lbs.	100	5	315	1·5	3·5	4·4
			300	11	721	3·6	7·4	10·4
			500	20	1130	5·6	11·4	16·4
2	The same cord.	*Elm.* 6 inches diameter, wt. 25 lbs.	200	18	443	——	——	——
3	The same cord.	*Guiacum.* 6 inches diameter, wt. 50 lbs.	200	16	466	2·8	13·2	14·8
4	The same cord.	*Guiacum.* 2 inches diameter, wt. 4½ lbs.	25	11	65½	——	——	——
			200	52	456½	——	——	——
5	Cord No. 2, of 15 threads in a yarn.	*Guiacum.* 6 inches diameter, wt. 50 lbs.	25	1¼	101¼	——	——	——
			100	6	256	——	——	——
			200	11	461	2·8	8·2	7·6
			500	24	1074	6·4	17·6	17·8
6	Cord No. 1,	The same as above.	100	3	253	——	——	——
			200	6	456	2·7	3·3	3·1

From this table, it will be seen that the method of Amontons, and that of Coulumb, furnish nearly the same results. The latter ascribes the difference, where greatest, to the circumstance of the cords having been more used previous to their being taken for one kind of experiment than for the other.

M. Coulumb made various other experiments, both on the rigidity of cords and the friction of axles, &c.; but as they involve in them the principle of motion, we must refer them, as before stated, to the term FRICTION in our general Alphabetical arrangement.

§ XXIV. *Of the resistance of solids.*

162. When a solid body is submitted to any strain, either parallel to its length, or in a direction perpendicularly or obliquely to its longitudinal direction, and that strain is so far excited that a consequent fracture takes place; the instant before this happens there ought to be an equilibrium between the resistance of the particles or fibres of the solid and the straining or exciting force to which it is exposed; and hence the theory of the resistance of solids belong properly to the doctrine of statics, and as such we shall treat of it in the present article.

With regard to the resistance of any body of uniform texture, whether fibrous or crystalized, when exposed to a force acting parallel to its length, and having a tendency to draw it asunder; it is obvious that this ought in all cases to be proportional to the area of its transverse section; and all experiments that have been hitherto made are found to give results conformable to this deduction; viz. if it will require (as upon a medium it is found to do) 27 tons to tear asunder an iron bar 1 inch square, it will require $13\frac{1}{2}$ tons to tear asunder a bar whose transverse area is $\frac{1}{2}$ an inch; and so on, in proportion for any other area of section, independent of the figure of that section or of the length of the bar.

163. The above is one of the most simple cases of the resistance of solids: we shall find much more difficulty in establishing the law of resistanee when the bar or beam we are considering is either rested on two props, and loaded in any part of its length, or when it is fixed at one end and loaded at the other. It may also in this latter case be fixed at different angles of inclination, ascending or descending, &c.; each of which cases requires a distinct investigation. Moreover, the resistance here does not depend entirely, as in the preceding, upon the area of fracture; but the form of that area also becomes an important consideration; it is obvious, likewise, that we have not in this case simply to consider the quantity of resistance of the solid at the moment of rupture, but the resistance which it may oppose to any degree of inflection arising from the exciting forces.

164. When a beam of timber or any other matter is submitted to a transverse strain, being either supported at its two extremities, and loaded in the middle, or fixed at one end in a wall and loaded at the other; it will not, as is generally assumed, turn about its upper or lower surface, but on a line within the area of fracture; which line, in what follows, is denominated the *neutral axis of rotation*, or *neutral axis*.

If the fibres of a beam, referring for instance to fig. 101, were wholly incompressible, then the beam when loaded at the end I, would turn about the point or line C; and every fibre of it from C to A would be in a state of tension. On the contrary, if the fibres were wholly inextensible, then if the beam turned at all, it must be about the point A, and every fibre from A to C would be in a state of compression. But we know of no bodies in nature that are either inextensible or incompressible; and, therefore, the rotation of the beam will neither take place about A nor C, but on an intermediate point *n*; and all the fibres above that line will be in a state of tension, and those below it in a state of compression; while those which are so situated as to coincide exactly with its plane, will be neither extended or compressed, but be in a state perfectly neutral with regard to both.

Figure of transverse section.

It is obvious that the fibres submitted to tension are more and more extended as they are situated farther from the point *n*, and at A their extension is the greatest. The same also obtains with respect to the fibres submitted to compression, this being greatest at C; and whatever may be the law of the forces necessary for producing these several degrees of tension and compression, or whatever may be the law of the resistances which they offer after they are excited; there must be some point, situated between A and *n*, into which if all the resistances to tension were united; and some point between *n* and C, into which, if in like manner all the resistances to compression were condensed, the re-action arising from these two aggregate forces would be the same as in the actual operation; which points are designated the *centres of tension and compression*. Let *t* and *c* denote these centres, then the mechanical resistance of the fibres may be exhibited as in fig. 102, where *w* denotes the sum of all the tensions acting at *t*, and *w′* the sum of all the compressions acting at *c*, the momenta of which two forces are obviously equal to each other, it being this equality which determines the beam to turn about the point *n*.

But the position of these two centres depends upon the law of resistances to tension and compression of the fibres according to their respective distances from the neutral axis *n*, and the determination of which can only be effected from experiment; the usual law attributed to the tension and compression of elastic bodies; viz. "ut tensio sic vis," would give the centre of tension a distance from the neutral axis equal to the quotient arising from the division of the product of the distances of the centres of gravity and oscillation of the area of tension, by the whole depth of that area, and the same with regard to the centre of compression; but as this law will not agree with the result of experiment, other conditions ought to be sought in which the theoretical and practical deductions will coincide.

This task has been undertaken by Mr. Barlow of the Royal Military Academy; and the conclusions which he has drawn from numerous experiments are as follows, viz.

The centre of tension, and the centre of compression, each coincided with the centre of gravity of its respective area; and the neutral line which divides the two, is so situated, that the area of tension into the distance of its centre of gravity, from the neutral axis, is to the area of compression into the distance of its centre of gravity, from the same line, in a constant ratio for each distinct species of wood, but approximating in all towards the ratio of 3 to 1.

Remarks on the centre of tension.

165. As the above results are entirely at variance with the deductions of Leibnitz and others, who have written on this subject; the author above mentioned, enters upon a physical explanation and examination of them; observing, 'that the hypothesis, "ut tensio sic vis," is founded on a supposition of perfect elasticity, and admits of no limit or termination of this law; but it is obvious, that in all the bodies we are acquainted with, and particularly in timber, it can only obtain to a certain extent; after which, an increase of force may be, and obviously is, accompanied by a decrease of re-action; thus, referring to fig. 101, when the fibre *b* A is so extended that its re-action is at its maximum, all the other fibres between *b* A and *n* fall short of that maximum value; but as more weight is applied at F, the other fibres, situated below *b* A, arrive at their maximum, while the re-action of that fibre is diminished; and as the beam is farther exerted, other lower fibres come successively to their maximum of re-action, while those that have already arrived at that state are undergoing a diminution. The law of re-action, till the fibres have attained their maximum, we may suppose to coincide with the state of perfect elasticity; that is, that the re-action is as the exciting force: but what may be the law by which that power is diminished when excited beyond that which produces the maximum, we shall not attempt to assume; it is sufficient for our present purpose, to establish on general principles, that such a decrease does obtain; and as a consequence of this, that the fracture of the beam does not take place immediately after the extreme fibre has passed its maximum; but that this event depends upon the sum of the re-actions of all the fibres having arrived at and passed that state.'

'This being admitted, we see the reason why the deflections, during part of the operation (viz. while the re-action of the fibres are all increasing), are as the exciting forces; and why, after that point, they proceed more rapidly than in the proportion of the weights; and also, why these deflections are greater and greater as the beams are more and more excited. It follows also, as a necessary consequence of the above reasoning, that the centre of tension cannot be situated in the point assigned to it by Leibnitz; because this depends upon what we have shown to be an erroneous hypothesis, viz. that the fracture happens immediately after the extreme fibre has passed its maximum of re-action.'

'As to the exact determination of the situation of this centre, it obviously depends upon the law of decrease of energy in the fibres, after passing their maximum value, as combined with the increasing effect of the other fibres before they arrive at it; but whatever the former law may be, it will evidently have a tendency to place that centre farther from the axis than accords with the hypothesis of Lesbnitz; and consequently to approximate it towards that assumed by Galileo. The results of our investigations point towards the latter; and though we have not found a perfect coincidence with either, yet it appears to be so near to that of the latter, that no sensible error will take place by supposing the centre of gravity and that of tension to be in one and the same point. It must be observed, however, that the coincidence which we have thus found between the result of Galileo's hypothesis and our deductions, is by no means a proof of the correctness of the former; but on the contrary, admitting the above explanation, it shows that coincidence to be merely accidental; for instead of supposing the fibre of greatest re-action in the breaking state of the beam, to be situated in or near the centre of the area of fracture, and the fibres both above and below this to have a diminished action, on the one side, in consequence of not being sufficiently excited; and on the other, from that excitation being carried too far, he supposed them all to have an equal action; which, though wholly erroneous as an hypothesis, gave very nearly or exactly the true results.'

On cent com sion

With regard to the coincidence between the centres of gravity and of compression, Mr. Barlow explains it on other principles; viz. that in the actual operation of breaking a beam, the compressed fibres always exhibit themselves in the form of an isosceles triangle, having its vertex in the neutral axis; which being the case, and assimilating the re-action of the fibres to that of a series of spiral springs, increasing in length from the neutral axis outwards, it is obvious, that the length of the spring is every where as the exciting force; and consequently the resistance, from the vertex to the base, will be the same in all parts; and therefore the centre of gravity and compression ought to be found in the same point.

This theorem being established, it is evident that we may thence, without any specific numbers for exhibiting the actual resistance of the fibres, compute the proportional strengths of differently formed beams; and of the same formed beams in different positions; of which we will give one example by way of illustration.

Exa

166. Let a square beam be fixed with one end in a wall, first in a direct position, viz. with its sides perpendicular and horizontal; and secondly with its diagonal vertical; to find the ratio of its strengths in these two positions:

Conceive ABCD (fig. 103) to denote the beam in its first position; EF the neutral axis; EABF the area of tension; and *t* the centre of tension, or centre of gravity of that area; EDCF the area of compression; *c* its centre of gravity; and G the centre of gravity of the whole area of fracture; and the same letters will denote the similar quantities in fig. 104, which represents the section of the beam in its second position.

Then, by the preceding theorem, we have, in fig. 103,

$$\textit{area}\ \text{AEBF} \times nt \times 3 = \textit{area}\ \text{EDCF} \times nc;$$

in fig. 104,

$$\textit{area}\ \text{EBF} \times nt \times 3 = \textit{area}\ \text{EFCDA} \times nc;$$

both which, from the property of the centre of gravity (art. 117), are reducible to

fig. 103, $\textit{area}\ \text{AEBF} \times 2nt = \textit{area}\ \text{ABCD} \times n\text{G};$

fig. 104, $\textit{area}\ \text{EBF} \times 2nt = \textit{area}\ \text{ABCD} \times n\text{G}.$

For the sake of simplifying the computation, let the side of the square $= 1$, $n\text{H} = x$, or $nt = \frac{1}{2}x$, then $n\text{G} = \frac{1}{2} - x$; the area AEBF $= x$, and the area ABCD $= 1$; whence our first equation gives

$$x^2 = \tfrac{1}{2} - x;$$

or

$$x^2 + x = \tfrac{1}{2};$$

whence

$$x = -\tfrac{1}{2} \pm \surd 3 = \cdot 366,$$

which denotes both the depth of tension H n, and the area of the same AEFB; consequently $\cdot366 \times \frac{\cdot366}{2} = \cdot066978$; the numerical expression for the resistance to tension, on which depends the strength of the beam, as is obvious by referring to fig. 102.

It remains now to compute the same for the second position of the beam, as in fig. 104. Here, if we denote nB by x, $nt = \frac{1}{3}x$, and the area EBF $= x^2$; also area ABCD $= 1$ as before; whence our second equation becomes

$$x^2 \times \frac{2}{3}x = \frac{1}{2}\sqrt{2} - x, \text{ or}$$
$$x^3 + \frac{3}{2}x = \frac{3}{4}\sqrt{2} = 1\cdot0606.$$

From which we readily obtain $x = \cdot578$ nearly, $x^2 = \cdot33408$, and $x^2 \times \frac{1}{3}x = \frac{1}{3}x^3 = \cdot06436$, which is the numerical value of the tension in this position of the beam.

The strength of the beam, therefore, in the latter position, is to that in the former in the ratio of $\cdot06436 : \cdot06697$; or as the numbers 643 : 669 *nearly;* which accords with experimental results.

167. As our object is to investigate the conditions of equilibrium between the resistance of solids, and the strains to which they may be exposed, it is necessary for us now to examine the latter without any reference to the quantity of the former, in order that we may ultimately, by equating the two, determine either; the other being supposed given; and for the sake of simplification, we shall here confine all our enquiries to the case of rectangular beams:

A beam of timber, as ACFI (fig. 101), fixed with one end in a wall, and loaded with a weight W at the other, will be deflected from its first horizontal position AH, into the oblique direction AF, supposing it in the present instance, for the sake of simplicity, inflexible in every point except in the section of fracture AC; and this deflection will take place, as before remarked, by means of a rotation performed about the neutral axis n.

Now the force which has a tendency to produce this rotation, will obviously be the weight W multiplied by the length of the lever nF, into the cosine of the angle nFB; and the force opposed to it will be the resistance of all the fibres in nC to compression, *plus* the resistance of all those in nA to extension; and the sum of these two forces in the instant before fracture, must necessarily be in equilibrio with the former, viz. lW cos nFB. Our present object, however, is not to consider the nature of the resisting forces, but of the exciting force; and this, as we have seen above, when expressed symbolically, is

$$F = nF \times \cos nFB \times W, \text{ or } F = lW\cos\Delta.$$

Where F denotes the force or strain nF $= l$; the weight $=$ W; and the angle nFB, of deflection $= \Delta$.

It will be observed, that nF is not the length of the beam, but the distance of the neutral axis from the point on which the weight is suspended; nor is the angle nFB actually the angle of deflection: but as the depth of beams are generally small in comparison of their length, and the depth of the neutral axis still smaller, we shall in what follows, (except the contrary be expressed) consider l as the length of the beam, and Δ as the angle of deflection, since it will simplify the investigation and can produce no sensible error.

When a beam, instead of being fixed at one end into a wall, is merely rested on a support at its middle point, and loaded at each end; the tension of the upper fibre is still the same as in the former case; supposing the length of the beam in the latter instance to be double what it is in the former; that is, supposing the beam FF′ (fig. 105), to be double AF (fig. 101), then the three weights being equal, the tension of the fibre A b, in both cases will be the same, excepting only so much of A as depends upon the cosine of the angle of deflection; which, in fig. 105, will be only half that in fig. 101. The same general expression, however, will apply in both cases, by merely changing l in the former, into $\frac{1}{2}l$ in the latter; so that we shall have, in this case,

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Transverse strain.

$$F = \tfrac{1}{2}lW\cos\Delta.$$

Now a beam resting on a fulcrum C in the middle of its length, as in fig. 105, and acted upon by two weights W, W′, has commonly been considered in the same state, with regard to the strain upon it, as the equal beam F F′ (fig. 106), which is rested on the two props F F′, and loaded with a double weight P at its centre: but it is evident that this is not a correct conclusion.

In the first place it is obvious, that the resistance of the fulcrum is not made in a direction parallel to that of the vertical weight P, but perpendicular to the arms of the lever F n, F′n; and therefore, that the beam is, with regard to its strain, kept in equilibrio by the action of the three forces FO, F′O, and OR; the former FO, F′O, being supposed perpendicular to F n, F′n.

The re-action of the fulcrums F, F′, will therefore be to the weight P, as FO to half OR, or OC; or as radius to the cosine of the angle FOn, or nFC; that is, as radius to the cosine of the angle of deflection.

Hence, when a beam is rested upon two fixed props, and loaded at its middle point by any weight P, the strain upon that middle point, arising from the re-action of the props, will be found by the following proportion, as

$$OC : OF :: \tfrac{1}{2}P : \frac{FO \times P}{2OC}, \text{ or}$$
$$\cos\Delta : \text{rad} :: \tfrac{1}{2}P : \frac{P \times \text{rad}}{2\cos\Delta}, \text{ or } \frac{P}{2\cos\Delta},$$

taking radius equal to unity; or if we call half P $=$ W, then, according to our former notation,

$$F = \frac{\frac{1}{2}l \times W}{\cos d} = \frac{l \times P}{4\cos d}.$$

This supposes the arms of the lever F n, F′n, to remain of the same length: but it is obvious that this is also an erroneous hypothesis, for the props or fulcrums being fixed, these arms, either by the stretching of the fibres, or by the pieces of wood slipping between the points of support, are more and more lengthened as the piece descends; viz. the length of the lever is to half the distance of the props as rad to cos Δ; and consequently the strain on this account is again increased in the ratio of $\frac{\text{rad}}{\cos\Delta}$, or $\frac{1}{\cos\Delta}$ to radius 1. Whence, by introducing this consideration, our former expression becomes

$$F = \frac{lP}{4\cos^2\Delta} = \frac{lP\sec^2\Delta}{4}.$$

Strain with the ends fixed.

197. Let us now endeavour to ascertain the strain upon the centre of a beam which is loaded at that point, having each of its ends fixed in a wall or other immoveable mass, as in fig. 107.

Strain with the ends fixed.

Here it is obvious that the whole weight is not employed in producing the strain upon the middle section, a part of it being required to produce the strain and deflection at the point of fixing; consequently, besides the weight necessary to cause a fracture, or to produce any given deflection in a beam which is merely supported, so much additional weight must be laid on, as will, when the beam is fixed at each end, deflect the two half lengths to the same degree; that is, referring to fig. 107, the weight W must be greater than would be required to deflect the supported beam, by as much as is necessary to deflect the two half beams AC, A′C, the same quantity, or to produce in them the same angle of deflection. Now if we refer to figs. 101 and 105, and supposing the latter to be in length just double the former; we shall see as above, that if weights W, W′ are appended at the two extremities as in the latter figure equal to the weight W in the former, the fibre b A, b A in each will experience a like tension; but it is obvious that still the deflection in fig. 105, will be only half that in fig. 101, the angle A n b in both figures being equal. Supposing therefore the deflections to be as the weights, it will require in fig. 101, only $\frac{1}{4}$ of the sum of the weights W, W′, in fig. 105, to produce an equal deflection; therefore to deflect the two half beams, fig. 107, will require half the weight that is necessary to cause the middle deflection; and consequently, if the weight W in the latter figure be supposed to be divided into six equal parts, four of those parts will be employed in producing the deflection of the centre; and one of each of the other two, in producing that at the two ends; so that of the whole weight necessary to break a beam thus fixed, only four parts out of six operate in straining the middle section: the strain therefore in this case, is to the strain on a similar beam only supported at its ends, as 4 to 6, or as 2 to 3; or the strength in the two cases will be as 3 to 2, which is conformable to experiment.

General results.

169. If now we collect together our results, we shall find that the strain upon a beam,

Fixed at one end, and loaded at the other, is

$$F = l\,W \cos \Delta\,;$$

When supported at each end, and loaded in the middle,

$$F = \tfrac{1}{4} l\,W \sec^2 \Delta\,;$$

When fixed at each end, and loaded in the middle,

$$F = \tfrac{1}{6} l\,W \sec^2 \Delta\,.$$

And the same formulæ will apply to the cases in which the beam, instead of being fixed horizontally, as hitherto supposed, is fixed at any angle of inclination; or if the force, instead of acting vertically, act in any other direction; by resolving it into two forces perpendicular to each other, as in art. 35.

When the weight is not in the centre.

170. Let us next consider the case in which the weight is not placed in the centre of the beam, but in any other part of its length, as in fig. 108. Here, since the tension of the fibre A B is the same, whether we estimate it towards F, or F′, we may suppose the weight W to be divided into two weights, which shall have to each other the ratio of IC to I′C; that is,

as $$\mathrm{II'} : \mathrm{IC} :: \mathrm{W} : \frac{\mathrm{IC} \times \mathrm{W}}{\mathrm{II'}},$$

$$\mathrm{II'} : \mathrm{I'C} :: \mathrm{W} : \frac{\mathrm{I'C} \times \mathrm{W}}{\mathrm{II'}}.$$

Then it is obvious, that whether we consider the first of these weights as acting at the point C of the lever CI′, or the latter as acting at the point C of the lever CI, or both of them as acting at the point C of the beam or compound lever II′, the strain or tension of the fibre A b will be the same, and will be expressed by

$$\mathrm{F} = \frac{\mathrm{IC} \times \mathrm{W}}{\mathrm{II'}} \times \mathrm{I'C} = \frac{\mathrm{IC} \times \mathrm{I'C} \times \mathrm{W}}{\mathrm{II'}}, \text{ or}$$

$$\mathrm{F} = \frac{\mathrm{I'C} \times \mathrm{W}}{\mathrm{II'}} \times \mathrm{IC} = \frac{\mathrm{I'C} \times \mathrm{IC} \times \mathrm{W}}{\mathrm{II'}}.$$

That is, if l be taken to denote the length of the beam II′, and m and n the two distances IC, I′C; then

$$\mathrm{F} = \frac{m\,n}{m+n}\,\mathrm{W} = \frac{m\,n}{l}\,\mathrm{W}.$$

That is, the strain varies as the rectangle of the two parts into which the beam is divided by the point of suspension: and hence it follows that the strain will be the greatest when the rectangle is the greatest; that is, when the weight acts at the centre.

We shall now take the case of two weights suspended from any two points of a beam, to determine the strain upon the beam at any given point.

Conceive FII′F′ (fig. 109), to be a beam resting on the two props F F′, and having two weights, equal or unequal, suspended from the two points DE; then, from the preceding formula, it appears that the strain at D is $\mathrm{F} = \dfrac{\mathrm{ID} \times \mathrm{DI'}}{\mathrm{II'}} \times \mathrm{W}$; and the strain at E is

$$\mathrm{F} = \frac{\mathrm{IE} \times \mathrm{EI'}}{\mathrm{II'}} \times \mathrm{W'}.$$

In order to find the strain at any other point C, we have only to make the following proportions; viz.

$\mathrm{DI'} : \mathrm{CI'} :: \dfrac{\mathrm{ID} \times \mathrm{DI'}}{\mathrm{II'}}\mathrm{W} : \dfrac{\mathrm{ID} \times \mathrm{CI'}}{\mathrm{II'}}\mathrm{W} =$ the strain at C as arising from that at D; and again

$\mathrm{EI'} : \mathrm{IC} :: \dfrac{\mathrm{IE} \times \mathrm{EI'}}{\mathrm{II'}}\mathrm{W'} : \dfrac{\mathrm{I'E} \times \mathrm{IC}}{\mathrm{II'}}\mathrm{W'}$ the strain at C as arising from that at E.

Consequently the whole strain at C, arising from both weights, will be expressed by

$$\mathrm{F} = \frac{\mathrm{ID} \times \mathrm{I'C} \times \mathrm{W} + \mathrm{I'E} \times \mathrm{IC} \times \mathrm{W'}}{\mathrm{II'}}.$$

From this general formulæ, we may readily deduce that for any particular case.

Suppose, for example, that the two weights W, W′, are equal to each other, and that C is the centre of the beam; then, since $\mathrm{I'C} = \mathrm{IC} = \frac{1}{2}\mathrm{II'}$, and $\mathrm{W} = \mathrm{W'}$; the general expression becomes in this particular case

$$\mathrm{F} = \frac{(\mathrm{ID} + \mathrm{I'E}) \times \mathrm{I'C} \times \mathrm{W}}{\mathrm{II'}} = \frac{\mathrm{ID} + \mathrm{I'E}}{2} \times \mathrm{W}.$$

And if we further suppose $\mathrm{ID} = \mathrm{I'E}$, then it becomes simply

$$\mathrm{F} = \mathrm{ID} \times \mathrm{W}.$$

Now if both weights acted at the centre, it appears from our preceding investigation, that

$$\mathrm{F} = \tfrac{1}{4}\mathrm{II'} \times (2\mathrm{W}) = \tfrac{1}{2}\mathrm{II'} \times \mathrm{W} = \mathrm{IC} \times \mathrm{W};$$

whence the strain in the two cases will be to each other as ID to IC; and hence the following practical deduction: viz.

When a beam is loaded with a weight, and that weight is appended to an inflexible bar or bearing, as DE (fig. 110), the strain upon the beam will vary as the distance ID, or as the difference between the length of the beam and the length of the bearing; for the bearing DE being inflexible, the strains will be excited in the points D and E exactly in the same manner as if the bearing were removed, and half the weight hung on at each of these points.

This remark may be worth the consideration of practical men in various architectural constructions.

en there several ights.

In the same manner it may be shown that if a beam be loaded with many weights W, W′, W″, W‴, &c. as in fig. 111, all equal to each other, and every two of which are equally distant from the centre, the strain excited on the middle point C, will be expressed by

$$F = (ID + ID' + ID'' + \&c.) \times W.$$

Hence, if the length of the beam be l, and the number of equal weights m, and the sum of all the weights W; then the above becomes

$$F = \left(0 + \frac{l}{m} + \frac{2l}{m} + \frac{3l}{m} + \&c.\ \frac{\frac{1}{2}ml}{m}\right) \times \frac{W}{m}, \text{ or}$$

$$F = \frac{lW}{m^2}\left\{1 + 2 + 3 + 4, \&c.\ \tfrac{1}{2}m,\right\} \text{ or}$$

$$F = \frac{W}{m^2}\left(\frac{\frac{1}{2}m + 1}{2}\right)\tfrac{1}{2}m = \frac{\frac{1}{4}lWm^2 + \frac{1}{2}lWm}{2m^2}, \text{ or}$$

$$F = \tfrac{1}{8}lW + \tfrac{1}{4}\frac{lW}{m}.$$

Therefore when the weight is uniformly distributed through the whole length, since the number of points of suspension then become infinite, the last term of the preceding expression $\frac{lW}{4m}$, vanishes in consequence of its denominations being infinitely large; and there results

$$F = \tfrac{1}{8}lW,$$

for the strain on the centre of a beam, when the weight W is uniformly distributed throughout its length; which is obviously half what it would be if it were all suspended from its middle point.

parison ain and istance.

171. Having thus determined the strain upon a beam under the various cases of the exciting force, it only remains to compare them with the resisting forces, in order to obtain the equation of their equilibrium.

Without enquiring at present into the numerical value of the resisting force, it obviously depends upon the number of fibres in the area of tension and compression, and the distance of those centres from the neutral axis; but the number of fibres depend upon the area, and the centres of compression upon the depth; therefore the whole resistance is as $a d^2$, where a is the breadth and d the depth of the beam. If, therefore, the resistance be determined from experiment for any given breadth and depth, we may readily find the same for any other given dimensions.

Suppose, for example, that the resistance ascertained by experiment on a rod of 1 inch deep and 1 inch in thickness were found = S, we should have $a d^2$ S for the resistance when the breadth was a inches and the depth d inches; so that $a d^2$ S is a general expression for the resistance of any rectangular beam; and hence, therefore, we obtain the following equations, whence W may be determined, S being supposed given; l being the length of the beam in inches.

1. *When the beam is fixed at one end,*

$$lW \cos \Delta = a d^2 S.$$

2. *When the beam is supported at each end, and loaded in the middle,*

$$\tfrac{1}{4}lW \sec^2 \Delta = a d^2 S.$$

3. *When the beam is supported at each end, and loaded at any distances n and m from each end,*

$$\frac{mn}{l} W \sec^2 \Delta = a d^2 S.$$

Comparison of strain and resistance.

4. *When the beam is fixed at each end, and loaded in the middle,*

$$\tfrac{1}{6}lW \sec^2 \Delta = a d^2 S.$$

5. *When it is fixed as above, but loaded at any part of its length,*

$$\frac{2mn}{3l} W \sec^2 \Delta = a d^2 S.$$

The value of S is of course different for different kinds of materials; and even in woods of nearly the same species considerable variation is found in the numerical value of the above quantity: this however being a subject entirely experimental, we shall not offer any farther remarks on it in this place; but shall merely observe, that in computing the strength of timber for building and other practical applications of it, as the deflections are always in these cases very inconsiderable, we may reject without any sensible error the quantities $\sec^2 a$, and $\cos a$, and then the preceding equations become

1. $\frac{lW}{ad^2} = S$; 2. $\frac{lW}{4ad^2} = S$;

3. $\frac{mnW}{lad^2} = S$; 4. $\frac{lW}{6ad^2} = S$;

5. $\frac{2mnW}{3lad^2} = S$;

which are sufficiently accurate for all practical purposes.

Primitive deflection.

172. It remains now to consider the resistance of solids, not as actually opposed to fracture, but to simple deflection; which is of more importance to the practical engineer than that which relates to the ultimate strength.

In this case, as our investigations will not extend beyond the first small deflections of a beam, we may without any appreciable error consider the latter as perfectly elastic, and the curve which it assumes when loaded as the elastic curve; the equation of which is, as we have seen (article 106),

$$\frac{-b\,\dot{x}\,\ddot{y}}{(\dot{x}^2 + \dot{y}^2)}\tfrac{3}{2} = P\,\{(y + c)\sin\gamma - x\cos\gamma\};$$

from which, by assuming the curve to differ but little from a right line, the angle $\gamma = 90°$, and $c = 0$, we might proceed to investigate the laws of deflection; but we believe it will be more satisfactory to deduce the same from other principles.

Let A m (fig. 112) denote the original horizontal position of the beam AD, which is supposed to be deflected into its present curvilinear form by the action of the weight W acting at D. In the first place, let us conceive AD to be divided into any number of equal parts m, as A a, $a\,a'$, $a'\,a''$, &c. and suppose the beam to be inflexible throughout each of these portions; or, which is the same, that it takes the polygonal form A, a, a', a'', &c. Then, since by the conditions of elasticity the deflections are as the exciting forces, we shall have the deflections $d\,a$, $d'\,a'$, $d\,a''$, &c. every where proportional to their distances from D; for the weight being constant, the strain will be as the lever by which it acts: whence, if we denote the first of these, which may be denominated the *element of deflection*, by δ, and assume

Primitive deflection.

the number of parts into which the beam AD is divided to be m; we shall have

$$\ldots\ldots\ldots\ldots\ldots \frac{m}{m}\,\delta = a\,d,$$

$$m : m-1 :: \delta : \frac{m-1}{m}\,\delta = a'\,d',$$

$$m : m-2 :: \delta : \frac{m-2}{m}\,\delta = a''\,d'',$$

$$m : m-3 :: \delta : \frac{m-3}{m}\,\delta = a'''\,d''',$$

$$\&c. \qquad \&c.$$

Also, according to our hypothesis,

$$n\,m = m \times a\,d = \frac{m^2}{m}\,\delta,$$

$$n\,o = (m-1)\,a'd' = \frac{(m-1)^2}{m}\,\delta,$$

$$o\,p = (m-2)\,a''d'' = \frac{(m-2)^2}{m}\,\delta,$$

$$\&c. \qquad \&c. \qquad \&c.$$

Whence the whole deflection $m\,D$ will be expressed by the series

$$m\,D = \frac{\delta}{m}\left\{m^2 + (m-1)^2 + (m-2)^2 + \&c.\ 1^2\right\}.$$

Or, by summation,

$$m\,D = \frac{\delta}{m}\left\{\frac{m^3}{3} + \frac{m^2}{2} + \frac{m}{6}\right\};$$

that is

$$m\,D = \delta\,(\tfrac{1}{3}m^2 + \tfrac{1}{2}m + \tfrac{1}{6}).$$

And in like manner we should find, if we supposed the weight W to be suspended from any other point of the same beam, as for example a'', the deflection $m'\,D'$ of that point would be

$$m'\,D' = \delta'\,(\tfrac{1}{3}m'^2 + \tfrac{1}{2}m' + \tfrac{1}{6});$$

where δ' denotes the element of deflection, arising from the strain of the weight W when applied to the point a'', and m' the number of parts or sides of the polygon between A and a''. Wherefore, the ratio of the two deflections will be as

$$\delta\,(\tfrac{1}{3}m^2 + \tfrac{1}{2}m + \tfrac{1}{6}) : \delta'\,(\tfrac{1}{3}m'^2 + \tfrac{1}{2}m' + \tfrac{1}{6});$$

or if we denote by l and l' the length of the pieces, and by λ, λ, the length of the equal sides of the polygon; we have $\dfrac{l}{\lambda} = m$, and $\dfrac{l'}{\lambda} = m'$; whence the above ratio becomes

$$\delta\,(\tfrac{1}{3}\frac{l^2}{\lambda^2} + \tfrac{1}{2}\frac{l}{\lambda} + \tfrac{1}{6}) : \delta\,(\tfrac{1}{3}\frac{l'^2}{\lambda^2} + \tfrac{1}{2}\frac{l'}{\lambda} + \tfrac{1}{6});$$

or,

$$\delta\,(2\,l^2 + 3\,l\,\lambda + \lambda^2) : \delta'\,(2\,l' + 3\,l'\,\lambda + \lambda^2).$$

Which (as λ becomes less and less, or as the number of sides become greater and greater) approaches towards the ratio of $\delta\,l^2 : \delta'\,l'^2$; which is its limit, when the sides are indefinitely small: that is, the deflections of different beams, are, *cæteris paribus*, directly as the element of deflection into the square of the length.

Now referring to fig. 101, it will be evident, from the nature of our hypothesis, that the extension of the fibre b A, is directly as the strain, that is as $l \times$ W, and reciprocally as the resistance, or as $a\,d^2$, a being the breadth of the beam, and d the depth: but the angle $b\,n$ A, by which the element of deflection is measured, is again reciprocally as n A, or as the depth; whence therefore, the element of deflection is directly as the length and weight, and reciprocally as the breadth and cube of the depth.

Hence, as we have seen that the entire deflection is as the element of deflection into the square of the length, or as $\delta\,l^2$; and as the former is as $\dfrac{l\,W}{a\,d^3}$, it follows that the deflection of two beams, of the same materials, varies as $\dfrac{l^3\,W}{a\,d^3}$: if therefore, we denote the deflection by δ, we shall have

$$\frac{l^3\,W}{a\,d^3\,\delta} = E, \textit{ a constant quantity};$$

which being determined experimentally, for any length, breadth, and depth, the same will answer for any other given dimensions, the materials being the same; and consequently δ may readily be determined, the other quantities being supposed given.

We have here supposed the beam to be fixed at one end; but the conclusion is exactly the same when it is supported at both, and loaded in the middle, except that the value of the constant quantity E is not then the same, being in the latter instance to that in the former as 32 : 1.

It should be observed, that this result will only obtain while the deflections are small, because we have supposed, throughout our investigation, the elasticity to remain perfect; and that $a\,d$, $a'd'$, $a''d''$, &c. (fig. 112) are proportional to their distances from D, and each drawn perpendicular to the original tangent Am; whereas strictly, each ought to be drawn perpendicular to its respective tangent; while the deflections, however, are small, this will not affect our ultimate deductions in any appreciable degree.

In a similar manner to that pursued above, it may be demonstrated, that if two equal beams are each fixed by one end in a wall, and the one loaded at its extremity with any given weight; and in the other, the same weight is uniformly distributed throughout the entire length: the deflection in the first case will be to that in the second as 8 to 3.

But if two equal beams, under similar circumstances, be rested upon two props, the deflection of that which has the weight collected at the centre, is to that in which it is uniformly distributed as 8 to 5.

172. The same relations, however, have not place in the ultimate deflection of beams: in this case, referring for example, to fig. 101, or fig. 105, it is obvious, that whatever may be the length of the piece, the ultimate extension of the fibre A b, will always be the same; and therefore the ratio of $\delta\,l^2$: $\delta\,l'^2$ becomes now l^2 : l'^2; that is, the ultimate deflections, when the depth is the same, will be as the square of the length; also, as the extension is the same, whatever may be the depth, the angle A$\,n\,b$, by which the element of deflection is measured, will be inversely as the depth: consequently, in beams of equal lengths and breadths, the ultimate deflections will be directly as the square of the length, and inversely as the depth; and are therefore wholly independent of the weight W, and the breadth of the beam; whence we have for this determination, the equation

$$\frac{l^2}{d\,\triangle} = U, \textit{ a constant quantity},$$

where $\triangle$ denotes the ultimate deflection.

173. It only now remains to explain the relation between the force of direct cohesion of the fibres of any body, and the weight or strain that is necessary to produce the fracture; and as we have shown the relative strains under different manners, of placing the beam; and also the relative resistances of the fibres of the solid, as they depend upon the magnitude and figure of the area of fracture; it will be sufficient for our present purpose simply to consider the case of a rectangular beam fixed at one end in a wall, and loaded with a weight at the other.

Let AFIC (fig. 102), denote a beam under those circumstances; n its neutral axis of rotation; t the centre of tension; and c the centre of compression. Now, denoting the depth of compression Cn, by D, and the whole depth AC by d; also the breadth by a, we shall have $(d-\text{D})\,a =$ area of tension, and $\text{D}\,a =$ area of compression; again, let C denote the strength of direct cohesion on a square inch, then $\text{C}\,(d-\text{D})\,a =$ the whole force of tension which is represented by the weight w, and this acting at the distance $n\,t = \frac{1}{2}(d-\text{D})$, its whole effect to oppose the motion of the lever about n will be $\frac{1}{2}\text{C}\,(d-\text{D})^2 a$; moreover, this effect is equal to the force of compression represented by w' acting at the distance $n\,c$, for it is the equality of these two forces which determines the motion to take place about n; whence $\frac{1}{2}\text{C}\,(d-\text{D})^2 a =$ half the entire resistance of the beam, or $\text{C}\,(d-\text{D})^2 a =$ the resistance. Now, if we denote the weight by W, and the length in inches, or in the same measure as a and b, by l, we shall have $l\,\text{W}\cos\Delta$ for the strain, or that force which has a tendency to produce the rotation; Δ denoting the angle of deflection. Hence we shall have, at the moment of fracture, when the power and resistance are supposed to be in equilibrio,

$$\text{C}\,(d-\text{D})^2 a = l\,\text{W}\cos\Delta;$$

whence C and W are readily determined, the one from the other; the other quantities d, D, a, and l, being given.

When the beam is supported at each end, and the weight applied in the centre, the strain is reduced to $\frac{1}{4}\,l\,\text{W}\sec^2\Delta$; in which case, therefore, the above becomes

$$\text{C}\,(d-\text{D})^2 a = \tfrac{1}{4}\,l\,\text{W}\sec^2\Delta\text{, or}$$

$$\text{C} = \frac{l\,\text{W}\sec^2\Delta}{(d-\text{D})^2 a}.$$

But we have seen that in this case

$$\text{S} = \frac{l\,\text{W}\sec^2\Delta}{4\,a\,d^2};$$

whence
$$\text{C} = \frac{\text{S}'\,d^2}{(d-\text{D})^2}.$$

Collecting now our several results, we shall find,

1. $$\frac{l^3\,\text{W}}{a\,d^3\,\delta} = \text{E, } \textit{a constant quantity,}$$

which applies to the case of small deflections, these being denoted by δ;

2. $$\frac{l^2}{d\,\Delta} = \text{U, } \textit{a constant quantity,}$$

Δ being the ultimate deflection before rupture;

3. $$\frac{l\,\text{W}\sec^2\Delta}{4\,a\,d^2} = \text{S}'\text{, } \textit{a constant quantity,}$$

which answers to the ultimate strength S′, or the weight W necessary to produce the fracture;

Primitive deflection.

4. $$\frac{\text{S}\,d^2}{(d-\text{D})^2} = \text{C, } \textit{a constant quantity,}$$

which is the strength of direct cohesion on a square inch; a, l, d, &c. being taken in the same measure.

It is therefore only necessary to establish a proper course of experiments, for determining the values of E, S′, U, and D, in various materials; from which it is obvious we may then proceed to compute all the circumstances relating to the primitive and ultimated resistance of solids under the several conditions investigated in the preceding part of this article.

Experiments on timber.

174. Such a course of experiments has lately been published by Mr. Barlow, in his "Essay on the Strength and Stress of Timber, and other Materials;" and we propose to conclude this chapter, by abstracting from the above work a few of the most important results.

The following table relates principally to the determination of the constant quantities E, U, S′, C, and D, above enumerated.

The pieces of timber on which they were made were each 8 feet in length, and 2 inches square; they were rested on two props, and loaded in the middle. The 1st column contains simply the number of the experiments; the 2d, the names of the woods and their dimensions; and the 3d, their respective specific gravities at the time of making the experiments.

The 4th and 5th columns contain,—the former the greatest weight, and the latter the greatest corresponding deflection, while the elasticity remained perfect. These numbers were found by loading the scale successively with different weights, and observing the deflection for each; and as soon as it was found that the deflection increased in a greater ratio than the weights, the operation was discontinued, and the previous weight and deflection registered as in the table: the piece was then turned with that side horizontal which was before vertical, and the deflection again obtained in the same manner; and hence there are two results for the elasticity of each piece.

The eighth column shows the depth of the neutral axis, from the upper side of the beam corresponding with the quantity we have denoted by D; which was not, however, in all cases deduced from the experiment itself, as in long pieces the indications of this point (which is commonly shown by an obvious crush of the upper fibres), are not always sufficiently distinct; but in these cases the fragments were again broken at a short length, in order to determine the above point with the greater accuracy.

The 9th, 10th, 12th, and 13th columns, exhibit the values of the four constants E, U, S′, and C, investigated above; and the 11th an approximate value of S, drawn from the formulæ given in art. 171, as sufficiently accurate for most practical cases; viz.

$$\text{U} = \frac{l^2}{d\,\Delta};\ \text{E} = \frac{l^3\,\text{W}'}{a\,d^3\,\delta};\ \text{S} = \frac{l\,\text{W}'}{4\,a\,d^2};$$

$$\text{S}' = \frac{l\,\text{W}\sec^2\Delta}{4\,a\,d^2};\ \text{and C} = \frac{\text{S}'\,d^2}{(d-\text{D})^2}.$$

TABLE OF DATA,

CONTAINING

THE RESULTS OF EXPERIMENTS ON THE ELASTICITY AND STRENGTH OF VARIOUS SPECIES OF TIMBER; SELECTED FROM HIS MAJESTY'S DOCK-YARD, WOOLWICH.

By P. BARLOW, R.M.A.

Mechanics.

No. of Experiments.	Names of the Woods, and Dimensions.	Specific Gravity.	Greatest weight and deflection while the elasticity remained perfect.		Breaking weight, in lbs.	Ultimate deflection, in inches.	Depth of neutral axis, in inches.	Value of U.	Value of E.	Value of S.	Value of S′.	Value of C.
			Weight, in lbs.	Deflection, in inches.								
1	Teak, 7 ft. by 2 in. sq.	742	300 300	1·065 1·093	1020	4·75	1·2					
2		749	300 300	1·150 1·130	975	4·20	1·2					
3		744	300 300	1·276 1·192	820	4·00	—					
	Mean Results	745	300	1·151	938	4·32	1·2	818	9657802	2462	2488	15550
4	Poon, 7 ft. by 2 in. sq.	600	150 150	·830 ·780	860	6·00	1.25					
5		570	150 150	·820 ·837	848	5·75	—					
6		568	150 150	·837 ·830	830	6·00	1·20					
	Mean Results	579	150	·822	846	5·92	1·225	596	6759200	2221	2266	14787
7	English Oak, 1st specimen, 7 ft. by 2 in. sq. inferior specimen.	986	150 150	1·420 1·420	470	6·00	1·3					
8		998	150 150	1·700 1·700	421	5·90	1·3					
9		925	150 150	1·650 1·650	460	5·80	1·3					
	Mean Results	969	150	1·590	450	5·90	1.3	593	3494730	1181	1205	9836
10	English Oak, D°, 2d specimen, 6 ft. by 2 in. sq. reduced to 7 ft.	942	200 200	1·260 1·280	640	7·90	1·2					
11		900	200 200	1·290 1·290	623	8·30	1·2					
12		960	200 200	1·275 1·285	649	8·10	1·2					
	Mean Results	934	200	1·280	637	8·10	1·2	435	5806200	1672	1736	10853
13	Canadian Oak, 7 ft. by 2 in. sq.	865	225 225	1·150 1·150	660	5·70	1·1					
14		885	225 225	1·009 1·011	708	6·20	—					
15		867	225 225	1·070 1·070	651	6·10	1·15					
	Mean Results	872	225	1·080	673	6.00	1·125	588	8596854	1766	1803	11428

TABLE—*continued.*

No. of Experiments.	Names of the Woods, and Dimensions.	Specific Gravity.	Greatest weight and deflection while the elasticity remained perfect.		Breaking weight, in lbs.	Ultimate deflection, in inches.	Depth of neutral axis, in inches.	Value of U.	Value of E.	Value of S.	Value of S′.	Value of C.
			Weight, in lbs.	Deflection, in inches.								
16	Dantzic Oak, 7 ft. by 2 in. sq.	767	200 200	1·710 1·690	520	5·00	1·2					
17		787	200 200	1·260 1·300	580	4·10	1·2					
18		713	200 200	1·855 1·715	580	5·50	—					
	Mean Results	756	200	1·590	560	4·86	1·2	724	4765750	1457	1477	7386
19	Adriatic Oak, 7 ft. by 2 in. sq.	941	150 150	1·070 1·070	560	6·00	1·2					
20		948	150 150	1·550 1·450	500	5·50	1·25					
21		1090	150 150	1·720 1·720	520	5·70	1·15					
	Mean Results	993	150	1·430	526	5·73	1·2	610	3885700	1383	1409	8808
22	Ash, 7 ft. by 2 in. sq.	760	225 225	1·270 1·250	777	9·00	1·35					
23		758	225 225	1·300 1·270	760	9·10	1·30					
24		762	225 225	1·240 1·270	780	8·66	1·25					
	Mean Results	760	225	1·266	772	8·92	1·3	395	6580750	2026	2124	17337
25	Beech, 7 ft. by 2 in. sq.	712	150 150	1·075 1·025	565	6·00	1·2					
26		688	150 150	1·009 1·024	600	5·70	—					
27		688	150 150	1·025 1·000	615	5·50	1·2					
	Mean Results	696	150	1·026	593	5·73	1·2	615	5417266	1556	1586	9912
28	Elm, 6 ft. by 2 in. sq. reduced to 7 feet.	583	125 125	1·620 1·610	368	7·00	1·2					
29		510	125 125	1·420 1·460	398	6·93	1·1					
30		535	125 125	2·070 1·930	394	6·86	—					
	Mean Results	553	125	1·685	386	6·93	1·15	509	2799347	1013	1042	5767
31	Pitch Pine, 7 ft. by 2 in. sq.	712	150 150	1·133 1·166	650	6·25	1·2					
32		628	150 150	1·140 1·110	595	5·75	1·2					
33		641	150 150	1.166 1·091	620	6·00	—					
	Mean Results	660	150	1·134	622	6·00	1·2	588	4900466	1632	1666	10415

TABLE—*continued.*

No. of Experiments.	Names of the Woods, and Dimensions.	Specific Gravity.	Greatest weight and deflection while the elasticity remained perfect.		Breaking weight, in lbs.	Ultimate deflection, in inches.	Depth of neutral axis, in inches.	Value of U.	Value of E.	Value of S.	Value of S′.	Value of C.
			Weight, in lbs.	Deflection, in inches.								
34	Red Pine, 7 ft. by 2 in. sq.	655	150 150	·825 ·825	473	5·70	1·3					
35		667	150 150	·700 ·725	530	5·83	1·25					
36		650	150 150	·725 ·730	530	5·96	1·25					
	Mean Results	657	150	·755	511	5·83	$1·26\frac{2}{3}$	605	7359700	1341	1368	10000
37	New England Fir, 7 ft. by 2 in. sq.	560	150 150	·862 ·862	446	4·50	1·36					
38		560	150 150	·970 ·970	403	4·70	1·30					
39		540	150 150	·960 ·960	411	4·78	1·33					
	Mean Results	553	150	·931	420	4·66	1·33	757	5967400	1102	1116	9947
40	Riga Fir, 1st specimen, 7 ft. by 2 in. sq.	730	125 125	·812 ·837	420	5·80	1·35					
41		765	125 125	·912 ·912	440	6·10	1·35					
42		763	125 125	·837 ·910	406	6·10	—					
	Mean Results	753	125	·870	422	6·00	1·35	588	5314570	1108	1131	10707
43	Riga Fir, 2d specimen, 6 ft. by 2 in. sq.	714	150 150	·794 ·794	567	5·50	—					
44		768	150 150	·907 ·909	367	6·00	—					
45		732	150 150	·950 ·950	467	6·50	—					
	Mean Results	738	150	·883	467	6·00	....	....	3962800	1051	1081	—
46	Mar Forest Fir, 1st specimen, 7 ft. by 2 in. sq.	715	125 125	1·560 1·500	360	5·50	1·3					
47		616	125 125	1·370 1·250	463	5·50	1·3					
48		684	125 125	1·370 1·370	465	7·00	1·3					
49		769	125 125	1·560 1·560	457	6·00	1·3					
	Mean Results	696	125	1·442	436	6·00	1·3	588	2581400	1144	1168	9539
50	Mar Forest, 2d specimen, 6 ft. by 2 in. sq.	720	150 150	1·150 1·150	600	7·00	1·3					
51		756	150 150	1·250 1·150	517	6·00	1·3					
52		603	150 150	0·675 0·675	567	6·25	1·3					
	Mean Results	693	150	1·006	561	6·42	1·3	403	3478328	1262	1310	10691

TABLE—*continued.*

No. of Experiments.	Names of the Woods, and Dimensions.	Specific Gravity.	Greatest weight and deflection while the elasticity remained perfect.		Breaking weight, in lbs.	Ultimate deflection, in inches.	Depth of neutral axis, in inches.	Value of U.	Value of E.	Value of S.	Value of S′.	Value of C.
			Weight, in lbs.	Deflection, in inches.								
53	Mar Forest, 3d specimen, 6 ft. by 2 in. sq.	700	150 150	1·150 1·150	561	6.5	1·3					
54		710	150 150	1·230 1·170	570	6·5	1·3					
55		698	150 150	0·675 0·675	552	6·25	1·3					
	Mean Results	703	150	1·006	561	6·42	1·3	403	3478328	1262	1310	10691
56	Larch, 1st specimen, 7 ft. by 2 in. sq.	504	125 125	1·930 1·910	300	8·60	—					
57		576	125 125	1·740 1·760	340	8·64	—					
58		514	125 125	1·970 2·000	336	8·54	—					
	Mean Results	531	125	1·885	325	8·58	....	411	2465433	853	890	
59	Larch, 2d specimen, 6 ft. by 2 in. sq.	552	125 125	0·750 0·750	300	6·00	—					
60		480	125 125	0·812 0·812	412	4·50	—					
61		534	125 125	0·875 —	398	4·50	—					
	Mean Results	522	125	0·812	370	5·00	....	518	3591133	832	850	
62	Larch, 3d specimen, 6 ft. by 2 in. sq.	546	150 150	0·750 0·750	417	4·70	1·25					
63		552	150 150	0·825 0·825	497	4·90	1·25					
64		552	150 150	0·750 0·750	537	5·00	1·20					
65		576	150 150	1·050 ·950	552	5·40	1·20					
	Mean Results	556	150	0·831	501	5·00	1·25	518	4210830	1127	1149	7655
66	Larch, 4th specimen, 6 ft. by 2 in. sq.	552	150 150	·831 ·831	500	4·8	1·2					
67		531	150 150	·900 ·864	515	5·2	1·2					
68		548	150 150	·762 ·798	515	5·0	1·2					
	Mean Results	560	150	·831	510	5·0	1·2	518	4210830	1149	1172	7352
69	Norway Spar, 6 ft. by 2 in. sq.	600	200 200	·800 ·800	667	4·0	1·65					
70		600	200 200	·760 ·740	617	4·0	1·25					
71		530	200 200	·840 ·860	680	4·0	1·30					
	Mean Results	577	200	·800	655	4·0	1·30	648	5832000	1474	1492	12180

The application of these tabular results to any practical case is so obvious, that we deem it useless to give any practical examples: we shall, therefore, proceed to give the results of certain other experiments, selected from the same work, on the strength of Iron.

Mechanics.

Experiments on iron.

EXPERIMENTS.

175. *Upon the direct strength of cohesion of malleable Iron made at Messrs. Brunton and Co's. patent chain cable manufactory, with an hydrostatic machine, or Bramah press, constructed by Mr. Fuller. By* THOMAS TELFORD, ESQ.

Experiment, No. 1.

Cylindrical bar of South Wales iron, manufactured by S. Homfrey, Esq.

		Feet.	Inches.
April 5, 1814.	Length of bar when put in	2	$2\frac{3}{4}$
	......ditto......when taken out...	2	$6\frac{7}{8}$
	Diameter when put in............	0	$1\frac{3}{8}$
	......ditto......when taken out...	0	$1\frac{1}{8}$

Torn asunder by 43 tons 11 cwt.

Experiment, No. 2.

Cylindrical bar of South Wales iron, manufactured by S. Homfrey, Esq.

		Feet.	Inches.
April 5, 1814.	Length of bar when put in	2	$3\frac{3}{8}$
	......ditto......when taken out...	2	$6\frac{5}{8}$
	Diameter when put in............	0	$1\frac{1}{2}$
	......ditto......when taken out...	0	$1\frac{1}{4}$

Torn asunder by 52 tons 15 cwt. 1 qr. 10 lb.
Time, 34 minutes.

Experiment, No. 3.

Square bar of Staffordshire iron.

		Feet.	Inches.
May 17, 1814.	Length of bar when put in	1	$5\frac{1}{8}$
	......ditto......when taken out...	1	$11\frac{1}{4}$
	Side of square when put in.......	0	$0\frac{3}{4}$
	......ditto......when taken out...	0	$0\frac{6}{10}$

Began to stretch with 12 tons; broke with 15 tons 5 cwt. 3 qrs. 4 lbs.
Time, $9\frac{1}{4}$ minutes.

Experiment, No. 4.

Square bar of Staffordshire iron.

		Feet.	Inches.
May 17, 1814.	Length of bar when put in	1	$7\frac{1}{4}$
	......ditto......when taken out...	1	$9\frac{1}{4}$
	Side of square when put in........	0	$1\frac{1}{12}$
	......ditto......when taken out...	0	$0\frac{5}{8}$

Began stretching with 32 tons; broke with 32 tons 6 cwt. 0 qrs. 4 lbs.
Time, 16 minutes.

Experiment, No. 5.

Square bar of Welsh iron, 1 inch square.

		Inches.	
May 5, 1817.	With 18 tons, stretched...	$0\frac{1}{4}$	
	Ditto 21 ...... ditto	$0\frac{1}{2}$	
	Ditto 23 ...... ditto	$0\frac{3}{4}$	
	Ditto 25 ...... ditto	1	
	Ditto 27 ...... ditto	$2\frac{1}{4}$	
	Ditto 29 ...... ditto	$2\frac{3}{8}$	broke with thisweight.

Experiment, No. 6.

Bar of Swedish iron, 1 inch square.

May 5, 1817.	Began to stretch with 17 tons.		
	Stretched with 20 do.	$0\frac{1}{10}$ of an in.	
	......ditto...... 27 do.	$0\frac{3}{8}$	
	......ditto...... 29 do.		broke at a flaw.

Note.—The preceding and following stretchings were measured on 12 inches in the middle of the bar.

Experiment, No. 7.

Bar of faggotted iron from scrap iron, manufactured by Mr. Howard, of Rotherhithe, 1 inch square.

May 5, 1817.	Began to stretch with 16 tons.		
	Stretched with 20 do.	$0\frac{3}{8}$ of an inch.	
	....ditto.... 25 do.	$0\frac{1}{4}$	
	....ditto.... 28 do.	$2\frac{3}{8}$	
	....ditto.... 29 do.		broke with this weight.

Note.—A similar bar began to stretch with 18 tons, and broke with the same weight as above; viz. 29 tons.

Experiment, No. 8.

Bar of common Staffordshire iron, 1 inch square.

May 5, 1817.	Began to stretch with 19 tons.		
	Stretched with 24 do.	$0\frac{1}{2}$ inch.	
	....ditto.... 28 do.	$0\frac{5}{8}$	
	....ditto.... 29 do.	$0\frac{5}{8}$	
	....ditto.... 30 do.	1	
	....ditto.... 31 do.		broke with this weight.

Experiment, No. 9.

Cylindrical bar of common iron, 2 inches diameter.

	Tons.		
May 21, 1817.	With 45		Began to stretch, about $\frac{1}{10}$th of an inch on 12 inches in the middle; the machine being relieved, the bar shortened $\frac{1}{40}$th of an inch.
	Tons.	Inches.	
	With 50 stretched	·125	relieved and shortened as before.
	Ditto 55..ditto..	·25	Ditto..ditto.
	Ditto 60..ditto..	·26.	
	Ditto 70.. ditto..	·375	recovered very little when the machine was relieved
	Ditto 75.. ditto..	·544	Ditto..ditto.
	Ditto $80\frac{1}{10}$ ditto..	·75	reduced in diameter to $1\frac{15}{16}$th inch.
	Ditto 85.. ditto..	·86	no perceptible change.
	Ditto 90.. ditto..	1·00.	
	Ditto 95.. ditto..	1·35	reduced in diameter to $1\frac{7}{8}$th inch.
	Ditto 100..ditto..	2·2.	Ditto..ditto to $1\frac{1}{2}$ nearly.

With the last weight the bar gave evident signs of fracture, and in a few minutes gradually gave way.

Note.—The whole length of the above bar was 2 feet, and its stretch in its whole length $2\frac{7}{8}$ inches; of which, $2\frac{1}{5}$ inches was in 12 inches in the middle part. The whole time of making this experiment was 3 hours; and it was performed with the utmost care.

The machine was frequently relieved, and when re-applied, constantly brought up the weight to what it was before, but never exceeded it: which is evidence of its accuracy.

Note.—It is a curious fact, and deserving the attention of philosophers, that frequently at the moment of rupture the bar acquired such a degree of heat in the fractured part, as scarcely to suffer a person to hold the bar grasped in his hand without a slight painful sensation of burning.

Reduction of the above to 1 inch square.

	Tons.	Cwt.	
No. 1 red^d to 1 inch sq. gives	29	6	Welsh.
— 2 .... ditto	29	16	Welsh.
— 3 .... ditto	27	3	Staffordshire.
— 4 .... ditto	27	10	Staffordshire.
— 5 .... ditto	29	0	Welsh.
— 6 .... ditto	29	0	Swedish.
— 7 .... ditto	29	0	Faggotted.
— 8 .... ditto	31	0	Staffordshire.
— 9 .... ditto	31	16	
	9)263	11	
Mean strength of an inch sq. bar	29	5⅔	

It should be observed, however, that from a similar course of experiments carried on at the patent chain cable manufactory of Captain Brown, the mean strength of an iron square inch bar, is found to be only 25 tons; but it seems probable that an inaccuracy in estimating the power of the two machines, is the principal cause of this difference. Messrs. Brunton's machine is, as we have seen, an hydraulic press; the power of which is estimated by the pressure on a small valve: but, as the friction of the piston-leather in high pressures is very great, and the power of the press being opposed both to the resistance of the bar and to the friction, while the power acting on the valve, has no counteraction; it follows that there is necessarily a tendency in the machine to over-rate, not its own power, but the effect of that part of it which is effectively opposed to the resistance of the bar; and exactly the reverse has place in the machine of Captain Brown: it is probable, therefore, that the medium of the two results may be considered as a fair valuation of the cohesive power of iron, viz. 27 tons to the square inch.

176. We learn also, from the same work, that the strength of direct cohesion on a square inch of

Blistered steel, is .... 14·27 tons.
Cast steel .... 27·92 tons.
Cast iron Welsh pig ...... 7·26 tons.

That iron wire $\frac{1}{10}$th inch in diameter, from a medium of many trials, required 630 lbs. to tear it asunder by a direct draw, which reduced to square inches, gives

$$630 \times 100 \times \cdot 7854 = 49480 \text{ lbs.} = 22\cdot 2 \text{ tons.}$$

A few experiments are also given from Mr. Bank's work, on the power of machines relative to the transverse strength of cast iron; which are as follows:

No.	Length.	Depth.	Breadth.	Breaking-wt.
1	3 feet	1 inch	1 inch	756 lbs.
2	3 do.	1 do.	1 do.	756 —
3	2 do. 6 in^s.	1 do.	1 do.	1008 —
4	3 do.	1 do.	1 do.	735 —
5	3 do.	1 do.	1 do.	963 —
6	3 do.	1 do.	1 do.	958 —
7	3 do.	1 do.	1 do.	994 —
8	3 do.	1 do.	1 do.	864 —
9	3 do.	1 do.	1 do.	874 —

In the above experiments the ends of the bars were supported on props, and the weights applied in the centre.

Mechanics.

Experiments on cast iron.

If we here compute the constant value of S, as in the preceding table, for the different woods, we shall have, for the mean

$$\text{of the first two,} \quad S = \frac{lW}{4ad^2} = 6804;$$

$$\text{No. 3.} \quad S = \frac{lW}{4ad^2} = 7560;$$

$$\text{No. 4.} \quad S = \frac{lW}{4ad^2} = 6619;$$

$$\text{mean of 5, 6, and 7,} \quad S = \frac{lW}{4ad^2} = 8748;$$

$$\text{mean of 8 and 9,} \quad S = \frac{lW}{4ad^2} = \underline{7821;}$$

Mean value of S in the 9 experiments, 7510;
Value of direct cohesion, 7·26 tons = 16262;
Specific gravity of 5, 6, and 7..... 6812.

Comparing these numbers with the corresponding ones for the second specimen of oak, it appears that cast iron has its specific gravity to that of oak, in about the ratio of 7 to 1; and that its transverse strength is in about the ratio of 4½ to 1; or if we compare the oak with the above numbers 5, 6, and 7, in the ratio of 5¼ to 1, while its cohesive power is not in a higher ratio than 1½ to 1, or as 3 to 2; which shows this substance to be much more incompressible than oak, and consequently, that its neutral axis is much higher. For a variety of other experiments, on the strength of solid and hollow cylinders; bent timber for naval purposes; as also on the effect of boiling and seasoning timber, &c. see Barlow's "Essay on the Strength and Stress of Timber."

§ XXV. *On the equilibrium of banks of earth and revetments.*

Revetments.

177. The subject of revetments is a part of practical mechanics that has much engaged the attention both of the practical engineer and the theoretical mechanic; and no subject, perhaps, ever gave rise to a greater number of plausible theories, the deductions of which are so widely different from each other: and it will therefore, we trust, not be uninteresting to the reader to have a slight sketch of some of the principal doctrines that have been laid down with a view of establishing, on theoretical principles, the proper thickness of revetments for supporting banks of earth composed of soils of different tenacity, and of various heights.

Let BPED (fig. 113) represent a bank of earth, and ABDC a wall, or other obstacle, by which it is supported, or which prevents a certain portion of the bank as BED, from slipping down, which would obviously be the case if the obstacle were removed; and the question is, to determine what ought to be the thickness or stability of the wall, which exactly counterbalances the push of that part of the earth which has a tendency to slip down. This is at least the theoretical question; although, practically, it will always be necessary to exceed very far what is absolutely required to establish a mere statical equilibrium; the latter, however, is the consideration which belongs to the theorist; and the former, that is, how much an actual revetment ought to exceed the computed thickness, appertains entirely to the province of the practical constructor.

Requisite data.

178. It is obvious, in the first place, that in order to establish any computation with relation to the above

determination, it is necessary to have certain data, which can only be obtained from experiment, viz.

Requisite data.

1. The specific gravity of different soils, in order thence to determine the weight of any quantity of them.

2. We ought to know what slope the various soils will assume when thrown loosely in a heap, in order thence to ascertain the quantity DBE which it is necessary to support.

3. It is important to determine the value of the friction between the two surfaces, that slide upon each other, the section of which is represented by the line DE.

4. As every soil, after being compressed, will acquire a certain degree of cohesion, a bank of earth of the figure we are supposing, will not always, when the obstacle is removed, slide down at the same angle as the loose soil thrown in a heap, as supposed in No. 2 above: some authors, therefore, have thought it farther necessary that we should ascertain experimentally, the natural slope which different soils will assume after receiving various degrees of compression; but as this is probably extremely irregular and uncertain, we doubt much the propriety of introducing it as a datum into our computation, although it is made to cut a very conspicuous figure in the theories of some of our continental neighbours. Every one must have observed the extreme irregularity of the pieces of cliffs on the sea coasts, which slip down after heavy rains, &c. many of which have a slope equal to that which the same soil would assume when thrown loosely on a heap, and the figure of which is commonly determined by the insinuation of water into natural or artificial fissures in the land: but as we conceive that in no case the bank can slip down at a greater slope than that which it naturally assumes when thrown up, as supposed in No. 2; we consider it much the safest plan to assume this as the possible limit in all cases, and to make our revetment always of superior strength to the maximum thrust. This assumption has two important advantages; viz. 1st, it adds very considerably to the simplicity of the problem; and 2dly, it gives a degree of security to our results, which can never be felt where we draw our conclusions from investigations founded upon uncertain data. We shall, therefore, reject that consideration which has reference to the cohesion of the soil; although it may not be improper to give a sketch of the solution which is made to depend upon it, in order that the reader may judge how far our reasons assigned for excluding it are correct.

At present, we have only referred to the experimental data that are requisite for determining the intensity of the force by which the bank tends to slide down, or the force which it impresses on the wall or opposing obstacle; but other experiments are also necessary for the purposes of ascertaining the natural resisting power of the opposing obstacle: these are, 1st, its weight, by which it resists being overturned, independent of its cohesion; and 2dly, its strength of cohesion, by which it resists fracture independent of its weight; for it is obvious that both these resistances must be overcome before the bank can slip out of its position.

With regard to the first of the above data, it is obtained without difficulty; because we can always ascertain the specific gravity of any material of which our wall or revement is composed: but the strength of cohesion of the cement or mortar, by which it is converted into one mass, is a subject which appears at present, to have entirely escaped the notice of constructors and engineers; at least we know of no experiments of this kind, on which we can place any reliance: the two or three cases mentioned by M. Mayniel, p. 200, of his "Traite de la Poussée des Terres," are too much at variance with each other to inspire us with any confidence.

179. The strength of cohesion of some building materials has been ascertained, but on a very limited scale: thus Gauthey, a German engineer, found that a prism of stone, which he denominates soft givry, one foot long and a foot square, when fixed by one end in a rock, required a weight of 5000 lbs. suspended from the other extremity, to produce the fracture; and that a similar prism of hard givry required 5600 lbs., under like circumstances, to break it; and it appears that brick, at a medium, when reduced to the same dimensions, requires 4500 lbs.; and since, in the preceding chapter, it has been shown that the strength of materials, submitted to a transverse strain, is inversely as the length, and directly as the breadth and square of the depth, we shall have in these three cases

soft givry $\dfrac{l\,W}{a\,d^2} = C = 5000,$

hard ditto $\dfrac{l\,W}{a\,d^2} = C = 5600,$

brick $\dfrac{l\,W}{a\,d^2} = C = 4500;$

the dimensions l, b, and d, being supposed to be all given in feet. It is, however, the mortar and cement of which it is most important to know the strength or resistance; but we are not aware of any experiments that have been made with a view to such determination.

180. We are much better supplied with experiments on the natural slope and thrusts of banks of earth; but they are, many of them, on too small a scale to inspire us with great confidence in their results: we shall, however, give the reader a sketch of all that are deserving of particular attention; in doing which we shall avail ourselves of the summary of them given by Mayniel, in his "Traité de la Poussée des Terres."

The first of these experiments which we think it necessary to mention, are those of Papacini D'Antoni, an Italian engineer, who, in his researches on the thrusts of earth, constructed a large box, open at top, and having one end hung on joints at the bottom, so that it might be set up or let down at pleasure; to this moveable end, by means of two bars, were fixed cords, which passed over two fixed pulleys behind the machine, and to the extremities of which were attached scales for the convenience of receiving weights. The apparatus being thus prepared, the end of the box was shut, and sufficient weights placed in the scale to keep the end of the box still up after the latter was filled with earth; then unloading each scale equally, by small weights at a time, it was at last ascertained what was the least weight that was required to resist the pressure of the internal earth; for, by reducing the weight still farther, the earth slipped, and the face of the box turned. We have not the particular detail of the numerical results of these experiments; and are only informed generally that the natural slope which the earth took was that of an angle of 45°, and that the thrust was equal only to ¼th of the absolute weight of the earth which slid down; that is to say, that ¾ths of the natural weight of the sliding part was destroyed by its friction against the part which remained unmoved.

Gauthey, to whom we have referred in the preceding article, had a chest constructed 2 feet 6 inches in length, as much in height, and a foot broad, open at top and at one end; to which at bottom was hung on joints, as we have described in speaking of D'Antoni's experiments, a plane or face to represent a revetment; and at one-third of its height from the bottom, he fixed on each side a cord, which passed over pulleys in the same manner as we have already described; and having filled the box with sand, he found that a weight of 35 lbs. was sufficient to prevent the end of the box from turning, although the weight of the sand which slid down when the face was removed weighed 320 lbs. The slope that the sand took in this trial was about 45°.

In prosecuting his experiments, however, it was found that the sand frequently took a less slope, and that, being very fine, it escaped between the ends and sides of the box. Mr. Gauthey therefore changed both his material and apparatus, reducing very considerably the dimensions of the chest, and substituting, instead of the sand, the metal dust of an iron foundery, with which he made several experiments of no importance to detail: we shall only observe, that wishing to know whether all the parts of the revetment were acted upon with forces proportional to the different trapezoids of earth each part from the bottom upwards had to support, he divided the height of his box into five equal parts; these he shut by means of five vertical planes, which slipped one over the other without touching, in order that they might not impede each other's motion; each of these had cords at its extremities passing over pulleys: and it was found that when the weights attached were in the ratio of the numbers 3, 8, 12, 16, 20, they all moved equally forward with the pressure behind them: these differ a little from the ratio of the solids by which they were moved, the latter being as the numbers 2, 6, 10, 14, and 18: the experiments however were made on too small a scale to give very satisfactory results.

Our author then, in order to determine the place of the resultant of all the forces (which had been differently assumed by authors who had treated of this problem theoretically), shut the box, and fixed two joints to the end at ⅓d of the height from the bottom, one on each side, so as to allow of its turning about those points; and having then filled his box, he found no tendency in the end to turn either from the bottom or top, but on the contrary, it appeared to remain perfectly in equilibrio with regard to the strain upon it; when he fixed the joints higher than at a third, the end fled away at bottom; and if lower than ⅓d, it opened at top; whence he concluded that the plane of the resultant of the thrust passes through a line at one-third of the height from the bottom, which is perhaps one of the most useful deductions that were made from this course of experiments.

By Rondelet.

Experiments of a nature somewhat similar were made by M. Rondelet: he constructed a box like that of Mr. Gauthey, 1 foot 4½ inches long, 1 foot broad, and 1 foot 5½ inches high, measured in the opening, and this end he could shut and open at pleasure. As his principal object was to ascertain the natural slope that the earth would take, he first filled the box, took away the moveable side, and then measured the angle of the slope, which he found to be, for dry sand and pulverized stone 34½° with the horizontal plane, and consequently 55½° with the vertical. We conceive the other experiments of this author to be too limited and partial to be deserving of any particular detail.

By Mayniel.

The next experiments we have to notice are those of M. Mayniel, which were made upon a larger scale: the coffer or box within which the sand and earth was placed, having been 3 metres or near 10 feet long, 1½ metre or 5 feet wide, and 5 feet deep, and of proportionate strength; the moveable side, or that which was to be thrown down with the pressure of the earth, was so constructed that the point of application of the resisting force might be changed at pleasure; and by an ingenious contrivance, which it is unnecessary to describe, the power of the sliding bank was very accurately ascertained.

The results of some of these experiments are given in the following table; a few only are omitted, because, to render the account of them intelligible, would require more room than we feel ourselves justified in dedicating to this particular subject.

MAYNIEL'S
EXPERIMENTS ON REVETMENTS.

No. of Experiments.	Weight of the Earth that slipped down, in kilogrammes.	Its resistance at different points of application.	Point of application in parts of the height from bottom.	Medium distance of the Upper Edge of the Standing Earth, from the Original Face of the Revetment.	REMARKS.
				Metres.	
1	5650	2963	$\frac{2}{3}$	0·73	Sand, very dry.
2	5335	935	$\frac{2}{3}$	....	Vegetable earth.
3	14800	2154	$\frac{2}{3}$	....	Ditto, heaped above the top of revetment, so as to assume its natural slope to the height of ·62 metres.
4	12950	2375	$\frac{1}{2}$	....	Vegetable earth, heaped as above.
5	15910	3375	$\frac{1}{3}$	....	Similar to the above.
6	11100	2806	$\frac{1}{3}$	0·76	Vegetable, level above.
7	16650	5545	$\frac{1}{3}$	0·69	Ditto. In this experiment, a profile was raised on the top, at an angle of 45°, to the height of ·60 metres.
8	15910	3569	$\frac{1}{2}$	0·73	Ditto, parapet as above.
9	9990	2105	$\frac{1}{2}$	0·70	Ditto, level at top.
10	16280	2765	$\frac{2}{3}$	0·71	Ditto, profile as in Nos 7 and 8.
11	10360	2205	$\frac{2}{3}$	0·73	Ditto, level.
12	13155	2135	$\frac{2}{3}$	0·73	Ditto, level, but loaded with a weight of 3905 kilogrammes.
13	5700	225	$\frac{2}{3}$	....	Ditto, but carefully bedded.
14	10413	445	$\frac{2}{3}$	0·36	Ditto, parapet of natural slope, bedded as above.
15	12540	1805	$\frac{1}{3}$	0·45	Ditto, formed and bedded as above.
16	8740	475	$\frac{1}{3}$	0·37	Ditto, bedded, but no parapet.
17	8170	145	$\frac{1}{2}$	0·35	Ditto, ditto, ditto.

Experiments by Col. Pasley.

181. The most extensive series of experiments, however, that have yet been made on the subject of revetments, are those of Col. Pasley, performed at Chatham; and of which the results are given at considerable length in that author's "Course of Military Instruction," vol. 3: but in order to understand such of them as we propose to select, it will be necessary to define some few of the military terms which occur in the detail of them.

1. When the wall which supports the bank is rectangular, it is called a rectangular revetment.

2. When the section of the wall is a trapezoid, having its side next the bank perpendicular, it is a sloping revetment.

3. When the vertical side is outwards, and the slope towards the bank, it is a counter sloping revetment.

4. When the revetment is rectangular, but is built leaning upon the bank, it is a leaning revetment.

5. Counter-forts are parts of brickwork, or of the same materials as the revetment, united with it, and projecting at certain distances from the wall into the bank.

Again, when the earth is only level with the top of the wall, it is a scarp revetment; when the bank is made to slope to any height above the level of the wall, it is a counter-scarp revetment; and the distance between the outward edge of the wall and the foot of the slope is called the *berm*; and therefore, when the slope begins from the outward face or edge of the wall, there is no berm.

Thus ABCD (fig. 113) is a rectangular scarp revetment; fig. 114, a sloping counter-scarp revetment, with a berm CD; and fig. 115, a counter-sloping counter-scarp revetment. The leaning revitment differs in no respect from that shown in fig. 113, except that it is built leaning, having about the same exterior slope as fig. 114.

These definitions being laid down, the reader will readily comprehend the following tabulated results; after the apparatus with which they were performed have been described. These, in the first place, consisted of a number of wooden cases, filled with rammed earth, so as to weigh about 84 lbs. per cubic foot. They were each 3 feet long, and most of them 2 feet 2 inches high; but their mean thickness and slopes differed, in order that they might represent revetments of various kinds and proportions. The experiments were mostly tried with loose shingle, perfectly dry and under cover; which, when thrown up to a height of 5 feet, assumed a slope whose base was about 5 feet 6 inches. At smaller heights the earth might be made to stand a little higher, but in no case at a less slope than that of 1 to 1. A cubic foot of this substance weighed 89 lbs. When any experiment was tried, one of the models was placed on its base upon a smooth plane of wood, with a thin batten nailed to the floor in front, to prevent the model from slipping forward. To the back of the top of the model was fixed one end of a cord, which after passing horizontally over a pulley in front, hung down and supported a scale for holding weights. After the model was thus arranged, a trial was first made, by means of the scale, to ascertain what weight was required to overset it, without any shingle or loose earth being applied behind it. This done, the model was then replaced in the same position, and backed with shingle to a certain height, according to the nature of the proposed experiment; and if it did not overset, in consequence of the pressure thus acting against it, weights were put into the scale as before, in order to ascertain the difference caused by the shingle in the stability.

It is to be observed, that a rough board, sloping at an angle of rather less than 45°, was placed about a foot behind the bottom of the revetment or model; so that in whatever mode the mass of shingle might be applied to the supposed revetment, it was always bounded or supported in the rear by the board. This was done in order to save the men employed from unnecessary labour, in throwing up a superfluous quantity of that material: for the same reason, the space in which the shingle was thrown up, was bounded on each side, by two smooth planes of wood work placed vertically and parallel to each other, their distance apart being about 3 feet 1 inch, viz. a small quantity more than the length of each model; and great care was taken that the model should not be jammed between the ends of it and the planes; in order to which, it was placed immediately in front of, but yet as near to them as possible.

It was ascertained, from several experiments made exclusively with a view to this particular determination, that till the shingle exceeded a certain quantity, the wall or revetment was increased in its stability; which was never (we believe) before noticed by any experimentalist, or in any theoretical view of the subject; although it is a necessary consequence of the method of resolving the forces adopted by M. Rondelet, and which we shall follow in the subsequent part of this article, as being the only correct one out of the numerous methods that have been proposed; although in this respect, our opinion is decidedly in opposition to that expressed by M. Mayniel, in the work before referred to.

The following table contains the result of a part of Col. Pasley's experiments.

EXPERIMENTS ON THE PRESSURE OF EARTH.

By Col. Pasley.

Number of Experiments.	Nature of the Revetments.	Without Shingle.				With Shingle.						
		The Revetments.				Height of Shingle.	Berm equal to mean thickness.	Stability in trials.				
		Height.	Mean thickness.	Weight.	Stability in various trials.			1st	2d	3d	4th	5th
		inches	inches	lbs.		inches						
1	Rectangular........	26	4	182	11	14	....	0	0	0	0	0
2					12½	6	....	0	0	0	0	0
3					12½	0	....	0	0	0	0	0
4	Sloping at ⅕........	26	4	182	24½	0	....	5	17	12½	8½	10½
5					24½	6	....	0	0	0	0	0
6					24½	7	....	0	0	0	0	0
7					23	9	....	0	0	0	0	0
8					26	6	Berm ..	0	0	0	0	0
9					23	7	Ditto ..	0	0	0	0	0
10					25	8	Ditto ..	0	0	0	0	0
11					24	9	Ditto ..	0	0	0	0	0
12	Leaning at ⅕	26	4	182	24	2	....	0	0	0	0	0
13					25	0	....	0	15½	16½	16½	0
14					26½	30	No Berm	0	0	0	0	0
15					26½	30	Berm ..	6½	4½	5½	0	0
16					26½	48	Ditto ..	0	0	0	0	0

TABLE—*continued.*

Number of Experiments.	Nature of the Revetments.	Without Shingle.				With Shingle.						
		The Revetments.				Height of Shingle.	Berm equal to mean thickness.	Stability in trials.				
		Height.	Mean thickness.	Weight.	Stability in various trials.			1st	2d	3d	4th	5th
		inches	inches	lbs.		inches						
17	Counter-sloping at ⅕	26	4	182	13½	3	....	0	0	0	0	0
18					13½	6	....	0	0	0	0	0
19					15½	0	....	7½	13½	11½	0	0
20					16½	6	No Berm	0	0	0	0	0
21					15½	9	Berm....	0	0	0	0	0
22					15½	10	Ditto....	0	0	0	0	0
23	Sloping at ⅕........	26	5⅕	236½	41	0	....	35	30	33	0	0
24					42	30	No Berm	0	12½	13½	11½	0
25					44	30	Berm....	16½	19½	15	23½	16
26					43	60	No Berm	10½	10½	10½	0	0
27					41	60	Berm....	15½	16½	17½	0	0
28	Counter-sloping at ⅕	26	5⅕	236½	23½	0	....	26½	33½	36½	0	0
29					23½	15	No Berm	0	0	0	0	0
30					26	24	Ditto....	0	0	0	0	0
31					26	30	Ditto....	13½	12½	13½	0	0
32					23½	30	Berm....	30½	18½	30½	0	0
33					22½	60	No Berm	18½	20½	20½	0	0
34					21½	60	Berm....	44½	26½	24½	23½	0
35	Rectangular........	26	6	273	28	0	....	10	2	7	4½	4½
36					27	18	No Berm	0	0	0	0	0
37					28	21	Ditto....	0	0	0	0	0
38					25	9	Berm....	0	0	0	0	0
39					26	18	Ditto....	0	0	0	0	0
40					28	30	Ditto....	2½	2½	0	0	0
41					28	36	Ditto....	0	0	0	0	0
42	Sloping at ⅕........	26	6	273	49	0	....	43	38	0	0	0
43					48	30	No Berm	29	30	32	0	0
44					48	30	Berm....	33	43	31	0	0
45					49	60	No Berm	41	41	45	0	0
46					50	60	Berm....	41	41	54	0	0
47	Leaning at ⅕........	26	6	273	55½	0	....	39½	40½	42½	0	0
48					53½	30	No Berm	36½	42½	39½	0	0
49					57½	30	Berm....	27½	28½	25½	0	0
50					56½	60	No Berm	16½	38½	44½	47	0
51					60½	60	Berm....	57½	46½	47	47	0
52	Counter-sloping at ⅕	26	6	273	28	0	....	31	34	40	35½	34
53					30	30	No Berm	14	14	10	36½	33½
54					31	30	Berm....	0	0	38½	14½	31½
55					31	30	....	0	0	38½	14½	31½
56					31½	60	No Berm	13	18	25	9½	8½
57					29	60	Berm....	16½	32½	32½	10	20
58	Sloping at ⅕........	26	6½	205¾	58	0	....	46	49	47	0	0
59					59	30	No Berm	37	31	25	0	0
60					60	30	Berm....	46	39	39	0	0
61					59	60	No Berm	37	31	27	0	0
62					61	60	Berm....	44	38	36	0	0
63	Counter-sloping at ⅕	26	6½	295¾	36	0	....	56	53	45	35	0
64					35	30	No Berm	39	28	27	45	35
65					37	30	Berm....	31	47	38	37	44
66					38	60	No Berm	58	42	39	16½	27½
67					39	60	Berm....	42	43	19	31	39½

TABLE—*continued.*

Number of Experiments.	Nature of the Revetments.	Without Shingle.				With Shingle.						
		The Revetments.			Stability in various trials.	Height of Shingle.	Berm equal to mean thickness.	Stability in trials.				
		Height.	Mean thickness.	Weight.				1st	2d	3d	4th	5t
		inches	inches	lbs.		inches						
68	Sloping at ⅕........	26	7$\frac{11}{30}$	335$\frac{3}{16}$	86	0	....	68	69	70	0	0
69					79	30	No Berm	63	57	59	0	0
70					78	30	Berm....	66	62	64	0	0
71					76	60	No Berm	60	57	57	0	0
72					78	60	Berm....	64	65	66	0	0
73	Counter-sloping at ⅕	26	7$\frac{11}{30}$	335$\frac{3}{16}$	[illegible]	0	....	89	87	86	0	0
74					42½	30	No Berm	[illegible]	[illegible]	[illegible]	0	0
75					43	30	Berm....	72	73	73	0	0
76					43	60	No Berm	87½	74½	81½	0	0
77					43	60	Berm....	91½	85½	89½	0	0
78	Counter-sloping at ⅕	26	8	364	96	0	....	89	87	86	0	0
79					96	30	No Berm	68	62	67	0	0
80					94	30	Berm....	72	73	73	0	0
81					93	60	No Berm	87½	74½	81½	0	0
82					97	60	Berm....	91½	85½	89½	0	0
83	Sloping............	26	8	364	96	0	....	93	89½	90½	0	0
84					96	30	No Berm	69	71½	72	0	0
85					94	30	Berm....	72	77	74	0	0
86					93	60	No Berm	82	68	74	0	0
87					97	60	Berm....	103	80	91	0	0
88	Leaning at ⅕........	26	8	364	90	0	....	69	71	77	0	0
89					87	30	No Berm	65	57	61	0	0
90					81	30	Berm....	78	68	59	0	0
91					86	60	No Berm	68	67	68	0	0
92					82	60	Berm....	94	78	70	0	0
93	Counter-sloping at ⅕	26	8	364	53½	0	....	89½	90½	100	0	0
94					54½	30	No Berm	117½	83½	85½	0	0
95					58½	30	Berm....	96½	94½	101	0	0
96					59	60	No Berm	101	118½	111½	0	0
97					57½	60	Berm....	140	139	138	0	0
98	Sloping at ⅕........	26	10	455	136	0	....	136	136	0	0	0
99					138	30	No Berm	132	125	132	0	0
100					125	30	Berm....	140	125	140	0	0
101					134	60	No Berm	133	125	129	0	0
102					138	60	Berm....	125	129	0	0	0
103	Counter-sloping at ⅕	26	10	455	87	0	....	160	164	164	0	0
104					94	30	No Berm	181	181	181	0	0
105					90	30	Berm....	163	160	160	0	0
106					88	60	No Berm	174	181	174	0	0
107					92	60	Berm....	157	178	180	0	0
108	Sloping at ⅕........	26	12	546	182	0	....	194	195	188	0	0
109					181	30	No Berm	188	181	187	0	0
110					181	30	Berm....	181	181	0	0	0
111					181	60	No Berm	181	195	208	0	0
112					177	60	Berm....	184	182	0	0	0
113	Counter-sloping at ⅕	26	12	546	129	0	....	266	321	265	0	0
114					132	30	No Berm	258	287	237	0	0
115					132	30	Berm....	223	322	286	0	0
116					131	60	No Berm	279	237	351	0	0
117					126	60	Berm....	335	293	349	0	0

TABLE—*continued.*

Number of Experiments.	Nature of the Revetments.	Without Shingle.				With Shingle.						
		The Revetments.				Height of Shingle.	Berm equal to mean thickness.	Stability in trials.				
		Height.	Mean thickness.	Weight.	Stability in various trials.			1st	2d	3d	4th	5th
		inches	inches	lbs.		inches						
118	Sloping at $\frac{1}{5}$........	26	20	910	441	0	....	461	510	475	0	0
119					447	30	No Berm	615	629	657	0	0
120					448	30	Berm....	482	463	476	0	0
121					457	60	No Berm	657	637	671	0	0
122					444	60	Berm....	469	482	482	0	0
123	Counter-sloping at $\frac{1}{5}$	26	20	910	363	0		550	517	546	0	0
124					370	30	No Berm	727	755	738	0	0
125					367	30	Berm....	685	573	629	0	0
126					349	60	No Berm	734	629	671	0	0
127					379	60	Berm....	615	573	570	0	0
128	Sloping at $\frac{1}{5}$........	52	8	728	120	0	....	45	45	0	0	0
129					118	24	No Berm	0	0	0	0	0
130					119	30	Berm....	15$\frac{1}{2}$	14$\frac{1}{2}$	13$\frac{1}{2}$	0	0
131					118	35	Ditto....	0	0	0	0	0
132					118	36	Ditto....	0	0	0	0	0
133	Counter-sloping at $\frac{1}{5}$	52	8	728	62	0	....	0	0	0	0	0
134	Sloping at $\frac{1}{5}$........	52	10$\frac{2}{5}$	946$\frac{1}{2}$	161	0	....	103	99	94	0	0
135					163	30	No Berm	34	31	37	0	0
136					165	30	Berm....	66	73	70	0	0
137					164	60	No Berm	56	57	55	0	0
138					164	60	Berm....	76	82	76	0	0
139	Counter-sloping at $\frac{1}{5}$	52	10$\frac{2}{5}$	964$\frac{1}{2}$	96	0	....	64	61	67	0	0
140					94	30	No Berm	28$\frac{1}{2}$	36$\frac{1}{2}$	42$\frac{1}{2}$	0	0
141					96	30	Berm....	79$\frac{1}{2}$	74$\frac{1}{2}$	65$\frac{1}{2}$	0	0
142					95	60	No Berm	45$\frac{1}{2}$	80$\frac{1}{2}$	83$\frac{1}{2}$	0	0
143					95	60	Berm....	83$\frac{1}{2}$	84$\frac{1}{2}$	84$\frac{1}{2}$	0	0
144	Rectangular........	52	12	1092	122	0	....	17$\frac{1}{2}$	20$\frac{1}{2}$	18$\frac{1}{2}$	0	0
145					123	10	No Berm	0	0	0	0	0
146					122	12	Ditto....	0	0	0	0	0
147					124	24	Berm....	0	0	0	0	0
148					123	27	Ditto....	0	0	0	0	0
149	Leaning at $\frac{1}{5}$.......	52	12	1092	205	0	....	170	171	188	0	0
150					209	30	No Berm	96	93	125	0	0
151					227	30	Berm....	104	127	138	0	0
152					206	60	No Berm	69	96	115	0	0
153					234	60	Berm....	106	101	148	0	0
154	Sloping at $\frac{1}{5}$........	52	13	1183	259	0	....	210	182	184	0	0
155					248	30	No Berm	131	139	125	0	0
156					256	30	Berm....	176	180	178	0	0
157					262	60	No Berm	124	117	125	0	0
158					269	60	Berm....	202	199	199	0	0
159	Counter-sloping at $\frac{1}{5}$	52	13	1183	168	0	....	160	168	185	0	0
160					168	30	No Berm	78	101	107	0	0
161					157	30	Berm....	148	160	125	0	0
162					167	60	No Berm	150	143	139	0	0
163					166	60	Berm....	181	165	158	0	0
164	Sloping at $\frac{1}{5}$........	104	29$\frac{7}{16}$	5363	1263	0	....	1158	1086	1218	0	0
165					1263	52	No Berm	1056	1046	0	0	0
166					1280	52	Berm....	1183	1064	1176	0	0
167					1345	104	No Berm	1154	1070	1056	0	0
168					1325	104	Berm....	1294	1280	1322	0	0

TABLE.—*continued.*

Number of Experiments.	Nature of the Revetments.	Without Shingle.				With Shingle.						
		The Revetments.				Height of Shingle.	Berm equal to mean thickness.	Stability in trials.				
		Height.	Mean thickness.	Weight.	Stability in various trials.			1st	2d	3d	4th	5th
		inches	inches	lbs.		inches						
169	Counter-sloping at $\frac{1}{5}$	104	$29\frac{7}{16}$	5363	734	0	....	1203	1235	1196	0	0
170					730	52	No Berm	1068	1019	1075	0	0
171					744	52	Berm ...	1182	1189	1042	0	0
172					760	104	No Berm	1003	956	1045	0	0
173					756	104	Berm ...	1107	1070	1098	0	0
174	* Sloping at $\frac{1}{5}$........	26	$5\frac{1}{5}$	282	60	0	No Berm	73	73	71	0	0
175					58	30	Berm ...	59	58	61	0	0
176					59	30	....	59	62	60	0	0
177					60	60	No Berm	66	58	69	0	0
178					57	60	Berm ...	71	68	70	0	0
179	* Counter-sloping at $\frac{1}{5}$	26	$5\frac{1}{5}$	282	34	0	....	62	73	68	0	0
180					36	30	No Berm	54	49	51	0	0
181					35	30	Berm ...	63	60	62	0	0
182					34	60	No Berm	60	66	63	0	0
183					35	60	Berm ...	68	61	65	0	0
184	* Sloping at $\frac{1}{5}$........	26	$6\frac{1}{2}$	$354\frac{1}{8}$	85	0	....	115	116	116	0	0
185					86	30	No Berm	117	123	119	0	0
186					85	30	Berm ...	118	119	123	0	0
187					85	60	No Berm	126	136	147	0	0
188					86	60	Berm ...	118	126	123	0	0
189	* Counter-sloping at $\frac{1}{5}$	26	$6\frac{1}{2}$	$354\frac{1}{8}$	51	0	....	110	117	121	0	0
190					52	30	No Berm	129	137	134	0	0
191					53	30	Berm ...	127	132	126	0	0
192					52	60	No Berm	156	149	145	0	0
193					52	60	Berm ...	115	125	119	0	0
194	* Sloping at $\frac{1}{5}$........	26	13	$1419\frac{1}{2}$	349	0	....	579	584	576	0	0
195					349	30	No Berm	576	594	584	0	0
196					347	30	Berm ...	587	573	580	0	0
197					348	60	No Berm	569	547	559	0	0
198					349	60	Berm ...	555	559	547	0	0

* In the last five experiments, counterforts were connected with the revetments, from 4 to 10 inches in length, width $4\frac{1}{2}$ inches, and 18 inches distant from each other.

heoretical vestiga- ns.

182. In applying the preceding experimental results to the practical solution of the problem of revetments, we shall, in the first place, render the investigation general, by representing the height of the wall by h', the height of the earth which is to be supported by h, the specific gravity of the former by s', and of the latter by s; let us also call the length of the slope, as for example BE (fig. 113) $= l$; and the breadth of it at top DE $= b$; and the ratio of the pressure to the friction, as $1 : f$.

If, now, we consider the part of the bank DBE as a solid, supported on the plane BE, and call the weight of it W; we know, that independent of the friction between the surfaces, the weight necessary to support this body on the plane will be found by the following proportion, viz.

$l : h :: \mathrm{W} : \dfrac{h\,\mathrm{W}}{l} =$ the force or weight which, acting at the centre of gravity G, in the direction FG, would maintain the solid DEB in equilibrio; and let us suppose that, experimentally, this weight is found to be reduced by friction to $\dfrac{h\,\mathrm{W}}{l\,f}$. Now, as we may suppose this force to be applied at any point in the line of its direction, let us conceive it to be applied at the point I of a line IA, drawn from the point A of the revetment perpendicular to the line of direction GI; then IA will denote the lever by which the weight $\dfrac{h\,\mathrm{W}}{l}$ acts, in order to turn the wall about the point A as a fulcrum.

And since the triangles AIK, LKF, and EDB are similar, we have, by denoting the breadth of the wall at bottom by x,

$$b : h :: x : \frac{hx}{b} = \text{LK}.$$

And since, by the property of the centre of gravity of a triangle, FB or AL = $\frac{1}{3}$ of DB, or $\frac{1}{3}$ of h, we have again

$$\frac{h}{3} - \frac{hx}{b} = \frac{bh - 3hx}{3b} = \text{KA};$$

also,

$$\text{BE} : \text{DE} :: \text{KA} : \text{IA};$$

or,

$$l : b :: \frac{bh - 3hx}{3b} : \frac{bh - 3hx}{3l} = \text{IA}.$$

And consequently, including the reduction from friction,

$$\frac{bh - 3hx}{3l} \times \frac{hW}{lf}$$

will denote the momentum of the bank, or its effect to turn the wall as a lever about its lowest point; and which must be counterbalanced by the resistance of the wall: this resistance will consist, first, of its natural weight supposed to be collected in its centre of gravity, and acting as on a lever; the length of the latter being determined by the distance of that point of its base on which a perpendicular, demitted from the centre of gravity, falls from the fulcrum at A; and the resistance of cohesion of the whole section of the wall at the same point.

If we denote the weight of the wall by W′, and call the distance of the perpendicular, demitted from the centre of gravity of the wall, from A = y; the resistance of the wall, as arising from its weight, will be $W'y$: and if C be the constant quantity which denotes the cohesion of the wall on an area of a square foot, we shall have, from the theory of the resistance of solids illustrated in the preceding chapter, $C a d^2$ for the resistance of any column of the same materials whose breadth is a, and depth d; these dimensions being all supposed to be estimated in the same linear units, as for example, in feet. In the present case it will be sufficient to take $a = 1$; that is, to estimate the pressure of 1 foot of the bank in length on 1 foot of the wall; and from our preceding supposition we have $d = x$: whence Cx^2 will denote the resistance from the effect of cohesion, independent of its weight; and $W'y$ will, as we have seen, denote the resistance arising from its weight, independent of its cohesion; consequently

$$Wy + Cx^2$$

will represent the entire effort of the wall to resist being turned upon the point or fulcrum A; we have, therefore, in case of an equilibrium between the push of the bank and the resistance of the wall,

$$\frac{bh - 3hx}{3l} \times \frac{hW}{lf} = W'y + Cx^2;$$

where all the quantities except x being supposed known, this may be immediately determined, by the solution of a quadratic equation.

This equation is general, whatever may be the form of the revetment; but it will require certain modifications, depending upon the latter consideration, in order to find substitutes for W′ and y, which are obviously unknown quantities, till the dimensions of the wall are determined. It may also be observed, that the lever IA was determined by supposing FB, or LA = $\frac{1}{3}$ DB, or $\frac{1}{3}h$; which is only correctly true when the sliding solid is a triangle, viz. when it is a scarp revetment: in the case of counter-scarp revetments, with or without berms, as in figs. 114 and 115, the direction of the centre of gravity of the sliding solid may not cut the wall at $\frac{1}{3}$d of its height; we shall therefore render our equation more general by substituting, instead of FB = $\frac{1}{3}h$,

$$\text{FB} = \frac{1}{n}h;$$

in which case it becomes

$$\frac{bh - nhx}{nl} \times \frac{hW}{lf} = W'y + Cx^2.$$

Rem

183. Before we proceed to show the application of this equation to the practical determination of the thickness of revetments, we propose to offer a few remarks relative to the method of resolving the forces that we have adopted; which, as we have said above, appears to have been first practised by M. Rondelet, and scornfully rejected by M. Mayniel, as incorrect and unscientific.

A variety of methods have been proposed relative to the determination of the resultant and direction of the forces of the particles forming the solid DBE; some authors having estimated it by the power or force which is requisite when acting horizontally against the centre of gravity, as M G, to support the body on the plane; and consequently, these have found the point of application to be at $\frac{2}{3}$ds of the height of the bank from the bottom of the revetment: others have found that point to be at one half of the height; and others again at $\frac{1}{3}$d. The latter has been the most general determination, and is obviously the necessary result of a correct theoretical examination and valuation of the direction of the forces; it has also been experimentally verified by Gauthey, as we have stated in article 180; but it is necessary to observe, that this refers to the interior face of the revetment. Still, however, a singular error was committed in the resolution of the resulting force, which was supposed to be made at the point F; so that KF being taken to denote the direction and intensity of the thrust of the bank at F, this was resolved into a horizontal and vertical force at that point, which we may denote by LF and FH; but the latter of these was rejected as having no efficacy, either in causing the wall to turn about the point A, or in resisting that motion: this would have been true had the wall been merely a line without breadth, because then the points A and B would have coincided, and the force FH would have had its direction passing through the fixed point B: in giving to the line AB, however, any dimension, or length, it is clear that while the product of LF × LA denotes the efficacy of that force to turn the wall about the fulcrum A; FH × AB will represent the contrary or opposite moment of the force FH, to resist that motion; and consequently, when these products are equal, the wall will have the same stability as before any earth was thrown at its back: but when the former product is the greater, the stability will be diminished; and when the latter is the greater, the stability will be increased: results which were found to obtain in various practical experiments reported by Col. Pasley, in his work above referred to.

We shall come to the same conclusion if we resolve our forces at the point K; in which case the vertical force, having its direction passing through the fixed point A, will truly become ineffective, and there will remain the product KH × KA, for the moment of the force by which the bank tends to overturn the wall: but here again it is obvious, that if the earth at the back of the wall be only so high, that the line GF produced meet the base of the wall in the point A, the stability is the same as before any pressure took place: and if the line FG produced cut the vertical CA, the stability will be diminished; but if it meet any point in the base AB it will be increased. We have before observed, that these deductions are exactly conformable to the experimental results of Col. Pasley; and as to the point which we have assumed, or rather theoretically determined, for our point of application, it has been verified by the experiments of Gauthey; and therefore, thus far experiment and theory go hand in hand; which is always satisfactory to observe, particularly in cases where they have hitherto been found to give incomparable and anomalous results.

184. It remains now for us to reduce our preceding general equation to the forms requisite for the practical solution of problems, relative to the proper dimensions of revetments, for the support of banks, according to the nature of the materials of which they are composed, and the figure of their transverse sections.

Rectangular revetments. Here, denoting the specific gravity of the wall by s'; h' being its height, and x the breadth at bottom, we have $y = \frac{1}{2}x$, $W' = x h' s'$; and consequently

$$W'y = \tfrac{1}{2} x^2 h' s'.$$

We may here also substitute $W = \frac{1}{2} h b s$, and $n = 3$; which values being substituted in our general equation

$$\frac{bh - nhx}{nl} \times \frac{hW}{lf} = W'y + Cx^2,$$

it becomes

$$\frac{bh - 3hx}{3l} \times \frac{h^2bs}{2lf} = \frac{x^2h's'}{2} + Cx^2;$$

or

$$(3l^2fh's' + 6l^2fC)x^2 + 3h^3bsx = h^3b^2s;$$

or

$$x^2 + \frac{h^3bsx}{l^2fh's' + 2l^2fC} = \frac{h^3b^2s}{3(l^2fh's' + 2l^2fC)}.$$

Let us, for example, suppose a rectangular revetment of the same height as the bank; and that the natural slope of the latter is 45°; that the specific gravity of the wall to that of the earth is as 156 to 125: let us also suppose the friction to reduce the force to one half, or $f = 2$; also $C = 500$; and the height of the bank twelve feet: to find the requisite thickness of the revetment.

Here $h' = h = 12$; also $b = 12$; and $l = \surd 288$; $s' = 156$; $s = 125$; $C = 500$; and $f = 2$: and substituting these numbers in the above equation, it becomes

$$x^2 + 1{\cdot}568\,x = 6{\cdot}272:$$

whence $x = -{\cdot}784 \pm \surd({\cdot}784^2 + 6{\cdot}272) = 1{\cdot}838$ feet, the thickness sought.

In a similar manner, our general equation may be converted into a numerical one; computing, first, the point of the centre of gravity of the sliding part (which will require a particular process when there is a parapet, or when the revetment has a counter-slope); and thence again, the length of the lever AI; or otherwise the point in which GF, drawn parallel to the slope, meets the vertical DB; whence the value of n may be found.

Again, from the proportion between the base and top of the wall, estimate the place of its centre of gravity: and hence also, the value of y and W' in terms of x; then making these substitutions, and introducing the given numerical values of the other quantities, a quadratic or cubic equation will be obtained; from which, in all cases, the value of x may be determined.

The above is, we conceive, the simplest solution that the problem of revetments will admit of: and if, by means of experiments, the value of C were ascertained for different kinds of mortar and cement; as also the value of f, for other soils besides shingle and sand, by a course of experiments somewhat similar, but not exactly the same as those of Col. Pasley; we are convinced that entire confidence might be placed upon the results thus obtained. It is proposed, however, in concluding this article, to give, as we have promised, a sketch of the theory of M. Coulumb, which is generally considered by French mathematicians, as containing the true principles of solution; a point which we do not intend to dispute: all we have said, or intend to say on the subject is, that it is founded on data which are very difficult to obtain from experiment, or indeed (we think we might add) impossible to obtain with accuracy; and as such can only be considered as a mere analytical speculation.

Theory of Coulumb.

185. M. Coulumb has considered, in this problem, the profile of a triangular prism of earth as representing a solid right angled triangle; one of the sides of which is vertical, and whose hypothenuse touches an inclined plane, down which it has a tendency to slide, arising out of its own weight or gravity; but that this tendency is counteracted or destroyed, and the solid maintained in equilibrio by a horizontal force, and by its cohesion and its friction along the inclined plane. He then obtains, from the principles of Statics, the force requisite for producing an equilibrium; observing, that in general, earths being homogeneous, they may separate in the case of rupture, according to a right line, or in any curve line whatever; and consequently, if we can find, amongst all the surfaces described in an indefinite vertical plane, that surface, which solicited by its gravity, and retained by its friction and cohesion, requires, in order to establish its equilibrium, to be supported by a horizontal force which is a maximum; the problem of revetments, will be completely resolved: for it is evident that every other figure or solid would require for its equilibrium, a less force than that which is supposed to be already determined. But as it had always been found from observation, that a bank, of the kind we are considering, slides down a line exactly or very nearly straight; M. Coulumb confined his researches only to the case of a right line; viz. he endeavoured to ascertain, in an indefinite vertical plane of a given height, that right angled triangle which required the greatest horizontal force to retain it in equilibrio.

This being premised, let FEAD (fig. 116) represent the wall, or a revetment, behind which is formed an indefinite *remblai*, or bank of earth CDAB, of which the triangular prism, represented by the profile DGA, would detach itself, and slide down on the hypothenuse GA, supposing the opposing obstacle FDAE were removed; this latter we may conceive to be a force Q applied

Theory of Coulumb.

to the wall, and of which the direction is perpendicular to DA. The prism or triangle is solicited by its gravity or weight P, to glide down GA; but it is retained by the force Q, by its cohesion on GA, and by its friction on the same plane. Now if we resolve each of these forces into two others, of which one represents the perpendicular pressure destroyed by the resistance of GA, and the other acting in a direction parallel to the same line or plane; we shall have for the components of P, the forces represented by PI and IH; and for those of Q, the lines QL and LH. Consequently the triangle can have no motion, except that which is due to IH; but LH, acting in the contrary direction, or from A towards G, opposes itself to IH, and is moreover increased by the cohesion and by the friction; the equation of equilibrium will therefore be

$$\text{IH} = \text{LH} + \text{cohesion} + \text{friction}.$$

Let AD $= h$, DG $= x$, AG $= \sqrt{(h^2+x^2)}$, PH $=$ P: QH $=$ Q, and we shall have

$$\text{AG} : \text{DG} :: \text{PH} : \text{PI} = \frac{\text{P}x}{\sqrt{(h^2+x^2)}};$$

Also $$\text{AG} : \text{PH} :: \text{DA} : \text{HI} = \frac{\text{P}h}{\sqrt{(h^2+x^2)}},$$

$$\text{AG} : \text{DA} :: \text{QH} : \text{QL} = \frac{\text{Q}\,h}{\sqrt{(h^2+x^2)}}.$$

Now if the density of the earth is denoted by δ, then $\text{P}=\frac{\delta\,h\,x}{2}$; and if γ represent the cohesion on any unit of surface, this cohesion having place on the whole line GA, the total cohesion will be $\gamma\sqrt{(h^2+x^2)}$. From experiments, elsewhere stated, the friction is proportional to the pressure, and is independent of the extent of surface; $\frac{1}{f}$ will therefore express the constant ratio of it to the pressure; but the perpendicular pressure on the plane GA is equal to

$$\text{QL} + \text{PI} = \frac{\text{Q}h + \text{P}x}{\sqrt{(h^2+x^2)}}.$$

Consequently $\frac{\text{Q}h + \text{P}x}{f\sqrt{(h^2+x^2)}}$, will be the expression for the friction.

From these data, there results the following equation of equilibrium:

$$\frac{\text{P}h}{\sqrt{(h^2+x^2)}} = \frac{\text{Q}x}{\sqrt{(h^2+x^2)}} + \frac{\text{Q}h+\text{P}x}{f\sqrt{(h^2+x^2)}} + \gamma\sqrt{(h^2+x^2)}.$$

And consequently

$$\text{Q} = \frac{f\delta h^2 x - \delta h x^2 - 2\gamma f(h^2+x^2)}{2(fx+h)}, \text{ or}$$

$$\text{Q} = \frac{\frac{\delta h x}{2}\left(h - \frac{x}{f}\right)\gamma(h^2+x^2}{x+\frac{h}{f}}. \qquad (1)$$

It only remains now to determine, from the method of maxima et minima, that triangle, of all those that can be formed in the indefinite plane CDAB, which exercises the greatest pressure. Putting therefore the latter expression into fluxions, and equating it with zero, we obtain

$$x^2(-2f\delta h - 4f^2\gamma) - (4\delta h^2 + 8fh\gamma)x + h^2(2f\delta h + 4f^2\gamma) = 0.$$

Whence

$$x = -\frac{h}{f} \pm h\sqrt{\left(1+\frac{1}{f^2}\right)} \qquad (2)$$

Substituting this value of x, in the above equation (1), we shall have

$$\text{Q} = h^2\frac{\left\{-\frac{\delta}{f} - \frac{\delta}{f^3} \pm \frac{\delta}{2}\sqrt{\left(1+\frac{1}{f^2}\right)} \pm \frac{\delta}{f^2}\sqrt{\left(1+\frac{1}{f^2}\right)}\right\}}{\sqrt{\left(1+\frac{1}{f^2}\right)}}$$

$$-\gamma h\frac{\left\{\frac{2}{f^2} + 2 \mp \frac{2}{f}\sqrt{\left(1+\frac{1}{f^2}\right)}\right\}}{\sqrt{\left(1+\frac{1}{f^2}\right)}}.$$

Making now

$$\frac{-\frac{\delta}{f} - \frac{\delta}{f^3} \pm \frac{\delta}{2}\sqrt{\left(1+\frac{1}{f^2}\right)} \pm \frac{\delta}{f^2}\sqrt{\left(1+\frac{1}{f^2}\right)}\Big\}}{\sqrt{\left(1+\frac{1}{f^2}\right)}} = a,$$

$$\frac{\frac{2}{f^2} + 2 \pm \frac{2}{f}\sqrt{\left(1+\frac{1}{f^2}\right)}}{\sqrt{\left(1+\frac{1}{f^2}\right)}} = b;$$

we shall have

$$\text{Q} = ah^2 - b\gamma h. \qquad (3)$$

The value of x, determined in equation (2), shows that the dimensions of the triangle which produces the greatest pressure, depends only on the friction; and that the cohesion of the soil has no influence in that determination, as no function of γ enters into the expression.

If the friction were zero, whatever might be the cohesion, then we should have $x = h$; that is, the triangle producing the greatest thrust would be isosceles.

The expression $\text{Q} = a\,h^2 - b\gamma\,h$, is that of the horizontal force capable of resisting a triangle exercising the maximum thrust on the vertical height (h): but in order to determine what the horizontal force must be that is capable of resisting any other height, we must put the equation into fluxions, with reference to h as variable, and we shall have

$$\dot{\text{Q}} = 2ah\dot{h} - b\gamma\dot{h}$$

for the expression of the different pressures on DM and DA; and calling the total height DM $= h'$, this elementary horizontal pressure, acting at the extremity of the lever AM $= h' - h$, will have for its moment about the point O, the base of the revetment

$$(2ah\dot{h} - b\gamma\dot{h})(h'-h) = (2ahh'\dot{h} - bh'\gamma\dot{h} - 2ah^2\dot{h} + b\gamma h\dot{h}).$$

And taking the fluent, we obtain for the sum of the moments on the height h, about the point O, the expression

$$a\left\{h^2h' - \frac{2h^3}{3}\right\} + b\left\{\frac{\gamma h^2}{2} - h'\gamma h;\right\}$$

and when $h' = h$, we shall have the sum of the moments on the total height h' equal to

$$\frac{ah'^3}{3} - \frac{b\gamma h'^2}{2}.$$

The above contains the whole theory of M. Coulumb, independent of what relates to the friction which the triangle suffers against the interior face of the revetment at the moment of rupture; a consideration that effects in no essential manner the practical application of the formulæ, and which may therefore be looked upon rather as a subject of curiosity than utility.

It is obvious, that this solution of the theorem of revetments, depends upon the correct numerical values of f and γ, as determined from a series of well conducted experiments; and we must add, that we have very considerable doubt whether, in any case, it would be possible to ascertain the latter co-efficient, in a manner sufficiently correct and determinate to be employed with confidence in the practical construction of these works.

DYNAMICS.

§ I. *Definitions, &c.*

itions. 1. We have already defined *Dynamics* to be that branch of Mechanics, by which we investigate and compute the action of solid bodies upon each other, when the result of that action is motion.

otion. 2. *Motion* is a simple idea; which does not, therefore, require or even properly admit of definition: we may, however, in conformity to long established custom, say, that it is that state in which a body is, when passing from one point of space to another. When equal spaces are passed over in equal times, it is called *uniform* motion; and it is estimated or measured by the space which a body passes over in any assumed unit of time, which is commonly one second; or generally, by the quotient arising from dividing the space passed over, by the time of description, and this measure of motion is called *velocity*.

Hence, if t denote the time or the number of seconds a body has been in uniform motion, s the space in feet or any other linear units described in that time, and v the velocity; we have from the above definition

$$v = \frac{s}{t}{}^{*} \qquad (1)$$

When the motion of a body in any equal portions of time, receives equal increases of velocity, the motion is said to be *uniformly accelerated;* and if it receive in equal portions of time equal decreases of velocity, it is called *uniformly retarded motion;* and if the increase or decrease of velocity be not equal in all equal portions of time, it is called *variable motion.*

The preceding equation, viz.

$$v = \frac{s}{t},$$

involves in itself all the conditions and laws of uniform motion; for from this we have

$$v = \frac{s}{t},\ t = \frac{s}{v},\ \text{and}\ s = t\,v; \qquad (2)$$

that is, the velocity *is as*, or is equal, to the space divided by the time; the time is equal to the space divided by the velocity; and the space equal to the product of the velocity and time.

lemark. *Note.* As time, space, velocity, &c. are quantities incomparable with each other, it has been usual to distinguish the relation between them by the expression *is as:* thus, it is said, the time *is as* the space divided by the velocity; the space *is as* the time into the velocity, &c. We see, however, no occasion for this kind of paraphrase, any more in Mechanics, than in mensuration, and other mathematical sciences: lines, surfaces, and solids, are as incongruous as time, space, and velocity; and yet no author thinks it necessary to say, that the area of a rectangle *is as* the product of its length and breadth, or that the solidity of a prism *is as* the product of the base and altitude. In these, and various other instances, we never scruple considering, or at least saying, that a thing is equal to the abstract number by which it is measured; nor can we see any reason for a greater restriction in Mechanics; and therefore, as it throws considerable facility into our investigations, we shall constantly employ the term *equal*, and the sign of equality, when we have occasion to make any comparison between a quantity, and the product or quotient of the abstract units of other quantities by which it is measured: and the reader will always bear in mind, that in these and all similar cases nothing farther is meant, than, that the abstract numbers which express the number of units of each are equal; or that the one is any multiple or part of another.

This expression of equality obviously involves in it the conditions of proportionality: we shall, therefore, not exclude by this means the relative connection implied by the words *is as;* for since $v = \frac{s}{t}$, it necessarily follows, that when the times are equal, the velocity is proportional to the space; or if the spaces are equal, the velocity is reciprocally proportional to the time; and so on in other similar cases.

Of forces. 3. Forces, as we have seen (art. 2, STATICS), are divided into *momentary* or *impulsive;* or those which act but for an instant, after which they cease to have effect; and *uniform* forces, which act constantly, or incessantly, with the same continued energy: and *variable* forces, which also act incessantly, but with unequal degrees of energy.

These are also called *accelerating* or *retarding* forces, according as they are employed in the production or destruction of motion; but we must be careful not to confound any that fall under the former general designation, with what we call absolutely *the accelerating force;* which latter, but for the inconvenience of the paraphrase, ought rather, perhaps, to be expressed by the term *rate of acceleration:* but we shall have occasion to revert to this subject in a subsequent article.

Of inertia. 4. Every body has a tendency to remain in its present state, whether it be in motion or at rest; that is, a body which is in motion will continue to move uniformly, unless its motion be destroyed by the operation of some external force; and a body at rest will continue for ever in that state without the operation of some power or agency to put it in motion. This inert and inherent property of matter, which renders it incapable of changing its present state, we call *inertia.*

With regard to the latter part of the preceding definition; viz. that matter at rest will continue for ever in that state unless motion be impressed upon it by some external force, it is a fact which we learn from daily and hourly experience: but the former, in which we assume that a body in motion will continue for ever to move with the same velocity, has not the same evidence, nor is it ever so readily admitted by a student; because, in no case do we ever see it verified by observation. It is obvious, however, that as we diminish or remove the causes which obstruct a body's motion, the longer and the more uniform that motion will become: but as these preventing causes can never be entirely annihilated, we are unable to verify from actual experiment or observation this property of matter: we cannot, however, at the same time, doubt of its existence; because, although we are unable to reduce it to observation, or demonstrate it by actual experiment, we see the reason why it does not obtain, and the causes which prevent it: we see also that as these causes are diminished, the body approaches nearer and nearer to that state we have supposed; and whence we may conclude, that if they could be entirely removed, the motion of the body would continue uniformly the same.

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* We may observe here, and it is important that the reader should constantly bear it in mind, that when we speak of dividing space by time, or of multiplying velocity and time, &c. it must be understood to imply simply, the division or multiplication of the abstract numbers, by which the units of those quantities are denoted

It follows from the above definitions, that if a body at rest be acted upon by any momentary impulse, which puts it into motion, the velocity which it thereby acquires will be uniform.

§ II. *Of motion uniformly varied.*

Of accelerating forces and velocity.

5. We have seen above, that a momentary or impulsive force, generates in a body a velocity which, independent of external causes, would continue uniformly the same. If, after any interval of time, another impulse, equal to the former, be impressed; an equal accession of motion will be generated, and the velocity will be doubled. The double velocity, like the former, would continue for ever, independent of any farther external operating causes; but if, after another equal interval of time, a third equal impulse be impressed, an equal additional increase of motion will be generated, and the velocity will be trebled; and so on for any number of impulses.

Whence it follows, that the velocity will be always proportional to the number of impulses; or, since these are supposed to take place at equal intervals, proportional to the time: and as this conclusion is wholly independent of the duration or lengths of the intervals at which the several impulses act, it will be true, if we suppose them to be indefinitely small; or, which is the same, if we suppose the force to be constantly and incessantly acting: and hence we conclude, *that the velocities generated by constant and uniform forces, are uniformly accelerated; and that, with the same force, the velocity is directly as the time of acting.*

It is obvious, from what is above stated, that if, after any time t, the uniform force ceases to act; the velocity v, which the body has acquired, will continue uniform: and it is this which we denominate the *last acquired velocity*, or simply the *velocity*.

We have here merely considered the relation between the time and velocity, independent of any particular or comparative measure of the intensity of the force, by which the motion is produced; this, which will be a very important consideration in our subsequent investigations, is treated of under article 11.

6. Let us now endeavour to investigate the relations between the time, velocity, and space, described by a body, when urged by a constant or uniform accelerating force.

Relations of spaces, times, &c.

For this purpose, let t denote any time, and v the velocity acquired at the end of that time; and let us conceive the time t to be divided into any number of equal portions m; and suppose that the body only receives its accessions of velocity at the beginning of each of those equal intervals of time. Then it follows, from the preceding article, that at the ends of the successive periods of time

$$\frac{t}{m}, \frac{2t}{m}, \frac{3t}{m}, \frac{4t}{m}, \text{ \&c. } \frac{mt}{m};$$

the corresponding velocities (since these are as the times) will be

$$\frac{v}{m}, \frac{2v}{m}, \frac{3v}{m}, \frac{4v}{m}, \text{ \&c. } \frac{mv}{m};$$

and since, by the supposition, each of these velocities will continue uniform during the same equal interval of time $\frac{t}{m}$, and by art. 2, in uniform motions the space is equal to the product of the time and velocity; it follows, that if we multiply each of the above velocities by the time of its duration $\frac{t}{m}$, we shall have the several spaces passed over in the successive intervals; the sum of which will therefore be the whole space passed over in the time t. Hence, calling this space s, we shall have

$$s = \frac{tv}{m^2} + \frac{2\,tv}{m^2} + \frac{3\,tv}{m^3} + \frac{4\,tv}{m^4} + \text{\&c. } \frac{m\,t\,v}{m^2}; \text{ or}$$

$$s = \frac{tv}{m^2}\{1 + 2 + 3 + 4 +, \text{\&c. } m;\}$$

or, by summing the series,

$$s = \frac{tv}{m^2}\left\{\frac{m^2 + m}{2};\right\}$$

that is

$$s = \frac{tv}{2} + \frac{tv}{2\,m},$$

which is a general expression for the space, independent of any particular value of m; and is therefore true when m is indefinitely great; or when the interval of time $\frac{t}{m}$, is indefinitely small; that is, when the force acts incessantly. But in this case, since m is indefinitely great, the second part of the above expression, viz. $\frac{tv}{2\,m}$, is indefinitely small, and vanishes out of the equation; and we have simply

$$s = \tfrac{1}{2}\,t\,v. \qquad (3)$$

that is, *the space passed over by a body from rest, with a uniformly accelerated velocity, is equal half the product of the time and last acquired velocity.*

We have seen (art. 2), that in uniform velocities, the space is equal to the product of the time and velocity: if therefore we suppose the above velocity were to continue uniform during the time t, we should have

$$t\,v = 2\,s;$$

whence it follows, that the velocity generated in a body by an uniform force, acting during any time, is such as, supposing it to remain uniform, would carry the body over double that space in the same time.

If now we denote by g, the space which a body passes over in a unit of time (as for example, a second) when urged by a constant force, $2g$ will denote its velocity at the end of that time; and since the velocity is proportional to the time (art. 5), the velocity due to any time t, will be $2\,g\,t$, or $v = 2\,g\,t$: substituting therefore for v and t, in terms of each other, in equation (3), we shall have

$$s = \tfrac{1}{2}\, t\, v = g\, t^2 = \frac{v^2}{4g}. \qquad (4)$$

which equations exhibit all the relations between the time, velocity, and space passed over by bodies moving with uniformly accelerated velocities; the body and force being supposed constant, or proportional to each other.

§ III. *Of the descent of heavy bodies in a vacuum, by the action of gravity.*

s of fall- bodies.

7. The first and most important application of the preceding formulæ, is in the case of the descent of heavy bodies from the action of gravity near the earth's surface, which is known to be an uniform force; that is, we have demonstrated above, that in uniform forces the spaces are as the squares of the time, or as the squares of the velocity: and, as by experiment it has been ascertained, that bodies, urged by gravity, observe the same laws, we conclude that this is a constant or uniform force; and it is always proportional to the mass or body on which its action is impressed; because all bodies, whether great or small, are always observed to fall through the same space in the same time.

It may, however, be proper to observe, that in the preceding statement we must confine ourselves to the same place, and to small heights above the earth's surface; for the force of gravity being inversely proportional to the square of the distance from the centre of terrestrial attraction, which is in or near the centre of the earth, and the latter body not being a complete sphere, but an oblate spheroid, having its surfaces at different distances from the centre, the action of gravity ought to vary with the latitude; viz. it ought to be greatest at the poles, and to diminish thence to the equator, where it is the least: theory requires that this diminution should be proportional to the square of the sine of the latitude, and the prediction has been verified by experiment

On the same principles, also, it follows, that the force of gravity is different in the same place, at different heights above the earth's surface; but the greatest height to which it can ascend being very inconsiderable in comparison with the radius of the earth, we may, without sensible error, suppose it constant: at least we may proceed in our investigations, upon the supposition of it being so; and afterwards examine the case upon the strict principles to which we have above referred.

Gravity, therefore, being supposed a constant force; if we know the space through which it urges a body in any given time, as for example, one second, we shall be able to ascertain the space that the same body will pass through in any other given time; its velocity at the end of those times; and every other circumstance relative to its space, time, and velocity.

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8. It has been ascertained, by numerous and very accurate experiments, that a heavy body, in the latitude of London, falls through, in the first second of time, from the action of gravity, $16\frac{1}{12}$ feet, or 193 inches; therefore g in our preceding equations, is, in this case, equal to $16\frac{1}{12}$ feet: consequently, if we substitute this value of g, we shall have all the proposed relations expressed or involved in the equations

$$s = \tfrac{1}{2}\, t\, v = g\, t^2 = \frac{v^2}{4g}. \qquad (4)$$

Let it be proposed, for example, to determine the space through which a heavy body will fall in the latitude of London, in four seconds.

Here we have $t = 4$, and $g = 16\frac{1}{12}$; whence the formulæ

$$s = g\, t^2,$$

becomes

$$s = 16\tfrac{1}{12} \times 4^2 = 16\tfrac{1}{12} \times 16 = 257\tfrac{1}{3} \text{ feet.}$$

If the velocity at the end of that time were required, we obtain it from the formulæ,

$$\tfrac{1}{2}\, t\, v = g\, t^2, \text{ or } v = 2\, g\, t;$$

that is, $v = 2 \times 16\frac{1}{12} \times 4 = 128\frac{2}{3}$ feet, which is equal to half the space described in the same time.

If the space be given to find the time, we have

$$s = g\, t^2, \text{ or } t = \sqrt{\frac{s}{g}};$$

and if the velocity be required, we have

$$s = \frac{v^2}{4g}, \text{ or } v = 2\sqrt{g\, s}.$$

For example, let the space be 400 feet; then

$$t = \sqrt{\frac{400}{16\frac{1}{12}}} = \sqrt{\frac{4800}{193}} = 4''\tfrac{76}{77} \text{ seconds;}$$

and $v = 2\sqrt{(16\frac{1}{12} \times 400)} = 2\sqrt{6433\frac{1}{3}} = 160\frac{32}{77}$ feet.

If the velocity be given to find the time, we have

$$\tfrac{1}{2}\, t\, v = g\, t^2, \text{ or } v = 2\, g\, t, \text{ and } t = \frac{v}{2g}.$$

Suppose, for example, the given generated velocity to be 100 feet, to find the time, we shall have

$$t = \frac{100}{2 \times 16\frac{1}{12}} = \frac{1200}{386} = 3''\tfrac{21}{193} \text{ seconds.}$$

Lastly, let the space passed over during any part of the time be given, to find the time and velocity.

Suppose that a body was observed to pass over 100 feet in the last second of its descent, to determine the time, space, and velocity.

Here, if we denote the whole time of descent by x, the time prior to the last second will be $x - 1$; and the given quantity, 100 feet, is obviously the difference between the spaces passed over in the times x, and $x - 1$.

Now the space passed over in the time x is,

$$s = g\, x^2;$$

and the space passed over in the time $(x - 1)$ is

$$s' = g\,(x - 1)^2;$$

whence $s - s' = g\, x^2 - g\,(x - 1)^2 = 100$;

or $\quad 2\, g\, x = 100 + g$;

therefore $\quad x = \dfrac{100 + g}{2\, g} = \dfrac{116\frac{1}{12}}{32\frac{1}{6}} = 3''\tfrac{235}{386}$;

and the time being thus known, the space and velocity are readily determined.

Bodies projected vertically.

9. If a body be projected perpendicularly downwards with any given velocity, its motion will partake of that uniform velocity of projection, combined with the accelerating velocity arising from gravity: and if, therefore, we denote the projectile velocity by u, the spaces passed over in any time t, will be expressed by the formula

$$s = t\, u + g\, t^2; \qquad (5)$$

and the time t, when s is given, will be found from the quadratic equation

$$t^2 + \frac{u}{g}\, t = \frac{s}{g};$$

whence $\quad t = -\dfrac{u}{2g} \pm \sqrt{\left(\dfrac{u^2}{4\, g^2} + \dfrac{s}{g}\right)}$: or

$$t = \frac{-u \pm \surd(u^2 + 4sg)}{2g}. \qquad (6)$$

Suppose, for example, that a body at the height of 400 feet were projected vertically downward with a velocity of 10 feet per second, and the time of its descent were required.

Here we should have $s = 400$, $u = 10$, and $g = 16\frac{1}{12}$; whence

$$t = \frac{-10 \pm \surd(100 + 6433\frac{1}{3})}{32\frac{1}{6}},$$

the time sought.

Also, t being thus determined, the velocity will be found from the equation

$$v = u + 2gt, \qquad (7)$$

10. The same laws and formulæ as those above investigated for uniformly accelerated velocities, will apply equally to uniformly retarded velocities: that is, the velocity lost in any given time by the action of an uniform retarding force will be as the time; the space passed through as the square of the time, or the square of the velocity; and so on, the same as in accelerating forces.

Whence, a body projected vertically upwards, will ascend to the same height, as a body falling freely would descend through to generate that velocity; and the time in both cases will also be the same: wherefore, if a body be projected as above, and afterwards descend, it will pass through every point in its ascent and descent with the same velocity; that is, whatever velocity it may have in its ascent, in passing through any point, it will pass through the same point with the same velocity in its descent; and consequently the whole time of ascent and descent from any point will be double that of either.

§ IV. *On the comparison of different forces.*

Forces compared.

11. Hitherto it has not been necessary to establish any particular measure of force, because our first investigations were made simply with reference to the several relations of space, time, and velocity, while any one and the same force continued in action; and consequently the particular intensity of that force was not necessary to be considered, nor did it enter into our enquiry. On these principles, we found the relations above referred to; viz. we found that the velocity is proportional to the time; the space proportional to the square of the time; and that the space is equal to half the product of the time, and last required velocity: whence the space through which a body passes, when urged by any constant force, in one second, or any other time, being given, the space and velocity corresponding to any other time are readily determined: and as that space is known for bodies falling freely from the action of gravity, the investigations to which we have referred, were found sufficient for establishing all the several laws of falling bodies: but they are no longer applicable, at least without certain modifications, when the effects of other constant forces are to be considered; or when the force of gravity is diminished or obstructed by different counteracting causes: we must then make the intensity of the forces enter into our investigation; and in order to this, it is necessary to establish some determinate measure of them, whereby an accurate comparison may be made.

When we contemplate forces, with reference to their actual intensity, we assume, as the most natural measure of them, the product of the body or mass on which they act, into the velocity generated by their action; whether that action be momentary, or continued for any given time.

For example: if a momentary impulsive force generate in a given body any determinate velocity; we assume, that a force of double the intensity of the former, will generate a double velocity; a treble force, a treble velocity, and so on; that is, the body remaining the same, the velocity is as the force impressed. If, now, we suppose two bodies, each equal to the former, then it is obvious, that in order to generate the same velocity, we must have two forces equal to each other, one acting upon each body; or, which is the same, if the body be supposed doubled, the force must also be doubled to generate the same velocity; and consequently, if the force is the same, and the body be doubled, only half the first velocity will be produced: if the body be three times greater, the velocity will be but one-third, and so on; that is, the velocity varies as the force directly, and the body reciprocally: consequently, if f denote any impulsive force; b the body on which it acts; and v the generated velocity; we have from the above,

$$\frac{f}{b} = v, \text{ or } f = bv.$$

And since a uniform force is nothing more than a constant succession of impulsive forces, the measure of this will be also correctly exhibited by the same equations; the time during which the constant force acts, being always the same.

Accel rating defin

12. We have seen that the velocity generated by gravity is always the same in the same time, being independent of the magnitude of the body: and we therefore take the velocity simply which it generates in any determinate unit of time, as the measure of its force; and hence, when we have occasion to compare with this, the effect of any other given force, we must also make the comparison by the generated velocity; in order to which, however, the body, in many cases, necessarily enters into our investigations: that is, we must divide the intensity of the force by the mass or body moved, and this quotient is what we are accustomed to call *the accelerating force*, as affecting the velocity only; while the absolute force, or that which generates the motion, and which is denoted above by f, and is independent of the body, is called the *motive force*.

If therefore we call, as a constant standard of comparison, the accelerating force of gravity 1; and any other accelerating force ϕ; denoting by ϕ the quotient $\frac{f}{b}$, and the space passed over in one second by the action of the former by g; we shall have

$$1 : \phi :: g : \phi g;$$

the space passed over in one second by the action of the accelerating force ϕ: substituting, therefore, ϕg instead of g, in our equations 4, article 8, they become

$$s = \tfrac{1}{2} tv = \phi g t^2 = \frac{v^2}{4\phi g}; \qquad (8)$$

from which all the circumstances of space, time, and velocity, due to any given accelerating force, may be readily determined.

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13. As an example, by way of illustration, let us suppose two bodies, a and b, to be connected together by a fine line, without weight, and suspended over a fixed pulley; to determine the circumstances of the

time, space, and velocity generated in the descending body:

In the first place it is obvious, that if the bodies a and b were equal to each other, the system would be kept in equilibrio by their opposite actions, and no motion would ensue. But if one of the bodies, as for instance a, be greater than b, the difference $a - b$ will be the force by which that body has a tendency to descend, while the whole mass or bodies moved by that force, will be $a + b$; our motive force, therefore, is $a - b$, and the accelerating force $\frac{a-b}{a+b}$, or $\phi = \frac{a-b}{a+b}$.

Any numerical values being now given to a and b, ϕ will be also numerically determined; and hence the space and velocity due to any given time t.

If for example, $a = 2$ and $b = 1$, then $\phi = \frac{1}{3}$, and $\frac{1}{3} g$ will denote space which the descending weight would pass through the first second, and $\frac{1}{3} g t^2$ the space due to any time t; the velocity acquired at the end of one second will be $\frac{2}{3} g$, and the velocity due to any other time t will be $\frac{2}{3} g t$, and so on.

As another example, let us suppose a body b to be placed upon a perfectly polished horizontal plane, so that it may slide upon it without friction; and let us imagine a fine line attached to the body b, and to another body a, which hangs over the edge of the plane, and which by the action of gravity descends and draws the other body b along the plane; to determine the circumstances of their motion:

Here, since no part of the gravity of a is counteracted by that of the supported body, a will be the motive force, being that which gives motion to the system; while the mass which it moves is $a + b$, and consequently the accelerating force will be $\frac{a}{a+b}$, which, supposing the same numbers as those above, will be $\frac{2}{3}$. The space, therefore, descended in one second will be $\frac{2}{3} g$; and the space in any time t, $s = \frac{2}{3} g t^2$; the velocity generated in one second $= \frac{4}{3} g$; and the velocity due to the time t, $v = \frac{4}{3} g t$: consequently if the space were given, and the time sought, we should have $t = \sqrt{\frac{3}{2} \frac{s}{g}}$; and so on in other cases.

Atwood's machine. 14. It is on the above principles, that the instrument well known under the appellation of Atwood's Machine, is constructed, for exhibiting and examining the several phenomena and circumstances attending the fall of heavy bodies. It consists of a principal upright shaft, on the top of which is a system of wheels, so constructed and arranged as to offer very little friction, and over one of them passes a fine silk line, to which at its extremities are attached two weights, so adjusted as to their magnitude, that the difference of their weights is to their sum in the ratio of unity to some convenient number, which latter may be chosen at pleasure: suppose for example, that one of the weights is $31\frac{1}{2}$ penny-weights, and the other $32\frac{1}{2}$ penny-weights; the difference between them will be 1, and their sum 64; whence the accelerating force will be $\frac{1}{64}$; if, therefore, we take the descent from gravity to be 16 feet in the first second, the descent of the preponderating weight will be $\frac{16 \text{ feet}}{64} = 3$ inches, in one second; 12 in two seconds; 27 in three seconds, and so on: whence, by means of a long graduated rod, and a clock attached to the machine beating seconds, a person may, by following the weight with his eye, see the descending weight pass successively each of the specified divisions, and thus have an ocular demonstration of the accuracy of the preceding deductions (see fig. 1, plate 1, DYNAMICS): and by varying the ratio of the two weights, any other degree of acceleration and its consequent velocity may be exhibited.

Atwood's machine.

In order to show the degree of velocity acquired at the end of any time, the two weights are first taken equal; for example, each $31\frac{1}{2}$ penny-weights; in which state the machine is in complete statical equilibrium; and then, in order to put it into motion, a small bar about 2 inches long, and weighing one penny-weight, is laid upon that weight which is intended to be the descending one; a brass ring of about $1\frac{1}{2}$ inch in diameter being previously fixed at any proposed division (as for example at 12 inches) in the vertical line of the descending weight: then as the latter passes through the ring, the bar is left behind; and the accelerating force being thus removed, the velocity ought afterwards to be uniform. In the case we have supposed, the velocity of the body, after descending two seconds, or 12 inches, should also be 12; and therefore, if a plate or disc be placed 36 inches below the ring, or 48 inches below the top of the machine, the weight ought to reach, first, the ring in two seconds after it begins to descend; and three seconds after that, the disc below; and observing attentively, and counting the beats of the clock, this will be found to obtain in the most accurate and satisfactory manner. The principles of this machine will be readily comprehended from the preceding explanation: but for a complete and full account of its construction, and the various experiments to which it may be applied, the reader should consult Atwood's "Treatise on the Rectilinear Motion and Rotation of Bodies."

§ V. *The motion of bodies on inclined planes.*

15. We have seen, in treating of the equilibrium of bodies on an inclined plane, that the absolute weight of the body, or the natural force of gravity, may be resolved into two forces, W sin a, and W cos a; the latter denoting the resistance of the plane, the angle of its elevation being represented by a; and the former, W sin a, the force or power which acting parallel to the plane, preserves the system in equilibrio, and which, therefore, necessarily also denotes the force with which the body has a tendency to descend: W sin a is, therefore, the motive force; but W is the mass or body moved: whence $\frac{W \sin a}{W} = \sin a$ is the accelerating force; or the accelerating force with which the body descends down an inclined plane, is to the force of gravity as sin a to radius. Inclined planes.

We have, therefore, only to introduce into our equations (8), sin a g, instead of ϕ g, and all the circumstances of time, space, and velocity, will be exhibited in the following formulæ: viz.

$$s = \tfrac{1}{2} t v = \sin a g t^2 = \frac{v^2}{4 \sin a g}; \qquad (9)$$

or since, by denoting the length of the plane by l, and the height by h, we have $\frac{h}{l} = \sin a$, the above equations will become

Inclined planes.

$$s = \tfrac{1}{2}\,t\,v = \frac{h}{l}\,g\,t^2 = \frac{l\,v^2}{4\,h\,g}; \qquad (10)$$

under which latter form it is sometimes more convenient to consider them.

16. From these equations, we draw immediately the following remarkable properties relative to the descent of bodies down inclined planes. If the time of descending through the whole plane were required, we should have $s = l$; whence

$$l = \frac{h}{l}g\,t^2, \text{ or } t^2 = \frac{l^2}{h\,g}\text{ ; or}$$

$$t = l\,\sqrt{\frac{1}{hg}} = \frac{l}{h}\,\sqrt{\frac{h}{g}};$$

but the time of descending through the space h, by the free action of gravity, is $= \sqrt{\dfrac{h}{g}}$.

16. Whence the time a body is in descending down an inclined plane, is to the time in falling freely through the height of the plane, as

$$\frac{l}{h}\sqrt{\frac{h}{g}} : \sqrt{\frac{h}{g}}; \text{ or as } \frac{l}{h} : 1, \text{ or as } l : h;$$

that is, *as the length of the plane to the height.*

17. If the last acquired velocity on the plane were required, it would be found from the equation

$$s = \frac{l\,v^2}{4\,g\,h}, \text{ or } l = \frac{l v^2}{4 g\,h};$$

that is,

$$v = \sqrt{4\,g\,h} = 2\,\sqrt{g\,h};$$

which is precisely the expression for the velocity acquired by a heavy body falling freely through the space h.

Whence, the velocity acquired by a body descending down an inclined plane, is equal to that acquired by a body falling freely through the height of the plane.

18. Hence also, the space descended down an inclined plane, is to the space descended freely in the same time, as the height of the plane to the length: for the spaces are as the forces; the times being equal.

Therefore, if AC (fig. 2, plate 1, DYNAMICS) be taken to denote any plane; and the perpendicular DB be drawn; the body will descend from C to D along the plane, in the same time as it would fall freely through the height CB; because

AC : BC :: BC : DC.

As this result is wholly independent of the angle of inclination of the plane, it follows that in any right angled triangle BDC (fig. 3), having its hypothenuse BC perpendicular to the horizon, a body will descend down any of its three sides BD, BC, DC, in the same time: or, if on the diameter BC a semicircle be described, the time of descending down any of the cords BD, BE, BF, or DC, EC, FC, will be the same as the time in falling down the diameter BC.

Various problems.

19. The following series of problems and properties, which are abstracted from Bridge's Treatise of Mechanics, flow immediately from the principles above investigated.

1. Let AC (fig. 4) be an inclined plane, whose length is AC, and perpendicular height AB; draw CE at right angles to AC, and produce AB to meet it in E; upon AB, AE, describe the semicircles ADB, ACE, and join DB, CE. Then, by art. 18, the times in falling down the chords AD, AC, are equal to the times in falling the diameters AB, AE, respectively: hence the intersection D of the semicircle ADB, with the plane AC, determines the part AC, which a body describes while another would fall freely through the height AB; and the intersection E, of the perpendicular CE, with AB produced, determines the space AE through which a body would fall freely, whilst another would describe the whole length of the plane AC.

2. If it were required to determine the point D (fig. 5), so that the time in falling from a given point P to the plane CA, shall be equal to the time in falling through the height AB; through P draw PE parallel and equal to AB, and upon it describe a semicircle cutting AC in D; and join PD, which will be the line required; for the time in PD = the time PE = the time in AB; the two latter being equal.

3. To find the point P (fig. 6), such that the time in falling from it to the given point B, shall be equal to the time in falling down AC. Draw CD at right angles to BC, and AD to CA; let them meet in D, and upon CD describe a semicircle which will pass through A: draw DE parallel to CB, and produce BA to meet it in E; upon BE describe a semicircle cutting AC in P; so shall P be the point required. For the time down PB = time down EB = the time down DC = time down AC.

4. To determine the point F (fig. 7) upon the plane AC, so that the time of falling from a given point P to it shall be equal to the time through AC. Describe the semicircle DAC as in the foregoing figure, and join DP; complete the parallelogram DCPE, and upon PE describe a semicircle cutting AC in F; so shall this be the point required. For the time down PF = time down PE = time down DC = time down AC.

5. AC, AD (fig. 8), are two inclined planes having the common altitude AB; draw CE at right angles to CB, and EA at right angles to CA, and let them meet in E; on CE describe a semicircle cutting AD in P; then P is the point in the plane AD, from which a body will fall to the given point C in the same time that another body descends down the plane AC. For join PC, then the time down PC = time down EC = time down AC.

6. Let it be required to determine upon the plane AC (fig. 9) a point P, such that the time in falling from P to B, shall be equal to some given time T. Draw BD at right angles to BC, and $= g\,T^2$ (the space described by gravity in the time T); upon BD describe a semicircle cutting AC in P, which will be the point required, as is obvious.

7. AB, CD (fig. 10), are two lines perpendicular to the horizon, placed at a given distance BD from each other; join AC, and let BD and AC, when produced, meet in the point E; then if F be taken such a point that $EF^2 = AE \times EC$, and AF, CF, be joined, the time down AF will be equal to the time down CF. For since EF^2, $AE \times EC$, a circle may be described that shall pass through the points A, C, and touch BD in F; draw FG at right angles to BD, and it will be the diameter; and the time down AF = time down GF = time down CE.

8. Suppose the base BC (fig. 11) of an inclined plane to be given, and that it were required to determine the height, so that the time down the plane shall be the least possible. Draw BF at right angles to BC, and take BA = BC; complete the square ABCD, and with the centre D and radius DC describe the circle CEE′, which will evidently touch BF, BC, in the points A, C;

join AC, and draw any other chords, CE, CE′, which produce to meet BF in F, F′. Now the time down E′C = time down AC = time down EC; therefore the time down FC or F′C (being greater than the time down EC or EC) is greater than the time down AC: whence AC is the *plane* of quickest descent from vertical BF to C.

9. If it be required to find the right line of quickest descent from the plane AC to B, take CD (fig. 12) equal to CB, and join DB, then DB will be the line required. For draw DE at right angles to DB, and upon EB describe the semicircle EDB, which will touch BC, CA, in B, D; then, since no other line but BD can be drawn from B to the plane AC, which does not cut the circle EDB; DB will be the *right line* of quickest descent from AC to B.

10. To find the right line of quickest descent from a given point P (fig. 13) to the plane AC: draw PG parallel to CB, and PH parallel to AB; take GD = GP, and join PD; PD is the right line required. For draw DE at right angles to PD, and upon PD describe the semicircle PDE, which, for the same reason as in the preceding article, will touch AC, PG in D and P; and as no other line but PD can be drawn from P to the plane AC which does not cut the circle, PD will be the right line of quickest descent from P to AC.

11. Let it now be required, to find the right line of quickest descent from a given point P, to the circle ACB (fig. 14). Take O, the centre of the circle ACB; draw the diameter AB perpendicular to the horizon, and PE parallel to AB; join PB, and let it cut the circle in C; join OC, and produce it to cut PE in Q; then the triangles PQC, COB, will be similar: and since OC = OB, QC will be equal to QP: a circle therefore PCK, described with the centre Q, and radius QC, will pass through P; and since OQ joins the centres of the two circles ACB, PCK, it will pass through the point of contact: hence, as the circle PCK touches the circle ACB in the point C, it can touch it in no other point; and, because no other line but PC can be drawn from P to the circle ACB, which does not cut the circle PCK, PC will be the line of quickest descent from P to the circle ACB.

12. Lastly, suppose two sides of a triangle (one of which is parallel to the horizon) were given, and it were required, to determine the third side, so that the time of falling down it should be a *minimum*. Let BC (fig. 15) represent the longer of the two given sides, and take BG equal to the shorter of them; then, with the centre B, and radius BG, describe the semicircle GHO; draw BH at right angles to CB, and join CH: the point A, where CH cuts the semicircle GHO, determines the position of the shorter side AB; so that the time down the third side AC, shall be a *minimum*. For draw CE perpendicular to CB, and produce BA to meet it in D; then (as in the preceding article) it may be shown, that if, with the centre D, and radius DC, a circle CAE be described, it will touch the semicircle GHO in the point A: and as no other line but CA can be drawn from C to the semicircle GHO, which does not cut the circle CAE, the time down AC will be the least, when the shorter of the two given sides is in the position AB.

VI. *Of variable motions and forces.*

20. We have avoided, at present, the introduction of considerations relative to the fluxional or differential calculus, into our investigations of the laws of motions and forces: but we must now, in examining the theory of variable forces, or such as change every instant their intensity, avail ourselves of the aid afforded by the doctrine of fluxions.

Variable motion.

When the intensity of an accelerating force varies every instant while it acts upon a body in motion, the velocity acquired in each unit of time will be variable likewise.

Forces which depend upon the velocity of the body; such as the resistance of fluids and friction, present examples of variable accelerating forces: and the same has place, when we consider bodies falling from great heights, so that the force of gravity is continually changing from the first instant of their fall, to the moment when they arrive at the surface of the earth: bodies falling within the earth, offer also another instance of variable motion and forces.

The different natures of variable accelerating forces, as compared with constant forces, and the consequent motions of the bodies on which they act, have been illustrated by Dr. Hutton in the following manner:—

21. Let two weights W *w* (figs. 16, 17, 18) be connected by a thread passing over a pulley at A, B, and C; and let the weight W descend perpendicularly down, while it draws the smaller weight *w* up the line AD, or BE, or CF; the first being a straight inclined plane, and the other two curves; the one convex, and the other concave to the perpendicular. Then the small weight *w* will always make some certain resistance to the free descent of the large weight W; and that resistance will be constantly the same in every part of the plane AD: the energy necessary to draw it up being the same in every point of it, because every part of it has the same inclination to the horizon or to the perpendicular: and consequently the accessions of velocity in the descending weight W, will be always equal in equal times; that is, in this case, W descends by a constant accelerating force.

But in the two curves BE, CF, the resistance or opposition of the small weight *w* will be incessantly altering as it is drawn up the curves: because every part of them has a different inclination to the horizon, or to the perpendicular; in the former curve, the direction becomes more and more upright, or nearer perpendicular as the small weight *w* ascends; and the opposition it makes to the descent of W becomes greater and greater: and consequently the accessions to the velocity of W will be always less and less in equal times; that is, W descends by a decreasing accelerating force, but in the latter curve CF, as *w* ascends, the direction of the curve becomes less and less vertical; and the opposition it makes to the descent of W becomes always less and less and consequently the accessions to the velocity of W will be always increased more and more in equal times; that is, W descends by an increasing accelerating force. So that, although the velocity continually increases in these cases; yet, whilst it increases in a constant ratio to the time of motion in the plane AD, it increases in a less ratio than the time in ascending up BE, and in a greater ratio in ascending up CF.

22. In whatever manner any motion is varied; if we consider it with relation to instants of time indefinitely small, we may imagine its velocity to be

invariable or constant during such indefinitely small instants. But when the motion is uniform, the velocity is expressed by the equation $\frac{s}{t} = v$ (art. 2); that is, by the quotient of the space s, divided by the time t, of describing that space. Therefore, when the velocity is only uniform during the evanescent instant $\dot{t}$, the velocity must be expressed by the indefinitely small space $\dot{s}$, described during this instant, divided by the instant itself; that is, we shall have

$$v = \frac{\dot{s}}{\dot{t}}. \qquad (1)$$

23. As this equation is of the highest importance, being that on which is founded the whole theory of variable motions; and as objections have been offered, not to the above deduction, but to the manner of obtaining it, we shall endeavour to establish it on other principles, to which the same objections will not apply.

For this purpose, let us suppose the time t to be divided into any number of equal portions n; the duration of each of which, will, therefore, be $\frac{t}{n}$. And let us imagine, that a force acts on the moving body, at the commencement of each of these equal intervals of time; and that it communicates to it, at each impulse, a certain finite velocity, and which thus increases that which the body had the preceding instant.

Let us denote by a, the velocity communicated to the body by the first impulse, or at the commencement of the first interval of time; by a', the velocity communicated at the commencement of the second interval; by a'', that due to the third interval; and so on, to $a^{(n-1)}$, which will represent the velocity communicated at the commencement of the nth, or last interval.

Then the several velocities due to, or with which the body moves in, each successive interval, will be, during the

1st .... a;
2d .... $a + a'$;
3d .... $a + a' + a''$;
4th .... $a + a' + a'' + a'''$;
&c. &c.
nth .... $a + a' + a'' + a''' + a^{iv} +$ &c. $a^{(n-1)}$.

And multiplying each of these velocities by the time of its continuance, we shall have the whole space described; viz.

$$s = at + a'\frac{(n-1)t}{n} + a''\frac{(n-2)t}{n} + a'''\frac{(n-3)t}{n} +, \text{ \&c. } a^{(n-1)}\frac{t}{n}.$$

Now if the last velocity

$$a + a' + a'' + a''' +, \text{ \&c. } a^{(n-1)}$$

had been uniform during the whole time, the space described would have been

$$at + a't + a''t + a'''t +, \text{ \&c. } a^{(n-1)}\, t;$$

which is obviously greater than the former; the velocity being supposed to be accelerated from the commencement of the motion to the end of it; or a, a', a'', a''', &c. being all supposed positive and addative quantities.

Whence it follows, that in every accelerated motion, the space passed over in any time is less than would be passed over in the same time with the last acquired velocity, continued uniform during that time: and as this result is wholly independent of the number of intervals n, or of their time of duration $\frac{t}{n}$, it will be true also, when that number becomes indefinitely great, or, when the interval $\frac{t}{n}$ is indefinitely small; that is, when the velocity changes every instant.

We have already seen, that when the velocity is uniformly accelerated; the one of those spaces is the double that of the other.

Let, now, s and v denote the space and the velocity due to any time t: and let $s = \text{F}.t$; that is, let F. t be that function of the time which expresses the space: then F. $(t + t')$ will be the space due to the time $t + t'$; and F. $(t + t') - \text{F}.t =$ the space passed over during the time t'. And here we may always assume t' so small, that the velocity shall be either accelerated or retarded during the whole of that time: that is, however variable the motion may be, we may assume t' so small, that there shall be no change from acceleration to retardation, or the contrary, during that interval: let us, for example, suppose it to be accelerated; and let u denote the velocity generated in the time t'; or, which is the same, let $v + u$ be the velocity at the end of the time $t + t'$. Then, from what is shown above, the space F. $(t + t') - \text{F}.t$, described during the time t', is less than $t'(v + u)$; and it is obviously greater than $t'v$; because v is the velocity at the end of the time t: hence, since

$$\text{F}.(t + t') - \text{F}.t \angle t'v + t'u, \text{ and}$$
$$\text{F}.(t + t') - \text{F}.t \; 7 \; tv;$$

we have $v + u$, and v, for the limits of the quotient

$$\frac{\text{F}.(t + t') - \text{F}.t}{t'}.$$

And here, since we are at liberty to assume t' as small as we please; let us suppose it to become indefinitely small; then the velocity u, generated in the time t', will be also indefinitely small, or less than any assignable quantity: consequently, in this case, the value of the above function approaches nearer to the value of v than any assignable quantity; that is, when t' is evanescent, we have

$$\frac{\text{F}.(t + t') - \text{F}.t}{t'} = v.$$

But in this case, t' is denoted by $\dot{t}$, or the fluxion of the time; and F. $(t + t') - \text{F}.t$ by $\dot{s}$, or the fluxion of the space: whence, then, we obtain

$$\frac{\dot{s}}{\dot{t}} = v;$$

the same result as before.

If we allow ourselves to introduce considerations relative to the expansion of functions, we should see more obviously, that the limit of

$$\frac{\text{F}(t + t') - \text{F}t}{t'},$$

is expressed by

$$\frac{ds}{dt}, \text{ or } \frac{\dot{s}}{\dot{t}} = v;$$

which is still an equivalent result.

24. In estimating the value of any variable accelerating force, which we may denote by ϕ, it is obvious that we must first assume some certain uniform accelerating

force as the standard of comparison; and this standard is, by universal assent, the force of gravity. Now, we have seen, that any two uniform accelerating forces, are to each other as the velocity that they generate in the same body, in the same time; and in this case, any time whatever may be employed. But when we have to compare a variable accelerating force with a uniform force of the same kind, it is evident that we are no longer at liberty to assume the time at pleasure; but must only take the indefinitely small interval $\dot{t}$, during which the variable accelerating force may be considered constant.

If, therefore, we denote by p any constant accelerating force, which communicates to a heavy body a velocity $2g$ in any unit of time; $2g\dot{t}$ will be the velocity due to the time $\dot{t}$: but, during the same instant, the accelerating force ϕ will communicate the velocity $\dot{v}$; for v being supposed to be the velocity at the end of the time t; $v + \dot{v}$ will be the velocity at the end of the time $t + \dot{t}$; consequently $\dot{v}$ will be the velocity generated in the time $\dot{t}$: hence, then, we have

$$\phi : p :: \dot{v} : 2g\dot{t};$$

from which we immediately deduce

$$\phi = \frac{p}{2g}\frac{\dot{v}}{\dot{t}}.$$

We may simplify this expression, by assuming p as the accelerating force, which we call our unit; and $2g$ as the linear unit: that is, by assuming for the unit of force, that which generates a velocity equal to a linear unit, in a unit of time. By this means our equation reduces to

$$\phi = \frac{\dot{v}}{\dot{t}}. \qquad (2)$$

And we have before found

$$v = \frac{\dot{s}}{\dot{t}}; \text{ whence we have } \dot{v} = \frac{\ddot{s}}{\dot{t}}:$$

and substituting this in the above equation 2, we obtain

$$\phi = \frac{\ddot{s}}{\dot{t}^2}. \qquad (3)$$

It has been before observed, that we universally assume the force of gravity as our unit of force; and a second as the unit of time: consequently, since the velocity generated in one second by this force is $2g = 32\frac{1}{6}$ feet; this will be our unit of length. If, therefore, we substitute, in the equations,

$$\phi = \frac{\dot{v}}{\dot{t}}, \text{ and } \phi = \frac{\ddot{s}}{\dot{t}^2},$$

by means of the above units, the time and the linear quantities, which enter into them; they will be converted into abstract numbers; expressing at every instant, the ratio of the accelerating force ϕ to that of gravity.

§ VII. *Application of the preceding general formulæ to different problems.*

25. When the accelerating force, which acts on a body in motion, is given in terms or functions of the time, the equations

$$\phi = \frac{\dot{v}}{\dot{t}}, \phi = \frac{\ddot{s}}{\dot{t}^2},$$

will discover, by simply taking the fluent, the acquired velocity, and the space passed over at any required instant. But this force is commonly given in functions of the space, or in functions of the acquired velocity, or even in functions of both: and in these cases, the above two equations become fluxional equations of the first and second order; and of which we must endeavour to obtain the fluents: which, however, present great, and frequently insurmountable difficulties; although, in other cases, the fluents may be obtained in a very direct and satisfactory manner.

It may not be improper to observe here, that although the above formulæ, and all our investigations and deductions, have been made exclusively with reference to isolated material points; yet they may evidently be applied to bodies of any figure and dimension, by supposing all the points of them to describe by the same motion parallel right lines; and that each of them move as if it were isolated, and independent of all the others.

Problem I.

Problems.

26. *To determine the vertical motion of a heavy body; having regard to the variation of gravity.*

Let the radius of the earth be denoted by r; the force of gravity by $2g$; a the distance of the body from the centre of the earth, when its motion commences; and s any space through which it has fallen in the time t; then its distance from the centre will be $(a - s)$, and the accelerating force ϕ at the same point, will be found by the analogy

$$\phi : 2g :: r^2 : (a - s)^2;$$

the force of gravity being inversely as the square of the distance: whence

$$\phi = \frac{2r^2 g}{(a - s)^2}.$$

But we have from equation 3, art. 24,

$$\phi = \frac{\ddot{s}}{\dot{t}^2};$$

whence again is obtained the equation of the bodies motion; viz.

$$\frac{\ddot{s}}{\dot{t}^2} = \frac{2gr^2}{(a-s)^2}.$$

In order to take the fluent, let us multiply each side of the above $2\dot{s}$, and we have

$$\frac{2\dot{s}\ddot{s}}{\dot{t}^2} = 4gr^2\frac{\dot{s}}{(a - s)^2};$$

taking the fluents

$$\frac{\dot{s}^2}{\dot{t}^2} = v^2 = \frac{4gr^2}{a - s} + C;$$

C being the correction, or a constant arbitrary quantity to be determined from the nature of the problem.

Now, if we suppose the velocity 0, or zero, at the origin of the motion; or at the commencement of the action of gravity upon the body; we shall have $v = 0$, when $s = 0$; or when $\frac{\dot{s}}{\dot{t}} = 0$: therefore

$$\frac{4gr^2}{a} + C = 0, \text{ or } C = -\frac{4gr^2}{a}.$$

Subtracting this equation from the former, it becomes

$$\left.\begin{array}{l}\dfrac{\dot{s}^2}{\dot{t}^2} = \dfrac{4gr^2}{a-s} - \dfrac{4gr^2}{a};\\ \text{or, since}\\ \dfrac{\dot{s}}{\dot{t}} = v, \text{ we have}\\ v^2 = \dfrac{4gr^2}{a} \times \dfrac{s}{a-s}:\end{array}\right\}\quad(1)$$

from which equation the velocity of the body may be found for every point in the vertical line of its descent.

In this equation, we never can have $s = a$; because the body can only arrive at the surface of the earth: but it appears from the above, if it were possible, or if we consider merely an attracting point, that the velocity v would be infinite, independent of any particular value of a; that is, from whatever height the body began to fall.

On the other hand, when the body arrives at the surface of the earth, then $a - s = r$, and $s = a - r$; whence the above equation, in this case, gives

$$v^2 = \frac{4gr^2}{a} \times \frac{a-r}{r}, \text{ or}$$

$$v = \sqrt{\left(4gr \times \frac{a-r}{a}\right)};$$

which, when a is infinite, becomes

$$v = 2\sqrt{gr}:$$

that is, the velocity acquired by a body falling from an infinite height to the surface of the earth, will be finite, and equal to $2\sqrt{gr}$.

If we assume $r = 3965$ miles, and substitute this value for r in the above equation, we shall find $v = 6{\cdot}9506$ miles per second, the greatest velocity, which is also uniform.

If it be required to determine the velocity which a body would have acquired when it arrives at the surface of the earth, supposing it to fall from the distance of the sun, and assuming this distance = 12000 of the earth's diameters, or $a = 24000\,r$, we shall have

$$v = \sqrt{\left(4gr \times \frac{23999}{24000}\right)} = 6{\cdot}9505 \text{ miles per second.}$$

And in the same manner, the velocity of a body, falling from the height of the moon, or from a distance equal to 60 semi-diameters of the earth, will be found = 6·8927 miles per second.

If we resolve equation 1, viz.

$$\frac{\dot{s}^2}{\dot{t}^2} = \frac{4gr^2}{a} \times \frac{s}{a-s},$$

with reference to $\dot{t}$, we shall have

$$\dot{t}^2 = \frac{(a-s)\,a\,\dot{s}^2}{4gr^2 s} = \frac{(a-s)^2\,a\,\dot{s}^2}{4gr^2 s\,(a-s)}, \text{ or}$$

$$\dot{t} = \frac{1}{2r}\sqrt{\frac{a}{g}} \times \frac{(a-s)\,\dot{s}}{\sqrt{(as-s^2)}}.$$

This again may be put under the form

$$\dot{t} = \frac{1}{2r}\sqrt{\frac{a}{g}}\left\{\frac{(\frac{1}{2}a-s)\,\dot{s}}{\sqrt{(as-s^2)}} + \frac{1}{2}a\frac{\dot{s}}{\sqrt{(as-s^2)}}\right\}$$

Now

$$\int\frac{(\frac{1}{2}a-s)\,\dot{s}}{\sqrt{(as-s^2)}} = \sqrt{(as-s^2)};$$

and

$$\int\frac{\frac{1}{2}a\dot{s}}{\sqrt{(as-s^2)}} = \text{arc rad. } \frac{a}{2} \text{ and vers } s:$$

whence

$$\int\frac{1}{2r}\sqrt{\frac{a}{g}}\left\{\frac{(\frac{1}{2}a-s)\,\dot{s}}{\sqrt{(as-s^2)}} + \frac{1}{2}a\frac{\dot{s}}{\sqrt{(as-s^2)}}\right\}$$

gives

$$\left.\begin{array}{l}\dfrac{1}{2r}\sqrt{\dfrac{a}{g}}\{\sqrt{(as-s^2)} + \text{arc rad. } \tfrac{1}{2}a\\ \text{and vers } s\},\\ \text{or}\\ t = \dfrac{1}{2r}\sqrt{\dfrac{a}{g}}\{\sqrt{(as-s^2)} + \dfrac{a}{2}\text{ arc}\\ \cos\dfrac{a-2s}{a};\end{array}\right\}\quad(2)$$

which needs no correction, because when $s = 0$, $t = 0$.

When $s = a - r$, that is, when the body arrives at the surface, this becomes

$$t = \frac{1}{2r}\sqrt{\frac{a}{g}}\{\sqrt{(ar-r^2)} + \text{arc rad. } \tfrac{1}{2}a \text{ vers } (a-1)\};$$

or

$$t = \frac{1}{2r}\sqrt{\frac{a}{g}}\left\{\sqrt{(ar-r^2)} + \frac{a}{2}\text{arc cos}\frac{a-2r}{a}\right\}$$

which is evidently infinite when a is infinite, although v is then only a finite quantity.

When the height above the earth's surface is inconsiderable with regard to the radius; then s is obviously also inconsiderable with regard to a, and our equation (1) reduces to

$$v^2 = 4gs, \text{ or } v = 2\sqrt{gs},$$

as we have found it for uniformly accelerated velocities.

With respect to equation (2), it may be observed, that $\text{arc cos}\dfrac{a-2s}{a} = \text{arc sin } 2\sqrt{\dfrac{as-s^2}{a^2}}$.

Now the $\sin 2\sqrt{\dfrac{as-s^2}{a^2}}$ being very small, because s is very small with respect to a, the sine and arc may be taken indifferently, the one for the other; we may also neglect the square s^2, as very inconsiderable with respect to the other term as; and if we put, moreover, r, in the place of a, our equation (2) reduces to

$$t = \frac{1}{2r}\sqrt{\frac{r}{g}}.2\sqrt{rs} = \sqrt{\frac{s}{g}}, \text{ or } s = gt^2;$$

as we found it for uniformly accelerated motions.

If we call the distance of the moon equal to 60 semi-diameters of the earth, and assume the latter = 3965 miles; we shall have $a = 60\,r$, and t will be found equal to 416806″ = 4 days, 19 hours, 46 minutes, and 46″; which is the time a body would be in falling from the moon to the earth's surface.

If the attracting body be considered as a point, so that s may become ultimately equal to a, then in this case our equation (2) is

$$t = \frac{1}{2r}\sqrt{\frac{a}{g}} \times \{\text{arc to cos} - 1\}\frac{a}{2}$$

$$t = \frac{\pi}{4r\sqrt{g}}a^{\frac{3}{2}},$$

where π is the semi-circumference of a circle, whose radius = 1; that is, the times in which bodies fall from quiescence to any centre of force, are as the $\frac{3}{2}$ power of the distances through which they descend.

PROBLEM II.

To determine the circumstances attending the fall of bodies, within the surface of the earth.

27. In this case, we know that the accelerating force ϕ varies directly as the distance from the centre. Let us, therefore, conceive a heavy body moving from the earth's surface, in the direction of a terrestrial radius, towards the centre. Let this radius be denoted by r; the force of gravity at the surface by $2g = 32\frac{1}{2}$ feet; and any variable distance from the surface $= s$: then $r - s$ will be the distance from the centre, and we shall have

$$\phi : 2g :: r - s : r,$$

or,

$$\phi = \frac{2g}{r}(r - s).$$

But from equation (3), art. 24, we have

$$\phi = \frac{\ddot{s}}{\dot{t}^2} = \frac{2g}{r}(r - s).$$

Multiply by $2\dot{s}$, and we have

$$\frac{2\dot{s}\ddot{s}}{\dot{t}^2} = \frac{4g}{r}(r - s)\dot{s}.$$

Taking the fluents, we obtain

$$\frac{\dot{s}^2}{\dot{t}^2} = v^2 = -\frac{2g}{r}(r - s)^2 + C.$$

With regard to the constant arbitrary C, supposing the body to depart from the surface, without any initial velocity, we shall have $v = 0$, when $s = 0$; whence

$$0 = -2gr + C, \text{ or } C = 2gr.$$

Consequently,

$$v^2 = \frac{2g}{r}\{r^2 - (r - s)^2\}, \text{ or}$$

$$v = \sqrt{\frac{2g}{r}} \times \{r^2 - (r - s)^2\}^{\frac{1}{2}}.$$

An equation from which we readily deduce the value of v, for any point in the line of the body's motion.

By making $s = r$, this equation becomes

$$v = \sqrt{2gr},$$

which is the velocity acquired by the body when it has passed through the entire radius, or when it has arrived at the centre of the earth.

From this point the body will, in consequence of its acquired velocity, ascend to the surface; and having run through an entire diameter, we shall have $s = 2r$; which substituted in the above equation, gives $v = 0$: that is, the velocity will be zero at the extremity of the diameter, as in the commencement of its motion. Consequently it will again return through the centre, and pass over again the entire diameter a second time in the contrary direction to the first; but in other respects it will be identically the same: and when arrived at its first point of departure, its velocity will be again equal to zero. The body being thus in the same state as at first, will again pass over the diameter and return again, and so on, making an infinite number of oscillations, each of equal duration.

In order to determine t in functions of s, we shall, by means of our equation, art. 24, viz. $\frac{\dot{s}}{\dot{t}} = v$; and by resolving the equation

$$\frac{\dot{s}^2}{\dot{t}^2} = \frac{2g}{r}\{r^2 - (r - s)^2\},$$

with reference to $\dot{t}$, we obtain

$$\sqrt{\frac{2g}{r}} \cdot \dot{t} = \frac{\dot{s}}{\sqrt{(r^2 - (r - s)^2}};$$

and taking the fluents,

$$\sqrt{\frac{2g}{r}} \cdot t = \text{arc to cos}\,\frac{r - s}{r};$$

which requires no correction, because when $s = 0$, $t = 0$. We have also, by inverting the equation,

$$\frac{r - s}{r} = \text{cos of arc}\left(\sqrt{\frac{2g}{r}} \cdot t\right);$$

whence

$$s = r - r\,\text{cos of arc}\left(\sqrt{\frac{2g}{r}} \cdot t\right).$$

And by substituting this general value of s in our equation, for the velocity, we shall have the velocity v in functions of the time; by which means the velocity and space at every instant of the motion will be determined.

When $s = r$, our equation for the time is

$$\sqrt{\frac{2g}{r}} \cdot t = \text{arc to cos} = 0 = \frac{\pi}{2},$$

π denoting the semi-circumference of a circle whose radius is 1; and hence we learn that the time the body employs in passing through the radius is

$$= \frac{\pi}{2}\sqrt{\frac{r}{2g}}.$$

And it will employ a double time in passing through the diameter; for by making $s = 2r$, we have

$$\sqrt{\frac{2g}{r}} \cdot t = \text{arc cos} = \frac{-r}{r} \text{ or } -1 = \pi,$$

or

$$t = \pi\sqrt{\frac{r}{2g}}.$$

Assuming $r = 3965$ miles, we find the time of falling from the surface to the centre

$= 21'7''\frac{1}{4}$; and in passing through the diameter $= 42$ $14''\frac{1}{2}$.

If we suppose the body to commence its motion from any other point in the terrestrial radius, instead of its extremity; it will be found, by proceeding as above, and making the requisite correction in the fluent, that the time will still be the same constant quantity: but we omit the process, because we propose to show generally, that when a body is attracted towards a centre, which is such that the force at every point is proportional to the distance, the time of arriving at the centre will always be the same, from whatever point the motion of the body commences.

PROBLEM III.

28. *To determine the circumstances of velocity, time, and space; with relation to a body which moves from quiescence, in consequence of an attracting force which varies directly as the distance from the centre of force.*

Since the force is directly as the distance of the body from the centre of force, let us conceive some distance

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d at which the force is f, and let a be the distance of the body from the centre in its quiescent state; also let s denote any space through which the body has passed in any time t, and let its velocity be v; then we shall have

$$d : a :: f : \frac{af}{d}, \text{ the force at } a;$$

and also

$$d : a-s :: f : \frac{(a-s)f}{d}, \text{ the force after}$$

the body has passed through the space s.

But

$$\frac{\ddot{s}}{\dot{t}^2} = \phi; \text{ whence}$$

$$\frac{\ddot{s}}{\dot{t}^2} = \frac{(a-s)f}{d}.$$

Multiply by $2\dot{s}$, and we obtain

$$\frac{2\dot{s}\ddot{s}}{\dot{t}^2} = \frac{2f(a-s)\dot{s}}{d};$$

taking the fluents,

$$\frac{\dot{s}^2}{\dot{t}^2} = v^2 = \frac{f}{d}(2as - s^2);$$

which requires no correction, because when $s = 0$, $v = 0$; therefore

$$v = \sqrt{\frac{f}{d}}.(2as - s^2)^{\frac{1}{2}};$$

which when $s = a$, that is, when the body arrives at the centre of force, becomes

$$v = \sqrt{\frac{f}{d}}.$$

It is therefore obvious from this result, that the velocity which a body acquires in falling from different points of the same line to a centre attracting every where with a force proportional to its distance, is as the distance of the point of quiescence from the centre.

If we now resolve the preceding equation with reference to t, we shall have

$$\dot{t}^2 = \frac{d\dot{s}^2}{f(2as - s^2)}, \text{ or}$$

$$\dot{t} = \sqrt{\frac{d}{f}} \times \frac{\dot{s}}{\sqrt{(2as - s^2)}};$$

whence

$$t = \sqrt{\frac{d}{f}}.\text{ arc to cos } \frac{a-s}{a} + \text{C}.$$

In order to find C, we observe that when $s = 0$, $t = 0$; whence

$$0 = \sqrt{\frac{d}{f}} \text{ arc to cos } 1 + \text{C}:$$

but arc to cos $= 1$ is 0; therefore $\text{C} = 0$, and

$$t = \sqrt{\frac{d}{f}}.\text{ arc to cos } = \frac{a-s}{a},$$

is the correct fluent; and when $s = a$, it becomes

$$t = \sqrt{\frac{d}{f}}.\text{ arc to cos} = 0, = \tfrac{1}{2}\pi, \text{ or}$$

$$t = \tfrac{1}{2}\pi\sqrt{\frac{d}{f}};$$

a result which is wholly independent of a; and we should therefore obviously arrive at the same conclusion from whatever distance the body is supposed to begin to fall. Whence we conclude, that when a body is attracted from a state of quiescence by any centre of force, the intensity of which is directly as the distance, the whole time of descent will be always the same constant quantity.

PROBLEM IV.

29. *To determine the rectilinear motion of a body or material point attracted towards two fixed centres, supposing the attraction of each to be in the inverse ratio of the square of the distance.*

Let us consider, first, the case in which the body is situated on a right line which joins the centres of the two bodies. Let AB (fig. 19) be this right line, its length $= l$, s any variable distance of the body from the point A, and consequently $l - s$, its distance from B. We may represent the two attractions by $\frac{a^2}{s^2}$ and $\frac{b^2}{(l-s)^2}$, a^2 and b^2 being two constant and positive co-efficients, which will denote the intensity of these forces at the unit of distance; intensities which may be to each other in any ratio at pleasure. Now the accelerating force, or that which determines the motion of the body to either of these centres, will obviously be the excess of one of these forces over the other, for example of the force $\frac{b^2}{(l-s)^2}$ over $\frac{a^2}{s^2}$; hence we have

$$\phi = \frac{\ddot{s}}{\dot{t}^2} = \frac{b^2}{(l-s)^2} - \frac{a^2}{s^2};$$

and by multiplying both sides by $2\dot{s}$, we obtain

$$\frac{2\dot{s}\ddot{s}}{\dot{t}^2} = \frac{2b^2\dot{s}}{(l-s)^2} - \frac{2a^2\dot{s}}{s^2};$$

the fluent of which gives

$$\frac{\dot{s}^2}{\dot{t}^2} = v^2 = \frac{2b^2}{l-s} + \frac{2a^2}{s} + \text{C}: \qquad (1)$$

v being the velocity of the body, and C the correction, or constant arbitrary quantity, which remains to be determined.

In order to determine the value of C, we shall assume that at the origin of motion $s = a$, and $v = k$; and we shall thus obtain

$$k^2 = \frac{2b^2}{l-a} + \frac{2a^2}{a} + \text{C}; \qquad (2)$$

and subtracting this from our first equation, it becomes

$$v^2 - k^2 = \frac{2b^2}{l-s} - \frac{2b^2}{l-a} + \frac{2a^2}{s} - \frac{2a^2}{a};$$

from which expression we know the value of v in function of s and known quantities, at every point of the right line which the particle describes.

Now there is necessarily a certain point C on our supposed right line joining the centres, where the two attractions are equal; if we call the distance of this point from A equal to l', we shall have

$$\frac{b^2}{(l-l')^2} = \frac{a^2}{l'^2};$$

an equation which obviously gives two values of l', of which the one answers to the point C, and the other to a second point of equal attraction, situated in the prolongation of AB; the two quantities a and b being a supposed positive quantities, we have, for the value of l', answering to the point C,

$$l' = \frac{l\,a}{a + b}; \qquad (3)$$

and while s is less than this value of l', the attraction of the centre A will exceed that of B; but when s is greater than l', the attraction of B will be greater than the attraction of A, as is obvious.

Let us suppose $a < l'$, or, which is the same, that the body commences its motion from some point between A and C, and that its initial velocity should be made towards C; it is evident that, on this supposition, two distinct cases may happen, viz. the value of v may become zero before the body reaches the point C, or it may not.

In the first instance, the body having attained the point where the velocity is zero, it will return again towards A; and in the second instance, if the velocity be such that the body passes through C, its motion will continue, and it will pass on to B; but if the velocity become zero just as the body arrives at the point C, it will, being there acted upon by equal forces, remain at rest between the two attracting points.

If the least value of the initial velocity k be required, which is necessary for the body's reaching the point C, we may determine it by assuming, that when $v = 0$, $s = l'$, which gives, from equation (2),

$$-k^2 = \frac{2\,b^2}{l - l'} - \frac{2\,b^2}{l - a} + \frac{2\,a^2}{l'} - \frac{2\,a^2}{a}; \qquad (4)$$

and substituting for l' its value, as determined in equation (3), we shall find, after the requisite reduction,

$$k = \sqrt{\left\{\frac{2\,a^2}{a} + \frac{2\,b^2}{l - a} - \frac{2\,(b + a)^2}{l}\right\}}. \qquad (5)$$

This is the least value of k that will enable the body to arrive at the point of equal attraction C; and therefore when k is less than this, the body will ascend to a certain point, and afterwards return: when it has precisely the above value, it will remain suspended in the point C; and when k is greater than this, the body will pass through C, and ultimately arrive at the point B.

As a practical application of the above formula, let the velocity be required with which a body must be projected from the surface of the moon, that it may pass through the common centre of attraction and fall to the earth, these two bodies being supposed at rest.

Here we must first determine the numerical values of a^2 and b^2. Now, we know that at the earth's surface the intensity of gravity is $2\,g$; and hence we have for its intensity, at any distance, $l - s$: as

$$(l - s)^2 : r^2 :: 2\,g : \frac{2\,gr^2}{(l - s)^2};$$

whence $b^2 = 2\,gr^2$: if now we assume the mass of the moon to that of the earth as 1 : 68·5, we shall have $a^2 = \frac{2\,gr^2}{68{\cdot}5}$.

Again, calling the distance of the centres of the earth and moon $= 60\,r$, and the lunar radius $= 1060$ miles, and that of the earth 3958 miles, we have 1060 miles $= a = \frac{1060}{3958}\,r = {\cdot}267812\,r$.

Substituting these values in the above expression for k, we have

$$k = \sqrt{\left\{\frac{4\,gr^2}{68{\cdot}5 \times {\cdot}267812\,r} + \frac{4\,gr^2}{(60 - {\cdot}267812)r} - \frac{4\,gr^2\,\left(1 + \frac{1}{\sqrt{68{\cdot}5}}\right)^2}{60\,r}\right\}};$$

or,

$$k = \sqrt{4\,gr} \times \sqrt{\left\{\frac{1}{68{\cdot}5 \times {\cdot}267812} + \frac{1}{59{\cdot}732188} - \frac{(\sqrt{68{\cdot}5} + 1)^2}{68{\cdot}5 \times 60}\right\}};$$

or, $k = \sqrt{4\,gr} \times {\cdot}22431$; or writing r and g both in feet, $k = (4 \times 16\tfrac{1}{12} \times 3958 \times 5280)^{\frac{1}{2}} \times {\cdot}22431 = 8224$ feet per second; and any velocity exceeding this will carry the body through the centre of equal attraction; after which, the terrestrial attraction prevailing, it will be urged with an accelerated motion towards the earth.

It still remains to express the value of the space in functions of the time, that is of s in terms of t, or of t in terms of s: with this view let us repeat again our equation (1), viz.

$$\frac{\dot{s}^2}{\dot{t}^2} = \frac{2\,b^2}{l - s} + \frac{2\,a^2}{s} + \text{C};$$

which being resolved with reference to $\dot{t}$, gives

$$\dot{t} = \frac{\sqrt{(s\,l - s^2)}\,\dot{s}}{\sqrt{\{2\,a^2\,l + (2\,b^2 - 2\,a^2 + \text{C}\,l)\,s - \text{C}\,s^2\}}}. \qquad (6)$$

The fluent of this expression cannot be taken generally, except by means of series; but if we can render the denominator under the radical a square, it will then submit to the common rules of operation: now this may be effected by making

$$(2\,b^2 - 2\,a^2 + \text{C}\,l)^2 = -\,8\,a^2\,\text{C}\,l;$$

or, which is the same,

$$\text{C}^2\,l^2 + 4\,\text{C}\,l\,(b^2 + a^2) + 4\,(b^2 - a^2)^2 = 0;$$

which requires that

$$\text{C} = \frac{-\,2\,(a \pm b)^2}{l}.$$

Now we have seen, that the value of C depends upon the initial velocity k, or that

$$k^2 = \frac{2\,b^2}{l - a} = \frac{2\,a^2}{a} + \text{C}.$$

If, therefore, we substitute in the latter equation the preceding double value of C, we shall have

$$k = \sqrt{\left\{\frac{2\,a^2}{a} + \frac{2\,b^2}{l - a} - \frac{2\,(b + a)^2}{l}\right\}};$$

or,

$$k = \sqrt{\left\{\frac{2\,a^2}{a} + \frac{2\,b^2}{l - a} - \frac{2\,(b - a)^2}{l}\right\}}.$$

And it is therefore only for one or other of these two cases, or initial velocities, that we can determine the time under a finite form: the first is that particular initial velocity which is requisite to carry the body to the point of equal attraction; the other obviously exceeds this limit, and a body being projected with it, will necessarily arrive at the point B; whereas, with the former, it will ascend to C, and there having lost all its motion, it will remain suspended between the equal action of the two attracting forces.

It is not necessary, however, to treat these as two separate cases; for by substituting for C its double value as above determined in equation (6), we have

$$\dot t = \sqrt{\frac{l}{2}} \cdot \frac{\sqrt{(l s - s^2)}\,.\,\dot s}{a l \mp (a \pm b) s}.$$

Now in order to render this rational, let us assume

$$l s - s^2 = s^2 x^2, \text{ or } s = \frac{l}{1 + x^2}:$$

whence $$\dot s = \frac{2 l x \dot x}{(1 + x^2)^2}.$$

The above equation therefore, with reference to this new variable, becomes

$$\dot t = -\sqrt{2 l^3}\, \frac{x^2 \dot x}{(1 + x^2)^2 (a x^2 \mp b)};$$

or, which is the same,

$$\dot t = \frac{-\sqrt{2 l^3}}{(a \pm b)^2} \left\{ \frac{(a \mp b x^2)\dot x}{(1 + x^2)^2} \pm \frac{2 b \dot x}{a x^2 \mp b} \right\}.$$

Now in taking the fluents, it is only necessary to observe, that in separating the formula into two cases answering to our former ambiguous or double sign; we must take all the upper signs in the one case, and all the lower in the other; and we shall thus have

$$\int \frac{(a \mp b x^2)\dot x}{(1 + x^2)^2} = \frac{\frac{1}{2}(a \pm b) x}{1 + x^2} + \tfrac{1}{2}(a \mp b) \operatorname{arc}(\tan = x);$$

$$\int \frac{\dot x}{a x^2 - b} = \frac{1}{2\sqrt{a b}} \cdot \log \left\{ \frac{x\sqrt{a} - \sqrt{b}}{x\sqrt{a} + \sqrt{b}} \right\};$$

$$\int \frac{\dot x}{a x^2 + b} = \frac{1}{\sqrt{a b}} \cdot \operatorname{arc} \left\{ \tan = x\sqrt{\frac{a}{b}} \right\}.$$

Whence we conclude, that in the case of the upper signs being used, that is for the value of C, answering to our first value of k, we have

$$t = \frac{\sqrt{2 l^3}}{(a + b)^2} \left\{ \frac{\frac{1}{2}(a + b) x}{1 + x^2} + \tfrac{1}{2}(a - b) \operatorname{arc}(\tan = x) + \tfrac{1}{2}\sqrt{a b}.\log\left(\frac{x\sqrt{a} - \sqrt{b}}{x\sqrt{a} + \sqrt{b}}\right) \right\} + \mathrm{C}'.$$

And when the lower signs are employed, that is for our second value of k, we obtain

$$t = \frac{\sqrt{2 l^3}}{(a - b)^2} \left\{ \frac{\frac{1}{2}(a - b) x}{1 + x^2} + \tfrac{1}{2}(a + b) \operatorname{arc}(\tan = x) - \sqrt{ab}.\operatorname{arc}(\tan = x)\sqrt{\frac{a}{b}} \right\} + \mathrm{C}''.$$

Substituting in these equations for x, its value in s, viz.

$$x = \frac{\sqrt{(l s - s^2)}}{s},$$

we shall have in both cases the value of t in functions of s; and the corrections C′ and C″, will be determined by assuming $t = 0$, when $s = a$, that is, at the commencement of the motion.

When the initial velocity does not agree with either of the above values of k, there is then no resource but that of infinite series for determining the value of t.

Problem V.

30. *To determine the vertical motion of a body, having regard to the resistance of the air, and supposing that resistance to vary as the square of the velocity.*

We must here distinguish two cases, viz. when the body is projected upwards, and when it falls. We shall first consider the latter case.

When a body falls, or is projected vertically downwards from any given height, the force of gravity will be opposed by the resistance of the medium; and consequently our accelerating force ϕ, will be the difference between these; and as we consider generally the resistance to be as the square of the velocity, we may denote this force by $m v^2$, v denoting the velocity at any time t, and m a constant co-efficient, which is to be determined from experiment, and which we may consider constant for the same body, and in the same medium; but which varies with the form of body, its density, and that of the medium.

If the body were to fall from a great height, it would be necessary to attend to the different degrees of density of the medium, or atmosphere, which we know is greater or less, according as it is nearer the surface of the earth, or higher above it; the effect of gravity also, in this case, would vary with the altitude. We propose, however, to consider only small heights, throughout which the action of gravity may be considered constant, and the atmosphere as of uniform density, which we may safely do as far as relates to any practical cases of projectiles.

Assuming then, as we have hitherto done, 2 g to denote the intensity of gravity, we shall have

$$\varphi = 2 g - m v^2;$$

or, since $$\phi = \frac{\dot v}{\dot t}, \text{ this becomes}$$

$$\frac{\dot v}{\dot t} = 2 g - m v^2.$$

Resolving this with reference to t, we have

$$\dot t = \frac{\dot v}{2 g - m v^2}; \text{ or,}$$

$$\dot t = \frac{1}{2\sqrt{2 g}} \left\{ \frac{\dot v}{\sqrt{2 g} + v\sqrt{m}} + \frac{\dot v}{\sqrt{2 g} - v\sqrt{m}} \right\};$$

the fluent of which gives

$$t = \frac{1}{2\sqrt{2 g m}} \log \left\{ \frac{\sqrt{2 g} + v\sqrt{m}}{\sqrt{2 g} - v\sqrt{m}} \right\}, \qquad (1)$$

which requires no correction, because we suppose the body to fall from quiescence; so that when $t = 0$, $v = 0$, which condition is involved in the above equation, for here taking $v = 0$, we have

$$t = \frac{1}{2\sqrt{2 g m}} \log 1 = 0.$$

If we multiply both terms of equation (1) by $2\sqrt{2 g m}$, we shall have

$$2 t \sqrt{2 g m} = \log \left\{ \frac{\sqrt{2 g} + v\sqrt{m}}{\sqrt{2 g} - v\sqrt{m}} \right\};$$

and if ε be taken to denote the base of the hyperbolic system of logarithms, we know, that $\log \varepsilon = 1$, whence the above may be written

$$2 t \sqrt{2 m g}.\log \varepsilon = \log \left\{ \frac{\sqrt{2 g} + v\sqrt{m}}{\sqrt{2 g} - v\sqrt{m}} \right\},$$

or $$\log \varepsilon^{2 t\sqrt{2 g m}} = \log \left\{ \frac{\sqrt{2 g} + v\sqrt{m}}{\sqrt{2 g} - v\sqrt{m}} \right\};$$

and taking the corresponding numbers,

$$\varepsilon^{2 t\sqrt{2 g m}} = \frac{\sqrt{2 g} + v\sqrt{m}}{\sqrt{2 g} - v\sqrt{m}}; \qquad (2)$$

the reciprocal of which gives

$$\frac{1}{\varepsilon^{2t\sqrt{gm}}} = \frac{\sqrt{2g} - v\sqrt{m}}{\sqrt{2g} + v\sqrt{m}}. \qquad (3)$$

From equation (2), the value of v may be determined at every instant of the body's motion, or, the velocity being given, the time will thence become known.

And from equation (3), we learn, that as the times augment, the denominator of the first term, $\varepsilon^{2t\sqrt{2gm}}$, approaches nearer and nearer to infinity, and consequently

$$\frac{\sqrt{2g} - v\sqrt{m}}{\sqrt{2g} + v\sqrt{m}}$$

nearer and nearer to zero; so that in fact we shall have, when t is infinite,

$$\frac{\sqrt{2g} - v\sqrt{m}}{\sqrt{2g} + v\sqrt{m}} = 0;$$

or, $\sqrt{2g} - v\sqrt{m} = 0$; or,

$$v = \sqrt{\frac{2g}{m}}, \textit{ a constant quantity.}$$

We see, thence, that a heavy body falling in a uniform resisting medium, tends continually towards a uniform velocity, but which it will not actually attain, till after an infinite time.

In order to determine the space, we must multiply our equation

$$\dot{t} = \frac{\dot{v}}{2g - mv^2}, \text{ by } v,$$

which gives $\dot{t}v = \dot{s} = \dfrac{v\dot{v}}{2g - mv^2}$;

the fluent of which is

$$s = -\frac{1}{2m}\log(2g - mv^2) + \text{C}.$$

And in order to determine C, we have $s = 0$, when $v = 0$, whence

$$0 = -\frac{1}{2m}\log 2g + \text{C};$$

and subtracting the latter from the former, there results

$$s = -\frac{1}{2m}\log\left(1 - \frac{mv^2}{2g}\right); \qquad (4)$$

from which s, or v, may be determined, the one from the other; after the co-efficient m has been ascertained from experiment.

Let us now investigate the case in which the body is projected vertically upwards.

Here, gravity and the resistance of the medium, both act so as to retard the motion of the body; and preserving, therefore, the same notation as above, we shall have

$$\phi = -2g - mv^2;$$

and consequently, the equation of motion will be

$$\frac{\dot{v}}{\dot{t}} = -2g - mv^2;$$

whence $-\dot{t} = \dfrac{\dot{v}}{2g + mv^2}$; (5)

the fluent of which gives

$$-t = \frac{1}{\sqrt{2gm}} \cdot \text{arc}\left(\tan = v\sqrt{\frac{m}{2g}}\right) + \text{C}.$$

In order to determine the value of C, let a be the initial velocity of projection, so that when $t = 0$, $v = a$; this gives

$$0 = \frac{1}{\sqrt{2gm}} \cdot \text{arc}\left(\tan = a\sqrt{\frac{m}{2g}}\right) + \text{C};$$

consequently

$$t = \frac{1}{\sqrt{2gm}} \cdot \left\{ \text{arc}\left(\tan = a\sqrt{\frac{m}{2g}}\right) - \text{arc}\left(\tan = v\sqrt{\frac{m}{2g}}\right) \right\}. \qquad (6)$$

Resolving, now, this equation with reference to v, we shall have the velocity of the projectile at each instant of the body's motion.

For the space, it is only necessary to multiply equation (5) by $-v$, and we obtain

$$\dot{t}v = \dot{s} = \frac{-v\dot{v}}{2g + mv^2};$$

and taking the fluents

$$-s = \frac{1}{2m}\log(2g + mv^2) + \text{C}.$$

To determine C, we have $s = 0$, when $v = a$; whence

$$s = \frac{1}{2m}\log(2g + ma^2) - \frac{1}{2m}\log(2g + mv^2);$$

or $$s = \frac{1}{2m}\log\left\{\frac{2g + ma^2}{2g + mv^2}\right\}. \qquad (7)$$

Consequently, as v may be determined from equation (6), for any given time t; we shall have, by substituting this value in equation (7), the space s, due to the same time.

If the greatest height h, of the projectile, were required, or that which answers to the case of $v = 0$; we should obtain it by substituting this value of v; which gives

$$h = \frac{1}{2m}\log\left(\frac{2g + ma^2}{2g}\right); \text{ or}$$

$$h = \frac{1}{2m}\log\left(1 + \frac{ma^2}{2g.}\right)$$

It is impossible to submit these formulæ to numerical computation till the value of m shall be determined from experiment; we shall only observe here, that this co-efficient, supposed constant for the same body in the same medium, will vary for bodies of the same form (for example spheres) in different mediums, in the inverse ratio of their radii and density; and directly, as the density of the medium: so that if r denotes the radius of any sphere, and D its density, also D′ the density of the fluid or medium, we shall have

$$m = \frac{\mu \text{D}'}{\text{D}r};$$

μ being a constant quantity, which will be the same for all spherical bodies, but which can only be determined from experiment.

It will appear, however, when we treat of practical Gunnery, that Dr. Hutton has shown, that the law which we have assumed above, viz. that the resistance is as the square of the velocity, is not general; but that this resistance varies as $mv^2 + nv$; namely, as a function of the velocity involving both its first and second power.

31. The problem above resolved, relative to the motion of a heavy body in a resisting medium, may be made to apply to the motion of bodies along inclined planes; the friction being supposed to vary as the square of the velocity. For, in order to accommodate our formulæ to this particular case, it is only necessary to

substitute, instead of the absolute force of gravity $2g$, the relative measure of it on the plane, viz. $\frac{2hg}{l}$ (where h is the height, and l the length of the plane). When the plane is horizontal, then $2g = 0$; and we must therefore substitute this value of $2g$, and we shall obtain the result sought.

It will, however, be easy to determine the circumstances of the latter motion, independent of our preceding investigations; for since, as we have seen in this case, $2g = 0$, our equation of motion will be simply

$$\frac{\dot{v}}{\dot{t}} = -mv^2:$$

whence, by taking the fluents and determining the variations, we have immediately

$$v = \frac{a}{1 + amt} \text{ and } s = \frac{1}{m}\log(1 + amt);$$

a being the initial velocity.

This value of v shows, that if we suppose the friction or resistance m to be zero, the velocity will be uniform; a result which it is obvious we ought to obtain: and also, that the greater m is, the greater will be the diminution of velocity in the same time; and lastly, that however small m may be, as the time increases the denominator increases, and the value of v diminishes, and will become finally $= 0$: this also is a result which we find continually verified from observation.

Problem VI.

32. *To determine the conditions of the motion of a body acted upon by a repulsive and attracting force.*

Suppose that a material point, or a very small globe, placed at A (fig. 20), is solicited by two forces: the one tending to make it move from A towards B, with a motion uniformly varied; the other tending, on the contrary, to push it back from A towards D; and let the circumstances of the motion of the particle be required, on the supposition that the repulsive power impress upon it an accelerating force, varying inversely as the distance from the point B.

Assume $AB = a$, $AN = s$, or the space passed over at the end of the time t; the acceleration that arises from the repulsion from A towards D, will be $= \frac{m}{a+s}$, m being a constant quantity depending upon the law, according to which the repulsive force acts. Lastly, let $2g$ denote the constant accelerating force, which tends to give the body any opposite direction. We shall thus have

$$\phi = \frac{\ddot{s}}{\dot{t}^2} = \frac{m}{a+s} - 2g:$$

multiplying by $2\dot{s}$, we have

$$\frac{2\dot{s}\ddot{s}}{\dot{t}^2} = \frac{2m\dot{s}}{a+s} - 4g\dot{s};$$

and taking the fluents,

$$\frac{\dot{s}^2}{\dot{t}^2} = v^2 = 2m\log(a+s) - 4gs + C. \qquad (1)$$

In order to determine the constant arbitrary C, we have, when $s = 0$, $v = 0$; whence

$$0 = 2m\log a + C.$$

Wherefore, by subtraction,

$$v^2 = 2m\log(a+s) - 2m\log a - 4gs;$$

or,

$$v = \pm\sqrt{\left\{2m\log\frac{a+s}{a} - 4gs;\right\}}$$

which equation determines the velocity of the body after it has passed over any given space s.

If we solve equation (1), with reference to $\dot{t}$, we shall have

$$\dot{t}^2 = \frac{\dot{s}^2}{2m\log(a+s) - 4gs + C};$$

or,

$$\dot{t} = \frac{\dot{s}}{\sqrt{\{2m\log(a+s) - 4gs + C\}}}.$$

If we suppose still the same correction, this becomes

$$\dot{t} = \frac{\dot{s}}{\pm\sqrt{\left\{2m\log\frac{a+s}{a}4gs\right\}}}:$$

it is obvious, however, that there will be considerable difficulty in obtaining the fluent of this latter quantity; and as it is not important, with respect to the application we propose to make of the problem, we shall not attempt the integration in this place.

As to the velocity, the problem above solved finds its application in the following case. If a heavy body, as a piston, is forced into a cylinder or vertical tube BD, open only at the extremity D, which the piston closely fits; and if the part AB is full of a compressed and elastic fluid, or an expansive vapour; then, not considering the friction of the piston against the sides of the tube, it is evident that this piston will be subjected to the action of gravity; which tends to make it descend, and impresses a constant accelerating force $2g$, and at the same time to the repulsive force of the elastic fluid. But this fluid having less elasticity as it is less compressed, viz. as the piston is farther distant from the extremity B, the accelerating force thence arising varies inversely as the distance of the moveable piston from the bottom of the tube.

Another example of this species of motion is in the case of balls, in guns or pieces of cannon, impelled by the inflammation of gunpowder. This produces instantaneously a great quantity of aeriform fluid, the repulsive force of which is inversely as the space it occupies.

In this instance, however, we may neglect the consideration of the weight of the ball, since it has but little effect upon the velocity up to the mouth of the piece; the weight being nothing in theory, when the axis of the gun is horizontal. We may therefore, in this case, make $2g = 0$; or, which amounts to the same, we may consider at the commencement of the calculation, the accelerating force $\phi = \frac{m}{a+s}$; consequently, $v = \sqrt{\left(2m\log\frac{a+s}{a}\right)}$.

And in this equation, assuming $s =$ the length of the gun, or rather, equal to the distance of the ball in its first position from the muzzle, which we may suppose $= l$, we shall have

$$v = \sqrt{\left(2m\log\frac{a+l}{a}\right)},$$

for the velocity of projection; that is, the velocity with which the ball issues from the gun.

It appears from this equation, that v will be greater as the length l is greater; but in practice it is said that this does not obtain: we must, therefore, attribute the defect to having neglected the friction of the ball against the interior of the bore of the gun. But this, and other practical considerations, will be resumed when we treat of Gunnery.*

§ VIII. *Of the curvilinear motion of a material point.*

33. At present we have only considered the circumstance of the motion of a body in a right line: we must now endeavour to investigate the same with respect to that motion being made in any curve line whatever.

When the motion of a material particle is not made in a right line, the deflection is obviously due to the action of two or more forces acting upon the moveable in different directions. The first question, therefore, which ought, in this investigation, to engage our attention, is to determine the velocity and the direction which a material point will take in consequence of the action of these different forces.

In order to resolve this problem in all its generality, it will be necessary for us to establish a new principle, viz. the *parallelogram of velocities;* which, though frequently confounded with what we have already treated of under the denomination of the parallelogram of forces, is, notwithstanding, a distinct principle, although there is a certain similarity and connection between them.

If, in any unit of time, two forces P and Q (fig. 21), communicate to a material point m, velocities denoted by mB, mC, the resultant R of the forces P, Q, will communicate to it in the same unit of time a velocity mD, which will be the diagonal of the parallelogram constructed on the velocities mB and mC. In fact, as we are always at liberty to represent a given force by any part of its direction, the force Q may be represented by mC; in which case the force P will be equally denoted by mB, because the velocities are proportional to the forces. Now, considering mBDC as the parallelogram of the two forces P and Q, the diagonal mD will denote the intensity and the direction of the resultant R, of P and Q; and it remains to determine the velocity that this resultant is competent to communicate to the material point, and which we shall see is equal to the diagonal of the parallelogram, of which the sides denote the velocities of the two original forces P and Q.

In order to accomplish this, let x represent the velocity which R is capable of communicating to the particle m; then the velocity being proportional to the forces, we shall have

$$P : R :: mB : x;$$

and the property of the parallelogram of forces gives

$$P : R :: mB : mD;$$

whence we obtain

$$mB : mD :: mB : x;$$

consequently, $x = mD$:

that is, we may regard the diagonal of the parallelogram, constructed on the two lines denoting the velocities due to the forces P and Q, as equal to, or as denoting the velocity which R will communicate to the same particle.

34. The parallelopiped of velocities follows as an immediate consequence of the parallelogram of velocities; for let P, Q, R (fig. 22), be three forces which communicate the velocities mp, mq, mr, to a material point m; and let us compound the velocities mp and mq into one mp'; it results, from the preceding article, that this velocity will be the same as that which will be communicated to the point m by the resultant P′ of the forces P and Q: and in the same manner the resultant ms, of the velocities mp', mr, will represent the intensity and direction of the velocity due to the force S, and the resultant of the two forces P′ and R, or of the three original forces P, Q, and R: therefore the diagonal ms of the parallelopiped of velocities, will denote the velocity, both in intensity and direction, communicated to the particle m by the three forces P, Q, and R.

Curvilinear motion.

35. It follows, from the preceding investigation, that while a moveable is acted upon by simultaneous uniform forces, or while it is moving with a uniform velocity, it will describe a right line; but if one or more of the velocities impressed upon the body be variable according to different laws, it will no longer preserve its rectilinear direction, but move in some curve, the nature of which will depend upon the relation of the different velocities with which it is impressed: let us, therefore, now consider the circumstances of the latter motion.

In order to this, we will suppose, in the first place, a material point m, at rest, to cede to an instantaneous impulsion, which causes it to describe the right line mA (fig. 23) in the time θ; and that at the end of this time it receives a second impulsion, capable of making it describe, in a second equal time θ, the right line AB: now the body will not obey entirely the force impressing upon it the direction AB, in consequence of the law of inertia, by which it tends, in consequence of the first force, to preserve its rectilinear direction, and by which it would, in the second time, describe the right line AC $=$ Am; but its actual direction, from what has been above illustrated, will be in the diagonal AD of the parallelogram ABCD. If it receive at the point D another impulsion, which would cause it to describe the space DG in the third time θ, it will, on the same principles, describe the diagonal DF, of the parallelogram constructed on DG, and the prolongation DE, of the side AD; and so on for any number of successive impulsions: so that at the end of the time composed of n equal intervals θ, the moveable will have described a polygon of n sides, each of which will be passed over with a uniform velocity, but each different from the preceding; the first side being described with the velocity due to the first impulsion, the second with the velocity due to the resultant of the first and second forces, the third with that due to the resultant of the three first forces, and so on; and we may easily conceive, that as the interval of time θ, between the successive impulsions become less and less, the path of the body will approach nearer and nearer to a continued curve, which it, in fact, ultimately becomes, when we consider these intervals as indefinitely small.

While we give to the time θ any sensible duration, the sides of the polygon will have a determinate magnitude; and if, in this case, when the body is describing any one of these sides, it should receive no new impulsion, it will fly off in the direction of that side, as is obvious; and if, as in the second case, we suppose the interval θ to be indefinitely small, and the accelerating force to cease at any instant, the body will fly off in the direction of the tangent of the curve at the point where it was at that instant.

And, according to the same hypothesis of θ being indefinitely small, we shall have $\theta = \dot{t}$, and what we have above considered as the side of the polygon will become the element or fluxion of the curve; and consequently, in order to obtain the velocity or the space which would be passed over in a unit of time in the direction of the tangent of the curve, on the supposition that the accelerating force cease to act, we must repeat $\dot{s}$ (which denotes the element of the curve) as often as $\dot{t}$ is contained in unity, which will give us a general expression for the velocity; that is, denoting the velocity by v, the curve by s, and the time by t, we shall have

$$v = \frac{\dot{s}}{\dot{t}}.$$

Equations of motion.

36. Having established this principle, let us return again to the case in which the particle is supposed to receive its successive increases of velocity at the points $m', m'', m''', m^{\text{iv}}$, &c. (fig. 24) of the polygon m, m', m'', m''', &c.; and let v, v', v'', v''', &c. be the velocities which it receives at the points m, m', m'', &c.; and θ, θ' θ'', &c. the time that it will employ in describing the successive sides $m\,m'$, $m'm''$, $m''\,m'''$, $m'''\,m^{\text{iv}}$, &c. As, by the hypothesis, the velocity is constant while the particle is describing each of those sides, we shall have, from the nature of uniform motion,

$$m\,m' = v\,\theta, m'm'' = v'\theta', \; m''m''' = v''\theta'', \text{\&c.};$$

and consequently the perimeter of the polygon $m\;m'\;m''\;m'''$, &c. will be expressed by

$$v\theta + v'\theta + v''\,\theta'' + v'''\theta''' +, \text{\&c.}$$

Now, if we project the sides of this polygon on the co-ordinate axes, and if we call

$$\alpha, \beta, \gamma; \; \alpha', \beta', \gamma''; \; \alpha'', \beta'', \gamma'', \text{\&c.}$$

the angles formed by the direction of the velocities

$$v, v', v'', v''', \text{\&c.}$$

with the co-ordinate axes, these velocities will have for their projections

$$v \cos \alpha, \; v' \cos \alpha', \; v'' \cos \alpha'', \text{ on the axis } x;$$
$$v \cos \beta, \; v' \cos \beta', \; v'' \cos \beta'', \text{ on the axis } y;$$
$$v \cos \gamma, \; v' \cos \gamma', \; v'' \cos \gamma'', \text{ on the axis } z;$$

and consequently the projection of the contour of the polygon $m\;m'\;m''\;m'''$, &c. on the axis x, will be expressed by

$$v \cos \alpha\,\theta + v' \cos \alpha'\theta' + v''\,\alpha''\theta', + \text{\&c.}$$

Whence it appears, that while the point m describes the polygon $m\;m'\;m''\;m'''$, &c. its projection n will necessarily describe the space $n\;n'\;n''\;n'''$, &c.

Now, if this projection were only the result of the solicitation of a single force X, directed according to the axis x, and if it were such that the point n described in the times $\theta, \theta', \theta'', \theta'''$, &c. the right lines $n\,n'$, $n'n''$, $n''n'''$, &c., with the velocities v, v', v'', v''', &c., the path of the projection, or the space that it passed over along the axis x, would be represented by

$$v \cos \alpha\;\theta + v' \cos \alpha'\theta' + v'' \cos \alpha''\,\theta'' +, \text{\&c.}$$

which is precisely the same as we found above; and the identity of these two expressions, shows that when the point m is moving in space, the projection of its motion on the axis n, is the same as if the other forces did not act; for in the calculation which led to our second conclusion, we have made no reference whatever to the action of any other force except that of X, that is to only one single force.

All that we have demonstrated above, with reference to the axis x, will obviously apply to the other two axes; and when the successive intervals θ, θ', θ'', &c. become indefinitely small, that is, when the polygon becomes a continued curve, the same conclusions may still be drawn; viz. when a particle is solicited by an accelerating force, such as will cause it to describe a curve in space, any one of its projections will move the same as if the other two did not exist. If, therefore, we denote by X, Y, Z, the components of the accelerating force ϕ, in the directions of the axes of the co-ordinates x, y, z, we may consider these as accelerating forces, which would impress on the corresponding projections of the body, motions independent of the other two.

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Hence, then, we conclude generally, that *if we resolve into three forces parallel to three fixed axes, any forces whatever, which acting on a material point produce in it a curvilinear motion; and if we consider as moveable points the projection of the material point on these axes; the motion on each axis will be that due to the forces parallel to this line, and will be exactly the same as if the other forces were zero, or had no action upon the body.*

In each particular case, therefore, we have only to determine the motion on each of the axes, from the equation of rectilinear motion given in the preceding chapter, and thus every problem relative to curvilinear motion, is reduced to the consideration of three rectilinear motions; or to two only, when, from the nature of the data, we are certain that the curvilinear motion is made in one and the same plane.

We may illustrate the latter case by the following simple example:

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Let us suppose a body placed in any point A (fig. 25) of an indefinite vertical plane, and let it be conceived to receive an impulsion which would project it forward in a horizontal line if that force alone acted; but that at the same instant it is submitted to the accelerating force of gravity: in this case, it is in the first place obvious, that the motion of the body will be made wholly in the vertical plane in which the horizontal force acts, for there is nothing to determine it to move either on one side or the other of that plane; and in the second place, the horizontal motion being uniform, $t\,v$ will denote the space described by its projection along the axis x, and by the laws of falling bodies $g\,t^2$ will represent the projection of its vertical motion on the axis y (t denoting any time, v the velocity of projection, and $g = 16\frac{1}{12}$ feet); and knowing thus the distance of the body at the end of the time t from the two given axes, its position on the plane will also be determined; and this, wholly independent of the nature of the line or curve which the body describes in that plane. In order to determine the equation of the curve, we observe, that if $\text{A}\,a = x = t\,v$, $\text{A}\,c = g\,t^2$, and if t' denotes the time the body would employ in arriving at the point a', we shall have $\text{A}\,a' = x' = t'\,v$, and $\text{A}\,c' = y' = g\,t'^2$; whence

$$\text{A}\,a : \text{A}\,a' :: t : t',$$

and

$$\text{A}\,c : \text{A}\,c' :: t^2 : t'^2;$$

but

$$(\text{A}\,a)^2 : (\text{A}\,a)^2 :: t^2 : t'^2;$$

therefore

$$\text{A}\,c : \text{A}\,c' :: (\text{A}\,a)^2 : (\text{A}\,a')^2.$$

This is a known property of the parabola, and which, therefore, is the trajectory or path of the projectile.

37. We may draw also from the preceding results the following general conclusion, viz.: that the relative motion of any number of bodies in a given space is the same whether that space be in motion or at rest. For let us consider an indefinite plane, in which any number of material points are in motion, arising from the action of different forces in that plane; and let us conceive that forces equal to each other are applied to all the points of this plane, and perpendicular to it, so that the plane itself may be put in motion in space, but remaining always parallel to itself. It is then obvious, that each of the moveable points will be submitted to the simultaneous action of the particular forces which were acting upon it at first, and to the force applied to every point of the plane; and the motion of each material point in space will be produced by the resultant of these different forces. Now it appears, from art. 36, that if we refer the motion to three fixed axes, of which one is perpendicular, and the other two parallel to the moveable plane, the motion of the projections of the bodies, on these two last axes, will not be changed by the action of the force parallel to the first; that is, the motion of each material point in the plane, which is supposed to have become moveable, will remain the same as if it were at rest: and hence we conclude, that the common motion of all the bodies of a system changes not the particular motion of these bodies; that is, they will preserve amongst themselves the same distances, and the same relative velocities, as if the common motion of the system did not exist.

38. In order to determine the analytical expressions of the accelerating forces, we must observe, that when the material point has described the space $\dot{s}$, its projections will have described the spaces $\dot{x}, \dot{y}, \dot{z}$; and consequently the velocity of the projections on the direction of the co-ordinate axes will be respectively

$$\frac{\dot{x}}{\dot{t}}, \frac{\dot{y}}{\dot{t}}, \frac{\dot{z}}{\dot{t}};$$

and as these accelerating forces are the fluxional coefficients of the velocities, taken with reference to the time, we shall have

$$X = \frac{\ddot{x}}{\dot{t}^2},\ Y = \frac{\ddot{y}}{\dot{t}^2},\ Z = \frac{\ddot{z}}{\dot{t}^2}. \qquad (1)$$

Such are the equations which will serve to determine the motion of a material point in a curve line, or of a heavy mass, considering its centre of gravity as the place of the body.

When the functions X, Y, and Z, are given from the conditions of the problem, if we can obtain the fluents of the preceding equations they will contain the four variables x, y, z, and t; and if afterwards we eliminate t, we shall still have two, containing the three variables x, y, and z; which equations will be those of the trajectory or curve resulting from the action of the accelerating forces.

If all the forces are in one plane, we may consider it as the plane of the two axes x, and y, and the variable z will disappear; in which case, therefore, we need only consider the two equations,

$$X = \frac{\ddot{x}}{\dot{t}^2},\ \text{and } Y = \frac{\ddot{y}}{\dot{t}^2}.$$

And when from the nature of the problem we shall have determined X and Y, if we can obtain the fluents of these equations, they will contain only the three variables x, y, and t; and when t is eliminated, we shall find one equation expressing x in functions of y, or y in functions of x, and which will contain the equation of the plane curve or trajectory of the body.

38. We have seen (art. 35) that the velocity of the moveable is given by the equation Of the velocity.

$$v = \frac{\dot{s}}{\dot{t}};$$

and we know from the principles of fluxions that the element of a curve in space, as referred to three rectangular co-ordinates, is

$$\dot{s} = \sqrt{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)};$$

if, therefore, we substitute this value in the preceding equation, we shall have

$$\left.\begin{aligned} v &= \frac{1}{\dot{t}}\sqrt{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}; \\ \text{or,}\quad v &= \sqrt{\left(\frac{\dot{x}^2}{\dot{t}^2} + \frac{\dot{y}^2}{\dot{t}^2} + \frac{\dot{z}^2}{\dot{t}^2}\right)}. \end{aligned}\right\} \qquad (2)$$

With regard to the angles that the velocities make with the axes, these being denoted by α, β, γ, they will be determined from the equations

$$v \cos \alpha = \frac{\dot{x}}{\dot{t}},$$

$$v \cos \beta = \frac{\dot{y}}{\dot{t}},$$

$$v \cos \gamma = \frac{\dot{z}}{\dot{t}}.$$

39. There is, however, another method by which we may determine the velocity, which consists in multiplying our equations, No. 1; the 1st, by $2\dot{x}$; the 2d by $2\dot{y}$; and the 3d by $2\dot{z}$; and then adding the results, which give Another method.

$$\frac{2\dot{x}\ddot{x} + 2\dot{y}\ddot{y} + 2\dot{z}\ddot{z}}{\dot{t}^2} = 2X\dot{x} + 2Y\dot{y} + 2Z\dot{z};$$

and observing that the first member in this equation is the fluxion of $\dot{x}^2 + \dot{y}^2 + \dot{z}^2$ divided by $\dot{t}^2$, we shall have

$$\frac{\text{f.}^{*}\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}{\dot{t}^2} = 2(X\dot{x} + Y\dot{y} + Z\dot{z}).$$

And now, replacing $\dot{x}^2 + \dot{y}^2 + \dot{z}^2$, by its equal $\dot{s}^2$, we shall have, regarding $\dot{t}$ as constant,

$$\frac{\text{f. } \dot{s}^2}{\dot{t}^2} = 2(X\dot{x} + Y\dot{y} + Z\dot{z});$$

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[*] We use here, as we have already explained, page 28, f to denote the fluxion of a compound quantity: the above expression therefore, f. $(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$, is what, according to the strict fluxional notation, would be written $(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{\cdot}$

or taking the fluents,

$$\frac{\dot{s}^2}{\dot{t}^2} = 2\int (X\dot{x} + Y\dot{y} + Z\dot{z}) + C;$$

and which again is equivalent to

$$v^2 = 2\int (X\dot{x} + Y\dot{y} + Z\dot{z}) + C; \qquad (3)$$

C being a constant arbitrary quantity or correction, to be determined from the condition of the problem.

When the formula

$$X\dot{x} + Y\dot{y} + Z\dot{z},$$

which is given above under the sign of integration, is an exact fluxion of the three variables x, y, and z; this last equation will give the velocity of the body in any point of the trajectory, provided only that we know the velocity at any determined point; in fact, in taking the fluent of this formula, we shall have v expressed in certain functions of x, y, and z, which we may denote by $f(x, y, z)$, and consequently that equation will become

$$v^2 = f(x, y, z) + C.$$

And if we denote by V, the velocity at the point of the trajectory of which the co-ordinates are a, b, c, we shall have, in order to determine C,

$$V^2 = f(a, b, c);$$

whence by subtraction

$$v^2 - V^2 = f(x, y, z) - f(a, b, c),$$

an equation from which v may be determined, that of V being supposed given, and the values of the co-ordinates a, b, c, and x, y, and z, being also known.

It appears from the above, and the result is very remarkable, that we can determine the difference of the squares of the velocity in two points of the trajectory, by means of the co-ordinates only, that is to say, without knowing the curve that the moveable describes in passing from one of those points to the other. We must not however forget, that this is only true when

$$X\dot{x} + Y\dot{y} + Z\dot{z}$$

is an exact and complete fluxion of the three variables x, y, and z, which is frequently not the case. If for example, the forces X, Y, and Z, proceed from friction, the resistance of a fluid, &c. they will involve in their values, the velocities

$$\frac{\dot{x}}{\dot{t}},\ \frac{\dot{y}}{\dot{t}},\ \frac{\dot{z}}{\dot{t}},$$

as we shall see in a subsequent article; and consequently the formula

$$X\dot{x} + Y\dot{y} + Z\dot{z},$$

will not then be an exact fluxion of any function of x, y, z, regarded as independent variables. In order to obtain the fluents in these cases, we must substitute the values of these variables, and their fluxions in functions of the time, which supposes the problem already solved.

Particular case.

40. There is, however, a particular and very extensive case in which the above formula is always a complete fluxion, viz. when all the accelerating forces acting on the moveable are directed towards fixed centres, and where the intensity of each of them is a function of the distance of the moveable from its centre of action.

In order to demonstrate this property, first, for one centre, we may observe, that the forces which act on the body being in this case reducible to one resultant only, passing through the fixed centre, and acting in the direction MC (fig. 26), it may be represented by any part of this line, as CD. Let us take the centre C for the origin; and denote by λ, its distance from the point M; and by α, β, γ, the angles that CM = λ makes with the co-ordinate axes: then the resultant R, forming the same angles, we shall have

$$X = R\cos\alpha;\ Y = R\cos\beta;\ Z = R\cos\gamma:$$

and, consequently,

$$\frac{X}{Y} = \frac{\cos\alpha}{\cos\beta};\ \frac{Y}{Z} = \frac{\cos\beta}{\cos\gamma};\ \frac{Z}{X} = \frac{\cos\gamma}{\cos\alpha}. \qquad (4)$$

Now if we call x, y, and z, the co-ordinates of the point M, where the moveable is situated, there will result

$$x = \lambda\cos\alpha,\ y = \lambda\cos\beta,\ z = \lambda\cos\gamma;$$

whence we draw the following proportions, viz.

$$x : y : z :: \cos\alpha : \cos\beta : \cos\gamma:$$

and hence again

$$\frac{\cos\alpha}{\cos\beta} = \frac{x}{y};\ \frac{\cos\beta}{\cos\gamma} = \frac{y}{x};\ \frac{\cos\gamma}{\cos\alpha} = \frac{z}{x}.$$

Substituting these values in equation (4), we obtain

$$\left.\begin{array}{l} yX - xY = 0 \\ zY - yZ = 0 \\ xZ - zX = 0 \end{array}\right\}$$

If here we put for X, Y, and Z, their values, as determined in equation (1), we shall find

$$y\frac{\ddot{x}}{\dot{t}^2} - x\frac{\ddot{y}}{\dot{t}^2} = 0,$$

$$z\frac{\ddot{y}}{\dot{t}^2} - y\frac{\ddot{z}}{\dot{t}^2} = 0,$$

$$x\frac{\ddot{z}}{\dot{t}^2} - z\frac{\ddot{x}}{\dot{t}^2} = 0.$$

Multiply the first of these equations by $\dot{t}$, and find the fluents by parts, and we shall have

$$\frac{y\dot{x} - x\dot{y}}{\dot{t}} = C. \qquad (5)$$

Proceeding in the same manner with the other equations, and then multiplying by $\dot{t}$, there will be obtained the following results, *i. e.*

$$y\dot{x} - x\dot{y} = C\dot{t},$$

$$z\dot{y} - y\dot{z} = C'\dot{t},$$

$$x\dot{z} - z\dot{x} = C''\dot{t}.$$

Multiply the first of these equations by z, the second by x, the third by y, and take the sum of the products, and we shall have

$$0 = (Cz + C'x + C''y)\dot{t}:$$

Whence again $Cz + C'x + C''y = 0$.

Since this equation is that of a plane passing through the origin, that is, through the centre of attraction; it shows that the body moves in a plane curve. Therefore, in resolving the problem anew, and placing the trajectory in the plane of the axes x, y, we shall have no occasion for the equation $Z = \frac{\ddot{z}}{\dot{t}^2}$, nor for the quantities Z and z, which in this case become zero.

We have therefore only to consider the equation

$$y\dot{x} - x\dot{y} = C\dot{t};$$

from which we deduce

$$\int(y\dot{x} - x\dot{y}) = Ct + C'; \qquad (6)$$

C′ being a constant arbitrary, arising out of this integration.

In order to obtain the above fluent, we may remark that $y\dot{x}$ being the fluxion or element of a plane curve surface, we may assume that surface to be comprised between the absciss $x = 0$, and $x = \text{CP}$ (fig. 27); wherefore the expression $\int y\dot{x}$ will be represented by the area LCPM. If we subtract from this surface the triangle CPM, there will remain

$$\textit{sector}\ \text{LCM} = \textit{area}\ \text{LCPM} - \textit{trian.}\ \text{CPM},$$

or $\textit{sector}\ \text{LCM} = \int y\dot{x} - \frac{xy}{2}.$

Taking here the fluxions, and reducing, we obtain

$$\text{f}\,\textit{sector}\ \text{LCM} = y\dot{x} - \frac{\dot{x}y + \dot{y}x}{2}, \text{ or}$$

$$\text{f}\,\textit{sector}\ \text{LCM} = \frac{y\dot{x} - \dot{y}x}{2}.$$

Again taking the fluents,

$$2\ \textit{sector}\ \text{LCM} = \int(y\dot{x} - x\dot{y});$$

whence equation (6) becomes

$$2\ \textit{sectors}\ \text{LCM} = \text{C}\,t + \text{C}'.$$

But since we may suppose that the time commences when the body is in the point L, the correction $\text{C}' = 0$, and we have simply

$$2\ \textit{sectors}\ \text{LCM} = \text{C}\,t,$$

or $\qquad \textit{sector}\ \text{LCM} = \text{A}\,t,$

(making $\text{C} = 2\,\text{A}$). From which result it appears, that when a moveable is solicited by a force which attracts it towards a centre C, it will describe a curve LM, such that the surface of the sector LCM is proportional to the time that the body employs in describing the curve.

This property is known under the name of the *principle of the conservation of areas*, and, as it regards the motions of the planets, was first discovered by Kepler from observation, being the second of the three principal laws discovered by that celebrated astronomer.

41. We have seen (art. 39) that the determination of the velocity depends upon our being able to determine the fluent of the formula

$$\text{Z}\dot{z} + \text{Y}\dot{y} + \text{X}\dot{x},$$

which, as we have stated, may always be effected when the forces are directed towards any number of fixed centres, and are functions of the distances of the centre of attraction of the body from those centres.

Let M (fig. 28) be the centre of attraction of any body which is attracted towards the centres C, C′, C″, &c. by the forces P, P′, P″, &c.; and let us call

x, y, z, the co-ordinates of the point M;
a, b, c, the co-ordinates of the centre C;
a', b', c', the co-ordinates of the centre C′;
a'', b'', c'', the co-ordinates of the centre C″;
p, p', p'', &c. the distances CM, C′M, C″M;
α, β, γ, &c. the angles formed by p,
α', β', γ', &c. the angles formed by p',
α'', β'', γ'', &c. the angles formed by p'',

with the co-ordinate axes.

The resultant of all these attractive forces will have for their components, in the direction of the co-ordinate axes,

$$\left.\begin{aligned} \text{X} &= \text{P}\cos\alpha + \text{P}'\cos\alpha' + \text{P}''\cos\alpha'' + \&\text{c.} \\ \text{Y} &= \text{P}\cos\beta + \text{P}'\cos\beta' + \text{P}''\cos\beta'' + \&\text{c.} \\ \text{Z} &= \text{P}\cos\gamma + \text{P}'\cos\gamma' + \text{P}''\cos\gamma'' + \&\text{c.} \end{aligned}\right\}. \qquad (7)$$

The projection of the right line CM on the axis of x being represented in (fig. 28), by BD, we have

$$\text{BD} = \text{AB} - \text{AD};$$

and observing, that AB and AD, are in fact, the co-ordinates x, and a, of the points M and C on the axis Ax, and BD being the projection of MC on the same axis, it will have for its analytical expression $p\cos\alpha$; and we shall find, by substituting these values in the preceding equation

$$p\cos\alpha = x - a;$$

and since what we have here shown to obtain with regard to the projection of MC on one of the axis, will equally have place with respect to the other two, we shall have, in order to determine the angles α, β, γ, the equations

$$p\cos\alpha = x - a;\ p\cos\beta = y - b;\ p\cos\gamma = z - c.$$

In the same manner, the angles α', β', γ', α'', β'', γ'', &c. will be given by the equations

$$p'\cos\alpha' = x - a';\ p'\cos\beta' = y - b';\ p'\cos\gamma' = z - c',$$
$$p''\cos\alpha'' = x - a'';\ p''\cos\beta'' = y - b'';\ p''\cos\gamma'' = z - c''.$$

Consequently, after eliminating the angles, equations (7) will become

$$\left.\begin{aligned} \text{X} &= \text{P}\frac{x-a}{p} + \text{P}'\frac{x-a'}{p'} + \text{P}''\frac{x-a''}{p''} + \&\text{c.} \\ \text{Y} &= \text{P}\frac{y-b}{p} + \text{P}'\frac{y-b'}{p'} + \text{P}''\frac{y-b''}{p''} + \&\text{c.} \\ \text{Z} &= \text{P}\frac{z-c}{p} + \text{P}'\frac{y-c'}{p'} + \text{P}''\frac{y-c''}{p''} + \&\text{c.} \end{aligned}\right\}(8)$$

Substituting these values in the formulæ

$$\int(\text{X}\dot{x} + \text{Y}\dot{y} + \text{Z}\dot{z}),$$

there will be obtained

$$\left.\begin{aligned} &\int(\text{X}\dot{x} + \text{Y}\dot{y} + \text{Z}\dot{z}) = \\ &\int\text{P}\left\{\frac{x-a}{p}\dot{x} + \frac{y-b}{p}\dot{y} + \frac{z-c}{p}\dot{z}\right\} \\ &+\int\text{P}'\left\{\frac{x-a'}{p'}\dot{x} + \frac{y-b'}{p'}\dot{y} + \frac{z-c'}{p}\dot{z}\right\} \\ &+ \qquad \&\text{c.} \qquad \&\text{c.} \qquad \&\text{c} \end{aligned}\right\}$$

Now the distances of the point M from the centre C, C′, C″, &c. being given by the equations

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = p^2,$$
$$(x-a')^2 + (y-b')^2 + (z^2-c')^2 = p'^2,$$
$$\&\text{c.} \quad + \quad \&\text{c.} \qquad \&\text{c.}$$

we shall find after taking the fluxion:

$$\frac{x-a}{p}\dot{x} + \frac{y-b}{p}\dot{y} + \frac{y-o}{p}\dot{z} = \dot{p},$$
$$\frac{x-a'}{p'}\dot{x} + \frac{y-b'}{p'}\dot{y} + \frac{z-b'}{p'}\dot{z} = \dot{p}'.$$
$$\&\text{c.} \quad + \quad \&\text{c.}$$

Substituting now these values in the above equations, we have

$$\int(\text{X}\dot{x} + \text{Y}\dot{y} + \text{Z}\dot{z}) =$$

$$\int(\text{P}\dot{p} + \text{P}'\dot{p}' + \text{P}''\dot{p}'' + \&\text{c.}) \qquad (9).$$

and since, by the hypothesis, P, P′, P″, &c. are functions of p, p', p'', &c. the expression

$$\int(\text{P}\dot{p}' + \text{P}'\dot{p}' + \text{P}''\dot{p}'' + \&\text{c.})$$

will contain only one variable in each of its terms, and consequently its fluent may be obtained, or referred to the quadratures of plane curved areas.

Example of the application.

42. In order to give an application of this theorem, let it be proposed to determine the velocity of a body, which is drawn towards one fixed centre only by a force P, whose intensity varies inversely as the square of the distance of the centre of attraction of the body from the fixed centre.

Let us place the axis of the z's, on the direction of this force; then denoting the fixed centre by C (fig. 29), we shall have

$$p = \mathrm{AC} - \mathrm{AM} = c - z;$$

and consequently

$$\dot{p} = -\dot{z}.$$

Call g the effect of the force P at the distance r, from the centre C, and P its effect at the distance p, then we shall have

$$g : \mathrm{P} :: \frac{1}{r^2} : \frac{1}{p^2};$$

whence

$$\mathrm{P} = \frac{g\,r^2}{p^2}.$$

The value of $\dot{p}$ being negative, $\mathrm{P}\,\dot{p}$ ought to be replaced by

$$-\frac{g\,r^2}{p^2}\,\dot{p};$$

now taking the fluents, we reduce equation (9), to

$$\int (\mathrm{X}\dot{x} + \mathrm{Y}\dot{y} + \mathrm{Z}\dot{z}) = \frac{g\,r^2}{p};$$

substitute this in equation (3), and we obtain

$$v^2 = \mathrm{C} + 2\,\frac{g\,r^2}{p}.$$

In order to determine the correction C, we shall suppose, that the body M begins its motion at a point, the distance of which from the centre C, $= a$; the velocity being therefore zero in that point, we shall have

$$0 = \mathrm{C} + 2\,\frac{g\,r^2}{a};$$

consequently the correct fluent will be

$$v^2 = 2\,g\,r^2\left(\frac{1}{p} - \frac{1}{a}\right).$$

§ IX. *Of the motion of projectiles in a vacuum.*

Projectiles. Definitions.

44. When a heavy body is impelled by an instantaneous force in any direction, either vertical, parallel, or oblique to the horizon, it is said to be *projected;* the body itself is called a *projectile;* and the curve or line which it describes is denominated the *path of the projectile.* These terms, however, are more generally supposed applicable to those cases in which the first projectile direction is in a line oblique or parallel to the horizon; as in the projection of balls and shells from mortars, or other pieces of ordnance.

It is evident, in the first place, that the curve described by a projectile, is comprised in a vertical plane drawn through the line of direction of the impelling force: let us, therefore, conceive this plane to coincide with the plane x, y of our rectangular co-ordinate axes; of which A x (fig. 30) is horizontal, and A y vertical; also the y's positive above the axis A x.

General equations.

Since the entire motion takes place in this plane, it will obviously be sufficient to introduce only our two first equations (art. 37); and moreover, because the projectile force is impulsive, the accelerating force denoted by X in those equations will be zero, and that represented by Y will be $= -2g$, where $2g = 32\frac{1}{6}$ feet, or the force of gravity; which is taken negative, because it has a tendency to retard the natural motion of the body; these equations therefore become

$$\mathrm{X} = \frac{\ddot{x}}{\dot{t}^2} = 0, \text{ and } \mathrm{Y} = \frac{\ddot{y}}{\dot{t}^2} = -2g;$$

whence, by taking the fluents, we obtain

$$x = b\,t + b', \text{ and } y = -g\,t^2 + c\,t + c';$$

c, c', b, b', being constant arbitrary quantities or corrections.

In order to determine them, let us place the origin of the co-ordinates x, y, at the point of projection, and estimate the time t from the instant the motion commences. By these means we shall have at the same time $x = 0$, $y = 0$, and $t = 0$; consequently, $b' = 0$ and $c' = 0$. Let us denote by a the velocity of projection; and conceive AB (fig. 30) to be the initial direction; call BA x the angle this line forms with the horizontal axis A $x = \alpha$: then the angle of direction, as referred to the other axis, will be $90° - \alpha$; and we shall have

$$a \cos \alpha, \text{ and } a \sin \alpha,$$

for the horizontal and vertical velocities of the body at the commencement of the motion.

Now, from what has been taught in the preceding sections, these velocities are expressed at any instant by $\frac{\dot{x}}{\dot{t}}$ and $\frac{\dot{y}}{\dot{t}}$; we have, therefore, by taking the fluxions of x and y in the above equations,

$$\frac{\dot{x}}{\dot{t}} = b,\ \frac{\dot{y}}{\dot{t}} = -2\,g\,t + c;$$

making afterwards $t = 0$, we obtain

$$b = a \cos \alpha, \text{ and } c = a \sin \alpha.$$

The corrections being thus determined, the values of x and y become

$$x = a\,t \cos \alpha,\ y = -g\,t^2 + a\,t \sin \alpha.$$

Again, eliminating t out of these equations, we find that of the curve or path of the projectile to be

$$y = x \tan \alpha - \frac{2\,g}{2\,a^2 \cos^2 \alpha}\,x^2;$$

which is obviously the equation of a parabola, having its principal axis vertical. At the vertex, which we may suppose to be at C, the tangent is horizontal, and we have $\frac{\dot{y}}{\dot{x}} = 0$; whence we have for the co-ordinates AD, CD, of this point,

$$x = \mathrm{AD} = \frac{a^2 \cos \alpha \sin \alpha}{2\,g},$$

$$y = \mathrm{CD} = \frac{a^2 \sin^2 \alpha}{4\,g}.$$

For the velocity in any point of the curve we shall have, denoting that velocity by v,

$$\frac{\dot{x}^2}{\dot{t}^2} + \frac{\dot{y}^2}{\dot{t}^2} = v^2 = a^2 - 4\,g\,y.$$

The ordinate CD of the vertex of the parabola represents the greatest height to which the projectile will ascend, and we therefore give to the line itself that denomination. After having attained this point, the body begins to descend, and as the descending branch of the curve is similar to the ascending one, it will cut again the axis A x, in a point, in which the absciss AE is double of that which answers to the vertex. This abscis is commonly called the *amplitude* or *range* of the projectile; and observing that

$$2 \cos a \sin a = \sin 2\,a,$$

we have

$$AE = \frac{a^2 \sin 2\,a}{2\,g}.$$

Whence we learn, that in order, with a given force, to obtain the greatest range, we must have $\sin 2\,a$ a maximum, which is obviously when the angle $a = 45°$. We may also obtain the same value of AE, by making $y = 0$, in the equation of the curve.

The velocity being supposed given, if it be required to determine the angle a, such that the projectile may pass through a given point, of which the co-ordinates are $x = \beta$, and $y = \gamma$, we must introduce these values into the general equation of the curve, and we shall thus have, in order to determine a, the following equation; viz.

$$\gamma = \tan a.\,\beta - \frac{2\,g}{2 a^2 \cos^2 a}\beta^2.$$

In order to simplify this expression, let h be the height due to the velocity a, or the height from which a body must fall to acquire that velocity, so that $a^2 = 4\,g\,h$; let also $\tan a = z$, and consequently

$$\frac{1}{\cos^2 a} = 1 + z^2,$$

and our equation will thus become

$$4\,h\,\gamma + \beta^2 - 4\,h\,\beta z + \beta^2 z^2 = 0;$$

whence we deduce

$$z = \frac{2\,h \pm \sqrt{(4\,h^2 - 4\,h\,\gamma - \beta^2)}}{\beta}.$$

It appears, therefore, from this equation, which expresses the value of the tangent of the angle of elevation, that there are always two angles given by the double sign in the numerator of the above fraction, which answer to the same range; except when h and β have such a relation that $4\,h^2 = 4\,h\,\gamma + \beta^2$; in which case there will obviously be but one: and if $4\,h\gamma + \beta^2 > 4\,h^2$, these two values are imaginary, and the problem under the given data is then impossible.

Such is the method of investigating the motion of projectiles, when treated according to the general principles developed in the preceding section; but like many other branches of Mechanics, it admits of a more simple solution when considered in a manner less general, and on principles peculiarly adapted to the nature of the problem: we propose, therefore, again to retrace our steps, by giving another solution of this problem, founded only upon the most simple and obvious rules of analysis and geometry; our object in the foregoing solution having been merely to point out to the reader one of the easiest applications of our fundamental formulæ, in order to render their use more familiar, when we arrive at investigations in which it is difficult to employ any but the most general and comprehensive principles of solution.

§ X. *On projectiles in a vacuum by geometry.*

Projectiles by geometry.

44. Let a body be projected from A (fig. 31, plate iii, DYNAMICS), in any direction, not vertical, as AD; and let AC, AD, be the spaces that the body would describe in the times t and t', from the uniform velocity of projection, and CE and DB the spaces through which it would descend by the action of gravity in the same times; then it is obvious, from the composition of the two motions, that the body will be found at the ends of those times in the points E and B.

Now by the laws of uniform motions (art. 2), $AC = t\,v$, and $AD = t'v$, therefore

$$AC : AD :: t : t';$$

and by those of falling bodies (art. 6),

$$CE : DB :: t^2 : t'^2;$$

whence

$$CE : DB :: AC^2 : AD^2,$$

which is a known property of the parabola; and as the same has place for every point in the projectile's path, it follows, *that the curve described by the projectile, is a parabola.*

It may be proper to observe, that this demonstration is founded on a supposition of the direction of gravity upon the body in every point of the curve being exerted in lines parallel to each other, which is not strictly true, because that action is always directed towards a certain point near the centre of the earth; and consequently the lines of direction all converge towards that point, forming angles with each other: but the small extent which we are able to give to the range of any projectile, being very inconsiderable in comparison with the terrestrial radius, it is obvious, that no sensible error will arise from the supposition of the parallelism of the several lines of direction.

Laws of a projectile.

In order now to investigate the laws of the body's motion, let A (fig. 32) be the point of projection, AB, or AB', the plane on which the body is projected, passing through A, and which also denotes the range.

Let AC be drawn parallel, and BCD perpendicular to the horizon; let the angle of elevation $CAD = a$, the angle of inclination of the plane $CAB = b$, the velocity of projection $= v$, the time of flight $= t$, and the range $AB = r$, also $16\frac{1}{12} = g$.

Then we know, from the laws of uniform motion, that the body at the end of the time t, if gravity did not act, would be found in the point D; while by the laws of falling bodies, it would, in the same time, pass through the perpendicular DB; consequently

$$AD = t\,v; \text{ and } DB = g\,t^2.$$

Whence we have

$$\cos b : \sin(a \pm b) :: t\,v : \frac{t\,v \sin(a \pm b)}{\cos b} = g\,t^2;$$

or,

$$\frac{g\,t}{v} = \frac{\sin(a \pm b)}{\cos b}. \qquad (1)$$

Again,

$$\cos a : \sin(a \pm b) :: r : \frac{r \sin(a \pm b)}{\cos a} = g\,t^2;$$

or,

$$\frac{g\,t^2}{r} = \frac{\sin(a \pm b)}{\cos a}. \qquad (2)$$

And equating the value of t, in equations (1) and (2), we obtain

$$\frac{r}{v^2} = \frac{\sin(a \pm b)\cos a}{g \cos^2 b}. \qquad (3)$$

From these three equations, all the relations between the time, velocity, range, and angle of elevation, are readily determined: for example, If the time and elevation be given to find the velocity and range; in this case,

$$\text{equat. (1) gives } v = \frac{g\,t \cos b}{\sin(a \pm b)};$$

$$\text{equat. (2) gives } r = \frac{g\,t^2 \cos a}{\sin(a \pm b)}.$$

If the range and elevation be given, to find the time and velocity, we have from

$$\text{equat. (2)} \;\ldots\; t = \sqrt{\frac{r \sin(a \pm b)}{g \cos a}};$$

$$\text{equat. (3)} \;\ldots\; v = \sqrt{\frac{r\,g \cos^2 b}{\sin(a \pm b) \cos a}}.$$

If the velocity and elevation be given, to find the time and range, we obtain from

$$\text{equat. (1)} \;\ldots\; t = \frac{v \sin(a \pm b)}{g \cos b};$$

$$\text{equat. (3)} \;\ldots\; r = \frac{v^2 \sin(a \pm b) \cos a}{g \cos^2 b}.$$

If any two of the above quantities be given to find the angle of elevation, we must substitute, instead of $\sin(a \pm b)$, its value; viz.

$$\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a;$$

whence the $\sin a$, or $\cos a$, may be obtained.

In all these cases we shall find $\sin(a \pm b) \cos a$, equal to a known quantity, which let be denoted by C, then

$$\cos a \sin a \cos b \pm \sin b \cos a^2 = C;$$

substitute $\sin a = x$, then $\cos a = \sqrt{(1 - x^2)}$; and we have

$$x \sqrt{(1 - x^2)} \pm \tan b\,(1 - x^2) = \frac{C}{\cos b}.$$

A quadratic equation, from which two values of x will always be determined; and whence we learn, that there are always two different angles of elevation which answer equally the conditions of the problem, except at least, under that particular relation of the co-efficients, in which the two roots of the equation are equal to each other; then only one angle will be found, which in this case will be $\sin a = \sin \frac{1}{2}(90° \pm b)$ under which limitation the range will be a maximum; and all angles of elevation equally above and below that which gives the maximum range, will give ranges equal to each other. This will appear more obvious by referring to our last determined value of r; viz.

$$r = \frac{v^2 \sin(a \pm b) \cos a}{g \cos^2 b};$$

where it is obvious, if we suppose v, and the angle b, to be given, the range will vary as

$$\sin(a \pm b) \cos a;$$

now

$$\sin(a \pm b) \cos a = \frac{1}{2}\sin(2a \pm b) + \frac{1}{2}\sin(\pm b);$$

and since the second part of this equation is constant, the range will vary as

$$\sin(2a \pm b);$$

and will therefore be greatest, when this quantity is the greatest; that is, when

$$2a \pm b = 90°,$$

or when

$$2(a \pm b) = 90° \mp b;$$

that is, the range is greatest, when

$$a \pm b = 45° \mp \tfrac{1}{2} b;$$

and since all angles equally above and below 90° have the same sine; it follows, that all angles equally above and below that which gives the maxium, will have equal ranges.

When the data of the problem give, or require, a greater value for $\sin(2a \pm b)$ than 1, that is, than the sine of 90°, it shows the problem, under the proposed conditions, to be impossible.

To find the greatest height of the projectile above the point of projection, we must observe, that the body will continue to ascend till the velocity of descent from gravity is equal to the uniform velocity of ascent from projection; that is, calling the time at which the height is the greatest t', we shall have $2\,g\,t' =$ the velocity of descent from gravity; and $v \sin a$ will obviously denote the uniform velocity of ascent from projection; whence we have

$$2\,g\,t' = v \sin a, \text{ or}$$

$$t' = \frac{v \sin a}{2\,g};$$

but the descent in the time t' is $g\,t'^2$, in which, substituting the above value of t', it becomes

$$g\,t'^2 = \frac{v^2 \sin^2 a}{4g};$$

and the ascent, in the same time from projection, is $t'\,v \sin a$, or substituting for t' as above, we have

$$t'\,v \sin a = \frac{v^2 \sin^2 a}{2g};$$

and, consequently, the difference of these will be the greatest height of the projectile above the point A; calling, therefore, the greatest height h, we have

$$h = \frac{v^2 \sin^2 a}{4g}, \text{ or}$$

$$\frac{4\,h\,g}{v^2} = \sin^2 a; \qquad (4)$$

whence, the angle of elevation and the velocity being given, the greatest height will be immediately determined; and if r, or t be given, the value of v^2 may first be determined from the proper equation; and hence again the value of h, from equation (4).

All the preceding equations are rendered much more simple if we suppose the plane AB to become horizontal; so that $b = 0$; and consequently $\sin b = 0$; and $\cos b = 1$; for after this reduction,

$$\text{equation 1 becomes } \frac{g\,t}{v} = \sin a;$$

$$\text{equation 2} \;\ldots\ldots\; \frac{g\,t^2}{r} = \tan a;$$

$$\text{equation 3} \;\ldots\ldots\; \frac{r}{v^2} = \tfrac{1}{2}\sin 2a;$$

$$\text{equation 4} \;\ldots\ldots\; \frac{4\,h\,g}{v^2} = \sin^2 a;$$

which involve all the conditions of a projectile while the plane is horizontal, and passes through the point of projection.

When the plane is not horizontal, it is obvious that in the formula in which $\sin(a \pm b)$ occurs, b must be accounted positive when the plane descends, and negative when it ascends.

If the velocity of the projectile in any point of the curve be required, or the velocity after any time t', it is obvious that this is compounded of the constant horizontal velocity of projection $v \cos a$, and the difference of the vertical velocities of ascent from projection $= v \sin a$, and that of descent from gravity $= 2\,g\,t'$, that is, calling the required velocity $=$ V, we shall have

$$V = \surd\,\{v^2 \cos^2 a + (v \sin a - 2\,g\,t')^2\}. \qquad (5)$$

45. In our preceding investigations we have supposed the plane on which the projection is made to pass through the point of projection: let us now consider the case in which the plane does not pass through that point.

Let A (fig. 33) be the point of projection; EF the direction of the plane; which, being supposed known, the points where it cuts the horizontal line AE, and the vertical AF, will also be determined. Let us then denote AF by m, and AE by n, the angle of elevation above the horizon by a, the velocity $= v$, the time $= t$, and $16\frac{1}{12} = g$, as before.

Here we have, in the first place, AC $= t\,v \cos a$, and CE $= n - t\,v \cos a$; whence,

$$n : m :: n - t\,v \cos a : \frac{m\,(n - t\,v \cos a)}{n} = \text{CB},$$

to which adding DC $= t\,v \sin a$, we have

$$\text{DB} = g\,t^2 = t\,v \sin a + m - \frac{m\,t\,v \cos a}{n}, \text{ or}$$

$$t^2 - \left(\frac{v \sin a}{g} - \frac{m\,v \cos a}{n\,g}\right) t = \frac{m}{g}; \qquad (6)$$

an equation whence the value of t or v may be determined, one of them being known; and, consequently also, the lines CB, AC; because

$$\text{CB} = \frac{m\,(n - t\,v \cos a)}{n}, \text{ and}$$

$$\text{AC} = t\,v \cos a.$$

But

$$\text{AB} = \surd\,(\text{AC}^2 + \text{BC}^2);$$

therefore,

$$\text{AB} = \surd\,\left\{(t\,v \cos a)^2 + \left(\frac{m\,(n - t\,v \cos a)}{n}\right)^2\right\}. \qquad (7)$$

There are six cases to this problem to which the above equations apply; viz.

1 and 2. When the plane rises and meets AC, either produced as at E, or between A and C; in both these cases m and n, in equations 6 and 7, are *positive*.

3. If the plane descend, and when produced, meet the plane AC, beyond A; then n is *negative*, and m *positive*, in both equations.

4. If it descend, and when produced, meet AC between A and C; then m is *negative*, and n *positive*.

5. When the plane is horizontal, n is *infinite*, and the term $\frac{m}{n}\,t\,v \cos a$ vanishes, and equation (6) becomes

$$t^2 - \frac{v \sin a}{g}\,t = \frac{m}{g}.$$

6. When the plane is vertical, then m is *infinite*, and all those terms in equation (6), in which it does not enter, vanish with respect to those in which it does; and we have, therefore,

$$\frac{t\,m\,v \cos a}{n\,g} = \frac{m}{g}, \text{ or } t = \frac{n}{v \cos a}.$$

Whence t or v being determined, in all these cases, the range will be found from equation (7).

When the point B on the plane is given, then join AB; and all the circumstances of velocity, time and elevation, may be determined from equations (1), (2), and (3).

By geometrical construction.

46. It may not be amiss, in concluding this article, to show the method of arriving at the same results as the above, by means of a geometrical construction.

With this view, let AB (fig. 34), be the range, AC, or AC′, the direction of projection; and let the velocity equal v, and $16\frac{1}{12} = g$: draw AD perpendicular to the horizon, and equal to $\frac{v^2}{g}$, or as it is called, equal to four times the impetus, or four times the height from which a heavy body must fall to acquire the velocity v.

If now t denote the time of flight, we shall have

$$\text{AC} = t\,v,\ \text{CB} = g\,t^2, \text{ and AD} = \frac{v^2}{g};$$

therefore,

$$\text{AD} : \text{AC} :: \text{AC} : \text{CB};$$

consequently, the triangles ADC and ACB, are similar, having the angles

$$\text{DAC} = \text{ACB};\ \text{ADC} = \text{CAB}, \text{ and ACD} = \text{ABC}.$$

And hence the following constructions;

1. *The velocity and angle of elevation being given to find the range.*

Take AD $= \frac{v^2}{g}$; and from D draw DC, making the angle ADC $=$ CAB, meeting the given direction AC in C; from C draw CB parallel to DA, meeting the given plane in B; and AB will be the range.

2. *The range and elevation being given to find the velocity.*

Through the given point B, draw BC parallel to DA, or perpendicular to the horizon; meeting the given direction AC in C: from C draw CD, making the angle ACD $=$ ABC, and meeting the indefinite perpendicular AD in D; so shall AD be given; and, consequently, since AD $= \frac{v^2}{g}$, v may hence be computed.

3. *The velocity and elevation being given to find the time.*

From the point A, erect the perpendicular to the horizon AD $= \frac{v^2}{g}$; draw from the point D, DC making the ADC $=$ ACB; and meeting the given direction AC in C; from C draw CB parallel to AD, meeting the plane in B; so shall CB be determined; and since

$$\text{CB} = g\,t^2,$$

t the time may hence be computed.

This construction is the same as the 1st; and by a construction similar to the 2d, the time may be determined when the range is given.

4. *The range and velocity being given to find the direction, or angle of elevation.*

From the given point B, draw the indefinite perpendicular to the horizon BC; and parallel to BC, draw AD $= \frac{v^2}{g}$; on which describe a segment, capable of containing an angle equal to the angle ABC, which will be cut by BC in two points C and C′; join AC or AC′, and it will be the direction required.

The demonstration is obvious; for in all these cases we have

$$\text{AD} : \text{AC} :: \text{AC} : \text{BC};$$

which is the only condition necessary to be fulfilled.

It appears from the last construction, that there are always two different angles of elevation which give the same range; unless when BC becomes a tangent to the segment, in which case there is but one; and that obviously such as to give the maximum range. It is also evident, that in the latter instance, AC bisects the angle DAB; and that in the former, the two lines of direction are equally above and below the line which bisects that angle. It may also be farther observed, that if the perpendicular BC does not meet the circle, the problem becomes impossible; the same results as we have already obtained analytically in (art. 43).

When the plane does not pass through the point of projection.

47. When the plane does not pass through the point of projection; let EF (figs. 35 and 36) be the plane, AC the direction, and AD $= \frac{v^2}{g}$ as before.

Produce AC to meet the plane also produced, if necessary, in F; draw DG (fig. 35), making the angle ADG = CFB; on FG describe a semicircle, and concentric with it another, passing through A; in the latter, apply AH = AG, and produce it to meet the other circle in I. Take AC = AI, and draw CB parallel to DA; so shall B be the point on the plane required.

For, by similar triangles,

AD : AG :: FC : CB;
by construction, AG : AI or AC :: IH or GC : FA;
by composition, AG : AC :: AC : FC;
consequently, AD : AC :: AC : CB;

therefore, the projectile passes through B, as shown (art. 46).

The same construction and demonstration apply to the case in which AC and EF meet each other on the opposite side of DA, as in fig. 36; except that DG is also then drawn on the opposite side of DA, and GC is made equal to AI.

Here, therefore, we have

AD : AG :: FC : CB;
by const. AG : AC or HI :: GC : FA;
by division, AG : AC :: AC : FC;
consequently, AD : AC :: AC : CB;

the same as before.

When the plane is parallel to the line of direction, CB is given, and AC is mean proportional between DA and CB. And when the plane is perpendicular to CB, then AC is given, and CD is a third proportional to DA and AC, as is obvious.

If the point B on the plane be given, all the other circumstances will be determined, as in the corresponding constructions in the preceding article.

48. It has been customary with authors on Mechanics, to treat the theory of projectiles, and to rest all their investigations and deductions relative to it, on the properties of the parabola, whence it is frequently called the parabolic theory of projectiles; but it is obvious from the preceding articles, that this curve is at least entirely superfluous in the theory, as every circumstance relative to the motion, time, range, &c. may be deduced without a single reference to the parabola. The theory of logarithms was formerly always considered with reference to the hyperbola; and all the early writers limited their investigations wholly to the properties of the latter curve, whence we have our hyperbolic logarithms.

But as the mathematical sciences are extended beyond all possible contemplated limits, it becomes more and more necessary to simplify its different branches, and to divest them of useless superfluities: it is on this account, we presume, that logarithms have long been considered only with reference to their analytical properties; and we cannot but think that mathematicians will see the propriety of dropping all considerations of the conic sections, in their investigations of the properties of projectiles.

Ri

49. *On the ricochet.* It is customary with authors who are writing on the projectile theory, to add a few observations relative to that species of motion which is called *ricochet;* and in conformity to this practice, rather than from the theoretical nature of the subject, we shall also subjoin a few remarks on this head.

The ricochet is a motion, by which a projectile, after having struck any obstacle whatever, is so reflected as to recommence a similar motion to that which it had at first.

The word signifies *duck and drake*, or rebounding; because, when a ball has this motion, it bounds along, striking the ground several times, resembling the motion of a flat stone along the surface of the water when thrown nearly horizontally. In what is termed ricochet firing, in practical gunnery, the ball is commonly thrown at an angle of elevation of between three and seven degrees.

The less the angle of elevation at which a ball is projected, the more *cæteris paribus* is it in the state for the ricochet; for then the force of projection being exercised more entirely in the horizontal direction, will take much more time before it is destroyed, by the resistance of the air and other obstacles.

If the projectile were entirely void of elasticity, and the surface on which it falls were horizontal and inflexible, there would be no ricochet; because the velocity of the projectile, when it arrives at C (fig. 37), according to any direction, MC may be decomposed into two others, of which the one, QC, perpendicular to the surface, will be simply destroyed without any restitution; while the other velocity PC subsists alone (abstracting from friction and the resistance of the air), and the body, instead of bounding, will slide along CZ.

But if there should be an eminence CE (fig. 38), at the point C, where the body strikes the surface, the motion, according to MC, may then be decomposed into a motion QC, perpendicular to the surface CE of this obstacle; and another motion, PC, in the direction of that surface; by which the ball will advance in the direction PE, and may describe, after quiting the point E, a new curve, of the same nature as that which it would describe, if it were projected from E with the same velocity; and this in such manner, that after it is elevated to a certain point, it may again descend, and strike the earth in a new point I; where, under similar circumstances, it may recommence a like motion, and so on.

If the object be flexible or moveable, as the earth, water, &c. there may be a ricochet, even when the surface is perfectly horizontal. For the body then tends with its vertical velocity QC (fig. 39), to plunge itself, more or less, according to the nature of the obstacle; while, with the velocity represented by PC, it ploughs its way, and forms a furrow, of which the depth continues to increase, till the vertical velocity QC is extinguished. Then, by virtue of the velocity remaining in the horizontal direction, it drives before it the matter which opposes its motion, and becomes gradually turned towards the side where it has the least resistance; and thus in its return from its greatest depth, the cavity of the furrow operates, with respect to the body, in the same manner as the obstacle CE (fig. 38) in the preceding case.

Now as the facility of the escape from the earth, or other obstacle, is greater as the total depth of the furrow is less, and as the depth depends upon the vertical velocity QC, which will be always smaller as the angle MPC is smaller, or the less the angle of projection; it is obvious how much the facility of the ricochet depends upon the magnitude of the angle of projection. The ricochet also depends much upon the figure of the projectile. If we want, for example, a ricochet on the water, and the projectile be globular, the velocity MC ought to be such, that the vertical velocity QC may be entirely consumed before the vertical diameter of the sphere be completely immersed; for if once it is covered with water, the resistance of the fluid will act equally on all sides of the direction of the projectile, in such a manner that it can only be turned in consequence of the action of gravity, which will, therefore, be a direct impediment to the ricochet.

As the plunging can only be made successively or gradually, it is manifest that, during the time in which it is going on, the centre of the ball will describe a curve line; because the direction in which the resistance is made, continually changes. If, for example, when the centre C (fig. 40), after having described any track whatever, PC tends to move along the prolongation CI of its actual direction, we conceive two tangents BR, DS, parallel to that direction; it is evident that only the part BVL is subjected to the resistance; and that the body being spherical, the resultant CK of all the resistances, offered to the different points of the surface BVL, has a direction which tends to elevate the body above CI; in such manner, that CEIK may be the parallelogram of forces, and CE will be the path the body will take, instead of CI abstracting from the effect of gravity.

Lastly, if both the projectile and the obstacle are flexible, these circumstances will still further contribute to the facility of the ricochet. To take a very simple instance: suppose that the projectile only is flexible and perfectly elastic, or that it restores itself with a force equal to that of compression; still abstracting from the consideration of gravity. At the instant in which the body moving in the direction AC (fig. 41), touches the surface, its velocity is resolved into a horizontal velocity QC, which subsists always the same (disregarding the friction, and the resistance of the medium in which the body moves): but the vertical velocity PC tends gradually to compress the body itself; becoming gradually extinguished, while that represented by QC remains unchanged: whence it is obvious, that the centre C approaches the plane HZ by degrees, which are continually diminishing, while the degrees by which it advances parallel to the plane remain constant; if, therefore, we conceive at every instant a parallelogram of forces constructed, of which the horizontal side is to the vertical side, as the horizontal velocity to that which remains in the vertical direction, the diagonal of this parallelogram, which will indicate each instant the path of the centre C, will be differently situated, so that the centre will approach to ZH, by describing a curve line CR, during the time of the compression: when the compression ceases, the centre C will move for an instant on a tangent parallel to HZ; after which the elastic force of the body, counteracting that of compression, restores to the body gradually the velocity by which it moves from the plane, in the inverse order of the degrees by which it approached it; and thus causes its centre to describe the second branch RO of the curve, equal and similar to the first CR. Finally, when it has arrived at the point O, whose distance from the plane HZ is equal to the radius CI, it will move on according to the tangent OT, situated in the same manner as AC. Hence, then, the oblique stroke of the elastic body upon an inflexible and immoveable plane, is made (abstracting from gravity) in such a manner, that the angle of reflection is equal to the angle of incidence; these two angles being measured by those contained between the horizontal plane and the tangents at the extremities C and O of the curve described by the centre, during the compression of the body, and its restitution to its original figure, and this curve is always the smaller as the compression and restitution approach more nearly to being instantaneous.

If we take gravity into the consideration, and BD be the line according to which the body is projected, it will describe the portion DC of the parabola, whose tangent is DB, at the instant when the body touches the plane; and when the compression has ceased, and the centre arrives at O, it will describe another portion SO of a parabola equal to the former, and similarly situated.

Friction also contributes to the facility of the ricochet, because it gives to the body a rotation, which enables it to surmount more easily the different obstacles it meets with.

Such are the causes and the principal circumstances of the ricochet; which, although as we have observed in the beginning of this article, they are so involved as nearly to bid defiance to theoretical computations, we have examined at length, because the same considerations and principles may be employed sometimes in cases where similar resistances may arise which admit of more decided numerical values, and consequently of determinate results.

§ XI. *Of projectiles in resisting mediums.*

Projectiles in resisting mediums.

50. Nothing can be desired more simple than the theory of projectiles, as delivered in the preceding chapter, where we have abstracted from the resistance of the medium; but, unfortunately, it is in no case strictly applicable to the actual motion of a projected body; and it more particularly fails in those instances in which it is most desirable, that we should be able to avail ourselves of the advantages of theoretical computation; that is, in the determination of the ranges of balls and shells thrown from pieces of ordnance. The resistance of the air is so great in some cases of this kind, that velocities, which would give ranges of more than twenty miles in a vacuum, will not produce an actual range of more than two or three miles at farthest; it is, consequently, highly important to introduce into our investigation, data which shall have a relation with the resistance of the atmosphere or other medium. We propose, therefore, to consider the subject under this point of view in the present chapter.

The resistance of any medium to the motion of a body in it, may be considered as a force acting in the direction of the tangent of the trajectory, or path, and in opposition to the motion. Such a force as this will not affect the body, so as to throw it out of the vertical plane, which contains the line of its primitive direction; it will, therefore, be sufficient here, as in article (44), to employ only the two axes x and y; we may, moreover, represent by s the arc P m (fig. 42) of the trajectory, comprised between the origin of the co-ordinates, or the point P of projection, and the point m, which answers to the co-ordinates x and y. Then the ratios $\frac{\dot{x}}{\dot{s}}$ and $\frac{\dot{y}}{\dot{s}}$ will denote the cosines of the angles, which the right line m T (the tangent of the curve at the point m) makes with the axes x and y; denoting, therefore, by R, the resistance of the medium, and observing, that this force acts according to the prolongation m T′ of the right line m T, we shall have

$$-\mathrm{R}\frac{\dot{x}}{\dot{s}} \text{ and } -\mathrm{R}\frac{\dot{y}}{\dot{s}}$$

for the horizontal and vertical components of that resistance; and it is obvious, that both in the ascending and descending branches of the curve, these values are positive or negative, according as the force R tends to augment or to diminish the values of x and y: including now, with the vertical resistance, the perpendicular action of gravity $2\,g$, the two equations of the motion will be

$$\frac{\ddot{x}}{\dot{t}^2} = -\mathrm{R}\frac{\dot{x}}{\dot{s}} \text{ and } \frac{\ddot{y}}{\dot{t}^2} = -2g - \mathrm{R}\frac{\dot{y}}{\dot{s}}.$$

Equation of motion.

51. These equations are those of a material point, but they will also apply to every point of the projected body, provided that all these points have the same motion in space, in such a manner, that they describe parallel curves throughout the entire motion, and that, at every instant, their velocities are all equal, parallel, and made in the same direction; such as we must conceive to be the case with the motion of a ball projected from a mortar, or other piece of ordnance. R, which we have assumed to denote the resistance of the medium, depends on the form, the dimensions, the density, and the velocity of the body, as also on the density of the medium; we may, however, in the case of projectiles from guns, suppose the latter density constant without any sensible error, because neither the height, nor the range in actual practice is sufficiently large to render it important to consider any change in the density to have place in the course of its flight. We shall, moreover, not take into our consideration the form of the body, nor its density; but suppose, that the resistance has been experimentally determined for some given velocity; we shall also farther assume, according to the most general hypothesis, that the resistance (which we have seen, varies as some function of the velocity) is proportional to its square; or, that R is always proportional to v^2, or to $\frac{\dot{s}^2}{\dot{t}^2}$; in short, that

$$\mathrm{R} = m\frac{\dot{s}^2}{\dot{t}^2};$$

m being a constant co-efficient to be determined from experiment.

52. The equations of the motion, therefore, which it remains for us to take the fluents of, will become, after introducing into them the preceding value of R,

$$\left.\begin{aligned} \frac{\ddot{x}}{\dot{t}^2} &= -m\frac{\dot{s}}{\dot{t}}\cdot\frac{\dot{x}}{\dot{t}} \\ \frac{\ddot{y}}{\dot{t}^2} &= -2g - m\frac{\dot{s}}{\dot{t}}\cdot\frac{\dot{x}}{\dot{t}} \end{aligned}\right\} \quad (1)$$

The first of these equations may be otherwise written

$$\frac{\ddot{x}}{\dot{t}} = -m\dot{s}\;\frac{\dot{x}}{\dot{s}}\cdot\frac{\dot{s}}{\dot{t}};$$

or

$$\frac{\ddot{x}}{\dot{t}} = -m\,\dot{s}\;\frac{\dot{x}}{\dot{t}}.$$

Whence we have, by dividing the first side by the second,

$$\frac{\;\frac{\ddot{x}}{\dot{t}}\;}{\frac{\dot{x}}{\dot{t}}} = -m\dot{s}$$

Taking the fluent of the latter fluxional equation,

$$h.\log\frac{\dot{x}}{\dot{t}} = -m s + \mathrm{C};$$

or,

$$\frac{\dot{x}}{\dot{t}} = \mathrm{C\,E}^{-ms};$$

E being the base of the hyperbolic logarithms of which the modulus is unity, and C a constant arbitrary, or correction.

To determine C, we observe, that at the point P, we have $s = 0$, at which point the horizontal velocity $\frac{\dot{x}}{\dot{t}} = a\cos\alpha$, ($a$ denoting the initial velocity of projection, and α the angle which that direction forms with the horizontal axis P x;
whence we obtain $\mathrm{C} = a\cos\alpha$; and consequently

$$\frac{\dot{x}}{\dot{t}} = a\cos\alpha.\,\mathrm{E}^{-ms}.$$

Let now p represent the trigonometrical tangent of the angle which the tangent m T, of the trajectory, forms with the axis P x, so that we may have $p = \frac{\dot{y}}{\dot{x}}$;

then we shall also have

$$\frac{\dot{y}}{\dot{t}} = p\frac{\dot{x}}{\dot{t}};$$

and taking the fluxion of this expression,

$$\frac{\ddot{y}}{\dot{t}^2} = \frac{\dot{p}}{\dot{t}}\cdot\frac{\dot{x}}{\dot{t}} + p\frac{\ddot{x}}{\dot{t}^2}.$$

Substituting now these values in our second equation (No. 1), it will be reduced (by involving also that of the first) to

$$\frac{\dot{p}}{\dot{t}}\cdot\frac{\dot{x}}{\dot{t}} = -2g.$$

Let us now divide this equation by the square of the preceding value of $\frac{\dot{x}}{\dot{t}}$, and it becomes

$$\frac{\dot{p}}{\dot{x}} = \frac{-2g\,E^{2ms}}{a^2\cos^2 a}. \qquad (2)$$

This new equation is such that its fluent may be found; for if we multiply the second side by $\dot{s}$, and the first by the equivalent value of the same, that is by $\dot{x}\sqrt{(1+p^2)} = \sqrt{(\dot{x}^2+\dot{y}^2)}$, we shall have

$$\dot{p}\sqrt{(1+p^2)} = -\frac{2g\,E^{2ms}\dot{s}}{a^2\cos^2 a};$$

where the values of p and s being separated, the fluents are attainable by the known methods, and we thence derive

$$p\sqrt{(1+p^2)} + h.\log\left(p+\sqrt{(1+p^2)}\right) = -C\,\frac{2g\,E^{2ms}}{a^2 m\cos^2 a}; \quad (3)$$

C being the correction.

The value of C may be found by observing that at the point of projection, we have, when $s = 0$, $p = \tan a$, which gives

$$C = \begin{cases} \tan a\sqrt{(1+\tan^2 a)} + \\ h.\log\{\tan a + \sqrt{(1+\tan^2 a)}\} + \dfrac{2g}{a^2 m\cos^2 a}; \end{cases}$$

but for the sake of abridging, we shall retain the letter C in the following transformations.

53. In order to determine x and y, we must eliminate E^{2ms}, by means of equations (2) and (3); whence we obtain

$$\dot{x} = \frac{\dot{p}}{m\{p\sqrt{(1+p^2)} + h.\log(p+\sqrt{(1+p^2)}) - C\}}; \quad (4)$$

and because $\dot{y} = p\dot{x}$, we have also

$$\dot{y} = \frac{p\dot{p}}{m\{p\sqrt{(1+p^2)} + h.\log(p+\sqrt{(1+p^2)}) - C\}}; \quad (5)$$

These equations will give, by the method of quadratures, the values of x and y in functions of p. We may also obtain a third equation which shall exhibit t, the time, in functions, of the same variable. We have seen for example, that

$$\frac{\dot{p}}{\dot{t}}\cdot\frac{\dot{x}}{\dot{t}} = -2g, \text{ or } \dot{t}^2 = -\frac{\dot{p}\dot{x}}{2g};$$

and substituting this value in equation (4), or, which is the same, multiplying by $\frac{-\dot{p}}{2g}$, and taking the square root, we have

$$\dot{t} = \frac{\dot{p}}{\sqrt{2mg}\{C - p\sqrt{(1+p^2)} - h.\log(p+\sqrt{(1+p^2)})\}^{\frac{1}{2}}}. \quad (6)$$

The sign $-$ being introduced here before the radical, because the angle of which p is the tangent, diminishes as the time augments; so that $\dot{p}$ and $\dot{t}$ must have contrary signs.

54. The general solution of the problem we have endeavoured to investigate, is given in equations (4), (5), and (6). If we could take the fluents of (4) and (5) under a finite form, we should have x and y in functions of p; and eliminating afterwards p, there would remain an equation in x and y, which would exhibit the equation of the curve or trajectory; but unfortunately no methods known at present will apply to fluxions of these forms, which indeed are too complicated to allow us to expect that the complete fluents of them will ever be obtained. We may, however, still avail ourselves of these results to construct the trajectory by points, as shown in what follows.

Approximation.

In order to abridge the operations, let us denote by Fp, the co-efficient of $\dot{p}$, in the value of $\dot{x}$, equation (4); so that we may write

$$\dot{x} = Fp\,\dot{p}, \text{ and } x = \int Fp\,\dot{p}.$$

This integral or fluent ought to be so comprised, that it may vanish at the point P, where we have $p = \tan a$; the absciss, therefore, of any point of the trajectory is equal to the sum of the infinite little values of $Fp\,\dot{p}$, comprised between $p = \tan a$, and the value of p answering to the point under consideration. Consequently we shall have an approximate value of x, by dividing the interval between the extreme values of p into a great number of equal parts, and calculating afterwards the values of Fp answering to these divisions; then, if we take the sum of these quantities, and multiply by the small difference of the consecutive values of p, which we have substituted for the infinitely small difference $\dot{p}$, our approximate results will be obtained.

Let, for example, δ denote a very small positive quantity, we shall then have the following series of corresponding values:

$$\begin{array}{ll} p = \tan a & ;\ x = 0, \\ p = \tan a - \delta & ;\ x = -F(\tan a - \delta)\delta, \\ p = \tan a - 2\delta; & x = -F(\tan a - \delta)\delta - F(\tan a - 2\delta)\delta, \\ p = \tan a - 3\delta; & x = -F(\tan a - \delta)\delta - F(\tan a - 2\delta)\delta, \\ \&c. = & \&c. \qquad\qquad\qquad\qquad -F(\tan a - 3\delta)\delta. \end{array}$$

It is evident that these values of x will be so much the more exact, as the difference δ is taken less and less. By continuing the operation to that point where we find $p = 0$, we shall have, to any required degree of accuracy, the abscisses of all the points of the ascending branch of the curve. From this point in the descending branch, the values of p become negative, but the continuation of the calculation will equally give all the abscisses of the descending branch; and by a similar method applied to equation (5), we shall have the co-ordinates answering to all the corresponding values of p. And taking afterwards the values of x and y belonging to each particular value of p, we may, as is obvious, construct the curve by means of these determined points.

The ordinate which corresponds to the value $p = 0$, will express the greatest height of the projectile; and the absciss, which answers to that value of p, where the ordinate changes from positive to negative, will be the range on the horizontal plane, passing through the point of projection. The direction of the tangent at any point of the trajectory will also be known, since it is from this, that we proceed in our determination of the other quantities. The length of the arc corresponding to each particular value of p, is also given by equation (3). With regard to the time that the body employs in describing this arc, we may determine it by means of equation (6), and by a similar proceeding to that which we have supposed for obtaining the values of x and y. It remains, therefore, only to compute the velocity or the value of v, answering to any given point of the trajectory; and this may be effected as follows:

Since v denotes the velocity in the curve, we have

$$v^2 = \frac{\dot{x}^2}{\dot{t}^2} + \frac{\dot{y}^2}{\dot{t}^2} = (1 + p^2)\frac{\dot{x}^2}{\dot{t}^2} = 2g(1 + p^2)\frac{\dot{t}^2}{\dot{p}^2};$$

and since we have found, art. (52,)

$$\frac{\dot{p}^2}{\dot{t}} \cdot \frac{\dot{x}}{\dot{t}} = -2g,$$

if we substitute for $\frac{\dot{t}^2}{\dot{p}^2}$ its value drawn from equation (6), the above will become

$$v^2 = \frac{g(1 + p^2)}{m\{C - p\sqrt{(1 + p^2)} - h\log(p + \sqrt{(1 + p^2)})\}}, \quad (7)$$

which is an equation expressing the value of v, in functions of p.

Remarkable property.

55. The above may be considered as a complete solution of the problem which we had originally proposed; viz. to determine the several circumstances attending the motion of a projectile in a resisting medium: but before we conclude this article, we shall bestow a few lines in pointing out a remarkable property of the curve, or trajectory, described by the body under the circumstances we have supposed.

It has been seen in the preceding chapter, that when a body is projected into a vacuum, the two branches of the curve, that is, the ascending and descending branches, are symmetrical; which is obviously not the case, when we take into our considerations the resistance of the medium; for, in the former case, the equality in the two branches depends upon the condition, that the horizontal velocity remains constant, and that the force of gravity is unimpeded; but in the present case, considering only the horizontal velocity, we shall find that it is diminished every instant, and the property to which we have alluded is, that notwithstanding this continued diminution, it will never be entirely destroyed; or at least only after an infinite time; although it will become such, that even in an infinite time the body cannot have passed a given vertical line; or, which is the same, that the descending branch of the curve will have a given vertical asymptote.

In order to demonstrate this property, let us transfer the origin of the co-ordinates to the point C (fig. 42), the vertex of the curve, and take the horizontal abscisses still in the same direction as before, but the vertical ordinates in a contrary one; viz. in such a manner that C x' and C y' shall be these new axes. Now x' and y' denoting the co-ordinates of any point in the curve referred to these axes, we shall have in comparing the co-ordinates of x and y of the same point,

$$x' = \mathrm{PD} + x, \text{ and } y' = \mathrm{CD} - y;$$

whence it follows, that

$$\dot{x}' = \dot{x}, \text{ and } \dot{y}' = -\dot{y}.$$

And if we make $\frac{\dot{y}'}{\dot{x}'} = p'$, we shall have also $p' = -p$; consequently

$$h.\log(p + \sqrt{(1 + p^2)}) = h.\log(-p' + \sqrt{(1 + p'^2)}) = -h.\log(p' + \sqrt{(1 + p'^2)}),$$

because

$$-p' + \sqrt{(1 + p'^2)} = \frac{1}{p' + \sqrt{(1 + p'^2)}}.$$

Equations (4) and (5) therefore become

$$\dot{x}' = \frac{p'}{m\{p'\sqrt{(1 + p'^2)} + h.\log(p' + \sqrt{(1 + p'^2)} + C)\}}$$

$$\dot{y}' = \frac{p'\dot{p}'}{m(p'\sqrt{(1 + p'^2)} + h.\log(p' + \sqrt{(1 + p')^2}) + C.\}}$$

Now, in the construction of the descending branch of the curve, by means of these equations, and according to the principles of the method indicated in the preceding article, we may give to p' positive values as great as we please; but when p' is very great, we shall obtain the approximate values of x' and y' in a manner more simple than that above explained; for in this case, p' and $\sqrt{(1 + p'^2)}$ may be considered equal to each other, and we may therefore employ the former in place of the latter; we may also here neglect the $h.\log(p' + \sqrt{(1 + p'^2)})$, which is very small in comparison of p'^2, remembering that p' is supposed to be a great number, and that the logarithm of a great number is small in comparison to the number itself, and particularly in comparison with its square: we may likewise neglect the correction C, with reference to the same quantity p'^2; and thus our preceding equations become simply

$$\dot{x}' = \frac{\dot{p}'}{mp'^2}, \text{ and } \dot{y}' = \frac{\dot{p}'}{mp'};$$

taking the fluents, we have

$$x' = b - \frac{1}{mp'}, \text{ and } y' = b' + \frac{1}{m}h.\log p';$$

b and b' being constant quantities or corrections.

The above value of y' has no limit; it increases indefinitely with p', although in a less ratio. But the value of x' has obviously for its limit, the constant quantity b; for when p' is indefinitely large, the fraction $\frac{1}{mp'}$ vanishes: x' therefore cannot exceed a certain value, and hence we conclude that the descending branch of the curve has a vertical asymptote; that is, if we take on the axis C x', from the point C, a part CE $= b$; and if we draw through the point E, the vertical line EF, the branch CB k of the trajectory, will approach this line nearer and nearer, but without ever coming rigorously in contact with it,

56. Again, writing $-p'$, instead of p' in equation (6), and making similar reductions to the above when p' becomes very great, we shall have

$$\dot{t} = \frac{\dot{p}'}{p'\sqrt{2mg}};$$

taking the fluents of this expression, we obtain

$$t = b'' + \frac{1}{\sqrt{2mg}}h.\log p';$$

b'' being a constant quantity.

If now we eliminate p' by means of the above equation, and that expressing the value of y', we shall have

$$y' = t\sqrt{\frac{2g}{m}} + \text{correction};$$

from which we learn that the motion of the projectile, while it approaches towards the vertical, approximates also towards a uniform velocity denoted by $\sqrt{\frac{2g}{m}}$; for the correction being independent of the time, is small in comparison with the other part of the expression $t\sqrt{\frac{2g}{m}}$.

56. When the angle of projection APx (fig. 43) is very small, the body will rise very little above the horizontal axis Px. In this case, we may find by approximation, the equation between x and y of the portion PCB of the trajectory, situated above this axis. For in all this portion of the curve it is nearly horizontal, which renders the quantity p very small, and therefore, neglecting its square, we shall have

$$\sqrt{(1 + p^2)} = 1;$$

whence it follows that

$$\dot{p} = \dot{x}\sqrt{(1 + p^2)} = \dot{x}, \text{ and } s = x.$$

If then we put x instead of s in equation (2) it becomes

$$\frac{\dot{p}}{\dot{x}} = \frac{-2ge^{2mx}}{a^2 \cos^2 a.};$$

also the angle a of projection being very small, we have very nearly $\cos a = 1$: moreover calling h the height due to the initial velocity a, that is making $a^2 = 4gh$, or

$$h = \frac{a^2}{4g},$$

we shall have

$$\dot{p} = -\frac{1}{2p} e^{2mx} \dot{x};$$

taking the fluent, and determining the constant arbitrary, from the condition that $p = \tan a$, at the point P, where we have $x = 0$, there is obtained

$$p = \tan a - \frac{1}{4mh}(e^{2mx} - 1).$$

Multiplying now this equation by $\dot{x}$, and observing that $p\dot{x} = \dot{y}$; then taking the fluent, and determining the correction in such a manner, that at the point P, $x = 0$ and $y = 0$, we shall have for our final result

$$y = x \tan a - \frac{1}{8m^2h}\left(e^{2mx} - 2mx - 1\right),$$

which is the equation sought.

We may here observe, that the first term of this value of y; viz. $x \tan a$, is the vertical ordinate of the right line PA, which the projectile would have described independent of the action of gravity; so that the second term expresses at every point of the curve the space through which the body has fallen, in consequence of gravity, combined with the resistance of the medium. By giving different values to x, the two last equations will show the corresponding values of y and p, and thence we may construct the curve PCB; the actual value of p will moreover make known the direction of the tangent, at each point. We may even, by means of the same equation, prolong the curve below the point B; for which purpose it will be sufficiently correct, while the value of p remains small, but beyond this point we can only proceed by means of our general equations (art. 53).

The greatest height of the projectile, or the greatest ordinate DC, may be found by a direct process; for at the vertex C, $p = 0$; if, therefore, we equate to zero the preceding value of p, and afterwards resolve the equation with reference to c^{2mx}, it will become

$$c^{2mx} = 1 + 4mh \tan a;$$

and consequently

$$x = \text{PD} = \frac{1}{2m} \log(1 + 4mh \tan a).$$

Substituting these values of e^{2mx} and x in the equation for y, we shall have the value of the ordinate DC.

As to the range PB, it may be determined by equating to zero the value of y, which gives

$$e^{2mx} = 1 + 2mx + 8hm^2 \tan a \,.\, x$$

a transcendental equation, from which we may obtain by approximation, the numerical value of x, or of PB, when the quantities h, a, and m are given.

§ XII. *On the elliptic motion of the planets.*

Motion of the planets.

57. In all the problems hitherto considered, we have supposed the nature of the forces to be known, and have investigated the condition of the resulting motion; we may also, in this case, assume the general law of gravitation, or rather the universal law of attraction to be given, and thence determine the circumstances of the planetary motions; or, we may otherwise suppose the law of the motions to have been determined from observation, and thence investigate the nature of the attracting force. We shall, after the method pursued by M. Poisson, examine both these problems, beginning with the latter; viz. we shall suppose, as actually was the case, that a knowledge of the law of the planetary motions preceded that of the universal law of attraction, and thence endeavour to discover the nature of that law; we shall then assume the principle of attraction to be known, and investigate the dependent conditions of the motion. The principles which regulate the motion of the planets in their revolutions about the sun, are generally known under the denomination of *Kepler's laws*, having been first discovered by that indefatigable and celebrated astronomer, who deduced them from innumerable observations. They are the three following:—

Kepler's Laws.

1. The planets move in plane curves, and their *radii vectores* describe about the sun, as a centre, areas which are proportional to the times.

2. The orbits or trajectories of the planets are ellipses, having the sun in one of their focii.

3. The squares of the times of the revolutions of the planets about the sun, are to each other as the cubes of the major axes of their orbits.

These laws appertain to the motion of the centre of gravity of each planet; and it is, therefore, the motion of this point that we shall have to consider in what follows; consequently, when we are speaking of the motion, velocity, or position of a planet, the centre of gravity of it is always to be understood, unless the contrary be specifically stated.

58. The first law shows, that the accelerating force which acts on a planet, and which is supposed to be applied to its centre of gravity, has its direction constantly coinciding with the right line joining the centres of gravity of the sun and planet. The direction of the force being thus known, let us see in what manner the second law may be employed to determine its intensity in the different positions of the planet, with respect to the attracting body.

Equation of the trajectory.

For this purpose, let ABDE (fig. 44) be the ellipse described by the planet, AB its major axis, C its centre, and O one of the two focii in which we suppose

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the sun to be placed. Let also m be the place of the planet at any instant, $r = Om$ its radius vector, v the angle $m\,O\,x$, contained between the radius $O\,m$ and fixed axis $O\,x$, which is drawn arbitrarily in the plane of the orbit; thus r and v will be the polar co-ordinates of any point of the ellipse, and its equation, as referred to these co-ordinates, will have the form

$$r = \frac{a\,(1 - e^2)\,[*]}{1 + e\,\cos(v - \omega)}; \qquad (1)$$

where $a = CA$ denotes the semi axis major; $a\,e = CO$ the distance from the centre to one of the focii; $a\,\surd\,(1 - e^2) = CD$ the semi axis minor; and ω the angle $AO\,x$ comprised between the major axis and the fixed line from which the angle v is estimated.

In astronomy, a is called the *mean distance* of the planet; $a\,(1 + e)$ and $a\,(1 - e)$ are the greatest and least distance; e is the *eccentricity* of the orbit. The variable angle v is the *longitude* of the planet estimated in the plane of the orbit; the constant angle ω is the longitude of the perihelion, or of the point of the orbit nearest to the sun; the *aphelion*, on the contrary, is the point of the orbit farthest from the sun. Lastly, the angle $v - \omega$ which expresses the angular distance of the planet from the perihelion, is called the *true anomaly* of the planet.

Equation of the motion.

59. We have seen, in treating of the motion of a body attracted towards a fixed centre, that if x and y denote the co-ordinates as referred to two fixed axes, we have

$$x\,\dot{y} - y\,\dot{x} = c\,\dot{t}.$$

And if in this equation we substitute for x and y their equals $r\cos v$ and $r\sin v$, we shall have

$$x\dot{y} - y\,\dot{x} = r^2\,\dot{v}.$$

The arc or angle denoted by $\dot{v}$, being infinitely small, the sector described by the radius vector, or the area included between the two radii vectores and the arc, may be considered as a circular sector, equal to $\frac{1}{2}\,r^2\dot{v}$; consequently we have

$$x\,\dot{y} - y\,\dot{x} = r^2\dot{v} = c\,\dot{t}; \qquad (2)$$

c being a constant quantity which expresses the double of the area described in the unit of time.

This being established, let us denote by x and y the rectangular co-ordinates of the planet, as referred to the axes $O\,x$ and $O\,y$, drawn through the centre of the sun and in the plane of its orbit; also let R denote the accelerating force which acts on the planet, and since this force has its direction coinciding with the right line $O\,m$, its components parallel to the axes of the co-ordinates will be $R\,\frac{x}{r}$ and $R\,\frac{y}{r}$, because $\frac{x}{r}$ and $\frac{y}{r}$ are the cosines of the angles $m\,O\,x$, and $m\,O\,y$. Moreover as the curve described by the body is concave towards the sun, it is evident that the force which produces the motion ought to be directed towards this point; its components will, therefore, tend to diminish the co-ordinates x and y: consequently the equation of the motion will be

$$\frac{\ddot{x}}{\dot{t}^2} = -R\,\frac{x}{r} \text{ and } \frac{\ddot{y}}{\dot{t}^2} = -R\,\frac{y}{r};$$

multiplying the first by $2\,\dot{x}$, and the second by $2\,\dot{y}$, we shall have, after adding the results,

$$\frac{2\,\dot{x}\,\ddot{x} + 2\,\dot{y}\,\ddot{y}}{\dot{t}^2} = -R\;\frac{2\,x\dot{x} + 2\,y\dot{y}}{r};$$

taking the fluents, and observing that $x^2 + y^2 = r^2$, and consequently that $2\,x\,\dot{x} + 2\,y\,\dot{y} = 2\,r\,\dot{r}$, we have

$$\frac{\dot{x}^2 + \dot{y}^2}{\dot{t}^2} = b - 2\int R\,\dot{r}; \qquad (3)$$

b being a constant arbitrary, or correction.

Again, since

$$x = r\cos v, \text{ and } y = r\sin v,$$

we have

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\,\dot{v}^2;$$

substituting this in equation (3), and eliminating $\dot{t}$, by means of equation (2), it will become

$$\frac{c^2\,\dot{r}^2}{r^4\,\dot{v}^2} + \frac{c^2}{r^2} = b - 2\int R\,\dot{r}. \qquad (4)$$

If the force R were given in functions of r, this equation would be that of the trajectory; and a comparison of it with equation (1) ought, therefore, to enable us to determine this force.

Now equation 1 may be written

$$\frac{1}{r} = \frac{1 + e\cos(v - \omega)}{a\,(1 - e^2)};$$

and taking the fluxion, and dividing the second side by $\dot{v}$, we have

$$\frac{\dot{r}}{r^2\,\dot{v}} = \frac{e\sin(v - \omega)}{a\,(1 - e^2)};$$

or squaring both sides,

$$\frac{\dot{r}^2}{r^4\,\dot{v}^2} = \frac{e^2\,\sin^2(v - \omega)}{a^2\,(1 - e^2)^2} =$$

$$\left\{\frac{e^2}{a^2(1 - e^2)^2} - \frac{e^2\cos^2(v - \omega)}{a^4\,(1 - e^2)^2}\right\};$$

but equation (1) gives also

$$\frac{e^2\cos^2(v - \omega)}{a^2\,(1 - e^2)^2} = \frac{1}{r^2} - \frac{2}{r\,a\,(1 - e^2)} + \frac{1}{a^2(1 - e^2)^2};$$

whence

$$\frac{1}{r^4}\cdot\frac{\dot{r}^2}{\dot{v}^2} = \frac{2}{r\,a\,(1 - e^2)} - \frac{1}{r^2} - \frac{1}{a^2(1 - e^2)^2};$$

which changes equation (4) into

$$\frac{1}{r}\cdot\frac{2\,c^2}{a\,(1 - e^2)} - \frac{c^2}{a^2(1 - e^2)^2} = b - 2\int R\dot{r};$$

taking the fluxions on both sides;

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[*] This equation is readily deduced from the known equation of the ellipse; viz.

$$y^2 = \frac{c^2}{a^2}\,(a^2 - x^2).$$

For here, observing that $CO = a\,e$, we have $a^2 - a^2e^2 = c^2$, also $y^2 = r^2 - r^2\cos^2\phi$, (writing $v - \omega = \phi$) and $x^2 = (r\cos\phi + a\,e)^2$; substituting these values, we obtain

$$r^2\,(1 - \cos^2\phi) = (1 - e^2)\,(a^2 - (r\cos\phi + a\,e)^2),$$

which readily reduces to

$$r^2 = \{a\,(1 - e^2) + r\,e\cos\phi\}^2; \text{ whence}$$

$$r = a\,(1 - e^2) + r\,e\cos\phi; \text{ or}$$

$$r = \frac{a\,(1 - e^2)}{1 - e\cos\phi}; \text{ or}$$

$$r = \frac{a\,(1 - e^2)}{1 - e\cos(v - \omega)}.$$

$$\frac{2c^2}{a(1-e^2)} \times \frac{-\dot{r}}{r^2} = -2R\dot{r},$$

Whence $$R = \frac{c^2}{a(1-e^2)} \times \frac{1}{r^2}, \text{ or}$$

$$R = \frac{u}{r^2};$$

(making, for the sake of abridging, $\frac{c^2}{a(1-e^2)} = u$) which denotes the intensity of the accelerating force at the unit of distance.

Here, since the numerator u is a constant quantity, it follows that the force R, which retains a planet in its orbit, is in the inverse ratio of the square of its radius vector. It is, therefore, by means of a combination of an accelerating force, which varies according to the above law, with a primitive impulsion, that each planet describes an ellipse about the sun, such that the latter body occupies one of the two focii. This, as we have seen, is one of the laws which Kepler deduced from observation. But, as far as we have at present pursued our investigation, it applies only to each planet separately; it remains, therefore, now to ascertain from Kepler's third principle, whether the force of gravity observe the same law in its action on the different planets of our system.

60. Let T denote the time of the revolution of any planet about the sun; then c T will represent double the area described by its radius vector in the time T, because the areas are proportional to the times in which they are described, and c has been assumed to denote double the area described in the unit of time. But during the time T, the radius vector will describe the entire surface of the ellipse, which is equal to the product of the two semi-axes into the decimal 3·14159, &c.; denoting, therefore, the latter by π, and the two axes by a and $a\sqrt{(1-e^2)}$, we shall have

$$\pi a^2 \sqrt{(1-e^2)};$$

for the area corresponding to the time T, and consequently

$$c\,T = 2\pi a^2 \sqrt{(1-e^2)};$$

whence again

$$c = \frac{2\pi a^2 \sqrt{(1-e^2)}}{T};$$

and substituting this value of c in the preceding equation, which exhibits the value of u, viz.

$$u = \frac{c^2}{a(1-e^2)};$$

it becomes

$$u = \frac{4\pi^2 a^3}{T^2}.$$

In like manner, if we consider another planet whose time of revolution is T′, and its mean distance a', we shall have

$$u' = \frac{4\pi^2 a'^3}{T'^2};$$

but, from the third law of Kepler,

$$T^2 : T'^2 :: a^3 : a'^3;$$

consequently $u' = u$.

That is, the accelerating force is the same at the unit of distance for all the planets of the system; and, as it respects the same planet in different positions, and the several planets in their different orbits, its intensity is always reciprocally proportional to the square of the distance from the sun. Consequently, if we conceive two planets, of whatever magnitudes, to be placed at equal distances from the sun, the accelerating forces acting upon them will be the same; and if left to the action of this force without any projectile velocity, they will each partake of the same motion, viz. each will pass in a right line from its first position to the attracting body, and each with the same contemporaneous velocity; which is precisely analogous to what we find to be the result of terrestrial attraction: all bodies placed at the same distance from the centre of the earth having equal motions impressed upon them in the same time, as we have already seen in treating of the laws of falling bodies.

Remark. We have taken the preceding order of investigation for the reasons already stated, namely, that it was from the laws of the planets' motions being first discovered from observation, that we were led to a knowledge of the nature of the forces by which they were retained in their orbits, or it was at least these laws which confirmed that deduction; but the most direct method of viewing this problem is to suppose the nature of the force to be given, and thence to investigate the conditions of the motion of the several planetary bodies.

Inverse problem. 61. Let us, then, still represent the radius vector of a planet at any instant by r, and the accelerating force estimated in the direction of this radius $\frac{u}{r}$; u being a constant and positive co-efficient. It is, in the first place, evident that a material point projected into space in the direction of any right line, being also at the same time submitted to the action of a constant force directed towards a fixed point, ought to move in the plane of the line of this direction, and that which joins it to the fixed point at the instant of projection; for whatever reason can be imagined for its moving in any other plane on the one side of this, will also apply to a plane symmetrically situated on the other side. The orbit, therefore, of each planet is a plane curve which coincides with the plane passing through the primitive direction of the impulsive and attracting forces, to which it is in the first instant submitted. Let us draw, therefore, in this plane, and through the centre of the sun, the two rectangular axes O x and O y (fig. 44), which will be those of the co-ordinates x and y of the planet; the components of the force $\frac{u}{r^2}$, parallel to these axes, will be $\frac{ux}{r^3}$, and $\frac{uy}{r^3}$;

and as they obviously tend to diminish the co-ordinates x and y, we shall have for the two equations of the motion,

$$\frac{\ddot{x}}{\dot{t}^2} = -\frac{ux}{r^3}, \text{ and } \frac{\ddot{y}}{\dot{t}^2} = -\frac{uy}{r^3};$$

$\dot{t}$ being the fluxion, or element of the time.

Multiply by $2\dot{x}$ and $2\dot{y}$, and we have

$$\frac{2\dot{x}\ddot{x}}{\dot{t}^2} = \frac{-2u\dot{x}x}{r^3}, \text{ and } \frac{2\dot{y}\ddot{y}}{\dot{t}^2} = -\frac{2u\dot{y}y}{r^3};$$

by addition

$$\frac{2\dot{x}\ddot{x} + 2\dot{y}\ddot{y}}{\dot{t}} = -2u\frac{x\dot{x} + y\dot{y}}{r^3};$$

but $x^2 + y^2 = r^2$; therefore $x\dot{x} + y\dot{y} = r\dot{r}$.

Whence $\dfrac{\dot{x}^2 + \dot{y}^2}{\dot{t}^2} = \displaystyle\int \dfrac{-2u\dot{r}}{r^2} = \dfrac{2u}{r}$;

or, introducing the correction,

$$\frac{\dot{x}^2 + \dot{y}^2}{\dot{t}^2} - \frac{2u}{r} + b = 0;$$

we have seen also (art. 59), that

$$x\dot{y} - y\dot{x} = c\dot{t}.$$

If we make use of the polar co-ordinates r and v, as in art. 58, the angle v being supposed to be estimated from the axis O x, we shall have

$$x = r \cos v, \text{ and } y = r \sin v;$$

and the above equations will become

$$\left\{\begin{array}{c} \dfrac{\dot{r}^2 + r^2\dot{v}^2}{\dot{t}^2} - \dfrac{2u}{r} + b = 0 \\ r^2\dot{v} = c\dot{t}. \end{array}\right\}. \qquad (5)$$

Eliminating $\dot{t}$, and abridging, by writing $z = \dfrac{1}{r}$, the reduced equation will be

$$c^2\frac{\ddot{z}}{\dot{v}^2} + c^2z^2 - 2uz + b = 0,$$

which is the fluxional equation of the trajectory.

Resolving this with reference to $\dot{v}$, we have

$$\dot{v} = \frac{c\dot{z}}{\sqrt{(2uz - b - c^2z^2)}} = \frac{\dfrac{c^2\dot{z}}{\sqrt{(u^2 - bc^2)}}}{\sqrt{\left\{1 - \left(\dfrac{u - c^2z}{\sqrt{(u^2 - bc^2)}}\right)^2\right.}}.$$

Taking the fluents, we obtain

$$v = \omega + \text{arc}\left(\cos = \frac{u - c^2z}{\sqrt{(u^2 - bc^2)}}\right);$$

ω a being a constant quantity or correction.

Reciprocally, we have

$$c^2z = u - \sqrt{(u^2 - bc^2)}.\cos(v - \omega);$$

or, replacing z by its value $\dfrac{1}{r}$,

$$c^2 = ur + r\sqrt{(u^2 - bc^2)}\cos(v - \omega). \qquad (6)$$

Because, the angle ω being arbitrary, we may change the sign of the radical $\sqrt{(u^2 - bc^2)}$, and take it with a positive sign, which is the same as augmenting ω by two right angles.

From this equation we see that the trajectory is a conic section; but it will be more obvious from the following transformations:

Equation of the orbit.

62. Let us draw through the centre of the sun, in the plane of the orbit, the rectangular axes O x' and O y', and conceive that the axis O x' forms with the right line O x, whence we estimate the angle v, an angle x' O x equal to the constant angle ω. Let x' and y' be the co-ordinates of the planet as referred to these new axes; the radius vector will make with the axis O x' an angle $v - \omega$; consequently we shall have

$$x' = r\cos(v - \omega),\ y' = r\sin(v - \omega),$$
$$r = \sqrt{(x'^2 + y'^2)};$$

whence equation (6) will become

$$c^2 = u\sqrt{(x'^2 + y'^2)} + x'\sqrt{(u^2 - bc^2)};$$

or, by transposing and squaring,

$$u^2y'^2 + bc^2x'^2 = c^4 - 2c^2x'\sqrt{(u^2 - bc^2)}.$$

The orbit is a conic section.

This equation belongs to the ellipse, the hyperbola, or parabola, according as the constant quantity b is positive, negative, or zero, and as the radius vector $\sqrt{(x'^2 + y'^2)}$ may be expressed under a rational form in functions of the absciss x', it follows that the origin of the co-ordinates, in all these cases, is in one of the focii of the curve. Hence, a material point attracted towards a fixed centre by a force which varies inversely as the square of the distance, will describe a conic section, of which that centre is in one of the focii; but the nature and dimensions of the conic section described will depend on the constant arbitraries b and c, while these themselves depend upon the initial conditions of the motion, as appears from the manner in which these quantities are determined.

63. At the origin of the motion let l be the distance of the body from the fixed centre; that is, at this point let $r = l$, let V denote the corresponding velocity, or $\mathrm{V} = \dfrac{\sqrt{(\dot{r}^2 + \dot{v}^2)}}{\dot{t}}$; lastly, let θ represent the angle that the direction of this velocity makes with the radius vector; then V $\cos\theta$ will be the component of that velocity in the line of the radius vector, or $\mathrm{V}\cos\theta = \dfrac{\dot{r}}{\dot{t}}$, to which it corresponds at this instant.

From these equations we readily deduce

$$\frac{\dot{v}}{\dot{t}} = \frac{\mathrm{V}\sin\theta}{l}.$$

And substituting these values of r, $\dfrac{\dot{r}}{\dot{t}}$, and $\dfrac{\dot{v}}{\dot{t}}$ in our equations (5), they become

$$b = \frac{2u}{l} - \mathrm{V}^2,\ c = \mathrm{V}l\sin\theta.$$

From which equations it appears, that the constant quantity b, will be positive, negative, or zero, according as V^2 is less or greater than, or equal to $\dfrac{2u}{l}$. It follows, therefore, that the nature of the conic section which the particle or body will describe, depends entirely on the initial velocity and distance, and that it is wholly independent of the direction of its first projection. But the angle θ will influence the dimensions of the curve, since it enters into the value of c; and when that angle is zero then $c = 0$, and the trajectory becomes a right line, which is the limit of all the conic sections.

From observation it has been ascertained, that the planets describe ellipses, of which the sun occupies one of the focii; and hence we conclude, that the force which solicits them towards the centre of the sun varies in the inverse ratio of the square of their radii vectores. We learn also, from analytical investigation, that the same accelerating force, combined with a proper primitive impulsion, may cause a material point to describe, not only an ellipse, but also an hyperbola or parabola: it is, therefore, possible that there are in nature bodies revolving in orbits of the latter kind, which are therefore necessarily only once visible in our system; such are, probably, many of the cometary orbits.

64. Let us consider, in particular, the circumstances of an elliptic motion, and determine for this case the ordinates r and v in functions of the time.

Eliminating $\dot{v}$ out of our equations (5), we find

$$\frac{\dot{r}^2}{\dot{t}^2} + \frac{c^2}{r^2} - \frac{2u}{r} + b = 0; \qquad (7)$$

whence we may immediately conclude, that the greatest and the least value of r, which obtain in the case where $\frac{\dot{r}}{\dot{t}} = 0$ will be given by a quadratic equation; viz.

$$r^2 - \frac{u^2}{b} r + \frac{c^2}{b} = 0.$$

If, therefore, we denote these values, as in art. 58, by $a(1+e)$ and $a(1+e)$; their sum $2a$, will be equal to the co-efficient $\frac{2u}{b}$, and their product $a^2(1-e^2)$ will be equal to the last term $\frac{c^2}{b}$; which identities give

$$b = \frac{u}{a} \text{ and } c^2 = ua(1-e^2).$$

We shall arrive at the same conclusion by comparing equations (1) and (6); in fact it is necessary, in order that these two equations of the trajectory may coincide, to have

$$\frac{c^2}{u} = a(1-c^2); \text{ and } \frac{\sqrt{(u^2-bc^2)}}{u} = e;$$

from which it is easy to deduce the preceding values of b and c.

If we substitute these values in equation (7), and then resolve it with reference to $\dot{t}$, it will become

$$\dot{t} = \frac{r\dot{r}\sqrt{\frac{a}{u}}}{\sqrt{(2ar - r^2 - a^2 + a^2e^2)}}.$$

By assuming $r = a(1 - e\cos\mu)$,
μ being a variable, the above radical disappears; for by this means we obtain

$$\dot{r} = ae\sin\mu\,.\,\dot{\mu};\ a^2e^2 - (r-a)^2 = a^2e^2\sin^2\mu;$$

and consequently,

$$\dot{t} = (1 - e\cos\mu)\frac{a\sqrt{a}}{\sqrt{u}}\,\dot{\mu}.$$

Or, for the sake of abridging, if we write $\frac{\sqrt{u}}{a\sqrt{a}} = n$, and then take the fluents, we have

$$nt = \mu - e\sin\mu + \text{correction}.$$

This equation gives the value of μ in functions of t, and we shall have afterwards the value of r from the equation

$$r = a(1 - e\cos\mu);$$

and finally the value of v will be given from the equation of the trajectory. The values of r and v being thus expressed in functions of the time, we may thence determine the position of the planet in its orbit at any proposed instant.

The eccentricity of the planetary orbits is, in general, a very small fraction, which allows of our converting the values of μ, r, and v, into infinite series, having a considerable degree of convergency, arranged according to the ascending powers of e.

If we limit the approximation to the first power, it will not be necessary to employ the auxiliary quantity μ, for the purpose of obtaining the values of r and v; they may be determined very simply as follows:

Neglecting the square of e, the equation of the elliptic orbit, viz. equation (1), gives

$$r = a^2\{1 - 2e\cos(v - \omega)\};$$

and substituting this value in equation (2) it becomes

$$r^2\dot{v} = c\dot{t} = \dot{t}\sqrt{(ua(1-e^2))}.$$

Or rejecting e^2, and for the sake of abridging, writing $\frac{\sqrt{u}}{a\sqrt{a}} = n$, we have

$$n\dot{t} = \{1 - 2e\cos(v - \omega)\}\dot{v};$$

whence by taking the fluents

$$nt + k = v - 2e\sin(v - \omega),$$

k being a constant arbitrary, or correction.

For the sake of simplifying, we may make the constant quantities k and ω, each equal to zero, or which amounts to the same, we may consider the angle v to commence from the right line Ox', which coincides with the part OA of the axis major, passing through the perihelion; and the time t to commence from the instant when the planet passes through this point; so that we may have $t = 0$, and $v = 0$, at the point A.

In this case we shall have

$$v = nt + 2e\sin v;$$

consequently, $v = nt$, when we neglect the quantity e; and, therefore, when we reject only e^2, we shall have

$$v = nt + 2e\sin nt.$$

The corresponding value of the radius vector r, deduced from the preceding expression for r^2, is

$$r = a(1 - e\cos nt).$$

And by means of these two equations we shall be able at any instant to determine the position of a planet in its orbit.

These approximations are obtained in consequence of the eccentricity of the planetary orbits being very small, and if it were consistent with the nature of this article, we might show, that when the eccentricity is very great, we have then also an easy method of obtaining a first approximation; but that all medium values of e would involve us in inextricable difficulties of computation; and it is remarkable, that the only celestial bodies whose motions fall within the contemplation of astronomers, have their orbits constituted under one or the other of these extremes; the planets, as we have seen, having their eccentricities very small, while the comets, on the contrary, have the eccentricities very great. It was this which led Lagrange to observe, that "Nature seems to have disposed these orbits on purpose to enable mathematicians to calculate them: thus the eccentricity of the planets is very small, and that of the comets very large: and without this disparity, so favourable to approximations, farewell to mathematicians—it would be impossible to do any thing."

Equation of the centre.

65. In order to represent the motion of the planets, astronomers conceive an imaginary star to move circularly about the sun in the plane of the orbit; which commences its motion from the perihelion at the same instant as the planet, and of which the angular distance from this point is always equal to the first term nt of the value of v. The radius vector of this imaginary star moves uniformly, and forms every instant with that of the planet an angle equal to $2e\sin nt$. This variable angle is called the *equation of the centre*. When the angle v is equal to two or to four right angles, the two angles v and nt are equal; consequently the imaginary star will pass the aphelion and return to the perihelion in the same time as the planet; but in the first half of the revolution, the planet precedes the star; and in the second, the star precedes the plane.

If we call T the time of one entire revolution, which is completed when the angle $n t$ is equal to four right angles, we shall have $n\,T = 2\,\pi$, π being the semi-circumference of a circle to radius 1; and in replacing n by its value $\frac{\sqrt{u}}{a\sqrt{a}}$, the above equation becomes

$$T^2 = \frac{4\,\pi^2\,a^3}{u},$$

which agrees with our former deduction, art. 60.

Particular deductions.

66. It appears from this conclusion, as also from what has been stated in the preceding part of this chapter, that the periodic revolution of a planet depends on its mean distance alone, and will be the same whether it describe a circle or an ellipse, having any degree of eccentricity whatever; if, therefore, we conceive the shorter axis DE, of the ellipse ADBE (fig. 45), to diminish continually, the longer axis AB remaining the same, then as the extremity of the invariable line DC moves from D towards C, the extremity C will move towards A, so that when D coincides with C′, C will coincide with A, and the ellipse will be transformed into a straight line AB, the length of which is equal to 2 CD. In all the successive ellipses formed by this gradual diminution of C′D, the periodic time remains unchanged, provided the force acting at C remains unchanged. Just before the perfect coincidence of D with C′, the ellipse may be conceived as indistinguishable from the right line AB; and the revolution in such ellipse, indistinguishable from the ascent of the body along the right line from A to B, and the subsequent descent in an equal time from B to A; consequently a body solicited by such a centre of force, will descend from B to A in half the time of the revolution in the ellipse ADBE, and the time of descending through any distance DC (supposing the projectile force extinguished) is half the periodic time of a body revolving at half that distance from the sun; therefore *the squares of the times of falling to the sun, or any other centre of force of which the intensity is inversely as the square of the distance, are as the cubes of the distances from which the bodies begin to fall*, as we have seen (art. 26).

To find the time of descent in particular instances, multiply half the periodic time by the square root of the cube of $\frac{1}{2}$, or by $\frac{1}{2}\sqrt{\frac{1}{8}}$, or $\frac{1}{8}\sqrt{2} = \cdot1767767$.

This process gives us 4 d. 20 h. for the time of descent from the Moon to the Earth;

Mercury to the Sun......	15 d. 13 h.
Venus................	30 d. 17 h.
The Earth.............	64 d. 13¾h.
Mars	121 d. 10½h.
Jupiter	765 d. 21⅔h.
Saturn	1901 d. 22½h.
Herschel	5433 d. 17 h.

And the same rule will apply to the descents of the satellites of Jupiter, Saturn, &c. to their respective primaries.

§ XIII. *Of the motion of a material point on a given curve.*

Motion on a curve.

67. When a material point is constrained to move on a given curve of double, or simple curvature, the resistance which the curve opposes to its motion, is equivalent to a force acting continually on the particle, in a direction perpendicular to the trajectory, such, that in adding to the given forces in each particular case, a new accelerating force to represent this resistance, we may afterwards make abstraction of the curve and consider the material point as perfectly free.

If we resolve each of the given accelerating forces which act on the particle into two others, the one class having their direction coinciding with that of the tangent to the curve, and the others perpendicular to it, the latter will be destroyed by the resistance of the curve, and the former only will be effective.

The resultant of the normal forces, which may be denoted by P, will express the pressure the given forces exert upon the curve in each of these points, and the resistance which counteracts this pressure will be an accelerating force equal and contrary to the force P. This pressure P would be the only one the curve would have to support, if the body were at rest; but its state of motion gives rise to another pressure, proceeding from the continual tendency of the body to fly off in the direction of the tangent of the curve.

Considering the curve as a polygon $m\,m'\,m''\,m'''$, &c. (fig. 46) of an infinite number of sides, the angle $t\,m'\,m''$ formed by the prolongation of the side $m\,m'$ with the consecutive side $m'\,m''$, is called the *angle of contingence*, which we may denote by ω; the plane $t\,m'\,m''$ of the two elements of the curve at this point is called the *osculatory plane* of the curve to the point m'. When the curve is of double curvature, this plane varies from one point to another; but when the curve is of simple curvature the osculatory plane is the same as the plane of the curve itself.

The particle m, solicited by a given force, receives a primitive velocity v, which causes it to describe the side $m\,m'$, but when it is arrived at m', it will be deflected from that course and pass over the side $m'\,m''$; and in this passage will lose a certain velocity which it is necessary to determine.

Let us represent the velocity v by the right line $m'\,q$, and resolve $m'\,q$ into two velocities, the one $m'\,n$, in the direction of the side $m'\,m''$, and the other $m'\,l$ perpendicular to this side; then we shall have

$$m'\,l = m'\,q \sin t\,m'\,m'' \;,\; m'\,n = m'\,q \cos t\,m'\,m'';$$

or,

$$m'\,l = v \sin\omega \;,\; m'\,n = v \cos\omega.$$

The component $v \sin\omega$ being destroyed by the resistance of the polygon, the velocity v will be reduced to $v \cos\omega$; consequently the velocity lost, which is equal to the primitive velocity *minus* the actual velocity, will be expressed by

$$v - v\cos\omega, \text{ or } v\,(1 - \cos\omega) = 2\,v \sin\frac{\omega^2}{2}.$$

When in place of the polygon we have a curve, the angle $t\,m'\,m''$ becomes indefinitely small, in which case the velocity $v\,(1 - \cos\omega)$, or $2\,v \sin\frac{\omega^2}{2}$ is an infinitely small quantity of the second order; the velocity, therefore, which is lost at each infinitely small side of the polygon being an infinitely small quantity of the second order, may be neglected with respect to the velocity $v \cos\omega$, which is of the first order; whence we may conclude, that a body constrained to move on a curve, will constantly preserve all the velocity impressed upon it in the first instant of its motion.

As to the force $v \sin\omega$ which acts on the curve, and is destroyed by its resistance, it varies at each element, because $\sin\omega$ is continually changing; we may therefore consider it as an accelerating force acting on the circulating body.

If, besides this force, we had several accelerating forces applied to the point m, by resolving them in the same manner, we should obtain the general resultant by adding all the other normal forces to $v \sin \omega$.

68. Let us conceive that at the point m (fig. 47), a normal force N is exerted, equal and directly opposed to the resultant of these forces; then the resistance opposed by the curve will be equal to and be measured by N. Call α, β, γ the angles which this normal accelerating force makes with three rectangular co-ordinates; then the components of N, as referred to these three axes, will be

$$N \cos \alpha,\ N \cos \beta,\ N \cos \gamma;$$

which must be added to the accelerating forces X, Y, and Z in the equation of the motion; so that we shall have

$$\left.\begin{aligned} \frac{\ddot{x}}{\dot{t}^2} &= X + N \cos \alpha \\ \frac{\ddot{y}}{\dot{t}^2} &= Y + N \cos \beta \\ \frac{\ddot{z}}{\dot{t}^2} &= Z + N \cos \gamma \end{aligned}\right\} \qquad (1)$$

To these equations we must add two others, which depend upon the relation existing between the angles α, β, γ; viz. first,

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 0;$$

and secondly,

$$\cos \alpha \cos \alpha' + \cos \beta \cos \beta' + \cos \gamma \cos \gamma' = 0 \qquad (2)$$

With regard to the latter we may observe, that when two right lines in space, including between them a right angle, form with the co-ordinate axes the angles α, β, γ and α', β', γ', this equation necessarily results; and in the present instance these two lines are the normal and tangent to the curve; the angles α, β, γ being those formed by the normal, and α', β', γ' the angles formed by the tangent; and therefore we have

$$\cos \alpha' = \frac{\dot{x}}{\dot{s}},\ \cos \beta' = \frac{\dot{y}}{\dot{s}} \text{ and } \cos \gamma' = \frac{\dot{z}}{\dot{s}},$$

$\dot{s}$ being the fluxion of the curve.

If, therefore, we substitute these values in equation (2), it will become

$$\frac{\dot{x}}{\dot{s}} \cos \alpha + \frac{\dot{y}}{\dot{s}} \cos \beta + \frac{\dot{z}}{\dot{s}} \cos \gamma = 0.$$

Multiplying equations (1) respectively by $2\dot{x}$, $2\dot{y}$ and $2\dot{z}$, and taking the sum, we obtain

$$2\dot{x}\frac{\ddot{x}}{\dot{t}^2} + 2\dot{y}\,\frac{\ddot{y}}{\dot{t}^2} + 2\dot{z}\frac{\ddot{z}}{\dot{t}^2} = 2(X\dot{x} + Y\dot{y} + Z\dot{z})$$
$$+ 2N(\dot{x} \cos \alpha + \dot{y} \cos \beta + \dot{z} \cos \gamma).$$

But the latter member of this equation is, from what we have seen above, equal to zero; whence, therefore, the above becomes simply

$$2\dot{x}\frac{\ddot{x}}{\dot{t}^2} + 2\dot{y}\,\frac{\ddot{y}}{\dot{t}^2} + 2\dot{z}\,\frac{\ddot{z}}{\dot{t}} = 2(X\dot{x} + Y\dot{y} + Z\dot{z});$$

Or, since the first side is equal to the fluxion of

$$\frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{\dot{t}^2}, \text{ we may write}$$

$$\frac{f(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}{\dot{t}^2} = 2(X\dot{x} + Y\dot{y} + Z\dot{z}).$$

Again we have $\dot{s}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$; therefore, taking the fluents

$$\frac{\dot{s}^2}{\dot{t}^2} = 2\int(X\dot{x} + Y\dot{y} + Z\dot{z});$$

or,

$$v^2 = 2\int(X\dot{x} + Y\dot{y} + Z\dot{z}). \qquad (3)$$

If we suppose the accelerating forces X, Y, and Z, each equal to zero; we have then

$$v^2 = \textit{a constant quantity.}$$

That is, a body void of gravity, moving on a curve, will preserve, always, the same velocity; which agrees with our previous determination (art. 67).

When the particle has weight.

69. If we suppose the body to have weight, then we shall have $Z = 2g$, $X = 0$, and $Y = 0$; therefore, equation (3) in this case gives

$$v^2 = 2\int 2g\dot{z} = 4gz + C.$$

And if v becomes V, when $z = 0$, then $V^2 = C$, and we obtain

$$v^2 = 4gz + V^2,$$

whence we deduce

$$v = \surd(4gz + V^2). \qquad (4)$$

This equation gives the velocity independent of the relations which may exist between the co-ordinates x, y, and z, and is therefore the same, whatever may be the form of the curve. The above equation, however, is not sufficient for determining the time and space; for by substituting in the usual way for $v = \frac{\dot{s}}{\dot{t}}$ we only obtain

$$\frac{\dot{s}}{\dot{t}} = \surd(4gz + V^2),$$

or

$$\frac{\surd(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}{\dot{t}} = \surd 4gz + V^2);$$

from which it is obvious no conclusion can be drawn; because, in order to effect the integration, it is necessary that by means of the equations of the curve, we should be able to reduce this equation to two variables only; for suppose the equations of the curve were

$$z = f(x, y), \text{ and } z = \phi(x, y).$$

If by means of these equations we eliminate two of the variables x, y, z, it will still remain to integrate, or find the fluents between one of those variables and $\dot{t}$.

70. The velocity given by equation (4) being determinable, without it being necessary to make use of the equations of the curve, we may conclude as above stated, that the velocity is independent of the form of the curve, and that it depends on the vertical z. Consequently, if from the point O (fig. 48), where $z = 0$, and $v = V$, we draw arcs of curves, OM, OM′, OM″, &c. which all terminate in a horizontal plane KL, all the ordinates z of these points being equal, it follows that if we send off different bodies from the point O, with the same velocity, they will have acquired equal velocities when they have respectively arrived at the points M, M′, M″, &c. situated on the horizontal plane.

Whatever may be the number of accelerating forces, when the equation is integrable, we may always determine v without it being necessary to assign the nature of the curve; for if, after having substituted for the accelerating forces X, Y, and Z, in the equation

$$v^2 = 2\int\left(X\dot{x} + Y\dot{y} + Z\dot{z}\right),$$

the integral or fluent be taken, it may be represented by

$$2f(x, y, z);$$

that is, v^2 will be some function of x, y, and z, plus correction, or

$$v^2 = 2f(x, y, z) + C.$$

And if $2f(a, b, c)$ be taken to denote this function when $v = 0$, we shall have $C = -2f(a, b, c)$: consequently the correct fluent will be

$$v^2 = 2f(x, y, z) - 2f(a, b, c):$$

an equation which depends only upon the actual values of a, b, c, and x, y, z, without any reference to their particular relation.

Normal force.

71. We have seen that the normal force N is derived from the components of the accelerating forces, estimated in the direction of the normal of the curve, and the normal force produced by the velocity. In order to compute this last force, let fall the perpendiculars $o\,n$, $o\,n'$ (fig. 49) upon the middle of the equal consecutive sides $m\,m'$, $m'\,m''$, of the polygon supposed to contain an infinite number of sides; then the angle $t\,m'\,m''$ formed by one of the sides, and the prolongation of the other, will be the angle which we have denoted by ω. Now the angles n and n' of the quadrilateral $n\,o\,n'\,m$ being right angles, we shall have

$$n\,o\,n' + n\,m'\,n' = 2 \textit{ right angles} = t\,m'\,m'' + n\,m'\,n';$$

therefore

$$t\,m'\,m'' \text{ or } \omega = n\,o\,n' = 2\,n\,o\,m'.$$

The angle $n\,o\,m'$ being indefinitely small, we may employ the sine of the arc instead of the arc itself; and as this sine is expressed by $\frac{m'\,n}{m'\,o}$, or by $\frac{m'\,n}{n\,o}$, since the right lines $m'\,o$ and $n\,o$ may be considered as equal, we shall find

$$\omega = \frac{2\,m'\,n}{n\,o} = \frac{m\,m'}{n\,o};$$

and passing from the polygon to the curve, which is its limit, the side $m'\,m$ will become the element of the curve; and consequently the preceding equation changes into

$$\omega = \frac{\dot{s}}{\gamma}.$$

Where γ denotes the radius of curvature, call ϕ the accelerating force, which is derived from the normal components of the velocity; then as every accelerating force is represented by the fluxion of the velocity divided by that of the time, and as in the present instance the fluxion or element of the velocity is $v \sin \omega$, we shall have

$$\phi = \frac{v \sin \omega}{\dot{t}}.$$

And because an infinitely small arc may be substituted for its sine, this expression will become

$$\phi = \frac{v\,\omega}{\dot{t}}.$$

Replacing now ω by its value, found as above, we shall have

$$\phi = \frac{v\,\dot{s}}{\gamma\,\dot{t}}, \text{ or } \phi = \frac{v^2}{\gamma}.$$

With regard to the normal pressure resulting from the other forces, we may determine it by the principle of the parallelogram of forces. Suppose, for example, the curve were a plane, and that the forces which are applied to the body act in the plane of the curve, we may, in this case, reduce all the forces to one force R directed in this plane; and calling θ the angle which this force makes with the normal, we shall have $R \cos \theta$ for the component of R in the direction of the normal. If this force act against the curve, it will be in an opposite direction to that of the force $\frac{v^2}{\gamma}$, which presses the material point on the curve, or which is the same thing, which tends to drive the body from the centre; we shall therefore have in this case

$$N = \frac{v^2}{\gamma} - R \cos \theta.$$

§ XIV. *Of the centrifugal force.*

72. When a body revolves about a fixed centre C (fig. 50), and describes the curve LMK, it would, if free, fly off, in consequence of its inertia, in the direction of the tangent LT of the curve; but being retained by its connection with the fixed centre C, it will, instead of taking the direction of the tangent, be drawn down to some point in the curve, as for instance to M.

Now the arc LM being supposed indefinitely small, the angle LCM will be so likewise, and consequently we may consider the lines LC and MC as parallel. In this case, therefore, we may replace CM by a line parallel to LC, drawn through the point M, that is to say, by MC′: then the fixed centre may be considered as if it were situated in C′. The body would, if free, have in the same time arrived at D; LD being one of the sides of the indefinitely small parallelogram LNMD; whereas, being retained by the fixed centre, it is found in M; the effect of the force which draws it towards C′ will therefore be measured by MD. On the other hand, the right line which draws the body towards the centre, will obviously experience a tension exactly equal to this force; and it will consequently be also measured by MD. This tension, or the tendency which the body has to fly off from the centre C, is what we call the *centrifugal force*, and is equal and directly opposed to that which solicits it towards the centre, which is denominated the *centripetal force.*

It follows, therefore, from what has been shown in the preceding chapter, that the centrifugal force is equivalent to what we have there denoted by the expression $\frac{v^2}{\lambda}$, λ denoting the radius of curvature.

73. In order to determine by a direct process the expression for the centrifugal force, we may replace the infinitely small arc, by the chord of the osculating circle to the point L (fig. 51), and the effect of the centrifugal force by the versed sine LN.

Then, from the property of the circle, we have

$$LN = \frac{\dot{s}^2}{2\lambda},$$

where $\dot{s}$ denotes the fluxion of the arc, and γ as before the radius of curvature, or the radius of the circle LOE; or, if instead of $\dot{s}$, we substitute its equal $v\,\dot{t}$, we obtain

$$LN = \frac{v^2\,\dot{t}^2}{2\lambda}. \qquad (1)$$

Let us endeavour now to obtain another expression for LN, by considering, that while the body moves over the space $\dot{s}$, that is in the time $\dot{t}$, it would by the constant action of the centripetal force, which we may denote by f, have described the line LN; this, therefore, gives us, by substituting f or $2g$, art. (4) equation (8),

$$\mathrm{LN} = \tfrac{1}{2} f \dot{t}^2;$$

consequently

$$f \dot{t}^2 = \frac{v^2 \dot{t}^2}{2\lambda}, \text{ or } f = \frac{v^2}{\lambda};$$

or denoting the radius of the circle by R, that is, supposing the body to revolve in a circle, we shall have

$$f = \frac{v^2}{\mathrm{R}}.$$

Let h be the height due to the velocity v, and we obtain from (art. 8)

$$v^2 = 4 g h;$$

and eliminating v^2 by means of this equation and the preceding, we obtain

$$\frac{f}{2g} = \frac{2h}{\mathrm{R}}. \qquad (2)$$

This shows that the centrifugal force is to gravity as double the height due to the velocity is to the radius of the circle described by the body, which corresponds with the comprehensive proposition first given by the Marquis de l' Hôpital, viz.

Find from what height a body must have fallen to acquire the velocity in the circle; then as the radius of the circle is to double that height, so is the weight of the body to its centrifugal force.

74. If a semicircle EAF (fig. 52) revolve about its diameter EF = 2 R, the point A, the middle of the arc EAF, will describe a circumference equal to 2 π R, and supposing this motion of the point A is uniform, and performed in the time T; and if we denote the velocity by v, we shall have

$$v\,\mathrm{T} = 2\pi \mathrm{R},$$

and eliminating v by means of equation (1), it becomes

$$f = \frac{4\pi^2 \mathrm{R}}{\mathrm{T}^2}.$$

In a similar manner, if we denote by f' the centrifugal force due to the rotation of another body in a circumference whose radius is R′, and calling T′ the time, we shall have

$$f' = \frac{4\pi^2 \mathrm{R}'}{\mathrm{T}'^2};$$

consequently

$$f : f' :: \frac{\mathrm{R}}{\mathrm{T}^2} : \frac{\mathrm{R}'}{\mathrm{T}'^2}. \qquad (3)$$

Whence it follows, that when the radii R, R′ are equal, the centrifugal forces are inversely as the squares of the times, and when the times are the same, the centrifugal forces are directly as the radii.

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75. In order to compute the centrifugal force at the equator, it is only necessary to substitute for T the time of one revolution, viz. T = 86164 seconds, and for R the equatorial radius of the earth = 20921185 feet, and π = 3·14159· which gives

$$f = \frac{4 \times 3{\cdot}14159^2 \times 20921185}{86164^2} = {\cdot}1112259,$$

the centrifugal force at the equator. Now the actual force of gravity at the equator, as determined by the laws of pendulums and falling bodies, is

32·08818; and consequently if the earth were at rest, we should have for the actual force at the equator,

$$32{\cdot}08818 + {\cdot}1112259 = 32{\cdot}1994059;$$

whence the centrifugal force at the equator is to the actual force of gravity as

·1112259 : 32·1994059; or as 1 : 289 *nearly*.

It appears from the preceding computation, that the fraction $\frac{1}{289}$ which enters into all our investigations relative to the figure of the earth depends upon a due estimation of the equitorial radius of the earth, and the force of gravity at the equator, which data are only determinable from experiments; viz. the first, by measuring an arc of the meridian in those regions, and the other, by observations on the vibration of pendulums: it is singular, therefore, that such experiments and observations have not been attended to during the great progress that physical astronomy has been making of late years; the preceding ratio of 1 : 289 so generally assumed, depending upon measurements made in the year 1744, since which time a degree of accuracy has been introduced in those operations unknown in the time above referred to.

Centrifugal force in different latitudes.

76. In order to compute the diminution of gravity as arising from the centrifugal force in any other point than in the equator, we must compute the centrifugal force in the direction of the vertical BZ (fig. 52) drawn through the point B, which we have to consider. If we consider the earth as a perfect sphere, then the latitude of the point B, being represented by the arc AB, will be measured by the angle

$$\mathrm{BOA} = \mathrm{ZBC} = \psi.$$

Let us denote by R the radius AO of the earth; and R′, the radius BD of the parallel which passes through B, and we shall have

$$\mathrm{R}' = \mathrm{R} \cos \psi;$$

whence the centrifugal force of the point B, in the direction of BC, will be

$$f' = \frac{4\pi^2 \mathrm{R} \cos \psi}{\mathrm{T}^2};$$

and resolving this force according to the vertical BZ in which it is to be estimated, we shall have

$$1 : \cos\psi :: \frac{4\pi^2 \mathrm{R} \cos\psi}{\mathrm{T}^2} : \frac{4\pi^2 \mathrm{R}}{\mathrm{T}^2} \cos^2\psi;$$

whence it follows, that *the centrifugal force at the equator is to the same in any proposed latitude, estimated according to the vertical at that place, as radius to the square of the cosine of the latitude.*

It will be observed, that this result is obtained upon the hypothesis of the earth being a perfect sphere, and consequently, that it will require a certain modification, or rather to be determined upon other principles, when we consider the earth, as it really is, an oblate spheroid; but this investigation would carry us too far in the present article; we shall, however, have reference again to this question in our treatise on physical astronomy.

We shall conclude this chapter by the solution of the following

PROBLEM.

Problem.

77. *Required the time of the diurnal revolution of the earth, such, that the centrifugal force at the equator may be equal to the force of gravity.*

It is only necessary here to substitute in the equation

$$f = \frac{4\pi^2 \mathrm{R}}{\mathrm{T}^2}, \text{ or } \mathrm{T} = 2\pi \sqrt{\frac{\mathrm{R}}{f}};$$

for f, the given value of gravity $= 32{\cdot}1994$, and for R its value $= 20921185$, and $\pi = 3{\cdot}14159$, which will obviously give the numerical value of T. But it will be more easy to refer to our equation or analogy (3); viz.

$$f : f' :: \frac{R}{T^2} : \frac{R'}{T'^2};$$

in this case R and R′ are equal to each other, we have, therefore,

$$f : f' :: T'^2 : T^2;$$

also by the supposition $f' = 289 f$, we have

$$1 : 289 :: T'^2 : T^2;$$

$$\text{or, } T' = \sqrt{\frac{T^2}{289}} = \frac{T}{17}.$$

Whence it appears, that if the velocity of rotation of the earth were seventeen times greater than it is, or the time of one rotation seventeen times less, the centrifugal force would be equal to gravity, and bodies placed at the equator, would appear to be without weight; and if a greater velocity were impressed upon the earth, all bodies would be thrown from its surface; and projected into space.

§ XV. *On the oscillation of pendulums.*

Pendulums. 78. Pendulums are either simple or compound; a simple pendulum is conceived to be a material point, attached to an inflexible line, without weight, having its other extremity fixed to an immoveable point, called the centre of suspension, upon which the pendulum oscillates.

The compound pendulum is any material body, or several material bodies connected with each other, and supposed to vibrate about a certain fixed point, or centre of suspension. At present, however, we propose only to treat of the simple pendulum, in circular arcs, in cycloids, &c.

When a *pendulum* vibrates in the arc of a circle, it is, as we have seen above, called a *simple pendulum;* when in a cycloid, a *cycloidal pendulum;* and if, instead of vibrating in a plane, as supposed in what is stated above, it is made to move in, or its lower point to describe a circle, whereby the pendulum itself describes a conical surface; it is called a *conical pendulum.* We propose to examine the conditions of motion in each of these.

Velocity. 79. *Of the oscillation of simple pendulums in circular arcs.* We have seen (article 70) that the velocity of a heavy body in a curve, is wholly independent of the form of the curve; and that bodies departing from O (fig. 48), with the same velocity down the several curves OM, OM′, OM″, will all also have the same velocity when they arrive at the horizontal plane KL; and if, therefore, we suppose one of those curves, as OM, to become a vertical right line, and the velocity at O $= o$, it will then be obvious that the velocity in each of those curves at the points M, M′, M″, &c. will be the same as that acquired by a heavy body in falling through the vertical OM; and this is universal for all bodies moving along curves, or any inclined planes; viz. that the velocity acquired by a body in descending from rest down any curve, is the same as that acquired by a body in falling down the same vertical height.

This being established, let a body be supposed to descend from rest through an arc MZ (fig. 53) of any curve CZ*m*, it will, from the velocity acquired at its lowest point, abstracting from the effects of friction, resistance, &c. ascend up the opposite branch Z *m*, till it arrives at the point *m*, whose distance from the horizontal line AX, is equal to MP, the distance of the point M from the same; which is true, even if the branches of the curve are not symmetrical, and if they are symmetrical, the time in the two branches will be equal. Having at the point *m* lost all its motion, it will begin to descend again through *m* Z, and at Z will have acquired a velocity such as will cause it to rise to the point M, where its motion first commenced, and where it will be again in a state of quiescence; and whence it will move a second time over the arc MZ *m*, and back again through *m* ZM; and so on continually.

These motions will obtain the same, whether the body run over a curve surface in consequence of the joint effect of gravity and the re-action of the curve surface, or whether it be made to describe the curve in consequence of being suspended by a line CM, whose centre is fixed. If we suppose the thread to be inextensible, and without weight, and the point M (fig. 54) to be a heavy material particle, the system, as we have seen, is called a simple pendulum; the motion of such a pendulum in one direction from a state of rest till it begins to return in the opposite direction, is called a *vibration* or *oscillation,* and the time employed in one such motion, the *time* of vibration or oscillation: as in the case we have now supposed, the two branches of the curve are symmetrical, the time of one entire vibration is obviously equal to double the time the body employs in descending down the arc MA.

80. In order now to determine the time of oscillation in any circular arc, let MA *m*, (fig. 54) be the proposed arc, and let the radius or length of the pendulum CM $= l$, A E $= b$, A $p = x$, $p\,o = y$, the variable arc $= s$, the accelerating force of gravity equal $2\,g$; then the velocity due to the height E *p*, or the velocity acquired by falling through M *o*, will be

$$v = 2\sqrt{(g(b-x))};$$

but $v = \dfrac{\dot{s}}{\dot{t}}$; whence

$$\frac{\dot{s}}{\dot{t}} = 2\sqrt{(g(b-x))};$$

or

$$\dot{t} = \frac{\dot{s}}{2\sqrt{(g(b-x))}}.$$

Now we know from the principles of fluxions that x being the absciss in a circle, or $x = \text{A}\,p$, of which the radius is l, we have

$$\dot{s} = \frac{-l\,\dot{x}}{\sqrt{(2\,l\,x - x^2)}},$$

which is taken negatively, because the arc diminishes as the time augments; whence, by substitution,

$$\dot{t} = \frac{-l\,\dot{x}}{2\sqrt{\{(2\,l\,x - x^2)g(b-x)\}}}$$

or

$$\dot{t} = \frac{-\dot{x}}{\sqrt{(b\,x - x^2)}} \times \frac{l}{2\sqrt{(2\,l\,g - g\,x)}};$$

$$\dot{t} = \frac{\sqrt{l}}{2\sqrt{2\,g}} \times \left(1 - \frac{x}{2\,l}\right)^{-\frac{1}{2}} \times \frac{-\dot{x}}{\sqrt{(b\,x - x^2)}} -$$

Now by expansion $\left(1 - \frac{x}{2l}\right)^{-\frac{1}{2}} =$

$$1 + \tfrac{1}{2}\frac{x}{2l} + \frac{1\cdot3}{2\cdot4}\cdot\frac{x^2}{4l^2} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{x^3}{8l^3} + \&c.$$

whence

$$\dot{t} = \tfrac{1}{2}\sqrt{\frac{l}{2g}} \times \frac{\dot{x}}{\sqrt{(bx - x^2)}} \times \left(1 + \tfrac{1}{2}\frac{x}{2l} + \frac{1\cdot3}{2\cdot4}\frac{x^2}{4l^2} + \&c.\right)$$

In order now to determine t, or the time of descent from M to A, we must find the fluent of each term in the series, in such a manner that they may vanish when $x = b$. Now we see at once that the variable factor of each term of this series will be of the form $\frac{-x^n\dot{x}}{\sqrt{(bx - x^2)}}$; that is, these factors will be

$$\frac{-\dot{x}}{\sqrt{(bx - x^2)}},\ \frac{-x\dot{x}}{\sqrt{(bx - x^2)}},\ \frac{-x^2\dot{x}}{\sqrt{(bx - x^2)}},\ \&c.$$

Whence it appears that taking $\pi : 1$ the ratio of c, the circumference of a circle to its diameter, the fluents of each of the terms taken between $x = o$, and $x = b$, are

$$\pi,\ \pi\frac{b}{2},\ \pi\frac{1\cdot3}{2\cdot4}b^2,\ \pi\frac{1\cdot3\cdot5}{2\cdot4\cdot6}b^3,\ \&c.$$

Substituting these values in that of t, it will become

$$t = \tfrac{1}{2}\pi\sqrt{\frac{l}{2g}} \times \left(1 + \frac{1^2}{2^2}\cdot\frac{b}{2l} + \frac{1^2\cdot3^2}{2^2\cdot4^2}\cdot\frac{b^2}{4l^2} + \frac{1^2\cdot3^2\cdot5^2}{2^2\cdot4^2\cdot6}\cdot\frac{b^3}{8l^3} +, \&c.\right)$$

This, therefore, is the time employed by the body in descending from M to A: but with the velocity acquired at A, the body would proceed along the equal branch A m of the curve, and would have its velocity extinguished at m after a time from A, equal to the time of descent, from M to A; consequently the time of complete oscillation will be double; that is,

$$t = \pi\sqrt{\frac{l}{2g}} \times \left(1 + \frac{1^2}{2^2}\cdot\frac{b}{2l} + \frac{1^2\cdot3^2}{2^2\cdot4^2}\cdot\frac{1^2\cdot3^2\cdot5^2}{2^2\cdot4^2\cdot6^2}\frac{b^3}{8l^3} + \&c.\right)$$

The relation of $\frac{b}{l}$ or $\frac{\text{AE}}{\text{AC}}$ expresses the versed sine of an arc of the same angular value as AM, having radius = 1.

And as the versed sine of small angles is very small, the series

$$1 + \frac{1^2}{2^2}\frac{b}{2l} + \frac{1^2\cdot3^2}{2^2\cdot4^2}\frac{b^2}{4l^2} + \frac{1^2\cdot3^2\cdot5^2}{2^2\cdot4^2\cdot6^2}\cdot\frac{b^3}{8l^3} + \&c.$$

will converge very rapidly.

So that in almost every case the two first terms will suffice, or we shall have *very nearly*

$$t = \pi\sqrt{\frac{l}{2g}} \times \left(1 + \frac{b}{8l}\right);$$

Therefore the times of vibrations in different small arcs are as $8l + b$, or as 8 times the radius *plus* the versed sine of the arc. Or, when the arc is very small, b being then also very small in comparison with l may be neglected, and we shall have for the duration of the oscillation

$$t = \pi\sqrt{\frac{l}{2g}}.$$

The quantity b does not enter into this value of t: consequently when the vibrations are made through very small arcs their duration may be considered as not depending on the magnitude of the arc; and the oscillations may be regarded as isochronal.

Mechanics.

81. To determine the numerical error which may arise by assuming the isochronism of the vibrations in small arcs, let us suppose the arc MA = 5°; the time of the oscillation of a seconds pendulum would be expressed by

Errors in pendulums.

$$t = 1 = \pi\sqrt{\frac{l}{2g}},\text{ or } l = \frac{2g}{\pi^2}; \qquad (1)$$

on the supposition of the arc being extremely small, or which amounts to the same, rejecting all the terms of the series except the first.

The versed sine of 5° = ·0038053 = $\frac{b}{l}$; and therefore $\frac{1^2}{2^2}\cdot\frac{b}{2l} = \cdot0004757$.

As to the third term, it is less than ·0000001. We have, therefore, for the time employed by the pendulum in moving through an arc of twice 5°, 1 × (1 + ·0004757), and the time of oscillating through an arc of 10°, differs not from an oscillation in an infinitely small arc more than ·0004757.

This multiplied into 86400, the seconds in 24 hours, gives *nearly* 41·1 seconds. So that a pendulum of the same length as that which vibrates seconds through an infinitely small arc, would lose 41·1 seconds per day if it described arcs of 5° degrees on each side the vertical CA.

If the arcs described on each side of the vertical were only 1°, of which the versed sine is ·0001523, it would be found by a similar process that the daily retardation would be about $1\frac{2}{3}$ seconds; and $\frac{1}{2}$ a degree on each side it would be about $\frac{2}{5}$ of a second.

Or, if D denote the degrees the pendulum describes on each side of the vertical, the time it would be retarded in a second, would be nearly expressed by $\frac{D^2}{51150}$, and consequently the time lost in 24 hours would be

$$24 \times 60 \times 60 \times \frac{D^2}{51150} = \tfrac{5}{3}D^2 \textit{ nearly.}$$

In like manner the time lost in 24 hours by describing δ degrees on each side of the vertical would be $\frac{5}{3}\delta^2$ nearly. Consequently, if a seconds pendulum keep true time in one of these arcs, the seconds lost or gained per day by vibrating in the other will be

$$\tfrac{5}{3}(D^2 - \delta^2).$$

Thus, for example, if a seconds pendulum measure true time in an arc of 3° on each side of the vertical, it will lose $11\frac{2}{3}$ seconds per day by vibrating through 4° on each side, and nearly 45 seconds per day by describing 6° on each side.

Thus, then, it appears that the vibrations in very small circular arcs may be regarded as sensibly isochronal; and whenever great accuracy is required, the necessary correction may be easily supplied as above.

82. From our equation (1) we readily deduce the following consequences; viz. let l and l' denote two pendulums, and t and t' the times of vibration; then from the equation above referred to, we have

General deductions.

$$t = \pi\sqrt{\frac{l}{2g}} \text{ and } t' = \pi\sqrt{\frac{l'}{2g}}.$$

Whence $t : t' :: \sqrt{l} : \sqrt{l'}$.

General deductions.

That is, the times of vibrations of different pendulums are as the square roots of their lengths; and reciprocally, the number of vibrations made by different pendulums in the same time, are inversely as the square roots of their lengths; for the numbers of vibrations in a given time are reciprocally as the duration of each. Therefore the length of any pendulum and its time of vibration being given, the time in which any other pendulum will vibrate, may be easily determined; or the time being given, the length may be found.

Again, we have seen, in treating of the laws of falling bodies, that $\frac{1}{2}s$, denoting any space, the time of falling through it from rest will be $t = \sqrt{\frac{s}{2g}}$, $=$; whereas the time of a pendulum vibrating, whose length is s, will be $t = \pi\sqrt{\frac{s}{2g}}$.

Therefore the time of vibrating is to the time of falling freely through half the length of the pendulum, as 3·14159, &c. : 1; *or as the circumference of a circle to its diameter; the force of gravity being the same.* If only the intensity of gravity changes, the times of vibration of the same pendulum, or of pendulums of the same lengths, will be reciprocally as the square roots of the numbers expressing the gravitating force.

Let us illustrate these conclusions by a few examples.

Examples.

1. Suppose that it had been ascertained by experiment that a heavy body, in one second, in the latitude of London, falls through $16\frac{1}{12}$ feet; to find the length of a pendulum in the same latitude vibrating seconds:

Here we have $g = 16\frac{1}{12}$; whence, since

$$t = 1 = \pi\sqrt{\frac{l}{2g}},$$

we obtain also

$$l = \frac{2g}{\pi^2} = \frac{32\frac{1}{6}}{3{\cdot}14159^2}\ \textit{feet} = \frac{193}{3{\cdot}14159^2} = 39\tfrac{1}{8}\ \textit{inches}.$$

2. Again, if the length of the seconds pendulum were given $= 39\frac{1}{8}$ inches; to find g, or the space through which a heavy body falls in a second, we should have

$$\frac{l\,\pi^2}{2\,g} = 1, \text{ or } g = \tfrac{1}{2}\,l\,\pi^2 = 16\tfrac{1}{12} \text{ feet.}$$

3. The length of a seconds pendulum being given $= 39\frac{1}{8}$ inches; required the length of a pendulum, which vibrates once in a *quarter* of a second, once in *half* a second, and once in *two* seconds:

Since the time is as the square root of the length, or the length as the square of the time, we have

$1 : (\frac{1}{4})^2 :: 39\frac{1}{8} : 2\frac{57}{128}$ inches $= \frac{1}{4}$ seconds,
$1 : (\frac{1}{2})^2 :: 39\frac{1}{8} : 9\frac{25}{32}$ inches $= \frac{1}{2}$ seconds,
$1 : 2^2 :: 39\frac{1}{8} : 156\frac{1}{2}$ inches $= 2$ seconds.

4. What is the length of that pendulum which vibrates as often in a minute as it has inches in length?

Put x for the length of the pendulum, and which, therefore, will also be the number of vibrations per minute.

Since the number of vibrations in a given time are reciprocally as the square roots of the lengths, we have

$$\sqrt{x} : \sqrt{39\tfrac{1}{8}} :: 60 : x.$$

Whence $$x^3 = 39\tfrac{1}{8} \times 60^2, \text{ or}$$

$$x = \sqrt[3]{(39\tfrac{1}{8} \times 60^2)} = 52{\cdot}03\ \textit{inches}.$$

Hatton's pendulum.

83. It is very difficult to ascertain the length of a seconds, or any other pendulum, from actual experiments, our computations being all made upon supposition of a perfectly inflexible line void of grav having a material particle attached to it, which weight but no sensible dimensions; circumstances wh it is impossible can be found to obtain in practice: which account the following method was proposed Mr. Hatton, and afterwards executed by Whitehu This consisted in the application of a moveable p of suspension to the same pendulum; which thus gi the absolute effect of two pendulums, the difference whose length is known, being the interval between points of suspension in the two cases; and the ratio their lengths is also known by observing the numbe vibrations performed by each in a given time. When there being two equations and two unknown quantit the actual lengths of the pendulums are easily deduc Thus we shall have

$$l : l' :: n'^2 : n^2,$$

and $$l - l = d,$$

d being the measured distance between the points suspension; consequently

$$l = \frac{d\,n'^2}{n'^2 - n^2}, \text{ and } l' = \frac{d\,n^2}{n'^2 - n^2}.$$

84. *Of simple pendulums vibrating in cycloidal arcs.*—Si our object in this article is to treat of the mechan properties of the cycloid, we shall not enter into an vestigation and demonstration of the geometrical p perties of this curve, that being a subject which belo to another department of this work; and shall theref merely show the principle of its generation, and e merate such of its properties as appertain to the ma at present under consideration.

If the circumference of a circle be rolled along a ri line, till any point A (fig. 55) of that circumferen which was in contact with the line, comes in cont with it again, that point A will describe a cu AGACA, which is called a *cycloid;* the line ABA, which the circle rolls, is the *base* of the cycloid; the diameter BA of that circle, which is perpendicu to the base when the circle touches its middle point the *axis* of the cycloid; and the circle which rolls al the line is called the *generating* circle.

85. The principal geometrical properties of this cu as appertaining to our present enquiry, may be e merated as follow:

If the generating or revolving circle be placed in middle of the cycloid, its diameter coinciding with diameter AB, and from any point there be drawn tangent CF, the ordinate CDE perpendicular to the a and the chord of the circle AD; then we shall have

1. The right line CD = circular arc AD;
2. The cycloidal arc AC = double the chord AI
3. The semi-cycloid ACA = double the diameter

and,

4. The tangent CF is parallel to the chord AD.
5. The involute of a semi-cycloid AOU (fig. 56 an equal semi-cycloid UPV, in an opposite directi the extremity of the base of the latter being in cont with the vertex of the former.

86. On the latter of these properties depends method of making a pendulum vibrate in the arc of curve. Thus, produce the axis VN, till VA = 2 V through A draw a line AC parallel to UW, and $= \frac{1}{2}$U on AC, as a semi-base, and with the axis CU descr a semi-cycloid AOU; and in like manner describe a

ther semi-cycloid AW, turned the contrary way; then a string, perfectly flexible, whose length is AOU, or AV, being fixed by one end at A, and by its motion coming gradually into contact with one or other of the cycloidal cheeks AOU, AW, will describe the cycloid UVW by its other extremity.

In order to find the time of a pendulum vibrating in the arc of a cycloid, let UVW (fig. 56) represent the curve; then, from the properties above enumerated, we have VN = NA, and AOU = AW = AV = $\frac{1}{2}$ UVW. The accelerating force in any point T is the same as if the body were placed upon the tangent T t, or VQ, to which T t is parallel. And by the property of the inclined plane, the accelerating force on the tangent T t or on QV is to the force of gravity as TG : T t; and because the triangles T t G and NVQ are similar, TG : T t :: VQ : VN.

In like manner, if any other point S in the cycloid be taken, we shall have the accelerating force in S to the force of gravity, as RV : VN; that is, the accelerating force in any point of the cycloid is always proportional to the corresponding chord in the circle NRV; or, since every arc of the cycloid TV, SV, &c. is always double the corresponding chord of the circle, it follows that the accelerating force in every point of the curve is proportional to the cycloidal arc between that point and the lowest point V; that is, the accelerating force continually varies directly as the distance or arc the body has to describe. This problem is therefore analogous to that investigated (art. 28), where we determined the circumstances attending the motion of a body attracted by a force which varies directly as the distance from the centre of force; consequently, in this case as well as that, the times of descent from any points on the arc where the body begins to move from quiescence to the lowest point V, are equal; and as equal forces, acting in opposite directions, will destroy equal quantities of motion in equal times, the time of ascent from V along the other branch of the curve, will all be equal likewise. *Hence the times of oscillation in a cycloid, whether through great or small arcs, are equal; which is what is to be understood by the isochronism of the cycloidal pendulum.*

We have seen (art. 82) that the time in which a pendulum vibrates in an indefinitely small arc of a circle, is to the time in which a body would fall freely through half the length of the pendulum, as the circumference of a circle to its diameter; and since an indefinitely small arc of a cycloid, at its vertex, coincides with a circular arc, whose radius is equal to the length of the pendulum, or double the axis; therefore the same ratio obtains for the time of vibrating in infinitely small cycloidal arcs. But all arcs in a cycloid, whether great or small, are performed in the same time; therefore,

The time of a pendulum's vibrating in any arc of a cycloid, is to the time in which a heavy body would fall freely through half the length of the pendulum, as the circumference of a circle to its diameter.

87. The isochronism of vibrations in cycloidal arcs, is demonstrated upon a supposition that the whole mass of the pendulum is concentrated in a point, or rather, that the pendulum consists of a material particle without sensible dimensions, suspended by a perfectly flexible line, void of gravity or weight; suppositions which cannot actually obtain in any vibrating body; and when the pendulum is of any given magnitude, there is no certain and fixed point in the body which determines the length of the pendulum; for what is called the centre of oscillation, which will be considered in a subsequent chapter, will not occupy the same place in the given body when describing different parts of the track it moves through, but will continually assume new situations with respect to the pendulum itself during its vibration; because the distance of the body from the point of suspension, always enters into the expression for the distance of the centre of oscillation from the same point. This circumstance has prevented any general determination of the time of vibration in a cycloidal arc, except in the imaginary case considered above; the property of isochronism, however, obtaining here, has given rise to the name of *Tautochrones* being applied to cycloids. Other reasons besides those above alluded to, have induced artists to abandon the use of the cycloidal pendulum, although it was commonly adopted for some time after its first invention, by Huygens. The principal of these are, the difficulty of giving to the metallic cheeks the true cycloidal form; the improbability of their long retaining it, supposing it once given; and the changes which the pendulum is susceptible of, in consequence of the expansion and contraction by heat and cold. These sources of error are such as the peculiar property of the cycloid cannot obviate; and as the variation from isochronism in small circular arcs is very trifling, the cycloidal pendulum is now wholly disused by artists.

Conical pendulum.

88. *Of the conical pendulum.* Huygens, to whom we are indebted for first making known the theoretical properties of the cycloidal pendulum, and which, considered as pure mechanical deductions, are highly curious and interesting; also discovered some properties of the conical pendulum which ought not to be omitted in this article.

We have seen that when a material particle is attached to an inflexible line, and a motion is given to the pendulum, such, that instead of vibrating in the arc of a circle in a given vertical plane, the point itself is made to describe a circle placed parallel to the horizon, so that the line or pendulum describes a conical surface; such a system is known by the term conical pendulum.

The conditions of motion, time, &c. of this pendulum, obviously depend upon the doctrine of central and centrifugal forces, investigated in art. 73 of the preceding chapter; in which, in fact, we have deduced the principal conditions of its motion, but which we abstained from enlarging upon there, in order to bring the subject again forward in this place, under the general consideration of the motion of pendulums. We have seen, in the article above referred to, that when a material point revolves in a circle, the centrifugal force is to gravity as double the height due to the velocity in the circle to the radius. Let CA (fig. 57), the length of the pendulum, be denoted by l; CS, the altitude of the cone, by a; the semi-diameter, or radius of the base, by r; and the periodic time by t: then, because the body is retained in the circle by three different forces; viz. the centrifugal force $\frac{f}{2g}$ in the direction SA; the force of gravity = 1, in a direction parallel to CS; and the tension of the thread, compounded of the two former, we shall have

CS : CA or $a : b$ :: the weight of the body at A) : (the force upon the thread at A).

But CS : SA :: $1 : \frac{f}{2g} :: a : r$.

Now we have seen (art. 73) that $f = \frac{v^2}{r}$; and v^2 may be denoted by the square of the space, divided by the square of the time; whence

$$v^2 = \frac{4\pi^2 r^2}{t^2};\ f = \frac{4\pi^2 r}{t^2} \text{ and } \frac{f}{2g} = \frac{4\pi^2 r}{2gt^2}.$$

Substituting this in the preceding analogy, we have

$$1 : \frac{4\pi^2 r}{2gt^2} : a : r$$

consequently,

$$2gt^2 = 4a\pi^2, \text{ or}$$
$$gt^2 = 2a\pi^2;$$

whence $$t = \pi\sqrt{\frac{2a}{g}}.$$

Time of gyrating.

Which shows that the periodic time t, varies as the square root of the altitude of the conical surface described by the pendulum, whatever may be the length of the pendulum, or the radius of the base of the cone.

Because gt^2, or its equal $2a\pi^2$, is the space through which a heavy body will descend by its own gravity in t seconds; and since

$$1^2 : \pi^2 :: 2a : 2a\pi^2 = gt^2;$$

it follows that

As the square of the diameter of any circle is to the square of its periphery, so is twice the altitude of the cone to the distance a heavy body will freely descend through in the time of a complete gyration of the conical pendulum.

We have seen that the time in which a pendulum, vibrating in a small circular arc, is expressed by $t = \pi\sqrt{\frac{l}{2g}}$ (art. 82); and by the last article, the time of a conical pendulum performing one gyration is $t = \pi\sqrt{\frac{2a}{g}}$, or $t = 2\pi\sqrt{\frac{a}{2g}}$.

Whence the time of gyration of a conical pendulum is exactly double the time in which a simple pendulum would vibrate in a small circular arc, the length of the latter being equal to the height of the former.

Vibration in a paraboloid

The following is one of the remarkable deductions which Huygens made from the preceding determination; viz. ABC (fig. 58) denoting the internal surface of a parabolic conoid, and CE its axis; if any number of pendulums be suspended from points in the axis as D, D′, &c. such as to become normals to the curve in P, P′, &c. the time of their revolution will always be the same; the sub-normal, which corresponds with the height of the conical surface, being in this curve a constant quantity.

Therefore, if a flexible line be made to encompass the involute of a parabola, having a heavy body attached to its extremity, if that involute be made to revolve, the body will always be found in the curve of a parabola, and the revolutions will therefore be isochronal.

The curve of swiftest descent, which most mechanical writers treat of in connection with the motion of pendulums, being only one of the numerous problems belonging to the class called *isoperimetricals*, which which fall properly under the calculus of variations, we propose to consider in our treatise on Analysis.

§ XVI. *Of D'Alembert's general principle of Dynamics.*

89. This general principle was first published by D'Alembert in his Dynamics, page 73; and which may be enunciated as follows:

In whatever manner several bodies change their actual motion, if we conceive the motion which each body would have in the succeeding instant, if it were quite free, to be resolved into two others, of which one is the motion which it really takes, in consequence of their mutual actions; the second must be such, that if each body were impelled by this force alone (that is, by the force which would produce this second motion), all the bodies would remain in equilibrio.

When we consider a system of bodies connected with each other in an invariable manner, this mutual connection will prevent them from obeying individually the forces with which they are respectively impressed, so that they will neither move with velocities corresponding to the exciting forces, nor in the direction of those forces; and in many cases it would (independent of the general principle above stated) be very difficult to assign either their velocities or their directions; but the difficulty vanishes as soon as this principle is once established.

Let us suppose, as a partial example, that we have several material points M, M′, M″, &c. (fig. 59) fixed to an inflexible right line AL moveable about the point A; it is obvious that in any time θ, gravity acting with equal energy and in the same manner on each of these points, they would, if they were free, describe equal vertical spaces; but as the points M, M′, M″, &c. are constrained to move with the line AL, they will describe the arcs MK, M′K′, M″K″, &c. in the same time; consequently the right lines KI, K′I′, K″I″, &c. will express the effects of gravity on the points M, M′, M″, &c. in the time θ. These lines being proportional to the arcs MK, M′K′, M″K″, &c. and consequently to the radii AM, A′M′, A″M″, &c. it follows that the effective velocities of the different points of the system, are so much the less as these points are nearer to the centre of motion A; whereas had these points been free, their velocities would have been all equal to each other.

The effective velocities being therefore different from those which have been communicated to the system, we can only be able to determine them, by expressing them in functions of the primitive velocities transmitted to the material points, and which are supposed to be given. It is the general method of exhibiting this expression which constitutes the principle of D'Alembert.

90. Let v, v', v'', v''', &c. denote the velocities which would be impressed on any system of bodies M, M′, M″, &c. if they were independent the one of the other, and let u, u', u'', u''', be the unknown velocities which they actually take in consequence of their mutual connection. Now one of the components of any velocity being arbitrary, we may take for one of the components of v, the velocity u, and let the other be supposed U; and operating the same with respect to all the other primitive velocities, we may, instead of the velocities v, v', v'', v''', &c. substitute the velocities

u and U the components of v,
u' and U′ the components of v',
u'' and U″ the components of v'',
&c. &c. &c.

then the quantities of motion which enter into the system, considered as free; viz. M v, M′ v, M″ v'', &c. will become

$$\text{M}\,u,\ \text{M}'\,u',\ \text{M}''\,u'',\ \text{M}'''\,u''',\ \&\text{c.}$$
$$\text{MU}\ \ \text{M}'\text{U}',\ \text{M}''\text{U}'',\ \text{M}'''\text{U}''',\ \&\text{c.}$$

Of which, however, only the former actually obtain, or are effective.

It is obvious, therefore, that the quantities MU, M′U′, M″U″, &c. would alone produce an equilibrium; and moreover, that these are the quantities of motion due to the velocities gained or lost.

For M v being the diagonal of a parallelogram, of which M u, MU, are the two sides, we see that if the component MU, become zero, we must have M u = M v; whence it is obvious that MU is a quantity of motion introduced into the system, by the change which has taken place; and it is the same with all the other quantities M′U′, M″U″, &c.

The first enunciation of this principle therefore is, that *the quantities of motion due to the velocities gained or lost would, if they alone acted on the system, produce an equilibrium;* for if this were not the case, there would be an alteration or change in the system, and u, u', u'', &c. could not be the effective velocities with which the points M, M′, M″, &c. are impressed.

We have seen that the velocity v has for its components U and u, and as there is always an equilibrium between three forces, of which the one is equal and directly opposed to the resultant of the other two, it follows that the three forces M u, MU and — M v ought to be in equilibrio. Again, if we consider MU as equal to, and affected with a contrary sign to the resultant of the other two forces, it is obvious that — MU will be the resultant of M u and — M v; and consequently MU will be the resultant of M v and — M u; and what we have said here with reference to MU will obviously apply to all the other analogous quantities M′U′, M″U″, &c.; therefore, by substituting the components of these forces for the forces themselves, we arrive at a second enunciation of this principle; viz.

There will be an equilibrium between the quantities of motion M v, M′ v', M″ v'', *&c. impressed on each of the bodies, and the effective quantities of motion* M u, M′ u', M″, u'', *&c. the latter being estimated in a sense directly opposed to that of their actual direction.*

The advantage of this second enunciation of the above general principle is, that it is not necessary to consider the velocities or quantities of motion lost or gained, viz.

$$\text{MU},\ \text{M}'\text{U}',\ \text{M}''\text{U}'',\ \text{M}'''\text{U}''',\ \&\text{c.}$$

but the equations of motion are immediately established between the actual velocities which would have place if the bodies were free, and those which actually obtain in consequence of the mutual connection of the different points in the system.

91. We propose to show the application of this principle to one or two problems in this place, and in some of our subsequent articles shall explain the method by which it becomes applicable in others.

As an example, let AB, AC, (fig. 60) be two inclined planes, on which are placed the two bodies M and M′, connected together by a line passing over a pulley at A, and let it be required to investigate the circumstances of their motion, the bodies not being such as to produce an equilibrium.

Let Mg be a right line passing through the centre of gravity of M, and which represents the force of gravity $2g$, then the component of $2g$ on the inclined plane AB is MR; and this is the only effective force which acts on M; and may be represented by

$$2\,g\cos\text{RM}\,g = 2\,g\cos\text{BAD} = 2\,g\,\frac{\text{AD}}{\text{AB}}.$$

In like manner the component of the other body M′ with reference to the plane AC, is $= 2\,g\,\dfrac{\text{AD}}{\text{AC}}$.

Call AD = h, AB = l, AC = l'; then the accelerating forces acting on the bodies, will be

$$\frac{2\,g\,h}{l}\ \text{and}\ \frac{2\,g\,h}{l'}. \qquad (1)$$

Now, in order to obtain the velocities which the bodies would have, if they were not connected the one with the other, we may observe, that the equation which determines the intensity of any force being generally

$$\phi = \frac{\dot{v}}{\dot{t}},$$

we draw from this equation

$$\dot{v} = \phi\,\dot{t}.$$

In the present case, the accelerating forces being as exhibited in equation (1), we have

$$\frac{2\,g\,h}{l}\,\dot{t}\ \text{and}\ \frac{2\,g\,h}{l'}\,\dot{t}.$$

Again, the velocities lost by the bodies being equal to the primitive velocities, *minus* those which they actually have, the former will be

$$\frac{2\,g\,h}{l}\,\dot{t} - \dot{v}\ \text{for M},$$
$$\frac{2\,g\,h}{l'}\,\dot{t} - \dot{v}'\ \text{for M}'.$$

By the principle above delivered, the quantities of motion which correspond to these velocities ought to produce between them an equilibrio; and as these velocities act in opposite directions, it will be sufficient to equate those quantities by making

$$\text{M}\left(\frac{2\,g\,h}{l'}\,\dot{t} - \dot{v}\right) = \text{M}'\left(\frac{2\,g\,h}{l'}\,\dot{t} - \dot{v}'\right).$$

Moreover, as in the present problem, we have necessarily $v = -v$, and, consequently, $\dot{v} = -\dot{v}$, we obtain from the above equation,

$$\dot{v} = \frac{\dfrac{\text{M}}{l} - \dfrac{\text{M}'}{l'}}{\text{M} + \text{M}'}\,2hg\dot{t};$$

taking the fluent, and reducing, we have

$$v = \frac{\text{M}\,hl' - \text{M}'\,hl}{ll'\,(\text{M} + \text{M}')}\,2\,g\,t + \text{C},$$

C being the correction, which, when the bodies begin to move from quiescence, = 0.

If we suppose the planes to become perpendicular, then $h = l = l'$, and the above reduces to

$$v = \frac{\text{M} - \text{M}'}{\text{M} + \text{M}'}\,2\,g\,t.$$

This is an analogous expression to that which we obtained on other principles (art. 13).

92. As another example, let it be proposed to determine the motion of two bodies attached to a wheel and axle (disregarding the inertia of the two latter).

Let M denote the body that is attached to the wheel, whose radius is R, and M′ the body attached to the axle, of which the radius $= r$. Then since, if the bodies were free, the velocities communicated to them in the time $\dot{t}$, would be $2g\dot{t}$; if we suppose their actual velocities to be $\dot{v}$, and $\dot{v}'$, the quantities of motion lost will be $M(2g\dot{t}-\dot{v})$ and $M'(2g\dot{t}-\dot{v}')$.

Now these quantities of motion, ought (according to the principle on which we are acting) to produce, by means of the wheel and axle, an equilibrium: we must have, therefore,

$$MR(2g\dot{t}-\dot{v}) = M'r(2g\dot{t}-\dot{v}').$$

From the nature of the machine,

$$r : R :: \dot{v} : \dot{v}' = \frac{R\dot{v}}{r}.$$

But since, as one body ascends, the other descends, these velocities ought to enter the above equation with contrary signs.

Substituting, therefore, $\dot{v}' = -\frac{R\dot{v}}{r}$, we find

$$MR(2g\dot{t}-\dot{v}) = M'r\left(2g\dot{t}+\frac{R\dot{v}}{r}\right);$$

which, by reduction, gives

$$\dot{v} = \frac{(MR-M'r)}{R(M+M')}2g\dot{t};$$

and, taking the fluents,

$$v = \frac{(MR-M'r)}{R(M+M')}2gt + C,$$

C being the correction, which is zero, when the motion of the body commences from quiescence; so that $t = 0$, when $v = 0$.

As it is easy to substitute any numbers in this equation for M, M′, R and r, we think it unnecessary to detain the reader with numerical examples.

§ XVII. *Of the motion of bodies about fixed axes.*

Preliminary remarks. 93. When a body, or a system of material points connected with each other in an invariable manner, turns uniformly about a fixed axis, which we shall suppose to pass through the point A (fig. 61), perpendicularly to the plane of the figure, if we suppose the body, or system of points, to be cut by an infinite number of planes $o'm'n'$, $o''m''n''$, &c. parallel to each other, and perpendicular to the fixed axis; the particles m, m', m'', &c. in one complete revolution, will describe about the fixed axis the circumferences mon, $m'o'n'$, $m''o''n''$, &c. and will describe, in equal time, arcs of an equal number of degrees. These arcs being proportional to their radii, it will be the same with the velocity of the several particles; so that, assuming any distance c A of the particle e, from the axis, as unity, and calling ω the velocity of this particle, the velocity of the several particles m, m', m'', &c. placed at the distances r, r', r'', &c. from the axis, will be respectively $r\omega$, $r'\omega$, $r\omega'$, &c. and we shall thus have, for the effective quantities of motion of the different particles,

$$mr\omega,\ m'r'\omega,\ m''r''\omega, \text{\&c.}$$

Let us denote, by v, v', v'', &c. the impressed velocities; then the quantities of motion actually received will be mv, $m'v'$, $m''v''$, &c.

The equilibrium of the forces. It follows, therefore, from the principle illustrated in the preceding chapter, that there must be an equilibrium between the impressed quantities of motion,

$$mv,\ m'v',\ m''v'', \text{\&c.}$$

and the actual effective quantities taken with a contrary sign, viz. $-mr\omega$, $-m'r'\omega$, $-m''r''\omega$.

In order to obtain the equilibrium between these quantities of motion, let us consider the first, and represent mv by a line mf, taken in the direction of this force, and proportional to its intensity; and let fall from the point f, the perpendicular fh, on the plane of the section omn, and make the angle $fmh = \phi$. We may then resolve mf into two forces;

$hf = mv \sin\phi$ parallel to the fixed axis;

$mh = mv \cos\phi$ in the plane omn.

The first will be destroyed by the resistance of the axis, and the second will have its effect.

Denoting in the same way, by ϕ' ϕ'' ϕ''', &c. the angles of the forces $m'v'$, $m''v''$, &c. with the planes $o'm'n'$, $o''m''n''$, &c. the quantities of motion impressed on the particles, will be

$$mv\cos\phi,\ m'v'\cos\phi',\ m''v''\cos\phi'', \text{\&c.}$$

These quantities of motions, as also the effective quantities

$$mr\omega,\ m'r'\omega,\ m''r''\omega, \text{\&c.}$$

will all be situated in the planes $o'm'n'$, $o''m''n''$, &c. Now, in order to establish an equilibrium between these quantities of motion, it may be observed that since they are all situated in planes perpendicular to the fixed axis, they ought to produce on this axis the same effect as if all the planes omn, $o'm'n'$, $o''m''n''$, &c. formed one plane only; consequently, in considering these forces as situated in the plane of the figure, it is necessary, in order that the equilibrium may subsist between them, that the sum of the moments which tend to cause the system to revolve about the fixed point A in one direction, be equal to those which tend to produce the rotation in an opposite direction.

The forces $mr\omega$, $m'r'\omega$, $m''r''\omega$, &c. which are derived from the common motion impressed on the system, will all tend to produce motion in one direction, and as these forces cause the points m, m', m'', &c. to describe the circumferences mno, $m'n'o'$, $m''n''o''$, &c. the radii r, $r'r''$, &c. will be the perpendiculars let fall on their directions, and consequently the sum of the moments of the effective forces is expressed by

$$mr^2\omega + m'r'^2\omega + m''r''^2\omega + \text{\&c.} = \omega(mr^2 + m'r'^2 + m''r''^2 + \text{\&c.})$$

Let $(mr^2 + m'r'^2 + m''r''^2 + \text{\&c.}) = \Sigma m, r^2$, then $\omega\Sigma mr^2$, will denote the sum of the moments of the effective forces, and it is this quantity which must make an equilibrium with the sum, or the differences of the moments of the forces

$$mv\cos\phi,\ m'v'\cos\phi',\ m''v''\cos\phi'', \text{\&c.}$$

In order to determine this second quantity, let A x (fig. 62) be the fixed axis, and assume the lines ml, $m'l'$, $m''l''$, &c. to denote the forces $mv\cos\phi$, $m'v'\cos\phi'$, &c. situated in the respective planes mno, $m'n'o'$, $m''n''o''$, &c. perpendicularly to the axis; let fall from the points A, A′ A″, &c. taken in these planes and in the fixed axis, the perpendiculars $Al = p$, $A'l' = p'$, $A''l'' = p''$, &c. on the direction of the forces $mv\cos\phi$, $m'v'\cos\phi'$, $m''v''\cos\phi''$, &c. then the moments of these forces will be

$$mvp\cos\phi,\ m'v'p'\cos\phi,\ m''v''p''\cos\phi'', \text{\&c.};$$

and representing this sum as above by the character Σ, viz. denoting it by $\Sigma\, mvp\cos\phi$, we shall have on the principles illustrated in the the preceding chapter,

$$\omega\,\Sigma\, mr^2 = \Sigma\, mvp\cos\phi;$$

and whence we have for determining the angular velocity

$$\omega = \frac{\Sigma m v p \cos \phi}{\Sigma m r^2}. \qquad (1)$$

94. This is the most general character that can be given to our investigation, and the result accordingly becomes more simple under certain modifications of the data; thus if the forces $m v$, $m' v'$, $m'' v''$, &c. all act in their respective planes, the angles ϕ, ϕ', ϕ'', &c. become zero, and consequently their sines will also be zero, and their cosines $= 1$; the above equation, therefore, in this case, becomes simply

$$\omega = \frac{\Sigma m v p}{\Sigma m r^2}. \qquad (2)$$

Again, if the several velocities v, v', v'', &c. be all equal, as will be the case when the body receives a single impulse which would send it forward in a right line, if it were not for the fixed axis, then we shall have

$$\omega = \frac{v \Sigma m p}{\Sigma m r^2}; \qquad (3)$$

because in this case

$m v p + m' v' p' + m'' v'' p'' + \&c. = v (m p + m' p' + m'' p'' + \&c.)$

Let us, in this last case, draw through the fixed axis Az (fig. 63) a plane AK, parallel to the forces $m v$, $m' v'$, $m'' v''$, &c. of which the directions are represented by the right lines $m l$, $m' l'$, $m'' l''$, &c.; then it will be observed that the perpendiculars p, p', p'', &c. let fall from the points A, A′, A″, &c. on the directions of these forces, are equal to the perpendiculars $m q$, $m' q'$, $m'' q''$, &c. demitted from the points $m, m' m''$, &c. on the plane AK; let these last perpendiculars be denoted by $q, q' q''$, &c. and the perpendicular let fall from the centre of gravity of the system on the plane AK by Q; let us also represent by M, the sum of all the particles of which the system is composed, then we shall have from the property of the centre of gravity (art. 117),

$$MQ = m q + m' q' + m'' q'' +, \&c.;$$

and since we have $p = q$, $p' = q'$, $p'' = q''$, &c. the preceding equation will become

$$MQ = m p + m' p' + m'' p'' +, \&c.$$

Substituting this value in equation (3), we have

$$\omega = \frac{v MQ}{\Sigma m r^2}. \qquad (4)$$

If it should happen that some number only of the particles m, m', m'', &c. are impressed with the common velocity v, then M will not denote the sum of all the particles of the system, but the sum of those on which the common velocity has been impressed; and Q must be made to denote the perpendicular let fall from the centre of gravity of that part of the system on the plane AK.

It remains now for us to show the method of determining the value of the expression $\Sigma m r^2$, which is called the *moment of inertia*, and which will form the subject of the following chapter.

§ XVIII. *Of the moment of inertia.*

95. The moment of inertia of a body being the sum of the products of each of the material points which compose it, into the square of their respective distances from the axis of rotation; we have, in the preceding article, in which the number of those points is supposed finite, expressed it by $\Sigma m r^2$; but if, in this expression, we replace m by $\dot{m}$, which will then denote the element or fluxion of the mass, then the moment of inertia will be found by taking the fluent of $r^2 \dot{m}$, and may, therefore, be expressed, according to our notation, by

$$\int r^2 \dot{m}. \qquad (1)$$

Examples.

In order to illustrate this expression by a few examples, let it be proposed to find the moment of inertia of the area of the circle CBD revolving about the point A as an axis (fig. 64).

Assume any variable distance $A m = x$, and call the radius of the circle $= a$; then if $\pi = 3{\cdot}14159$, we shall have $4 \pi x$ for the circumference described by the point m, and for the surface of the same circle πx^2. The fluxion of this expression, viz. $2 \pi x \dot{x}$, is equivalent to what we have denoted by $\dot{m}$; and multiplying this by x^2 (the square of the distance), we have

$$\int r^2 \dot{m} = \int 2 \pi x^3 \dot{x} = \tfrac{1}{2} \pi x^4;$$

which, when $x = a$, becomes $\frac{1}{2} \pi a^4$ for the moment of inertia sought.

In the above example we have merely considered a plane circle, but the result would evidently be analogous for a cylinder revolving on its axis, viz. we should only have to multiply the above by the length of the cylinder (l), which would give $\frac{1}{2} \pi l a^4$ for the moment of inertia in a cylinder revolving on its axis, whose length is l, and radius of its base a.

To find the moment of inertia of a sphere revolving on its axis:

Let RADB (fig. 65) represent a centre section of the sphere, passing through its axis RD; draw CA perpendicular to RD, and $s p r$ parallel to the same line. Put $C r = a$, $C p = z$, then $p r = \surd (a^2 - z^2)$, and the surface of the cylinder generated by $s r$ revolving about RD is

$$4 \pi \surd (a^2 - z^2),$$

whence

$$\dot{m} = 4 \pi z \dot{z} \surd (a^2 - z^2);$$

and multiplying this by the square of the distance z^2, we have

$$\int r^2 \dot{m} = \int 4 \pi z^3 \dot{z} \surd (a^2 - z^2).$$

In order to take the fluent, put $a^2 - z^2 = y^2$; then

$$z^4 = a^4 - 2 a^2 y^2 + y^4;$$

therefore

$$z^3 \dot{z} = - a^2 y \dot{y} + y^3 \dot{y};$$

whence

$$4 \pi z^3 \dot{z} \surd (a^2 - z^2) = 4 \pi (- a^2 y^2 \dot{y} + \dot{y}^4 y);$$

now

$$\int (- a^2 y^2 \dot{y} + y^4 \dot{y}) = - \tfrac{1}{3} a^2 y^3 + \tfrac{1}{5} y^5 + C.$$

In order to find C we may observe, that when $z = 0$, the fluent ought to become zero, but when $z = 0$, $y = a$; consequently

$$C = \tfrac{2}{15} a^5;$$

and therefore the correct fluent is

$$4 \pi (\tfrac{2}{15} a^5 - \tfrac{1}{3} a^2 y^3 + \tfrac{1}{5} y^5);$$

which, when $y = 0$, or $z = a$, becomes

$$4 \pi \times \tfrac{2}{15} a^6,$$

the moment of enertia sought.

96. We shall not multiply examples of this kind, which may be extended at pleasure by the reader;

When the axis passes not through the centre of gravity.

but it may be important to remark, that when we know the moment of inertia of a body with reference to an axis which passes through its centre of gravity, we may readily determine the same with respect to any other axis parallel to the first.

In order to this, let GF and CK (fig. 66) be two parallel axes, of which the former passes through the centre of gravity G of the body; let us place at the point G the origin of the co-ordinates, and take the axis GF for that of z; also draw through any point m of the body a plane $m\,k$ F parallel to that of $x\,y$; this plane will cut the axes GF, CK, in the two points F and K, and the distances of the point m from these axes will be measured by the right lines $m\,k$, and m F, which let be denoted by r and r'. Let fall from the point m the perpendicular m E on the plane of $x\,y$. Now it is evident that the triangles ECG and m KF are equal, being formed by the parallel sides. We may therefore substitute the sides of the first of these triangles instead of those of the other.

This being established, let us call

α and β the co-ordinates GD, DC of the point C;
x and y the co-ordinates GP, PE of the point E;
and a the distance CG of the two axes;

then we shall have

$$a^2 = \alpha^2 + \beta^2 \text{ and } r'^2 = x^2 + y^2.$$

On the other hand, considering the right line CE, which passes through the two points, of which the co-ordinates are respectively x, y, and α, β; the value r of EC, will be given by the equation

$$r^2 = (x - \alpha)^2 + (y - \beta)^2,$$

or by squaring

$$r^2 = x^2 - 2\,\alpha\,x + \alpha^2 + y^2 - 2\,y\,\beta \times \beta^2;$$

reducing this by means of the above values of a and r', we shall have

$$r^2 = r'^2 - 2\,\alpha\,x - 2\,\beta y + a^2.$$

Multiplying by $\dot{m}$, and then taking the fluent, we find

$$\int r^2\,\dot{m} = \int r'^2\,\dot{m} - 2\,\alpha\int x\,\ddot{m} - 2\,\beta\int y\dot{m} + a^2\int \dot{m}. \quad (2)$$

The expressions $\int x\,\dot{m}$ and $\int y\,\dot{m}$ which enter into this equation are zero, as will be seen immediately from the following considerations.

Let x and y be the co-ordinates of the element $\dot{m}$ of the mass M; then the moments of this element, with respect to the axes x and y, will be respectively $y\,\dot{m}$ and $x\,\dot{m}$; and consequently, the co-ordinates x' and y of the centre of gravity M, will be determined by the equations,

$$\text{M}\,x' = \int x\,\dot{m} \quad \text{M}\,y' = \int y\,\dot{m}.$$

In the present case the co-ordinates x' and y' are zero, because they have their origin in the centre of gravity; we have, therefore,

$$\int x\,\dot{m} = 0 \quad , \quad \int y\,\dot{m} = 0.$$

Which values obviously reduce our preceding equation (2) to

$$\int r^2\,\dot{m} = \int r'^2\,\dot{m} + \text{M}\,a^2. \quad (3)$$

Here $\int r'^2\,\dot{m}$ being the moment of inertia, with reference to the axis GF, which passes through the centre of gravity; we may conclude that when we know this moment of inertia, we may always determine the moment of inertia $\int r^2\,\dot{m}$, taken with reference to any other axis CK, of which the distance from the axis GF is known.

Putting equation (3) under the form

$$\int r^2\,\dot{m} = \text{M}\left(\frac{\int r'^2\,\dot{m}}{\text{M}} + a^2\right);$$

or, for the sake of abridgment, denoting $\frac{\int r'^2\,\dot{m}}{\text{M}}$ by K^2, we shall have, for the general expression for the moment of inertia, with reference to any axis parallel to that passing through the centre of gravity,

$$\int r^2\,\dot{m} = \text{M}\,(k^2 + a^2).$$

Rot fixed

§ XIX. *On the motion of a body which revolves in any manner whatever about a fixed axis.*

97. If the different accelerating forces act on the points of any system of bodies, so as to cause it to revolve with a variable motion about the fixed axis A z (fig. 67), every point m will describe about that axis, a circle $m\,n\,o$, which will be perpendicular to this axis, and cut it in the point C.

Let ϕ be the accelerating force which acts on the particle m, and δ the angle T m P, which it forms at the point m, with the element of the curve $m\,n\,o$. It is obvious we may resolve ϕ into three forces; the first parallel to the fixed axis, which will necessarily be without effect; the second in the direction of the radius mC, which will therefore be destroyed by the fixed point C; the third will be in the direction of the element of the curve, and will be expressed by $\phi \cos \delta$; this last will be the only one of the three components of ϕ that will have a tendency to produce a motion of rotation of the point m, about the fixed axis A z.

Let ω denote the angular velocity which has place at the end of the time t, and r the distance C m of the particle m from the axis of rotation. Then the velocity of ^{m} at the end of this time will be expressed by $r\,\omega$, and in the instant $\dot{t}$ the velocity will be increased by that which the accelerating force would generate in the particle in the same time; that is, the velocity during this instant will be increased by the quantity $\phi \cos \delta\,\dot{t}$; for since generally the accelerating force is equal to $\frac{\dot{v}}{\dot{t}}$, and since in our case the accelerating force $=$ $\phi \cos \delta$, we have

$$\phi \cos \delta = \frac{\dot{v}}{\dot{t}};$$

consequently $\dot{v} = \phi \cos \delta\,\dot{t}$.

The particle $\dot{m}$ therefore at the end of the time $t + \dot{t}$, would fly off in the direction of the tangent of the curve with a velocity equal to

$$r\,\omega + \phi \cos \delta\,\dot{t};$$

but $\dot{m}$ being connected with the system, its effective velocity at the end of the time $t + \dot{t}$, will be expressed by

$$r\omega + r\dot{\omega}.$$

Therefore, in the time $t + \dot{t}$, the quantity of effective motion of $\dot{m}$ is $(r\,\omega + r\,\dot{\omega})\,\dot{m}$.

What we have shown above with reference to the particle $\dot{m}$, is equally applicable to all the other particles of the system; and we may therefore, according to our form of notation, denote the sum of all the actual quantities of motion by the expression

$$\Sigma\,(r\,\omega + \phi\cos\delta\,\dot{t})\,\dot{m},$$

and the sum of all the effective velocities or quantities of motion by

$$\Sigma\,(r\,\omega + r\,\dot{\omega})\,\dot{m};$$

which two quantities ought, according to the principle of D'Alembert, to produce an equilibrium, observing to change the signs of the latter; that is to say, considering them as acting in a direction opposed to that of the motion of the system.

Now, in order that these two quantities of motion, acting in opposite directions, may produce an equilibrium on the fixed axis, it is necessary that their moments taken with reference to this axis may give equal products; and as the forces act in the direction of the elements of the circles described by the material points, these forces may be considered to be acting perpendicularly to the radii of the circumferences described by these elements; whence it follows that it will be sufficient to multiply the preceding expressions by these radii in order to have the moments sought; forming therefore the equation of the moments, we shall have

$$\Sigma\,(r^2\omega + r\,\phi\cos\delta\,\dot{t})\,\dot{m} = \Sigma\,(r^2\omega + r^2\,\dot{\omega})\,\dot{m};$$

which obviously reduces to

$$\Sigma\,(r\,\phi\cos\delta\,\dot{t})\,\dot{m} = \Sigma\,r^2\,\dot{\omega}\,\dot{m};$$

and here the time $\dot{t}$, and the angular velocity $\dot{\omega}$ entering the same into all the terms of the respective sums, they may be taken out of the parentheses; whence the above becomes

$$\dot{t}\,\Sigma\,r\,\phi\cos\delta\,\dot{m} = \dot{\omega}\,\Sigma\,r^2\dot{m};$$

also in the general case in which we have to find the sum of a series of infinitely small quantities, we may replace the sign Σ, which denotes the sum of a finite series by the sign of integration $\int$; whence we have

$$\dot{t}\int r\,\phi\cos\delta\,\dot{m} = \dot{\omega}\int r^2\dot{m},$$

which becomes by division

$$\frac{\dot{\omega}}{\dot{t}} = \frac{\int r\,\phi\cos\delta\,\dot{m}}{\int r^2\dot{m}}.$$

We may here, without any impropriety, call $\frac{\dot{\omega}}{\dot{t}}$ the angular accelerating force; whence the above equation may be expressed in words, by saying that *the angular accelerating force is found by dividing the sum of the momenta, or quantity of motion of the moving forces, or of their resultant, by their momentum of inertia.*

§ XX. *Of the centres of oscillation.*

98. *The centre of oscillation* is that point in a vibrating body into which, if all the matter of the body were collected, the vibrations would be performed in the same time; or, since the action of gravity is always proportional to the mass on which it acts, we may define the centre of oscillation to be that point in which if a single material particle were placed, the point of suspension remaining the same, the vibrations would still be performed in the same time as with the compound body or system.

The centre of oscillation of a compound body may be otherwise defined, that point in a vibrating body into which if all its matter were collected, the angular accelerating force would be the same as in the compound body: as there may be an infinite number of points in a body situated at the same distance from the axis of motion, some writers, amongst whom is M. Poisson, say that a body may have an infinite number of centres of oscillation; in what follows, however, we refer but to one, which we conceive to be that situated in the right line forming the centres of suspension and gravity.

After the preceding definitions, it is obvious, that the determination of the centre of oscillation of any compound body, or system of bodies, is nothing more than finding the length of a simple pendulum, which will perform its vibrations in the same time; for that length, being applied from the point of suspension along the line which passes through that point and the centre of gravity of the body, will be the centre of oscillation sought.

Distance from the axis of motion.

99. We have seen, in the preceding article, that the angular accelerating force of any compound mass or system of material points, solicited by various accelerating forces ϕ, ϕ', ϕ'', &c. is expressed by the equation

$$\frac{\dot{\omega}}{\dot{t}} = \frac{\int r\,\phi\cos\delta\,\dot{m}}{\int r^2\,\dot{m}};$$

and since, in this case, the several forces ϕ, ϕ', ϕ'', &c. denote the action of gravity on the several particles, they are equal to each other, being equal to $2\,g$; whence the above, as it relates to the vibration of bodies, becomes

$$\frac{\dot{\omega}}{\dot{t}} = \frac{2\,g\int r\cos\delta\,\dot{m}}{\int r^2\,\dot{m}}.$$

Now if our object were merely to express the angular accelerating force of a single particle $\dot{m}$ vibrating at the distance l, from the axis of motion, we should have

$$\frac{\dot{\omega}}{\dot{t}} = \frac{2\,g\,l\cos\delta\,\dot{m}}{l^2\,\dot{m}};$$

and equating this with the former, we have

$$\frac{2\,g\int r\cos\delta\,\dot{m}}{\int r^2\,\dot{m}} = \frac{2\,g\,l\cos\delta\,\dot{m}}{l^2\,\dot{m}};$$

whence
$$\frac{\int r\,\dot{m}}{\int r^2\,\dot{m}} = \frac{1}{l};$$

or
$$l = \frac{\int r^2\,\dot{m}}{\int r\,\dot{m}}.$$

That is, *the length of a simple pendulum vibrating in the same time as a compound mass, or the distance of the centre of oscillation of the latter from the point of suspension, is found by dividing the fluent of the expression of each particle into the square of its distance, by the fluent of each particle into its simple distance from the point of suspension.*

Or, since the latter fluent is equal to the product of the mass, by the distance of its centre of gravity from the point of suspension, and the former equivalent to what we have defined to be the moment of inertia; we may otherwise say, that the distance of the centre of oscillation of any compound body, from the axis of suspension, is equal to the quotient arising from dividing the moment of inertia with reference to that axis; by the product of the mass into the distance of its centre of gravity from the same axis. If, therefore, C be taken to denote the point of suspension of any vibrating body, O its centre of oscillation, and G its centre of gravity, our expression may be put under the form

$$\mathrm{CO} = \frac{\int \dot{m} r^2}{\text{Body. CG}}.$$

Partial deductions.

100. The following deductions flow immediately from our preceding investigation: 1. AB (fig. 68) being conceived to denote any body vibrating about the point C, it may, as far as regards the time of vibration, be considered as a simple pendulum, whose length is CO, O being the centre of oscillation.

2. If a body be turned about its centre of gravity, in a direction perpendicular to the axis of motion, the place of the centre of oscillation will remain unaltered; for the quantity $\int r^2 \dot{m}$ will not be at all effected by such a motion of the body.

3. The distance of the centre of gravity from that of oscillation (if the plane of the body's motion remain unaltered) will be reciprocally as the distance of the former from the point of suspension. If, therefore, that distance be found when the point of suspension is in the vertex, or so situated that the operation may be most simple, the value of the same, in any other position of that point, will likewise be given by a simple proportion.

4. The product of the distances of the centre of gravity, and that of oscillation, from the axis of motion, is manifestly a constant quantity for the same plane of vibration; *if, therefore, the centre of oscillation be made the point of suspension, the point of suspension will become the centre of oscillation.*

5. Hence also, if upon the plane of vibration, passing through the centre of gravity of any body, two concentric circles be described, having the common centre G, and radii GC, GO, the body suspended from any point, in the periphery of either circle, will perform its oscillations in the same time.

6. It follows also from our preceding investigation, that in a compound pendulum consisting of several bodies revolving about a common axis, the centre of oscillation may be determined by adding together the several products of each mass into the distances of the respective centres of oscillation and gravity, from the axis of motion, and dividing that sum by the product of the whole system, into the distance of the common centre of gravity from the axis of motion.

For with respect to any single body B, we have (denoting by C, G, and O′, the point of suspension, the centre of gravity, and centre of oscillation),

$$\mathrm{CO}' = \frac{\int r^2 \dot{m}}{\mathrm{B.\,CG}};$$

and with regard to any other body B′ of the system,

$$\mathrm{CO}'' = \frac{\int r'^2 \dot{m}}{\mathrm{B'\cdot CG}};$$

and so on of others; whence, according to our notation,

$$\Sigma \int r^2 \dot{m} = \Sigma\, \mathrm{CO \cdot B \cdot CG};$$

that is, the sum of all the $r^2 \dot{m}$ in the whole pendulum is equal to the sum of all the products in each part, which arise from multiplying each part into the product of the distances of its centres of gravity, and of oscillation, from the common axis of motion; and the former sum is also equal to the product of the whole mass into the distance of its centre of gravity, and common centre of oscillation, from the point of suspension. If, therefore, the sum ΣCO·B·CG be divided by the product of the compound mass into the distance of the common centre of gravity from the axis of motion, the quotient will be the distance of the common centre of oscillation from the same axis.

Exa

101. Let us now illustrate our preceding investigations relative to the centre of oscillation of a compound body by a few examples, in doing which we shall avail ourselves of those given by Dr. Gregory in his Treatise of Mechanics.

Of a line.

1. *To find the centre of oscillation of a slender cylinder or right line:*

Let CP (fig. 69) be a right line, or very slender cylinder, moving freely about its extreme point C as a centre of suspension; then if CA, considered as variable, be denoted by x, the force of $\dot{x}$ particles at A will be expressed by $x^2 \dot{x}$, and we shall have $\int x^2 \dot{x} = \frac{1}{3} x^3$ for the moment of inertia of all the particless in CA, and this, when CA becomes equal to CP, will be $\frac{1}{3}\,\mathrm{CP}^2$.

In this case also the body will be CP, and the distance of its centre of gravity from $\mathrm{C} = \frac{1}{2}\,\mathrm{CP}$; consequently we have

$$\frac{\int r^2 \dot{m}}{\int r \dot{m}} = \frac{\int x^2 \dot{x}}{\int x \dot{x}} = \frac{\frac{1}{3} x^3}{\frac{1}{2} x^2} = \frac{2}{3} x;$$

which, when $x = a$, or CP, becomes $\frac{2}{3} a$, or $\frac{2}{3}$ CP; whence O being supposed to denote the centre of oscillation, we shall have

$$\mathrm{CO} = \tfrac{2}{3}\,\mathrm{CP};$$

that is, the centre of oscillation of a right line from the point of suspension, when that point is at its extremity, is at the distance of two-thirds of the line from the axis of motion.

The susp at a poin

2. *To find the centre of oscillation of the same line, when the point of suspension is at any point S:*

Let $\mathrm{CP} = a + b$, viz. let $\mathrm{CS} = b$, $\mathrm{SP} = a$, and $\mathrm{SO} = x$.

Then the moment of inertia of the particles in SC $= \frac{1}{3} b^3$, and the moment of inertia of in SP $= \frac{1}{3} a^3$; the sum of which therefore is, $\frac{1}{3}(a^3 + b^3)$.

Again, the body in this case is $a + b$, and the distance of the centre of gravity from $\mathrm{S} = \frac{1}{2}(a - b)$, therefore what is denoted in our formula by
Body × CG $= \frac{1}{2}(a^2 - b^2)$;
whence

$$\frac{\int \dot{m} r^2}{\text{Body} \times \mathrm{CG}} = \frac{\frac{1}{3}(a^3 + b^3)}{\frac{1}{2}(a^2 - b^2)} = \frac{2}{3} \times \frac{a^2 - a b + b^2}{a - b};$$

that is, $\mathrm{SO} = \frac{2}{3} - \frac{a^2 - a b + b^2}{a - b}$.

If $b = \mathrm{CS} = o$, then S and C coincide, and we have $\mathrm{CO} = \frac{2}{3} a$, *as before.*

If $b = \frac{1}{2} a$, or $b = \frac{1}{3}(a + b)$, then

$$\mathrm{SO} = \frac{2}{3} \times \frac{a^2 - \frac{1}{2} a^2 + \frac{1}{4} a^2}{\frac{1}{2} a} = \frac{2}{3} + \frac{3}{2} a = a;$$

that is, SO = SP, as it ought to be, because when the centre of oscillation is made the point of suspension, the latter ought to become the centre of oscillation.

In order to find the ratio of a to b, the whole length of the line $a + b$ being given, such that SO may be a minimum, or such that the vibrations may be performed the quickest possible, we must substitute $a + b = c$; then the above expression may be written

$$\text{SO} = \tfrac{2}{3} \times \frac{(a+b)^3 - 3\,(a+b)\,a\,b}{(a+b)^2 - 2\,a\,b - 2\,b^2}.$$

Or, calling $b = y$, and $a = c - y$, also, as above, $a + b = c$, we have

$$\text{SO} = \tfrac{2}{3} \times \frac{c^3 - 3\,c\,y\,(c-y)}{c^2 - 2\,(c-y)\,y - 2\,y^2},$$

$$\text{SO} = \frac{2\,c^2 - 6\,c\,y + 6\,y^2}{3\,c - 6\,y} = a \textit{ minimum};$$

which, put into fluxions, and reduced, gives

$$y = \tfrac{1}{2}c \pm \tfrac{1}{2}c \surd \tfrac{1}{3}.$$

When S is taken between A and G, the lower sign obviously obtains; in which case,

$$y = \tfrac{1}{2}c\,(1 - \surd \tfrac{1}{3}) = \cdot 211325\,c.$$

3. *Let* ACB (fig. 70) *be an angular pendulum, composed of two equal thin cylindric wires* AC, CB, *vibrating upon the point of suspension* C, *in the plane of the figure, to find its centre of oscillation* O.

Bisect AC and CB, in g and γ, and bisect $g\,\gamma$ in G, so shall g, γ, and G, be the centres of gravity of AC, CB, and of AC + CB.

Denote AC, or CB, by a, and the angle ACG = $\tfrac{1}{2}$ACB by α: let also o and o' denote the centres of oscillation of the separate masses AC, CB; then $\text{C}o = \text{C}o' = \tfrac{2}{3}\,a$,

and
$$\text{CG} = \frac{a}{2 \sec \alpha};$$

whence we have (by N° 6, art. 100)

$$\text{CO} = \frac{\text{AC} \cdot \tfrac{1}{2}\text{AC} \cdot \tfrac{2}{3}\text{AC} + \text{CB} \cdot \tfrac{1}{2}\text{CB} \cdot \tfrac{2}{3}\text{CB}}{(\text{AC} + \text{CB})\dfrac{a}{2 \sec \alpha}};$$

or,
$$\text{CO} = \frac{2\,(a \cdot \tfrac{1}{2}a \cdot \tfrac{2}{3}a)\,2 \sec \alpha}{2\,a^2} = \tfrac{2}{3}\,a \sec \alpha.$$

The time of vibration of an angular pendulum may be increased *sine limite*, while the length of the arms or branches remain constant; because the time of vibration is a function of the angle ACO.

When this angle vanishes, then $\sec \alpha = 1$; and we have $\text{CO} = \tfrac{2}{3}\,a$, or $\tfrac{2}{3}$AC. When the angle $\tfrac{1}{2}$ACB = 90°, because in this case $\sec \alpha$ is infinite, CO is also infinite, and the time of vibration so likewise; that is, the pendulum will then become a right line, and will rest upon its centre of gravity C, in all positions.

When the lengths AC = CB are given, and the time of vibration, the angle ACB may be determined; for the above expression for CO, gives

$$\sec \alpha = \frac{3\,\text{CO}}{2\,a}.$$

Thus, when AC = CB = 15 inches, and the time of vibration is 1 second; then, since $\text{CO} = 39\tfrac{1}{8}$ inches (the length of a seconds pendulum), we have,

$$\sec \alpha = \frac{3 \times 39\tfrac{1}{8}}{30} = \sec \text{ of } 75^\circ\, 11'\tfrac{1}{2} \textit{ nearly}.$$

If, while AC = 15, the time of vibration should be 5 seconds, then $\text{CO} = 25 \times 39\tfrac{1}{8} = 978 \cdot 125$;

and
$$\sec \alpha = \frac{3 \times 978 \cdot 125}{30} = 97 \cdot 8125 = \sec 89^\circ\, 24'\tfrac{3}{4};$$

consequently $\text{ACB} = 178^\circ\, 49'\tfrac{1}{2}$ *nearly*.

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Of any solid of rotation.

4. *Let the solid formed by the rotation of the curve* VmM (fig. 71), *about its absciss* VA, *vibrate on an axis passing through* C, *to find the centre of oscillation.*

Put CV = d, the variable absciss Va = x, its corresponding ordinate $a\,m = y$, VA = a, AM = r, $3 \cdot 14159 = \pi$.

Now, first, in order to find the moment of inertia of the circle $m\,n$, let us suppose some variable radius $a\,z = z$; then the corresponding double ordinate will be $2\,\surd(y^2 - z^2)$; and the square of this distance from the axis C, will be $(d + x)^2 + z^2$;

whence $\quad 2\,\surd\,(y^2 - z^2)\{(d + x)^2 + z^2\}$

will be the momentum of inertia of the double ordinate, with reference to the axis C; wherefore

$$\int 2\,\surd\,(y^2 - z^2)\{(d + x)^2 + z^2\}\,\dot{z},$$

taken from $z = o$ to $z = y$, gives the moment of inertia of the semicircle; and, consequently,

$$2 \int 2\,\surd\,(y^2 - z^2)\{(d + x)^2 + z^2\}\,\dot{z}$$

will be the momentum of inertia of the circle $m\,n$; in which it will be observed y and x are both constant. Taking the fluent, this becomes, when $z = y$,

$$\pi\,(d + x)^2 \times y^2 + \tfrac{1}{4}\,\pi\,y^4.$$

Having thus the moment of inertia of the circle $m\,n$, multiply by $\dot{x}$, and we have

$$\pi\,\dot{x}\,\{(d + x)^2 y^2 + \tfrac{1}{4}\,y^4\};$$

whence
$$\text{CO} = \frac{\pi \int \{(d + x)^2\, y^2 + \tfrac{1}{4}\,y^4\}\,\dot{x}}{\text{body} \times \text{CG}}.$$

In order to accommodate this expression to any particular case, it is only necessary to substitute for y its value in functions of x, as determined from the equation of the curve; and the fluent will then give the distance CO required.

Of a cone.

Suppose, for instance, we wished to find the centre of oscillation in a cone, whose altitude is a, and radius of its base r. Here the relation of x and y is expressed by the equation $y = \dfrac{r\,x}{a} = n\,x$; and, substituting this value of y in the general expression, it becomes

$$\text{CO} = \frac{\pi \int \{(d + x)^2\, n^2 x^2 + \tfrac{1}{4}\,n^4 x^4\}\,\dot{x}}{\textit{body} \,.\, \text{CG}};$$

taking the fluent,

$$\text{CO} = \frac{\pi\,(\tfrac{1}{3}\,d^2 n^2 x^3 + \tfrac{1}{2}\,d\,n^2 x^4 + \tfrac{1}{5}\,n^2 x^5 + \tfrac{1}{20}\,n^4 x^5)}{\textit{body} \,.\, \text{CG}}.$$

But the body $= \pi\,\tfrac{1}{3}\,n^2 x^3$, and $\text{CG} = d + \tfrac{3}{4}\,x$; whence body × CG $= \pi\,(\tfrac{1}{3}\,d\,n^2 x^3 + \tfrac{1}{4}\,n^2 x^4)$; and therefore, by division,

$$\text{CO} = \frac{20\,d^2 + 30\,d\,x + 12\,x^2 + 3\,n^2 x^2}{20\,d + 15\,x},$$

which, for the whole cone (when $x = a$, and $n\,x = y = r$), is reduced to

$$\text{CO} = \frac{20\,d^2 + 30\,a\,d + 12\,a^2 + 3\,r^2}{20\,d + 15\,a}.$$

When V coincides with C, that is, when the cone is suspended at its vertex, then, as $d = o$, we have

$$\text{CO} = \text{VO} = \frac{4\,a^2 + r^2}{5\,a} = \tfrac{4}{5}\,a + \frac{r^2}{5\,a}.$$

If, also, in this case, $a = r$, then

$$\text{CO} = \text{VO} = \frac{4\,a^2 + a^2}{5\,a} = a.$$

That is, *if a right-angled cone be suspended at its vertex, its* *centre of oscillation will coincide with the centre of its base, and the cone will vibrate in the same time as a simple pendulum, whose length is equal to the altitude of the cone.*

This conclusion flows also very naturally from the geometrical method of finding the centre of oscillation of a cone suspended at its vertex; which is as follows:

Bisect AN, the radius of the base (fig. 72) in F; join VF, and make IF $= \frac{1}{5}$ VF; then IO drawn perpendicular to VF, will meet VA, or VA produced in O, the centre of oscillation.

For $\quad \text{VF} = \sqrt{(\text{VA}^2 + \text{AF}^2)} = \sqrt{(a^2 + \frac{1}{4} r^2)}$,

and $\quad \text{VI} = \frac{4}{5} \sqrt{(a^2 + \frac{1}{4} r^2)}$.

But the triangles VAF and VIO are similar; therefore,

$$\text{VA} : \text{VF} :: \text{VI} : \text{VO};$$

that is,

$$a : \sqrt{(a^2 + \tfrac{1}{4} r^2)} :: \tfrac{4}{5} \sqrt{(a^2 + \tfrac{1}{4} r^2)} :$$

$$\frac{\frac{4}{5} a^2 + \frac{1}{5} r^2}{a} = \tfrac{4}{5} a + \frac{r^2}{5\,a} = \text{CO};$$

the same expression as before; and when $a = r$, then CO $= a$; as we have already fonnd in our analytical process.

It is obvious from the above deductions, that if cones be made to vibrate like pendulums about their vertices as points of suspension, the time of vibration may be increased *sine limite*, while their sides remain constant. In practice, however, this cannot be carried so far in conical pendulums, as in angular pendulums, noticed in our 3d example.

Whether a right-angled cone be suspended at its base, or at its vertex, it will vibrate in equal times; because the centre of oscillation being made the point of suspension, the point of suspension will become the centre of oscillation (No. 4. art. 100).

Centre of oscillation in various bodies.

102. We shall not pursue these examples to a greater length; as the reader may find them in nearly every treatise on the Fluxional or Differential Calculus; but shall merely enumerate a few of the most useful cases.

1. In surfaces, when the vibration is flatwise or perpendicular to the plane of the figure, the distance CO of the centre of oscillation from the point of suspension, will be as below:

In an isosceles triangle, suspended at its vertex	$\frac{3}{4}$ *altitude.*
In a circle, suspended at its circumference	$\frac{3}{4}$ *radius.*
In the common parabola, suspended at its vertex	$\frac{5}{7}$ *axis.*
Any *parabola,* suspended at its vertex	$\frac{2m+1}{3m+1}$ *axis.*

2. When the surfaces are moved edgewise, or in the plane of the figure:

In a circle, suspended at its circumference ...	$\frac{3}{4}$ *diameter.*
In a rectangle, suspended at one angle	$\frac{2}{3}$ *diagonal.*
In a parabola, suspended at its vertex	$\frac{5}{7}$ *axis* + $\frac{1}{3}$ *param.*
Ditto, centre of its base	$\frac{3}{7}$ *axis* + $\frac{1}{2}$ *param.*

In a square pyramid, whose altitude $= a$, and side of its base $= 2\,b$; $\text{CO} = \frac{4}{5} a + \frac{4\,b^2}{15\,a}$.

In a sphere, whose radius is r, when suspended at the distance d from its centre,

$$\text{CO} = d + \frac{2\,r^2}{5\,d}.$$

If the point of suspension be at the circumference, then $d = r$, and CO $= \frac{7}{5}$ radius.

When CO is given $= \delta$, and the radius $= r$, to find d, we obtain it from the equation

$$d = \tfrac{1}{2} \delta \pm \sqrt{(\tfrac{1}{4} \delta^2 - \tfrac{2}{5} r^2)}.$$

Whence it follows, that a sphere may be suspended at two different distances from the axis of motion, and still vibrate in the same time; when $\frac{1}{4} \delta^2 = \frac{2}{5} r^2$, then the radical part vanishes, and $d = \frac{1}{2} \delta = r \sqrt{\frac{2}{5}} = \cdot 632495\, r$, is the only value of d.

3. If the weight of the thread is to be taken into the account, we have the following distance between the centre of the ball and that of oscillation; where B denotes the weight of the ball, d the distance between its point of suspension and the centre, r the radius of the ball, and w the weight of the thread or wire, viz.

$$\text{GO} = \frac{(\frac{1}{3} w + \frac{2}{5} \text{B})\ 4\,r^2 - \frac{1}{6} w\,(2\,d\,r + d^2).}{(\frac{1}{2} w + \text{B})\,d - r\,w.}$$

4. If W, W′ represent two weights fixed at the extremities of a rod, of given length WW′, C being the centre of motion between them; then if CW $= d$, and CW′ $=$ D, and m be taken to denote the weight of an unit in length of the rod, we shall have

$$\text{CO} = \frac{m\,\text{D}^3 + 3\,\text{W}'\text{D}^2 + m\,d^3 + 3\,\text{W}\,d^2}{m\,\text{D}^2 + 2\,\text{W}'\text{D} - m\,d^2 - 2\,\text{W}\,d};$$

the radii of the balls being supposed very small in comparison with the length of the rod.

5. In the bob of a clock-pendulum, supposing it to be two equal spheric segments joined at their bases, if the radii of those bases be each $= \rho$, the height of each segment v, and d the distance from the point of suspension to G, the centre of the bob, then is

$$\text{GO} = \tfrac{1}{3}\, \frac{\rho^4 + \frac{1}{2} \rho^2 v^2 + \frac{1}{10} v^4}{d\,(\rho^2 + \frac{1}{3} v^2)};$$

which shows the distance of the centre of oscillation below the centre of the bob.

If the radius of the sphere be known, the latter expression becomes

$$\text{GO} = \frac{\frac{2}{3} r^2 v - \frac{1}{2} r\,v^2 + \frac{1}{10} v^3}{d\,(r - \frac{1}{3} v)}$$

Cent oscill by e ment

103. We shall now conclude this article, by showing the method of determining the centre of oscillation of a compound mass or body experimentally.

Suspend the body proposed very freely by the given point, and make it vibrate in small arcs; count the number n of vibrations which it makes in a minute by a good stop watch; so shall the distance CO of the centre of oscillation and point of suspension be denoted by

$$\text{CO} = \frac{140850}{n^2}.$$

For the length of the seconds pendulum vibrating seconds being $39\frac{1}{8}$ inches; and the lengths of pendulums being reciprocally as the square of the number of vibrations made in a given time, we have

$$n^2 : 60^2 :: 39\tfrac{1}{8} : \frac{60^2 \times 39\frac{1}{8}}{n^2} = \frac{140850}{n^2}$$

for the length of the pendulum which vibrates n times in a minute, or the distance CO in the compound pendulum.

§ XXI. *Of the centre of gyration.*

Cent gyrat

104. *The centre of gyration,* is that point in which if all the matter contained in a revolving system be col-

lected, the same angular velocity will be generated in the same time, by a given force acting at any place, as would be generated by the same force acting similarly and in the same place in the body or system itself.

We have seen (article 97) that the expression for the angular accelerating force of any revolving body or system, is denoted by

$$\frac{\dot{\omega}}{\dot{t}} = \frac{\int r \cos\phi \, \delta \dot{m}}{\int r^2 m}.$$

In the present instance, the body is all supposed to be collected into one point; therefore, if we denote the distance of that point from the axis by CR, and the body by B, our denominator $\int r^2 \dot{m}$ will be expressed by $CR^2.B$; while the force represented by the numerator will remain the same; and since the angular motion in the two cases is equal, we shall have

$$\frac{\int r \phi \cos \delta \dot{m}}{\int r^2 \dot{m}} = \frac{\int r \phi \cos \delta \dot{m}}{CR^2.B};$$

whence

$$CR^2.B = \int r^2 \dot{m};$$

or

$$CR = \sqrt{\frac{\int r^2 \dot{m}}{B}}.$$

That is, CR, *the distance of the centre of gyration from the axis of rotation, is expressed by the square root of the quotient arising from dividing the moment of inertia by the mass or body.*

105. Several remarkable relations obtain between the distances of the centres of oscillation and gyration in the same bodies; of which we shall enumerate a few of the most important.

1. The distance of the centre of gyration from the axis of motion, is a mean proportional between the distances of the centres of oscillation and gravity from the same axis.

For, denoting the body by B, the distance of the centre of gravity by CG, the distance of the centre of oscillation by CO, and of gyration by CR, we have

$$\text{for} \quad CR^2 = \frac{\int r^2 \dot{m}}{B}, \text{ and } CO = \frac{\int r^2 \dot{m}}{CG.B};$$

multiplying the latter by CG, we have

$$CO \times CG = \frac{\int r^2 \dot{m}}{B}, = CR^2;$$

consequently,

$$CR = \sqrt{(CO \times CG)};$$

or,

$$CO : CR :: CR : CG.$$

2. The principal axis of gyration is that which passes through the centre of gravity; and the distance between the principal centre of gyration and the centre of gravity, is a mean proportional between the distances of the centre of motion and oscillation, from the centre of gravity.

For, referring to fig. 69, we have

$$PC^2 = PG^2 + CG^2 \pm 2\,PG \cdot GC,$$

therefore,

$$CO = \int \frac{PG^2 + CG^2 \pm 2\,PG \, . \, GC}{B.CG} \dot{m}.$$

But from the nature of the centre of gravity,

$$\int PG\,\dot{m} \text{ on each side of G are equal;}$$

and because 2 GG is constant,

$$\int \pm 2\,PG.GC\,\dot{m} = o;$$

also because

$$\int \dot{m}\, CG^2 = B.CG^2,$$

we have

$$OC = GC + \int \frac{PG^2 \dot{m}}{B.CG},$$

and

$$OG = \int \frac{PG^2 \dot{m}}{B.GC};$$

whence the truth of the proposition is manifest.

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3. The time in which a body vibrates will be the least possible, when the axis of motion passes through the principle centre of gyration. For if D and d denote the distances from the centre of gravity to the point of suspension, and to the centre of gyration, it will be

$$D : D + d :: D + d : \frac{(D + d)^2}{D},$$

the distance from the point of suspension to the centre of oscillation; and this latter expression will obviously be a minimum when $D = d$.

4. If it be required to determine what quantity of matter M must be placed at any other distance AC from C, so that the inertia may remain the same, we must have

$$AC^2.M = CR^2.B, \text{ or } M = \frac{B.CR^2}{AC^2}$$

5. And for the same reason, if any number of bodies B, B′, B″, &c. be put in motion about a common axis passing through C, by a force acting at A, the system will have the same angular velocity, if instead of those bodies placed at the distances CB, CB′, CB″, &c. there be substituted bodies equal to

$$\frac{CB^2}{CA^2}B, \; \frac{CB'^2}{CA^2}B', \; \frac{CB''^2}{CA}B'', \text{ \&c.}$$

these being all concentrated at the point A.

106. Let us now illustrate our preceding deductions by the solution of a few examples.

Examples.

1. *To find the centre of gyration of a right line, or very slender rod.*

Let CP (fig. 69) represent the right line in question, revolving on the point C, and let R be its centre of gyration; also let $CA = x$, be any variable distance; then will $\int x^2 \dot{x}$ denote the moment of inertia, and we shall have (art. 104)

$$CR = \sqrt{\frac{\int x^2 \dot{x}}{B}} = \sqrt{\frac{x^3}{3B}}.$$

Now, if we call the whole line $CP = l$, we shall have $B = l$; and, therefore, when $x = l$, the above becomes

$$CR = \sqrt{\tfrac{1}{3} l^2} = l\sqrt{\tfrac{1}{3}} = \cdot 57735\, l.$$

2. *To determine the centre of gyration of a plane circle, or wheel, of uniform thickness, revolving on its centre.*

Let $3{\cdot}14159 = \pi$, the radius of the wheel $= r$, and any variable radius $= x$; then the area of the wheel will be πr^2, and of the variable circle πx^2; consequently, the moment of inertia will be denoted by $\int 2\pi x^3 \dot{x}$, and we shall therefore have

$$CR = \sqrt{\frac{\int 2\pi x^3 \dot{x}}{B}} = \sqrt{\frac{\int 2\pi x^3 \dot{x}}{\pi r^2}} = \sqrt{\frac{\pi x^4}{2\pi r^2}}.$$

nics. When $x = r$, this becomes

$$CR = \sqrt{\frac{r^2}{2}} = r\sqrt{\tfrac{1}{2}} = \tfrac{1}{2}r\sqrt{2};$$

which is the distance of the centre of gyration from the axis of motion.

3. *To determine the centre of gyration of an uniform circular ring, revolving on its centre, as* ADED (fig. 73).

Let the external and internal radii AC, C a, be denoted by R and r, then will $\pi(R^2 - r^2) =$ the ring, which is denoted by B in our formula. Let x be any variable radius greater than r, then $\pi(x^2 - r^2)$ will denote the variable area, and its fluxion $= 2\pi x\dot{x}$: multiply this by the square of the distance x^2, and we have for the moment of inertia, $\int 2\pi x^3\dot{x} + C$.

Now this expression ought to become zero, when $x = r$; whence $C = -\frac{1}{2}\pi r^4$; and the correct fluent is, when $x = R$,

$$\int 2\pi x^3\dot{x} + C = \tfrac{1}{2}\pi(R^4 - r^4);$$

and, therefore,

$$CR = \sqrt{\left(\frac{\frac{1}{2}\pi(R^4 - r^4)}{\pi(R^2 - r^2)}\right)};$$

that is,

$$CR = \sqrt{\tfrac{1}{2}(R^2 + r^2)}.$$

When r vanishes, or the ring becomes a circle, we have $CR = \sqrt{\frac{1}{2}R^2} = \frac{1}{2}R\sqrt{2}$, as in the preceding example.

The sectors C ab, CAB, being to each other as the areas of their respective circles, if $\frac{1}{n}\pi$ be put for the corresponding part of the circumference, the same result will be obtained with respect to the sectors as to the whole circles; so that, if the part ABba revolve about the centre C, its centre of oscillation will in like manner be determined from the expression

$$CR = \sqrt{\tfrac{1}{2}(R^2 + r^2)}.$$

4. *To find the centre of gyration of a cone which revolves about its vertex.*

Here we may avail ourselves of the property given (art. 104), viz. that the distance of the centre of gyration from any axis is a mean proportional between the distances of the centres of oscillation and gravity from the same axis.

Now the centre of gravity of a cone is at $\frac{3}{4}$ths of the altitude from the vertex, or $CG = \frac{3}{4}a$, a denoting the altitude; and the distance of the centre of oscillation (art. 101, ex. 4), $CO = \frac{4}{5}a + \frac{r^2}{5a}$, r being the radius of the base; whence we have

$$CR = \sqrt{\left\{\tfrac{3}{4}a\left(\tfrac{4}{5}a + \frac{r^2}{5a}\right)\right\}};$$

or

$$CR = \sqrt{\left(\frac{12a^2 + 3r^2}{20}\right)}.$$

When $r = a$, this becomes

$$CR = \sqrt{\frac{15a^2}{20}} = a\sqrt{\tfrac{3}{4}}, \text{ or } \tfrac{1}{2}a\sqrt{3}.$$

The results in a few other useful cases, are as follow:

In *the periphery of a circle*, revolving about the diameter ... $\Big\}$ $CR = \text{rad.}\sqrt{\frac{1}{2}}$

In the plane of a circle, revolving about the diameter. .. $\Big\}$ $CR = \frac{1}{2}\text{rad.}$

In the surface of a sphere.ditto $CR = \text{rad.}\sqrt{\frac{2}{3}}$

In a solid sphere ....ditto $CR = \text{rad.}\sqrt{\frac{2}{5}}$

In a cone, revolving on its axis.......... $\Big\}$ $CR = \text{rad.}\sqrt{\frac{3}{10}}$

In a straight lever, whose arms are B and b ...... $\Big\}$ $CR = \sqrt{\frac{B^3 + b^3}{3B + 3b}}$

If the matter of any gyrating body were actually to be placed as in the centre of gyration, it ought either to be disposed in the circumference of a circle, whose radius is CR, or into two points, R and R, diametrically opposite to each other, and at distances from the centre, each equal to CR.

Ballist pendul

107. The properties we have been investigating, find their application in enquiries connected with the theory of gunnery; viz. in determining the initial velocity of balls thrown from pieces of ordnance, which are made to impinge on a massy pendulous block of wood, denominated the *ballistic pendulum*. This machine consists of a large block of wood annexed to the end of a strong iron stem, having a cross steel axis at the other end, placed horizontally, about which the whole vibrates together. When the machine is at rest, a ball, discharged horizontally from a piece of ordnance, strikes and enters the block, and causes the pendulum to vibrate, more or less, according to the velocity of the projectile; and by the extent of the vibration, the velocity of the ball, at the time of impact, becomes known.

When a pendulum is at rest, if the body impinge on it in a horizontal direction, the same velocity will be communicated to the point of impact, as if the mass of the pendulum were removed, and instead of it an equivalent mass were concentrated in the point of impact; the quantity of the equivalent mass being to that of the pendulum, in a duplicate ratio of the distances of the centre of gyration and the point of impact from the axis of motion.

For let CADE (fig. 74) represent a pendulum whose axis of motion passes through C, and let any impact be impressed upon the point P, in a horizontal direction, and perpendicular to the vertical plane ABCD, the pendulum being at rest.

Let Q be the quantity of matter, which being concentrated in P, the same angular velocity will be communicated by the impact, as when B, the whole body, is concentrated into its centre of gyration R. Then since the resistance arising from the inertia of the two masses at their respective distances must be equal, in order that their angular velocities may be equal, we must make, as in No. 4, art. 105,

$$B \cdot CR^2 = Q \cdot CP^2;$$

whence

$$Q = \frac{B \cdot CR^2}{CP^2}.$$

From which determination, combined with the laws of the impact of bodies, the velocity of the ball or impinging body becomes known.

We shall conclude this article relative to the centr of gyration with the following propositions, wherel that centre may be found experimentally:

109. If motion be communicated by a hangi weight to a system revolving on a fixed axis passi through the centre of gravity, and the moving fo act always at a given distance from the axis of moti it will generate in the revolving system the same mo as the weight would acquire in the same time by fal freely by its gravity from a state of quiescence.

DGB (fig. 75) represent a body or system moveable upon a horizontal axis passing through its centre of gravity G; let R be the centre of gyration, P the weight of the body which gives motion to the system whose weight is W, by means of the cord P *d* C wound round the circumference *d* CE. If GR be denoted by *r*, G *d* by *d*, the inertia of the whole system will be equivalent to the weight $W \frac{r^2}{d^2}$ uniformly diffused through the periphery *d* CE, every point of which manifestly moves with the same velocity as P.

The moving body P being supposed destitute of inertia, or indefinitely less than the weight of the system, the accelerating force ϕ will be $\frac{2 g P d^2}{W r^2}$; consequently in the time *t* the velocity generated will be $\frac{2 g P d^2 t}{W r^2}$; which multiplied into the quantity of matter $W \frac{r^2}{d^2}$, gives $2 g P t$ for the whole quantity of motion generated (2 *g* denoting here the force of gravity $= 32\frac{1}{6}$). Now, the velocity generated by a body falling freely in the time *t* is $2 g t$, and this multiplied by P, the quantity of matter gives $2 g P t$, the same quantity of motion as before.

Whence it follows that the permanence of motion, estimated by the product of the quantity of matter and velocity, obtains in bodies which revolve on fixed axes.

If P be supposed to possess inertia, the quantity $P d^2$ must be added to the denominator of the fraction expressing the accelerating force, which will then become

$$\frac{2 g P d^2}{P d^2 + W r^2},$$

which leads to an useful practical method of determining the centre of gyration by experiment.

Let *s* be the space described by the weight P descending from rest in *t* seconds; then the accelerating force being as above, we have

$$s = g t^2 = \frac{g P d^2 t^2}{P d^2 + W r^2};$$

whence

$$r = \sqrt{\left(\frac{g P t^2 d^2 - s P d^2}{s W}\right)};$$

in which equation, if the numerical values of *s*, *d*, P and W, as resulting from the experiment, be used, *r* will be determined.

scent es hem.

110. Let the body DB (fig. 75), whose centre of gravity G, is the centre of the circle CE *d*, be attached to a fixed point A, by means of a cord ACE *d*, wound about CE *d* as an axle; and let O be the centre of oscillation, when C is the point of suspension, or momentary centre of motion of the body; then, if while the body descends by the force of gravity, it is made to turn round by the unwinding of the cord ACE *d*, the space actually descended by the body DB, is to the space which would be described in the same time by a body falling freely as CG to CO.

Through the point of contact C, and the centre of gravity G, draw the horizontal line CGO, and suppose the motion of the body to commence when these centres are thus situated; then the angular velocity of the body at such commencement of its motion will be the same as if the whole were placed in O; and if a body were placed in O, its initial velocity would be the same as that of a body falling freely. Drawing, therefore, C *g o* indefinitely near CGO, and the small arc O *o* and G *g*, having the common centre C, we shall have

veloc. at O : *veloc.* at G :: O *o* : G *g* :: OC : GC;

but the velocity of the descending body is that of its centre of gravity; therefore the

veloc. in free descent : *veloc of the body* :: OC : GC.

Now, since the points C, G, O, are always in a horizontal line, and the radius CG is given, as well as the distance CO, the velocity of a body falling freely, and of the body unwinding from the thread, is always in the ratio of CO to CG, and consequently when CO : CG is a constant ratio, the whole spaces described in a given time will be as stated in the proposition. We may also draw from the above, the following conclusions, viz.

1. The weight of the body DB, is to the tension of the cord AC, as CO to OG.

For if the body were supported at O, as well as C, G being the centre of gravity, or estimated place of the body, we should have

weight of DB : *pressure at* C :: CO : GO;

but if the point O be set free, the force acting there will generate a motion about C, while the pressure there, and consequently the tension of the cord, will be neither increased nor diminished.

2. This motion of the body DE, by unwinding the thread, is uniformly accelerated, and the tension of the cord is invariable through the whole descent.

3. If the body, whose transverse sections are circular, run down an inclined plane, while the cord or ribband CE *d* unfolds, as if such round body roll down an inclined plane, being hindered by friction from sliding; the space it describes in any time, is to the space described by a body sliding down freely without friction, as CG : CO.

For the forces which generate the motion are both decreased in the same ratio; that is, as the absolute gravity to the relative gravity upon the plane; therefore the spaces described will, in either case, remain in the ratio of CG to CO.

Remark.

111. The force by which spheres, cylinders, &c. are caused to revolve, as they move down an inclined plane instead of sliding, is the adhesion of their surfaces, occasioned by the pressure against the plane; this pressure is part of the body's weight; for the weight being resolved into its components, one in the direction of the plane, and the other perpendicular to it, the latter is the force of the pressure; which (while the same body rolls down the plane) will be expressed by the cosine of the plane's elevation. Hence, since the cosine decreases, while the arc or angle increases, after the angle of elevation arrives at a certain magnitude, the adhesion may become less than what is necessary to make the circumference of the body revolve fast enough; in this case, the body descends partly by sliding, and partly by rolling. And the same may happen in smaller elevations, if the body and plane are very smooth; but at all angles, the body may be made to roll by the uncoiling of a thread or ribband wound about it.

Descent of different bodies.

If W denote the weight of a body, *s* the space described by a body falling freely, or sliding freely down an inclined plane; then the spaces S, described by rotation in the same time by the following bodies, will be in these proportions:

1. *A hollow cylinder*, or *cylindric surface*, $S = \frac{1}{2}s$ tension of the cord in the 1st case $= \frac{1}{2}$ W.

2. *In a solid cylinder*, $S = \frac{2}{3}s$; tension $= \frac{1}{3}$ W.

3. *In a spheric surface*, or *thin spherical shell*, $S = \frac{3}{5}s$; tension $= \frac{2}{5}$ W.

4. *In a solid sphere*, $S = \frac{5}{7}s$; tension $= \frac{2}{7}$ W.

The reader may see an investigation of these and various other similar deductions in Bezout's Mechanics (art. 732.)

§ XXII. *Of the centre of percussion.*

Centre of percussion defined.

112. *The centre of percussion* is that point in a body revolving or vibrating on an axis, at which, if it struck an immoveable obstacle, all its motion would be destroyed, or it would not incline either way; so that if at the instant of impact the supports of the axis were annihilated, the body would remain in absolute rest.

If, while a system of bodies is vibrating about a fixed axis passing through C (fig. 76), such an obstacle be opposed at the point O, as entirely to destroy the motion of that point, every other particle in the system will have a tendency, in consequence of its inertia, to proceed in the direction of its motion; that is, of the tangent of the circular arc it was describing at the instant the obstacle was opposed at O.

These forces, therefore, will act on the system to turn it round O; and if the sum of the forces on each side of O, should be unequal, the motion of the system will not be destroyed when O is stopped; but, since the forces which act upon the pendulous body between O and C, have a tendency to continue the motion of the system contrary to those which are impressed on the other side of O, if this point O be so situated that the sum of the forces to turn the system round that point be exactly equal, then the instant in which O is stopped, the whole motion of the system will be destroyed.

Centre of percussion investigated.

In order to investigate the position of the centre of percussion of a body revolving on a fixed axis, supposing that centre be required on the plane passing through the axis of motion and centre of gravity, we may proceed as follows. Let CBE (fig. 76) be a plane passing through the centre of gravity G of the body, and perpendicular to the axis of suspension which passes through C; and conceive the whole body to be projected upon this plane in lines perpendicular to it, or parallel to the axis of motion; for then, as each particle will fall at the same distance from the axis as in the body itself, the effect from the rotatory motion will not be changed, neither will the place of the centre of gravity. Through C and G draw the line CGON, and let P be the place of one of the particles $\dot{m}$ composing the system.

Since the angular motion of all the particles is the same, the absolute velocity will be proportional to the distance from the axis of motion; and if at the unit of distance the velocity is supposed to be the unit of velocity, the velocity of $\dot{m}$ will obviously be denoted by PS, and its quantity of motion will be $\dot{m}$. PS, which will act in the direction PR perpendicular to PC. If now we produce PR, till it meets OD, drawn parallel to PC in D, then the force $\dot{m}$. PC acting in the direction PR, will act upon O in the same manner as if it had the advantage of the lever OD; and, consequently, $\dot{m}$. PC. will represent the force of the particle $\dot{m}$ to move system round O.

But from the similarity of the triangles CPR, O we have

$$RC : CP :: RO : OD = \frac{CP \cdot RO}{RC} = CP \frac{CO -}{CR}$$

and if PA be perpendicular to CO, we shall have

$$RC : CP :: CP : CA = \frac{CP^2}{CR};$$

whence the entire force $\dot{m}$. PC·OD becomes

$$\dot{m}. PC^2 \cdot \frac{CO - CR}{CR} = \dot{m} \cdot CA \cdot CO - \dot{m}. PC^2.$$

But when O is the centre of percussion, the sum o the $(\dot{m}. CA \cdot CO - \dot{m}. PC^2)$ must be equal to zero

$$\int \dot{m}. CA \cdot CO = \int \dot{m}. PC^2 = \int \dot{m}\, r^2;$$

whence it follows, that

$$CO = \frac{\int \dot{m}\, r^2}{\int \dot{m}\, CA} = \frac{\int \dot{m}\, r^2}{B \cdot CG};$$

which is precisely the result obtained for the dete nation of the centre of oscillation; and whenc follows, that if the body be symmetrical with regar the plane BGE, or if it be a solid of rotation, the ce of percussion, found in the axis of the body, will c cide with the centre of oscillation. If the centr percussion be required for a plane which does pass through G, as for instance C*o*, we must pro as follows.

From G, the centre of gravity, let fall on C*o* perpendicular G*g*, and by the same argument above,

$$Co = \frac{\int \dot{m}\, r^2}{\int \dot{m}\, CA} = \frac{\int \dot{m}\, r^2}{B \cdot Cg}.$$

Now $Co : CO : CG : Cg$;
and hence the angles *o*OC and G*g*C are equal, the former is a right angle.

It follows, from the above deduction, that a body *several centres of percussion*, according to the pos of the plane passing through the axis of motion, in w the impact is made; and the right line O*o* perpe cular to OC, is their locus.

113. If CB (fig. 77) be the axis on which a bod volves, CAO a plane perpendicular to CB, and pas through the centre of gravity G, *p*A a perpendic let fall from any particle *p* or $\dot{m}$ of the body, o plane AC, and P the centre of percussion, then

$$PO = CB = \frac{\int \dot{m}\, Ap \,.\, AG}{Body \,.\, CG}.$$

For the sum of all the forces, with which the bo liable to be turned in one direction round PB, a axis, is equal to the sum of all the

$$\dot{m} \,.\, CA\,(CB - Ap) = \int \dot{m} \,.\, CA\,(CB - Ap)$$

and the sum of all the forces which tend to turn the contrary direction, is equal to the sum of all t

$$\dot{m} \,.\, CA\,(Ap - CB) = \int \dot{m}\, CA\,(Ap - CB);$$

therefore, in the case where there is no motion e ways, we have

$$\int \dot{m} \,.\, AC \,.\, CB = \int m \,.\, Ap \,.\, AC;$$

but $\int \dot{m} \,.\, CB \,.\, AC = \int \dot{m}\, CB \,.\, CG + \int \dot{m} \,.\, CB \,.$

and $\int \dot{m} . CB . GA = 0$, from the property of the centre of gravity.

In like manner,

$$\int \dot{m} . \dot{p}A . AC = \int \dot{m} . pA . AG + \int \dot{m} . pA . CG,$$

and $$\int \dot{m} . pA . CG = 0;$$

consequently

$$\int \dot{m} . CB . CG = \int \dot{m} . pA . AG;$$

whence

$$CB = PO = \frac{\int \dot{m} . pA . AG}{\int \dot{m} . CG} = \frac{\int \dot{m} . pA . AG}{\text{Body} . CG},$$

as stated in the proposition.

Therefore the centre of percussion of a body turning round the axis CB, is determined by these conditions: 1st. It is in a line PO, passing through the centre of oscillation, and parallel to CB. 2dly. Its distance from the centre of oscillation is

$$PO = \frac{\int \dot{m} . pA . AG}{\text{Body} . CG}.$$

114. From our definition and investigations relative to the centre of percussion of bodies, we readily derive the following conclusions:

1. When a pendulum, vibrating with a given angular velocity, strikes an obstacle, the effect of the impact will be the greatest, if it be made at the centre of percussion; for, in this case, the obstacle receives the whole revolving motion of the pendulum; whereas, if the blow be struck in any other point, a part of the motion of the pendulum will be employed in endeavouring to continue the rotation.

2. If a body, revolving on an axis, strike an immoveable obstacle, at the centre of percussion, the point of suspension will not be affected by the stroke.

We may practically ascertain this property when we strike a smart blow with a stick. If we give it a motion round the joint of the wrist only, and holding it at one extremity, strike smartly with a point considerably nearer, or more remote than ⅔ds of its length, we feel a painful wrench in the hand; but if we strike with that point which is exactly at ⅔ds of the length, no such disagreeable strain will be felt. If we strike the blow with one end of the stick, we ought to make the centre of motion at ⅓d of its length from the other end, by which means the wrench will be avoided.

115. It is stated by Dr. Gregory in his Treatise of Mechanics, and the same remark will be found in our historical sketch, that many of the early writers, while the science of Mechanics was still but little known, investigated the properties of the centre of percussion, upon a supposition, that it always coincided with the centre of oscillation; and, in fact, it does so in many cases; but there are others in which this coincidence does not obtain; a simple and obvious instance of which, is given by the author above mentioned. This is as follows:

Let two equal balls, B and B′, (fig. 78) (of evanescent magnitude with respect to their distance B, B′ from each other) be connected with an axis of motion, CC′, by inflexible lines CB = 3, and C′B′ = 7, void of gravity, perpendicular to CC′ = 6, and parallel to each other; to find the respective distances of the centres of gravity, oscillation, and percussion, of this system, from each other:

Join BB′, and bisect it in G, the centre of gravity, so shall $Gg = 5$; then (N° 6, art. 100), Mechanics.

$$Og = \frac{B \cdot BC^2 + B' \cdot B'C'^2}{(B + B')\,Gg} = \frac{9 + 49}{2 \times 5} = 5{\cdot}8,$$

the distance of the centre of oscillation from the axis; whence $GO = {\cdot}8$.

Through O, draw $b\,b'$ parallel to CC′, then (art. 113) the centre of percussion will be found somewhere in this line; and by the same article we find

$$PO = \frac{B \cdot BA \cdot AG + B' \cdot B'A' \cdot A'G}{(B + B')\,gG} = \tfrac{12}{10} = 1{\cdot}2.$$

Here the products 6 and 6 are added together, because, while B and B′ are on opposite sides of Gg, they are also on contrary sides of a line passing through G, parallel to CC′; so that if BA be positive, B′A must be accounted negative; and BA·AG and B′A′·A′G are both affirmative products. Now the centre of percussion P, must be so situated, that

$$B \cdot BC \cdot bP = B' \cdot B'C' \cdot b'P;$$

whence, since B = B′, we have

$$B'C' : BC :: bP : Pb' = \tfrac{3}{7}\,bP = \tfrac{3}{10}\,b'b = 1{\cdot}8;$$

consequently, $OP = 3 - 1{\cdot}8 = 1{\cdot}2$, and

$$GP = \sqrt{({\cdot}8^2 + 1{\cdot}2^2)} = 1{\cdot}4422;$$

the triangle GOP being right-angled at O.

The following problems will shew the method of applying the preceding principles to practical cases.

Problem I.

116. Let AB (fig. 79) denote an axle turning on two fulcrums at A and B; RS a wheel of given diameter, to which is attached, by a cord, wound round its circumference, a given weight W; conceive p, p', p'', p''' to be given weights fixed to the axle by inflexible lines or wires Cp, Cp', &c. and let it be required to determine the circumstances of the motion of the descending weight. Problems.

We shall, in the first place, consider this problem under its most simple form; viz. we shall suppose the wheel RS, the axle AB, and the lines or wires Cp, Cp', &c. as divested of inertia, whereby W, and the four masses p, p', p'', p''', will be the only weights in the system, the latter being all equal, placed at equal distances from the axle C, and at right angles to each other.

Let Cp, Cp', &c. $= r$; the radius of the wheel $= r'$; the sum of $p + p' + p'' + p''' = P$, and the given weight = W; then the moment of inertia of P will be $r^2 P$, and the moment of inertia of the weight $= r'^2 W$; consequently the accelerating force will be

$$\phi = \frac{2g' r^2 W}{r'^2 W + r^2 P};$$

the velocity of W after any time t will be

$$\frac{2g r'^2 W}{r'^2 W + r^2 P} . t;$$

and the space descended in t seconds will be

$$S = \frac{g r'^2 W}{r'^2 W + r^2 P} . t^2.$$

Let us now suppose the wheel and axle to be possessed of inertia, and let the weight of the former = W′, that of the latter = W″, also call the radius of the axle $= r''$; then

The moment of inertia of the wheel $= \frac{1}{2} r'^2 W'$,
and of the axle $= \frac{1}{2} r''^2 W''$;

the centre of gyration in both cases being at the distance of $r \surd \frac{1}{2}$ from the geometrical axis.

The motive force being here still the same as in the preceding case, we shall have for the accelerating force

$$\phi = \frac{2\, g\, r'^2\, \mathrm{W}}{r'^2\, \mathrm{W} + \frac{1}{2}\, r'^2\, \mathrm{W}' + \frac{1}{2}\, r''^2\, \mathrm{W}'' + r^2\, \mathrm{P}}.$$

The numerical values of all these quantities being supposed known, the space, time, velocity, &c. are determinable the one from the other, as in the cases of the simple descent of a body by uniformly accelerating forces.

If, instead of the weight W, we conceive the system to be put in motion by an equivalent force, but divested of inertia, as muscular power, the action of steam, &c. then the quantity r^2 P must be omitted in the denominator, as such a force adds nothing to the inertia of the system.

If we suppose the system of small weights p, p', p'', &c. to be replaced by a solid body of revolution, as in figure 80, the principles of calculation are still the same; viz. we must estimate the moment of inertia of the solid, by supposing all the matter of it to be collected into a circle, whose radius is equal to the distance of its centre of gyration from the geometrical axis.

Thus, in (fig. 80), let D denote a sphere whose radius is 3 feet, and weight 500 lbs.; the weight W = 50 lbs.; and the radius of the wheel = 6 inches, or $\frac{1}{2}$ a foot, whose weight and that of the axle are supposed inconsiderable with respect to the other part of the system; and let it be required to determine the time in which the weight W will pass through any given space, as, for example, 50 feet.

The centre of gyration of a sphere being $r' \surd \frac{2}{5}$ from the centre (r' denoting the radius), our expression for the accelerating force will be

$$\phi = \frac{2\, g\, \mathrm{W}\, r^2}{\mathrm{W}\, r^2 + \frac{2}{5}\, \mathrm{W}'\, r'^2},$$

where W′ = 500, W = 50, $r' = 3$ feet, $r = \frac{1}{2}$ foot, and $g = 16\frac{1}{12}$ feet;

$$\text{whence} \quad \phi = \frac{32\frac{1}{6} \times 50 \times \frac{1}{4}}{50 \times 9 + \frac{2}{5} \times 500 \times 9} = \frac{402\frac{1}{12}}{1359}:$$

$$\text{wherefore,} \quad \frac{402\frac{1}{12}}{1359}\, t^2 = s = 50,$$

$$\text{and} \quad t = \surd\left(\frac{50 \times 1359}{402{\cdot}083}\right) = 13'' \textit{ nearly}.$$

Problem II.

Problem II. 117. Let ABC (fig. 81) represent a wheel and axle, its weight w having a given weight q applied to the circumference of the axle, and p applied to the circumference of the wheel, in order to raise q; it is required to assign the space described by the elevated weight q from rest in any given time.

Let the radius of the axle $= r$, of the wheel $= r'$, and the distance of the centre of gyration of the wheel from the geometrical axis $= \rho$; we shall have, for the moment of inertia of the whole system,

$$w\rho^2 + p r'^2 + q r^2.$$

Again, the actual force applied at the point D is p; but this is resisted by the force q acting at A, and which, when applied at the distance SD, would be $\frac{q r}{r'}$;

$$\text{whence} \quad p - \frac{qr}{r'} = \frac{p r' - q r}{r'},$$

will express the force which impels the point D or the single weight, which, if applied at D, would produce the same effect as the two opposing weights. If, therefore, this weight be multiplied by the square of the distance at which it acts, and the product be divided by the inertia of the system, we shall have

$$\frac{p r'^2 - q r' r}{w\rho^2 + p r'^2 + q r^2},$$

for the accelerating force which urges the point D, the force of gravity being considered as unity; or calling, as we have hitherto done, the force of gravity $2g$, we shall have for the force which urges the point D, or the descending weight p,

$$\phi = 2g \cdot \frac{p r'^2 - q r' r}{w\rho^2 + p r'^2 + q r^2}. \qquad (1)$$

Now the force which accelerates the descending weight is to that which urges the ascending weight in the ratio of the respective radii; whence

$$r' : r :: 2g \cdot \frac{p r'^2 - q r' r}{w\rho^2 + p r'^2 + q r^2} :$$

$$2g \cdot \frac{p r' r - q r^2}{w\rho^2 + p r'^2 + q r^2}, \qquad (2)$$

the accelerating force by which the weight q ascends.

Substituting, in this fraction, any numerical values of the several quantities which enter into it, and calling the result F, we shall have for the velocity v, and space s, due to any time t,

$$v = 2g\, \mathrm{F}\, t,$$

$$s = g\, \mathrm{F}\, t^2.$$

Suppose, for example, $p = 20$ lb. $q = 10$ lb. radius of the axle = 1 foot, and that of the wheel = 3 feet, also the weight of the latter = 100 lbs.; that is, $p = 20$, $q = 10$, $r = 1$, $r' = 3$, and $w = 100$; then $\rho = r' \surd \frac{1}{2} = 3 \surd \frac{1}{2}$, and $\rho^2 = 4\frac{1}{2}$.

Substituting these numbers in the above, it becomes

$$2g \cdot \frac{20 \cdot 3 - 10}{100 \cdot 4\frac{1}{2} + 20{\cdot}9 + 10} = 2g\, \frac{5}{64}.$$

If, therefore, the proposed time were 3 seconds, we should have

$$v = 32\tfrac{1}{6} \times \tfrac{5}{64} \times 3 = 7\tfrac{1}{2} \text{ feet nearly.}$$

$$s = 16\tfrac{1}{12} \times \tfrac{5}{64} \times 9 = 11\tfrac{1}{4} \text{ feet nearly.}$$

If SD = SA, or $r = r'$, the force which accelerates either weight will be

$$2g \cdot \frac{(p - q)\, r'^2}{w\rho^2 + (q + p)\, r'^2}; \qquad (3)$$

and if the ratio of $r' : r$ were as $n : 1$, so that $r' = n r$, also $\rho^2 = \frac{1}{2} n^2 r^2$; then the expression becomes

$$2g \cdot \frac{(p n - q)}{\frac{1}{2} w n^2 + p n^2 + q}, \qquad (4)$$

If the inertia of the wheel be, in this last case, evanescent, or too small to have any sensible effect, then the same becomes

$$2g \cdot \frac{p n - q}{p n^2 + q}; \qquad (5)$$

and if the inertia of the moving force be also equal to zero, then it reduces still farther to

$$2g\, \frac{p n - q}{q}; \qquad (6)$$

and, in the last three cases, by multiplying the several numerators by n, we shall have the accelerating force of the descending weight.

If the mass moved have no weight, but possess inertia only, then q in the numerator disappears, and equation (5) becomes

$$2g\frac{pn}{pn^2+q}; \qquad (7)$$

and for the point D, or for the descending weight,

$$2g\cdot\frac{pn^2}{pn^2+q}. \qquad (8)$$

118. Whenever motion is communicated to a body, a certain resistance must have been overcome by the moving force; which resistance is of various kinds, as

1. The inertia of the mass moved, whereby it has a tendency to persevere in its state of quiescence, or of uniform motion in a right line.

2. That of a weight, or other absolute force, opposed to the action of the moving power.

3. Obstacles upon which the moving body impinging is retarded in its progress; such, for example, is the resistance which arises from the particles of a fluid through which a body moves. The estimation of these resistances, and their effects in retarding the motion of bodies, acted on by a given force, are deducible from the laws of motion; and constitute a part of the solution of almost all problems relating to the motion of bodies.

The moving forces, also, are of various kinds; as for instance, the force of gravity, muscular power, the impact of bodies solid or fluid, &c.; and the effects of these, exclusive of the resistance opposed to them, depend on the various circumstances under which they act, the time of action, the spaces through which they pass, or are impelled, &c.

These considerations are urged to show, that from the great variety of indeterminate conditions which may enter into mechanical problems, there must be, in course, various methods of producing the same mechanical effect; and it is a very material part of the art, considered either in a theoretical or practical point of view, to proportion the means to the end, and to effect this with all the advantages of which the nature of the case is capable. It is the due observation of these particulars which contributes to render mechanic instruments complete; while, on the other hand, the neglect of them will, as assuredly, lead to defective construction.

The proper choice of means to produce mechanical effect is frequently the result of long and continued experience, independent of all theory; the knowledge of which, however, when immediately applied to practice, would save the artist much time and trouble, and be productive of other advantages, of which an uninformed practitioner must be destitute.

PROBLEM III.

119. Let ABC (fig. 81) represent a wheel and axle of given weight, moveable round a horizontal axis, which passes through S; and suppose a known weight q, which is applied to the circumference of the axle, to be raised by a given force p, applied to the circumference of the wheel; to assign the proportion of the radii of the wheel and axle, so, that the time in which the weight q ascends through any given space, shall be a minimum:

Since only the ratio of the wheel and axle is required, let the radius of the axle $= r$, of the wheel $= nr$, the ratio being that of 1 to n, the weight of the wheel $= w$, and the distance of the centre of gyration $= \rho$ or $\rho^2 = \frac{1}{2}n^2r^2$; then, by equation (4) of the preceding problem, the accelerating force which urges the point A, or the weight q, is

$$2g.\frac{pn-q}{(\frac{1}{2}w+p)n^2+q}.$$

And, consequently,

$$s = gt^2.\frac{pn-q}{(\frac{1}{2}w+p)n^2+q};$$

whence we have

$$t = \sqrt{\left(\frac{s(\frac{1}{2}w+p)n^2+qs}{g(pn-q)}\right)} \text{ a minimum.}$$

Or, since s and g are given quantities, and also when any quantity is a minimum, its square is so likewise, we must have

$$\frac{p'n^2+q}{pn-q} = \text{a minimum},$$

where p' is for the sake of abridging, made equal to $p + \frac{1}{2}w$.

This being put into fluxions, gives

$$2p'n\dot{n}(pn-q) - p\dot{n}(p'n^2+q) = o;$$

whence

$$2p'pn^2 - 2p'qn = pp'n^2 + pq;$$

or

$$pp'n^2 - 2p'qn = pq;$$

that is,

$$n^2 - 2\frac{q}{p}n = \frac{q}{p'};$$

wherefore,

$$n = \frac{q}{p} \pm \sqrt{\left(\frac{q^2}{p^2} + \frac{q}{p'}\right)}.$$

In the case in which the inertia of the wheel becomes evanescent, we shall have $p' = p$, and the above becomes

$$n = \frac{q \pm \sqrt{(q+pq)}}{p}:$$

And if $q = p$, that is, when the moving force is equal to the weight moved, we shall have in the last case

$$n = 1 \pm \sqrt{2}.$$

Example.

Suppose ABC to represent a cylindric wheel, the radius of which is required, but of which the weight is given = 20 lb.; and let the radius of axle = 1 inch; the weight $q = 100$ lb.; and moving force $p = 33$ lb.; to find the radius of the wheel:

Here $p + \frac{1}{2}w = p' - 43$ lb.; therefore,

$$n = \frac{100}{33} \pm \sqrt{\left(\frac{100^2}{33^2} + \frac{100}{43}\right)} = 6{\cdot}43 \text{ nearly.}$$

Consequently, since the radius of the axle = 1, the radius of the wheel = 6·43 inches.

We shall enter more into this kind of enquiry in treating of the maximum effect of machines.

PROBLEM IV.

Problem IV.

120. In the wheel and axle, when a given weight P, acting at the distance R, raises a weight W, acting at the distance r from the geometrical axis, it is required to assign the pressure sustained by the axis; the weight of the wheel and axle, and the friction of the cord not being considered.

Suppose P and W to be at their respective extremities of the horizontal line passing through the centre of motion, and in that situation let v and g be the

Pressure of a wheel and axle in motion.

respective distances of their centres of oscillation and of gravity from the centre of motion. Then, since the whole system revolves with the whole angular velocity, as if P + W were placed at the distance *o* from the centre of motion; and since the accelerative forces are as the velocities generated in a given time, or as the distance from the centre of motion, we have

$$o : g :: \mathrm{P} + \mathrm{W} : \frac{g}{o} + (\mathrm{P} + \mathrm{W}),$$

the force with which the centre of gravity descends.

But the distance of centre of oscillation from the axle is

$$o = \frac{\mathrm{P{\cdot}R}^2 + \mathrm{W}\cdot r^2}{\mathrm{P{\cdot}R} - \mathrm{W}\cdot r},$$

and the distance of centre of gravity from the same is

$$g = \frac{\mathrm{P{\cdot}R} - \mathrm{W}.r}{\mathrm{P} + \mathrm{W}};$$

whence

$$\frac{g}{o} = \frac{(\mathrm{P{\cdot}R} - \mathrm{W}\cdot r)^2}{\mathrm{P{\cdot}R}^2 + \mathrm{W}\cdot r^2} + \frac{1}{\mathrm{P} + \mathrm{W}};$$

or

$$\frac{g}{o} \times (\mathrm{P} + \mathrm{W}) = \frac{(\mathrm{P{\cdot}R} - \mathrm{W}\cdot r)^2}{\mathrm{P{\cdot}R}^2 + \mathrm{W}\cdot r^2}.$$

Now the force with which the centre of gravity tends to descend, is manifestly that by which the pressure upon the axis P + W, arising from the weight, is diminished by the motion; consequently, the pressure upon the axis is

$$\mathrm{P} + \mathrm{W} - \frac{(\mathrm{P{\cdot}R} - \mathrm{W}\cdot r)^2}{\mathrm{P{\cdot}R}^2 + \mathrm{W}\cdot r^2} = \mathrm{P{\cdot}W}\,\frac{(\mathrm{R} + r)^2}{\mathrm{P{\cdot}R}^2 + \mathrm{W}\cdot r^2}.$$

If R = *r*, as in the case of the single fixed pulley, then the pressure is

$$\frac{4\,\mathrm{P{\cdot}W}}{\mathrm{P} + \mathrm{W}}.$$

The same reasoning will also obviously apply to the pressure of the fulcrum upon the straight lever.

Problem V.

Problem V.

121. Let A, B (fig. 82) represent a single moveable and fixed pulley, by means of which the power P elevates the weight W; then, having given P and W, together with the weights of the cylindric pulleys A and B, it is required to assign the space which the descending weight P describes in a given time, the weight of the moveable pulley being included in the weight W.

The absolute force which impels P in its descent is its own weight, or P; but since the weight W acts against this descent with a force $= \frac{1}{2}$ W, the whole force with which P has a tendency to descend, is P $- \frac{1}{2}$ W.

The inertia which resists the communication to P, depends on the mass contained in P, together with that contained in the two pulleys and the weight W. Let the weight of each pulley be Q, the inertia of P is its own mass P, and that of the pulley A $= \frac{\mathrm{Q}}{2}$; the inertia of the pulley B, is the same as if $\frac{\mathrm{Q}}{2}$ were uniformly accumulated into its circumference; but since the velocity of this circumference is less than the velocity of P, in the proportion of 2 to 1, its inertia will be equivalent to a mass $\frac{\mathrm{Q}}{2\cdot 4} = \frac{1}{8}$ Q descending with P; for the same reason, since the weight W moves with a velocity equal to one half of that with which P descends, its inertia referred to P's motion will be $\frac{1}{4}$ W; and therefore the inertia of the whole mass will be

$$\mathrm{P} + \tfrac{1}{2}\mathrm{Q} + \tfrac{1}{8}\mathrm{Q} + \tfrac{1}{4}\mathrm{W} = \frac{8\,\mathrm{P} + 5\,\mathrm{Q} + 2\,\mathrm{W}}{8};$$

and because the impelling force is $\frac{2\,\mathrm{P} - \mathrm{W}}{2}$,

the accelerating force

$$= \frac{2\,\mathrm{P} - \mathrm{W}}{2} \times \frac{8}{8\,\mathrm{P} + 5\,\mathrm{Q} + 2\,\mathrm{W}} = \frac{8\,\mathrm{P} - 4\,\mathrm{W}}{8\,\mathrm{P} + 5\,\mathrm{Q} + 2\,\mathrm{W}},$$

that of gravity being unity; if, therefore, we assume, as before, 2 *g* for the latter force, we shall have, for the velocity and space due to the time *t*,

$$v = 2\,gt\,.\,\frac{8\,\mathrm{P} - 4\,\mathrm{W}}{8\,\mathrm{P} + 5\,\mathrm{Q} + 2\,\mathrm{W}},$$

$$s = g\,t^2\,.\,\frac{8\,\mathrm{P} - 4\,\mathrm{W}}{8\,\mathrm{P} + 5\,\mathrm{Q} + 2\,\mathrm{W}}.$$

If W should be greater than 2 P, W will descend; and by the same method of reasoning, the force which accelerates the descent of W will be found =

$$\frac{2\,\mathrm{W} - 4\,\mathrm{P}}{2\,\mathrm{W} + 5\,\mathrm{Q} + 8\,\mathrm{P}}.$$

§ XXIII. *Of the centre of spontaneous rotation.*

Sponta neous rotatio

122. Hitherto our investigations have been principally confined to the circumstances connected with the rotation of bodies about fixed axes; but it is necessary, in order to complete this part of Dynamics, to pay some attention to the rotation of bodies in free space as depending upon the action of one or more forces, whose direction does not pass through the centre of gravity of the body. The point about which this rotation takes place, or that point which for an instant we may suppose to remain at rest when the body is first struck, is called the centre of *spontaneous rotation.*

It is obvious that if a body in free space receive a single impulse, the direction of which passes through its centre of gravity, no rotation can be communicated, but the body will move forward in space in the direction of the impelling force. But when the force has any other direction than that of the line passing through the centre of gravity, a motion of rotation will ensue; at the same time the body will also be urged forward in a rectilineal direction.

When a body B (fig. 83), of any shape whatever, receives an impulse, the direction of which does not pass through the centre of gravity, and takes, in consequence, the two motions spoken of above, it is evident that, for an instant of time, we may consider it as having only one motion, that is, a motion of rotation about a point or fixed axis C, which may be either within the body or out of it, according to its shape and the distance GS between the centre of gravity and the direction of the impact. If, while the line GS is carried parallel to itself from GS to G′S′, we imagine it to turn about the moveable point G; since the particles of the body have greater or less velocities, as they are more or less distant from G, it is manifest that there is upon SG, a certain point C which will be found to describe from C′ towards C an arc equal to GG′, which during an evanescent instant may be regarded as a right line: in that case the point C will have been carried as far backwards by its motion of rotation as it will have been advanced parallel to GG′ by the velocity common to all the parts of the body; the point C, has, therefore, during this instant, been actually at rest in C′, and may consequently be considered as a fixed point, about which the body during such an instant has a rotatory motion.

123. The arcs CC′, SI, which the points C and S describe during an instant, being regarded as right lines perpendicular to GS or parallel to GG′, the similar triangles CC′G, G′S′I, give

$$\mathrm{GS} : \mathrm{GC} :: \mathrm{SI} : \mathrm{GG}'.$$

Now the velocity GG′ will be expressed by $\frac{\phi}{\mathrm{B}}$, ϕ denoting the force, and the velocity S′I by $\frac{\phi\,\mathrm{SG}^2}{\int \dot{m}\, r^2}$; that is, the velocity of the centre of gravity is as the force divided by the body, and the velocity of rotation is equal to the moment of the force divided by the moment of inertia; therefore

$$\mathrm{GS} : \mathrm{GC} :: \frac{\phi.\mathrm{SG}^2}{\int \dot{m}\, r^2} : \frac{\phi}{\mathrm{B}}.$$

Consequently

$$\mathrm{GC} = \frac{\int \dot{m}\, r^2}{\mathrm{B.GS}};$$

the several distances r, r', &c. being reckoned from G. Hence

$$\mathrm{CS} = \mathrm{GS} + \frac{\int \dot{m}\, r^2}{\mathrm{B{\cdot}GS}} = \frac{\mathrm{B{\cdot}GS}^2 + \int \dot{m}\, r^2}{\mathrm{B.GS}};$$

which value of CS corresponds with the value of CO for the centre of percussion, art. 113; for having the moment of inertia with regard to an axis passing through the centre of gravity of any body, we have the same for any other axis parallel to the former, by adding to it the product of the mass into the square of the distance GS, between the two axes (art. 96).

Whence we may conclude that, *the centre of spontaneous rotation is the same with the centre of suspension corresponding to the centre of percussion, the latter being that point where the body is struck.*

The same conclusion may otherwise be deduced thus: The action of the body against an immoveable obstacle in the centre of percussion, must have the same effect upon the body, as if the latter had been at rest and the obstacle had struck it; in which latter case the centre of suspension would not be affected, and therefore it becomes the centre of spontaneous rotation.

Since the force ϕ does not enter into the expression for the centre of spontaneous rotation, it is obvious that the position of that centre is independent of the magnitude of the impact. We may also observe that the line CG is always greater as the force ϕ, or the resultant of all the forces, acts nearer to the centre of gravity, and *vice versa:* so that when ϕS coincides with GG, GC will be infinite, and there will be no motion of rotation, as is indeed otherwise obvious.

It follows also from the above, and a comparison with our deductions for the centres of oscillation and percussion, that

If an impact be made on any point of the axis of a symmetrical body, or a solid of rotation, and that point be considered as the point of suspension, the corresponding centre of oscillation will be the centre of spontaneous rotation.

Since the expression $\frac{\phi}{\mathrm{B}}$, denoting the velocity of the centre of gravity, is always the same for the same force and body, it appears, that to whatever point of a right line, passing through the centre of gravity of a body, the impact be applied, the velocity of the centre of gravity will be the same; and the same will obviously obtain for any number of forces ϕ, ϕ', ϕ'', &c. *Hence, in rotations about a centre of gyration the permanency of the quantity of motion obtains.*

Particular deductions.

124. Let the quantity of matter of the impinging body $= b$, and its velocity $= v$; or $\phi = b\,v$, and conceive the body B to be struck in the direction ϕS, which passes through the centre of the body b; let the velocity of the centre of gravity of B be V, and the centre of spontaneous conversion at C, then

$$\mathrm{CG} : \mathrm{CS} :: \mathrm{V} : \frac{\mathrm{V{\cdot}CS}}{\mathrm{CG}} = \textit{vel. of point}\ \mathrm{S};$$

consequently, $v - \mathrm{V}\frac{\mathrm{CS}}{\mathrm{CG}} =$ the velocity lost by b in the direction ϕS;

therefore, $b\,(v - \mathrm{V}\frac{\mathrm{CS}}{\mathrm{CG}}) = \mathrm{B{\cdot}V}$;

or $$\frac{b.v.\,\mathrm{CG} - b.\,\mathrm{V{\cdot}SC}}{\mathrm{CG}} = \mathrm{B{\cdot}V};$$

whence $$\mathrm{V} = \frac{b{\cdot}v{\cdot}\,\mathrm{CG}}{\mathrm{B{\cdot}CG} + b{\cdot}\mathrm{CS}}.$$

That is, when the direction of impact passes through the centre of the impelling body, the centre of gravity of the body struck, will move with a velocity equal to the product of the quantity of motion in the impelling body, into the distance of the centre of gravity, from that of spontaneous rotation; divided by the sum of the products of the impelling body into the distance of the point of impact from the centre of spontaneous rotation, *plus* that of the impelled body into the distance between the centres of spontaneous rotation and of gravity. If the inertia of the impelling body be evanescent, the velocity V will become

$$\mathrm{V} = \frac{b\,v}{\mathrm{B}} \text{ or } \frac{\phi}{\mathrm{B}},$$

(because in this case, $b{\cdot}$CS vanishes); which is the same as would be generated in the body B, if the impinging body were to act immediately at the centre of gravity of B, or in a line passing through that centre.

125. In the last case, also, the velocity of the centre of the system about the centre of gravity, is equal to the moment of the impelling body, divided by twice the product of the mass of the impelled body and the distance CG, into the periphery of a circle, whose diameter is unity.

For if a fixed axis passed through C, the centre of gravity would describe a circle, whose radius is CG, with the velocity $\frac{b\,v}{\mathrm{B}}$. But, in the present case, the motion of the system will be compounded of the uniform rectilinear motion of the centre of gravity, in the direction GG′, perpendicular to CS, and the angular motion $x\,G'\,p = GCG'$, generated round the centre of gravity. Let $\pi = 3{\cdot}14159$, &c. or the circumference of a circle, whose radius is $\frac{1}{2}$, or diameter 1; the circle, whose radius is CG, will be $2\,\pi.\mathrm{CG}$, and we have therefore, the analogy

$$\frac{b\,v}{\mathrm{B}} : 2\,\pi.\mathrm{CG} :: 1^s : \frac{2\,\pi.\mathrm{B.CG}}{b\,v},$$

the time of one revolution in seconds.

Whence it follows, that the number of revolutions, or parts of a revolution in a second, or the angular velocity, will be expressed by

$$\frac{b\,v}{2\,\pi.\,\mathrm{B{\cdot}CG}}.$$

And since C is the centre of percussion to S, considered as a centre of motion, if Q be the centre of gyration with respect to G, as a centre of motion, we have (art. 105)

$$\mathrm{GC{\cdot}GS} = \mathrm{GQ}^2,\ \text{or}\ \mathrm{CG} = \frac{\mathrm{GQ}^2}{\mathrm{GS}};$$

which value of CG being substituted for CG in the preceding equation, and taking U to denote the angular velocity, it becomes

$$\mathrm{U} = \frac{b.v.\mathrm{GS}}{2\,\pi.\ \mathrm{B{\cdot}GQ}^2}.$$

126. From the preceding investigations it also appears, that the *centre of spontaneous rotation, during the motion of the system, describes the common cycloid.* For the motion of any point in the system is compounded of the uniform rectilinear motion of the centre of gravity, and of the angular motion generated round that centre; but the velocity with which the centre of spontaneous rotation would move round the centre of gravity, if there only existed a rotatory motion in the system, would be equal to that with which the centre of gravity would move round it, if the centre C were fixed; consequently, since the centre C has both a rotatory and progressive rectilinear motion, each of which is equal to that of the centre of gravity, it will describe a cycloid.

PROPOSITION.

Proposition.

127. If a body or system B, to which, when quiescent, motion has been communicated by the impulse of a force ϕ, without inertia, that is, rectilinear motion to the centre of gravity, measured by the space V, which that centre would describe uniformly in any given time, and angular motion, measured by the revolutions U, which it would describe uniformly about G, in the same time; then, if the notation in the preceding article be retained, and Q be the principal centre of gyration, when the system revolves about its centre of gravity, the perpendicular distance from the centre of gravity, at which the impelling force must act so as to have generated these progressive and rototary motions, will be

$$\mathrm{GS} = \frac{2\,\pi.\ \mathrm{U{\cdot}GQ}^2}{\mathrm{V}}.$$

Let ϕS be the direction of the impulse, and let ϕ be equal to the momentum of an evanescent body, b, moving with the velocity v, B being the weight of the system; then, from what is shown above, the angular velocity communicated to the system will be

$$\mathrm{U} = \frac{b.v.\ \mathrm{GS}}{2\,\pi.\,\mathrm{B.\ GQ}^2}.$$

But from the laws of uniform rectilinear motion,

$$\mathrm{V} = \frac{\phi}{\mathrm{B}} = \frac{b\,v}{\mathrm{B}},\ \text{or}\ b.v = \mathrm{B{\cdot}V};$$

substituting, therefore, for $b.v$, we have

$$\mathrm{U} = \frac{\mathrm{B{\cdot}V.\ GS}}{2\,\pi.\mathrm{B.\ GQ}^2};$$

whence $$\mathrm{G\,S} = \frac{2\,\pi.\ \mathrm{U.\ GQ}^2}{\mathrm{V}}.$$

In the above expression, GQ denotes the distance of the principal centre of gyration from the axis of motion; and, therefore, when the body in question is a sphere, whose radius is r, we shall have $\mathrm{GQ}^2 = \frac{2}{5}r^2$; and if u be the absolute velocity of rotation of an equator of the sphere, we have

$$\mathrm{U} = \frac{u}{2\,\pi\,r};$$

whence the preceding value of GS is transformed to

$$\mathrm{GS} = \tfrac{2}{5}\,r\,.\,\frac{u}{\mathrm{V}}.$$

128. By means of this formula, we readily determine the distances from the centre at which the sun and planets may have received the single impulses which gave them both their motions of revolution in their orbits, and rotation on their axes; considering them, as they really are, very nearly spherical bodies.

Taking first the Earth, we know that any point in its equator passes over 25020 miles in one sidereal day by its rotation; and if we assume the mean distance of the earth from the sun at 95 million miles, it passes over about 596904000 miles by its orbicular motion in a year; which consists of 366 sidereal days; whence $\frac{596904000}{366} = 16308852$ miles in one day; we have therefore $\mathrm{V} = 16308852$, and $u = 25020$; which gives

$$\mathrm{GS} = \tfrac{2}{5}\,r\ \frac{u}{\mathrm{V}} = \tfrac{1}{163}\,r:$$

so that if an impulse be impressed on a quiescent sphere, and the direction of the force should be at a distance GS, from its centre of gravity of about $\frac{1}{163}$d part of its radius, the angular motion of that sphere, and the absolute motion of its centre of gravity, will have the same relation to each other as those which actually obtain in the Earth. By the same method of computation, we obtain for the distance GS,

For the Moon $\frac{1}{535}\,r$,
Earth $\frac{1}{163}\,r$,
Mars $\frac{1}{195}\,r$,
Jupiter $\frac{1}{2{\cdot}812}\,r$,
Saturn $\frac{1}{2{\cdot}588}\,r$.

We have not data for determining the same for the sun; but the very circumstance of this body having a motion of rotation, makes it highly probable that it has, with all the planets of the system, a general motion of translation in celestial space, perhaps round some system of still more general and extensive gravitation.

A revolving body which has no motion of translation, cannot have been brought into that state by the action of a single force; it necessarily requires at least two forces to produce this effect, and these must act simultaneously, and be equal and opposite to each other. The equality and opposition of the forces is necessary for stopping all progressive motion: if one of them has acted at the centre, the rotatory motion has been the effect of the other only; but if they have acted on opposite sides, they conspired with each other in producing the rotation, but have opposed each other if they acted on the same side. Two such forces must, therefore, have acted on the sun in order to produce a rotatory motion without a motion of translation, and these two forces must have been perfectly equal and opposite; a concurrence of circumstances which has infinity to unity for the measure of the probability. This seems, therefore, to corroborate the conjectures of philosophers, and the observations of Herschel and other astronomers, who think that the solar system is approaching to that quarter of the heavens in which is situated the constellation *Aquila*.

§ XXIV. *Of the permanent axes of rotation in bodies revolving in space.*

129. In the preceding article, we have only attempted to investigate the centre of momentary or spontaneous rotation, or that point about which the body has a tendency to revolve, at the instant the external force is first impressed; but as all circular motions, such as those of rotation, are accompanied by centrifugal forces, it is necessary, in order that the first axis of motion should remain permanent, that the centrifugal forces appertaining to the component particles of the body, exactly balance each other; for if this condition do not obtain, the axis will be continually changing, till at length the rotation is made about an axis in which that equilibrium has place, and after this no other variation will occur; on this account the latter is called *a permanent axis of rotation*.

130. The position of the axes of permanent rotation, and the number of such axes, depend upon the figure of the body; thus, in a uniform sphere, any diameter will be a permanent axis, the centrifugal forces being obviously in equilibrio, and it is the same with a cylinder revolving on any axis passing through its centre perpendicular to its length, as well as in several other bodies; *and every body, however irregular, has three such axes passing through the centre of gravity at right angles to each other*, forming at that point the solid angle of a cube, about which the centrifugal forces are accurately balanced; and therefore about either of these a motion of rotation being once begun, it will be continued; that is, they will become permanent axes.

The first enunciation of this very remarkable theorem was made by Segnor, in his "De Motu Turbinum," published in 1755; but the first demonstration of it we owe to Albert Euler, in 1760; since which time it has engaged the attention of most of the celebrated mathematicians of Europe.

We propose, before concluding this article, to give a sketch of the celebrated theorem of Frisi, relative to the permanent axes of rotation of any body in space; but let us, in the first place, examine some circumstances relative to the centrifugal forces of the particles of a body, being a solid of rotation, in which we shall avail ourselves of the elegant investigation published on this subject by Landen, forming the fourth article of his first volume of "Mathematical Memoirs."

131. Let p be a particle of matter firmly connected with the plane DOEFQG (fig. 84), in which the line OCQ is situated; $p\,q$ being a perpendicular from p to the said plane; let the distance $p\,q$ be denoted by u; also the line $q\,l$ being at right angles to O l CQ, let the distance $p\,l$ be denoted by h. Then the said plane, with the particle p, being made to revolve about O l CQ, as an axis, with the angular velocity e, measured at the distance a from the axis, the velocity of p will be equal to $\frac{h\,e}{a}$, and its centrifugal force from l will be $\frac{h e^2}{a^2} \times p$. Whence, by resolving that force into two others, one in the direction $q\,p$, and the other in a direction parallel to $l\,q$, it appears that, in consequence of the centrifugal force of p, the point l of the plane DOEFQG will be urged in a direction at right angles to that plane by a force $= \frac{u e^2}{a^2} \times p$, let the distance of that plane $l\,q$ be what it may.

Centrifugal force of the particles.

The particle p being connected with the plane DOEF, &c. as stated above, and the distance C l being denoted by v; if p be urged directly from the said plane by a force $= f u \times p$, the efficacy of that force to turn the plane about the line HCI, drawn at right angles to OCQ, will (by the property of the lever) be equivalent to the force $\frac{f u v \times p}{g}$ acting on the same line, at right angles to the plane OCQ, at the distance g from the point C: and it is moreover obvious that, *cæteris paribus*, the efficacy will be the same, whatever may be the distance from q to l.

Let q coincide with l, and let C k (fig. 85) be a line in the plane C $l\,p$ continued (which plane will be at right angles to the plane DOEFQG); also $p\,k$, being at right angles to C k, let the lines $p\,k$ and C k be denoted by w and $x - k$, respectively, k and x denoting the respective distances of the points C and k from the given point V, in the line passing through those two points. Then, the sine and cosine of the angle kCO, being denoted by m and n, we shall have

$$\frac{f u v \times p}{g} = \frac{f p}{g}\left\{ m n \left(w^2 - (x-k)^2 \right) - (m^2 - n^2) w (x-k) \right\}. \qquad (1)$$

Consequently, if each particle of any solid body, through which a line HCI, and a plane DOEIF, &c. may be conceived to pass, be urged from that plane by a force expressed by $f u \times p$, as above, the force which, acting on the line OCQ, at the distance g from C, would be equivalent to the efficacy of all the forces acting on the several particles of that body, to turn the same about the line HCI, will be obtained by computing the sum of all the forces exhibited on the second side of our equation (1), and which, according to our mode of notation, may be written

$$\int \frac{f p}{g}\left\{ m n \left(w^2 - (x-k)^2 \right) - (m^2 - n^2) w (x-k) \right\}. \qquad (2)$$

The computation of such equivalent force will, in many cases, be abridged, by observing, that if $p\,k$ be continued to p'', so that $k\,p''$ be $= k\,p$, the efficacy of the force on the particle p'' to turn the body about the line HCI, in opposition to the force on the particle p, will be represented by the equivalent force

$$\frac{f p''}{g} \times \left\{ m n \left((x-k)^2 - w^2 \right) - (m^2 - n^2) w (x-k) \right\}$$

acting on the line OCQ, at the distance g from C; and, therefore, that the efficacy of the two forces on

p and p'', to turn the body about HCI, will be represented by the equivalent force

$$\frac{2fp}{g}\left\{mn\left(w^2 - (x - k)^2\right)\right\} \qquad (3)$$

acting on the line OCQ, at right angles to the plane DOEIF, &c. at the distance g from C.

Body of any form.

132. The body being of any shape, such, that the section hi (fig. 86) passing through p and p'' at right angles to the line Ck, is a circle whose centre is k; and every other section parallel to the former, a circle whose centre is in the line passing through C and k; the ordinates corresponding to the abscisses kp, kp'', in the circular section hi, will each be parallel to that diameter HCI of the circular section passing through C, about which the body will be urged to turn; C being the centre of gravity of the body; and each of these ordinates will be expressed by

$$\sqrt{(y^2 - w^2)};$$

y being the radius of such section. Therefore, writing

$$p = 2\dot{x}\sqrt{(y^2 - w^2)} \times \dot{w},$$

our equation (3) becomes

$$\frac{4f\sqrt{(y^2 - w^2)}}{g}\left\{mn\dot{x}\left(w^2 - (x - k)^2\right)\right\}\dot{w};$$

the whole fluent of which is

$$\frac{\pi f}{g}\left\{mn\dot{x}\left(\tfrac{1}{4}y^4 - (x - k)^2 y^2\right)\right\} \qquad (4)$$

generated, while w ($= kp = kp''$) from o, becomes equal to the radius y ($\dot{x}$ and y being both considered as invariable).

The above equation (4) will express the value of (E), the force which acting on the line OCQ, at the distance g from C, would be equivalent to the force of all the particles in the said section, whose thickness is denoted by the indefinitely small quantity $\dot{x}$, the distance Ck being denoted by $(x - k)$, and 3·14159, &c. by $\pi = 4 \times {\cdot}7854$, &c.

Of the spheroid.

133. In the spheroid, whose axis is $2b$ (fig. 87), and equitorial diameter $2r$; taking $k = b$, we have

$$y^2 = \frac{r^2}{b^2}(2bx - x^2),$$

and

$$\tfrac{1}{4}y^4 - (x-k)^2y^2 = \frac{r^4}{4b^4}(2bx - x^2) - \frac{r^2}{b^2}(x-b)^2(2bx - x^2)$$

$$= \frac{r^2(r^2 + 4b^2)}{4b^4}.(4b^2x^2 - 4b^3x + x^4) - r^2(2bx - x^2);$$

and the fluent

$$\int\frac{r^2(r^2+4b^2)}{4b^4}.(4b^2x^2 - 4b^3x + x^4)\dot{x} - \int r^2(2bx - x^2)\dot{x}$$

generated while x from o becomes $= 2b$, is

$$\frac{4br^2}{15}(r^2 - b^2);$$

we find, therefore,

$$\frac{4\pi fbr^2}{15g} \times mn(r^2 - b^2) = \frac{fmn}{5g}(r^2 - b^2)\,\mathrm{S},$$

for the value of the force (E), which, acting at the distance g from the centre C of the spheroid, would be equivalent to the efficacy of the forces acting as above on all the particles of the spheroid, to turn it about a diameter of its equator; observing that S denotes the content of the spheroid.

Hemi-spheroid.

134. In the hemi-spheroid cut off at the equator from the spheroid of the preceding article, taking $k = \frac{5b}{8}$, we have $\frac{1}{4}y^4 - (x - k)^2.y^2 =$

$$\frac{r^4}{4b^4}(2bx - x^2)^2 - \frac{r^2}{b^2}\left(x - \frac{5b}{8}\right)^2(2bx - x^2).$$

Consequently the fluent of

$$\{\tfrac{1}{4}y^4 - (x - k)^2\ y^2\}\dot{x},$$

generated while x from o becomes $= b$, being

$$\frac{br^2}{480}(64r^2 - 19b^2),$$

we obtain

$$\frac{fmn}{320g}(64r^2 - 19b^2) \times \tfrac{1}{2}\mathrm{S},$$

for the value of (E), or the force, which acting at the distance g from the centre of gravity of the hemi-spheroid, would be equivalent to the efficacy of the forces acting as above on all the particles of the hemi-spheroid, to turn it about a diameter of the circular section in which the centre is situated; $\frac{1}{2}$ S denoting the mass or solidity of the body.

Para cono

135. In the parabolic conoid, the equation of whose generating curve is $px = y^2$, if the height be equal to b, and V be at the vertex, k will be $= \frac{2b}{3}$, and

$$\tfrac{1}{4}y^4 - (x - k)^2y^2 = \frac{p^2x^2}{4} - p\dot{x}\left(x - \frac{2b}{3}\right);$$

consequently the fluent of

$$\left\{\frac{p^2x^2}{4} - px\left(x - \frac{b}{3}\right)^2\right\}\dot{x},$$

generated while x from o becomes $= b$, being

$$\frac{pb^2}{36}(3p - b),$$

the force (E) will be

$$\frac{fmn}{18g}(3pb - b^2)\,\mathrm{M} =$$

$$\frac{fmn}{18g}(3r^2 - b^2)\,\mathrm{M};$$

r being the radius of the base, and $\mathrm{M} = \frac{1}{2}\pi pb^2$, the solidity of the conoid.

Doub parab conoi

136. In the solid, consisting of two parabolic conoids joined together at their bases; if the dimensions of each conoid be denoted as in the preceding article, and k be taken $= o$, we shall have

$$y^2 = p(b - x),$$

and

$$\tfrac{1}{4}y^4 - (x - k)^2y^2 = \frac{p^2(b - x)^2}{4} - px^2(b - x);$$

consequently, twice the fluent of

$$\left\{\frac{p^2(b - x)^2}{4}\ px^2(b - x)\right\}\dot{x},$$

generated while x from o becomes $= b$, being

$$\frac{pb^3}{6}(p - b),$$

we shall have for the force E,

$$\frac{fmn}{bg}(r^2 - b^2)\,\mathrm{N};$$

where N denotes the mass of the double conoid.

Con

137. In the cone, the radius of whose base is r, and perpendicular height b, if V be at the vertex, k will be $= \frac{3}{4}b$, and y being $= \frac{rx}{b}$,

we shall have

$$\tfrac{1}{4} y^4 - (x - k)^2 y^2 = \frac{r^4 x^4}{4 b^4} - \frac{r^2 x^2}{b^2} (x - \tfrac{3}{4} b)^2;$$

consequently, by taking the fluent of

$$\left\{ \frac{r^4 x^4}{4 b^4} - \frac{r^2 x^2}{b^2} (x - \tfrac{3}{4} b)^2 \right\} \dot{x},$$

generated while x from o becomes $= b$, and multiplying the result by $\frac{\pi f m n}{g}$, as in the preceding cases, we have the force (E) $=$

$$\frac{\pi f m n r^2 b}{80 g} (4 r^2 - b^2) = \frac{3 f m n}{80 g} (4 r^2 - b^2) \text{M};$$

$\text{M} = \frac{\pi r^2 b}{3}$ denoting the content of the body.

138. In the solid, consisting of two cones joined together at their bases; if the dimension of each cone be denoted, as in the preceding example, the force (E), by proceeding as above, will be found $=$

$$\frac{f m n}{20 g} (3 r^2 - 2 b^2) \text{N};$$

N denoting the content of the double cone.

139. In the cylinder, whose length is b, and diameter $2 r$, taking $k = \frac{b}{2}$, we have

$$\tfrac{1}{4} y^4 - (x - k^2) y^2 = r^2 \{\tfrac{1}{4} r^2 - (x - \tfrac{1}{2} b)^2\},$$

y being equal to r.

Consequently the fluent of

$$\{\tfrac{1}{4} r^2 - (x - \tfrac{1}{2} b)^2\} \dot{x},$$

generated while x from o becomes $= b$, being

$$\frac{r^2 b}{4} - \frac{b^3}{12};$$

we find, by multiplying that fluent by $\frac{\pi f m n r^2}{g}$, the value of $\text{E} = \frac{f m n}{12 g} (3 r^2 - b^2) \text{M};$

M denoting the content of the cylinder.

The forces denoted by E in the preceding articles are what we have hitherto denominated the resultants of the several forces of the system, and may be considered as motive forces. The corresponding accelerating forces are computed in the following manner:

140. The body being a spheroid, whose centre is C, and whose principal axis PN (fig. 87) is equal to $2 b$, and equitorial diameter AB $= 2 r$; let ϕ be the accelerating force of a particle of the body at the distance g, from the axis about which the body revolves; this axis being a diameter of its equator, when the motive force is such as we have considered above.

Denote C k and k i, by $x - k$ and y as before; and let the absciss k p, and its corresponding ordinate (parallel to the latter axis) in the circle whose radius is k i, be denoted by w and t respectively. Then considering the body as urged to turn about that diameter of its equator, which is at right angles to AB, the accelerative force of every particle in that ordinate will be

$$\frac{\sqrt{\{w^2 + (x - k)^2\}}}{g} \times \text{F};$$

and the motive force of all the particles in the same ordinate will be

$$\frac{\sqrt{\{w^2 + (x - k)^2\}}}{g} \times \text{F} t \dot{w} \dot{x} = \frac{\sqrt{\{w^2 + (x - k)^2\}}}{g} \times \text{F} \dot{w} \dot{x} \sqrt{(y^2 - w^2)};$$

to which, by the property of the lever, a motive force equal to

$$\frac{w^2 + (x - k)^2}{g^2} \times \text{F} \dot{w} \dot{x} \sqrt{(y^2 - w^2)},$$

acting at the distance g from the centre C, would be equivalent.

Therefore, considering x and y as constant, and w only as variable, $\frac{4 \text{F} \dot{x}}{g^2}$, multiplied by the whole fluent of

$$\dot{w} \sqrt{(y^2 - w^2)} \times (w^2 + (x - k)^2),$$

will denote a force which, acting at the distance g from C, would be equivalent to the motive force of all the particles in the circular section h i, whose radius is k i, and thickness the indefinitely small quantity $\dot{x}$.

This fluent is expressed by

$$\tfrac{1}{4} \pi \{\tfrac{1}{4} y^4 - (x - k)^2 y^2\} = \frac{\pi r^2}{4 b^2} (2 b x - x^2) \times \left\{ \frac{r^2}{4 b^2} (2 b x - x^2) + (x - b)^2 \right\},$$

in our spheroid, k being taken equal to NC $= b$; consequently

$$\frac{\pi r^2 \text{F}}{b^2 g^2} \times \int \dot{x} (2 b x - x^2) \times \left\{ \frac{r^2}{4 b^2} (2 b x - x^2) + (x - b)^2 \right\}$$

will denote the motive force which, acting at the distance g from C, at right angles to a radius drawn from it, would be equivalent to the whole motive force, urging the spheroid to turn as above mentioned. Such equivalent force will therefore be $=$

$$\frac{\text{F}}{5 g^2} (r^2 + b^2) \text{S}; \text{ S being the solidity};$$

which being put equal to

$$\frac{f m n}{5 g} (r^2 - b^2) \text{S},$$

the value for the same force, as determined (art. 133), we find

$$\text{F} = f g m n \times \frac{r^2 - b^2}{r^2 + b^2};$$

which will become

$$\text{F} = \frac{g m n e^2}{a^2} \times \frac{r^2 - b^2}{r^2 + b^2},$$

if f be taken $= \frac{e^2}{a^2}$, agreeably to the computation (art. 131).

Or we shall have

$$\text{F} = \frac{c d}{r} \times \frac{r^2 - b^2}{r^2 + b^2},$$

if we take $1 : e :: m : d :: n : c$, a and g being each equal to r.

141. It is evident, from the deductions in the preceding article, that the circle h i, whose radius is k $i = y$, being the section of any solid of revolution, whose principal axis coincides with CK, if C be the centre of gravity of the body, and C $k = x - k$,

$$\frac{\pi \text{F} \dot{x}}{g^2} \times \left\{ \tfrac{1}{4} y^4 + (x - k)^2 y^2 \right\}$$

will denote a force which, acting at the distance g from C, as above described, would be equivalent to the motive force of the particle in the said circular section, whose radius is y, and its thickness the indefinitely small quantity $\dot{x}$.

It follows, therefore, from article 133, and what is stated above, that

$$\frac{\pi F}{g^2} \times \int \dot{x} \left\{ \tfrac{1}{4} y^4 + (x - k)^2 y^2 \right\}$$

will be equal to

$$\frac{\pi f m n}{g} \times \int \dot{x} \left\{ \tfrac{1}{4} y^4 - (x - k)^2 y^2 \right\};$$

whence we have this theorem:

$$F = f g m n \times \frac{\int \dot{x} \{ \frac{1}{4} y^4 - (x - k)^2 y^2 \}}{\int \dot{x} \{ \frac{1}{4} y^4 + (x - k)^2 y^2 \}};$$

the value of *k*, and of the fluents, being so taken as to comprehend the whole of the body under consideration, according to the examples given in the preceding articles.

Particular deductions.

142. Proceeding on these principles, we find in the

hemi-spheroid $F = \frac{c d}{r} \times \frac{64 r^2 - 19 b^2}{64 r^2 + 19 b^2}$,

parabolic conoid $F = \frac{c d}{r} \times \frac{3 r^2 - b^2}{3 r^2 + b^2}$,

double conoid $F = \frac{c d}{r} \times \frac{r^2 - b^2}{r^2 + b^2}$,

cone $F = \frac{c d}{r} \times \frac{4 r^2 - b^2}{4 r^2 + b^2}$,

double cone $F = \frac{c d}{r} \times \frac{3 r^2 - b^2}{3 r^2 + b^2}$,

cylinder $F = \frac{c d}{r} \times \frac{3 r^2 - b^2}{3 r^2 + b^2}$,

a and *g* being taken equal to *r*, $f = \frac{e^2}{r^2}$;

$1 : e :: m : d :: n : c$, and reference being had to the respective articles for the quantities denoted by *e*, *m*, *n*, *b* and *r*.

Now as it appears from these formula, that $F = o$, when, in the

hemi-spheroid, the height $b = \frac{8 r}{\sqrt{19}}$;
parabolic conoid, ditto $b = r \sqrt{3}$;
double conoid, the half-length $b = r$;
cone, the height $b = 2 r$;
double cone, the half-length $b = r \sqrt{\frac{3}{2}}$;
cylinder, the length $b = r \sqrt{3}$.

It is manifest, therefore, that each of those bodies (with respect to its own particles) will, undisturbedly, revolve about any axis whatever, passing through its centre of gravity, the same as the sphere; it is also obvious that by means of what is done above, that other bodies, which are solids of revolution, or frustrums of such solids, of various forms, may be found having the same property; nothing farther being requisite than that the dimensions of the body be so proportioned, that

$$\int \dot{x} \{ \tfrac{1}{4} y^4 - (x - k)^2 y^2 \} = o;$$

k being taken equal to the distance from the centre of gravity of the body to one extremity of the axis, about which the solid is generated, and *x* being considered as flowing from *o*, till it becomes equal to the whole length of that axis.

Consequently, *every axis of such body passing through its centre of gravity is a permanent axis of rotation.*

143. In a preceding article of this section, we have slightly referred to a celebrated theorem of Frisi, which has given great facility to the theory of the rotation of bodies in free space; and by means of which, easy and elegant solutions may frequently be obtained of the most difficult theorems. The enunciation of this theorem may be stated as follows.

Frisi's theore

"If a body revolve about an axis passing through its centre of gravity, with the angular velocity U, while this axis is carried round another axis, also passing through its centre of gravity, with the angular velocity *u*; these two motions compose a motion of every particle of the body round a third axis, lying in the plane of the other two, and inclined to each of the former axes in angles whose sines are inversely as the angular velocities round them; and the angular velocity *v* round this new axis, is to that round one of the primitive axes, as the sine of inclination of the two primitive axes is to the sine of the inclination of the new axis to the other primitive axis."

Let us conceive a body to turn about an axis AGA′ (fig. 88), passing through its centre of gravity G, with the angular velocity U, while this axis is carried round another axis BGB′, with the angular velocity *u*; and if we take

GD : GE :: U : *u*,

(the points B and E being taken on that side of the centre where they are moving towards the same side of the plane of the figure) and the line DE be drawn, the whole and every particle of the body, will be in a state of rotation about a third axis CGC′, parallel to DE, lying in the plane of the other two; and the angular velocity *v*, about the axis CGC′, will be to U and to *u*, as DE is to GD and to GE.

For, let P be any particle of the body, and suppose a spherical surface, whose centre is G, to pass through P, and draw PR perpendicular to the plane of the figure: then is PR the common section of the circle of rotation IPI′ round the axis AA′, and the circle KPK′ of rotation about the axis BB′. Let F and O be the centres of these circles of rotation, and II′ and KK′ their diameters; draw the radii PF, PO, and the tangents PM, PN: these tangents are in a plane MPN, which touches the sphere in P, and cuts the plane of the axis in a line MN, to which a line drawn from G through R would be a perpendicular.

Now, conceive PN to represent the velocity of rotation of the point P about the axis BB′, while PF′ denotes its velocity of rotation about AA′, and complete the parallelogram PNT′F′: then is PT′ the direction and velocity of the resultant of the composition of PN and PF′, and is therefore necessarily in the same plane as these components. Let perpendiculars F′F, T′T, be drawn to the plane of the axis, and the parallelogram PNT′F′ will be orthographically projected on that plane, its projection being also a parallelogram RNTF.

If now the diagonal RT be drawn; because PR is perpendicular to the plane of the primitive axis, PRT′T is so likewise; and consequently the compound motion PT′ is in a plane of a circle of revolution about some axis situated in the plane of the other two. Produce TR and draw GC intersecting it perpendicularly in H, and let LPL′ be the circle of rotation, its diameter being LL′ = 2 LH; then is PT′ a tangent, and perpendicular to PH; and it will meet RT in some point Q of the line MN. The particle P is in a state of rotation about the axis CGC′, and its velocity is to the velocities round AA′ or BB′ as PT′ to PF or to PN.

Since the tangent PN is perpendicular to OP, and PR is perpendicular to ON; therefore

$$OP : PN :: PR : RN = \frac{PN \cdot PR}{OP}.$$

But the velocity of P about the axis BB' $= u \cdot OP$; whence

$$RN = \frac{u \cdot OP \cdot PR}{OP} = u \,.\, PR.$$

And in like manner $RF = U \cdot PR$; consequently,

$$RF : RN :: U : u :: GD : GE.$$

But

$$NT : RN :: \begin{cases} \sin NRT : \sin NTR, \\ \sin GED : \sin GDE. \end{cases}$$

Again, since NR is perpendicular to EG, and NT is parallel to IF, and therefore perpendicular DG, we have the angle RNT = EGD. Hence TR is perpendicular and CC' parallel to ED, and the rotation of the particle P is about an axis parallel to ED. Also, since RN, RF, RT, are as the velocities u, U, v, about these different axes, and vary respectively as EG, DG, DE, we have

$$v : U : u :: ED : DG : GE;$$

which are the relations stated in the proposition.

144. This theorem or general principle bears a strong analogy to that which relates to the composition of forces demonstrated in (art. 25, *Statics*), and like that, gives rise to many important and interesting deductions.

In the first place we learn from it, that if every particle of a body, whether solid or fluid, receive at the same instant two separate impulses, the one competent to the production of a motion of the particle round an axis, with a certain angular velocity, and the other competent to the production of a rotatory motion about another axis, with a certain velocity: the combined effect of all these impulses will be a motion of the whole system about a third axis given in position, with an angular velocity which is also given; and this motion will obtain, without any separation or disunion of the parts, except such as may be occasioned by the action of the centrifugal forces resulting from the rotation.

Again, if a body be turning round any axis, and every particle, in one and the same instant, be affected precisely with such an impulse as is competent to produce a given angular velocity round another axis, the body will revolve about a third axis given in position, and with a given angular velocity.

Lastly, when a rigid body acquires a rotation about an axis by an impulse on one part of it, and either at the same time, or afterwards, receives an impulse in any part, which alone would produce a certain rotation about another axis, the joint effect of these impulses will be a rotation about a third axis. For when the rigid body acquires a motion about an axis, not by the simultaneous impulse of the precisely competent force on each particle, but by an impulse on one part; there has been propagated to every particle (by means of the cohesive force) an impulse precisely competent to the production of that motion which the particle actually acquires; and when a rigid body, already turning round an axis AA', receives an impulse which makes it actually turn about another axis CC', there has been propagated to each particle a force precisely adequate to the production, not of the motion, but of the *change* of motion which takes place, in that particle; that is, a force which, when compounded with the inherent force of its primitive motion, produces the new motion; that is, a force which alone would have caused it to turn about a third axis BB', with a rotation making the other component of the actual rotation about CC'.

§ XXV. *Of the impact of bodies.*

Definitions.

145. All bodies in nature are more or less compressible: and when from any cause they are compressed, they will have a greater or less tendency to restore themselves to their original form; and this tendency is what we term *elasticity*. A body is said to be *perfectly elastic* when it regains its former figure as soon as the cause which compressed it is removed; when the body only in part resumes its original form, it is said to be *imperfectly elastic*. If a body have no tendency after compression to restore itself to its primitive form, it is called a *soft body*; and if on the contrary, we suppose it to resist entirely the compression, it is called a *hard body*: but in regard to our present enquiry, these are both brought under one general class, viz. *non-elastic bodies*. We know of no substances that are either perfectly hard, soft, or elastic; therefore all bodies, properly speaking, fall under that class which we have denominated *imperfectly elastic*. But still, as there are many which approach very nearly in character to non-elastic bodies, and others to a state of perfect elasticity, it is customary to treat only of these two extreme cases: we propose, however, in what follows, to make a slight reference to the impact of imperfectly elastic bodies, while investigating the circumstances appertaining to the two former classes.

The impact of bodies non-elastic.

146. In order to begin with the most simple case, we shall suppose the bodies to be perfectly non-elastic; that they are homogeneous spheres, of which the two centres move in the same right line, and that the several points in them are describing lines parallel to that of their centres. According to this supposition, every thing being the same on all sides of the right line in which the centres move, it is evident that their impact ought not to produce any rotation in either body; so that after the blow, their different points will continue to describe parallel right lines, with a velocity common to both bodies: this velocity we here propose to determine; that with which they first begin to move, and the ratio of the masses, being given.

Let A and A' (fig. 89) represent two homogeneous *non-elastic* spheres; C and C' their centres; BD the right line on which the two centres move. Let us suppose that the two bodies are moving the same way; from B towards D; and that the velocity of A is greater than that of A', so that the former will untimately overtake the latter. Now when two compressible bodies meet, they are mutually compressed, in consequence of the difference of their velocities; and this compression will continue till the two velocities are become equal, which will take place in a greater or less time, according as the bodies are more or less compressible; this change, therefore, would be instantaneous, if the bodies were perfectly hard: still, however, it is necessary always to admit that in the hardest bodies we may find a certain degree of compressibility, and to suppose that in impinging on each

Mechanics. other they are compressed during a time, although that time is inappreciable.

Thus in the impact, the velocity of A will be diminished, and that of A′ augmented, till the two become equal, after which the two bodies will discontinue their action on each other; for since we have supposed them void of elasticity, they will each retain the form they have assumed in the compression, and will continue to move with a common velocity, the same as if they formed only one and the same body.

Let us denote this common velocity by u, and call the velocity of A before impact equal to v, and that of A′ equal to v'; we may also denote the masses respectively by m and m'. Then we shall have $v - u$ for the velocity lost by A in the impact, and $u - v'$ for that gained by A′; whence, according to the principle of D'Alembert, an equilibrium ought to obtain between the two masses m and m' when impressed with the velocities $v - u$, and $u - v'$; that is, we shall have the equation

$$m(v-u) = m'(u-v');$$

by means of which we readily determine the value of u in terms of the other quantities v, v', m, and m'; that is, we find

$$u = \frac{mv + m'v'}{m + m'}. \qquad (1)$$

Motion after impact. *Hence, the common velocity of two non-elastic bodies after impact, when they both move the same way, is found, by dividing the sum of the products of each body into its respective velocity, by the sum of the bodies.*

When the bodies move in contrary directions; that is, when they meet each other, then v' must be accounted negative, or, at least, v and v' must be supposed to be affected with contrary signs; so that in this case, we have

$$u = \frac{mv - m'v'}{m + m'}. \qquad (2)$$

Therefore, when the bodies meet each other, we must divide the difference of the products of each mass into its velocity, by the sum of the bodies, for the common velocity after impact.

The latter case, however, is obvioulу included in the first, by considering the velocity of v' to be negative, as stated above.

As to the direction of the motion after impact, it will obviously depend upon the magnitudes of the products mv, $m'v$; that is, if $mv \; 7 \; m'v'$, the motion will continue in the direction of m, and the contrary if $m'v'$ be greater than mv.

The same, otherwise. 147. It appears from the above, that the only effect of the mutual impact of two non-elastic bodies is to unite their masses into one; which is found to have the same velocity as if the forces, which put these two bodies separately into motion, acted simultaneous on the total mass, in the respective direction of each.

For let F be the force which impressed on the mass m, the velocity v, or $F = mv$; and let F′ be the force which impressed on the mass m', the velocity v', or $F' = m'v'$. Then, from what have been shown (art. 11), the force F acting on the mass $m + m'$, would have impressed a velocity $= \frac{mv}{m + m'}$, and the force F′ would in like manner have produced the velocity $\frac{m'v'}{m + m'}$.

The velocity, therefore, due to the two forces will be

$$u = \frac{mv}{m + m'} + \frac{m'v'}{m + m'} = \frac{mv + m'v'}{m + m'},$$

when the forces act both in the same way; and

$$u = \frac{mv}{m + m'} - \frac{m'v'}{m + m'} = \frac{mv - m'v'}{m + m'},$$

when they act in opposite directions. In the latter case, the velocity will obviously take place in the direction of the greater force; and be the same as that which would arise from the impact of the bodies on each other, as determined above.

If one of the masses, as for example m', were at rest before the impact, then the velocity denoted by v' will be zero; and substituting $v' = 0$, into either of our equations (1) or (2), we shall have

$$u = \frac{mv}{m + m'} \qquad (3)$$

Of e bodi 148. Let us now examine the case of *elastic bodies*, beginning with that in which a body of this kind is made to impinge upon a hard and fixed plane. It has been demonstrated experimentally, and it is involved in our definition of elastic bodies, that they are compressible; and that they have a tendency to restore themselves to their original form as soon as the compression ceases. Hence the moment an elastic body comes in contact with the plane on which it impinges, it begins to experience a compression, in consequence of which its velocity is diminished, and is ultimately entirely destroyed; at this instant the natural elasticity begins to operate, and the velocity of the body is gradually restored to it again by the same degrees as it was destroyed: so that supposing it perfectly elastic, it will, at the moment it quits the plane, have acquired the very same velocity it had when it first came in contact with it. Whence it appears, that the natural and general effect of the elasticity of bodies is to restore to them, after impact, the same velocities they had before, but in an opposite direction.

Hence, again, it follows that a heavy elastic body falling freely from the action of gravity in a vacuum, upon a horizontal plane, will have its velocity first destroyed by the action of the plane; after which the elasticity will restore to it its former velocity, in consequence of which it will rise to the same height from whence it fell.

Ang inci = refle If the body, instead of impinging perpendicularly on the plane, meet it in an oblique direction, as BA (fig. 90); then calling v the velocity, and denoting the angle BAN by α, we may resolve v into the two velocities $v \sin \alpha$, and $v \cos \alpha$, of which the latter, being parallel to the plane, will not be affected by the impact, but will remain the same in direction and of the same magnitude after impact as before; while the velocity $v \sin \alpha$ may be considered as a velocity perpendicular to the plane; this, therefore, will be restored to the body in the opposite direction AE from the impact, and the velocity after the stroke will consist of the two velocities $v \sin \alpha$, and $v \cos \alpha$; consequently the body will rebound from the plane at the same angle at which it came in contact with it, that is the angle BAN = DAO; hence, when an elastic body impinges obliquely on a plane, *the angle of incidence is equal to the angle of reflection.*

If the body be only imperfectly elastic, then a part of its velocity will be actually destroyed, and it will ascend to a height greater or less, in comparison with that it fell from, according to its elastic quality; and we have thus a very simple means of estimating the degree of elasticity of different bodies.

149. Let a denote the space through which a body falls in the first instance, and let us denote the ratio of perfect elasticity to the imperfect elasticity of the body by $1 : m$; then the space through which the body will re-ascend will be as $1^2 : m^2$; for the velocity is as $1 : m$, and the spaces are as the squares of the velocities; and therefore, conversely, if the height through which a body falls, and the space it re-ascends, be given, the ratio of perfect to the imperfect elasticity of that particular body will be determined, being as the square roots of the two spaces. Thus, for example, if a body, after falling through 16 feet, impinge on a hard and fixed plane, and rebounds only through 9 feet, we should say that the elasticity of that body was to perfect elasticity, as $\sqrt{9} : \sqrt{16}$, or as $3 : 4$.

Suppose a body has its elasticity to that of perfect elasticity, as $m : 1$, and let the height from which it falls in the first instance be a; to find the whole space that it will describe before its motion ceases, and the time of that motion:

Since the velocity with which the body rebounds is as the elasticity, viz. $1 : m$, and the spaces being as the squares of the velocity, we shall have for the whole space

$$a + 2m^2 a + 2m^4 a + 2m^6 a, \text{ \&c. } ad\ infin.$$

or $$S = 2a\,(1 + m^2 + m^4 + m^6 +, \text{ \&c.}) - a,$$

the ratio in this series being m^2, the sum of

$$1 + m^2 + m^4 + m^6 +, \text{ \&c.} = \frac{1}{1 - m^2};$$

whence $$S = \frac{2a}{1 - m^2} - a = \frac{1 + m^2}{1 - m^2}\,a,$$

the space required.

The time is as the velocity; that is, as $1 : m$; and the time of falling through a feet is $t = \sqrt{\frac{a}{g}}$; therefore, the time of ascent and descent, after the first impact, will be $2tm$, $2tm^2$, $2tm^3$, &c.; whence we shall have, for the whole time,

$$T = 2\sqrt{\frac{a}{g}}\left\{1 + m + m^2 + m^3 +, \text{ \&c.}\right\} - \sqrt{\frac{a}{g}}.$$

Here the ratio of the series being m, the infinite sum $=$

$$\frac{1}{1 - m};$$

whence

$$T = 2\sqrt{\frac{a}{g}} \times \frac{1}{1 - m} - \sqrt{\frac{a}{g}};$$

or

$$T = \frac{1 + m}{1 - m} \times \sqrt{\frac{a}{g}}.$$

As a numerical example, let the ratio of $1 : m$ be as $1 : \frac{15}{16}$, and the height $a = 100$ feet; then we should have for the whole space

$$S = \frac{1^2 + (\frac{15}{16})^2}{1 - \frac{15}{16}} \times 100 = 1551 \text{ feet},$$

and

$$T = \frac{1 + \frac{15}{16}}{1 - \frac{16}{16}} \times \sqrt{\frac{100}{16\frac{1}{12}}} = 77{\cdot}19 \text{ seconds}.$$

150. Let us now consider the case of two perfectly elastic bodies, A and A′, impinging directly on each other; the ratio of their masses as m to m', and their velocities before impact v and v'; to find their respective velocities after the stroke:

Impact of elastic bodies.

Suppose, in the first instance, as the simplest case, the two bodies to meet each other; that their velocity and masses are such that the two products $m\,v$ and $m'\,v'$ are equal to each other; and the point E (fig. 91) that point of the line BD where they come in contact. Conceive through this point a plane MN to pass perpendicular to the right line BD, which plane we may consider as fixed (since it is equally acted upon by both bodies), and that they are each compressed against it, losing their velocities v and v'; which are afterwards restored by their respective elasticities, exactly in the same manner as in the fixed plane (art. 148); the two bodies, therefore, in this case, will rebound after the stroke with velocities precisely equal to those with which they approached and came in contact with each other.

Now, whatever may be the velocities v and v' before the stroke, we may conceive each of those velocities to be made up of two others; the one of which shall be common to both, and which may be assumed at pleasure, and the other equal to the difference between the actual velocities and that which we have supposed to be arbitrarily assumed. Let us denote the latter by u, and, instead of the velocity v, substitute the two velocities u, and $v - u$; and instead of v', the two u and $v' - u$, or $-(u - v')$. Let also V and V′ denote the velocities of the two bodies after impact; and in order to reduce the two cases of the bodies following or meeting each other into one, we shall consider the velocities which are made in the direction of the mass m as positive, and what are made in the opposite sense as negative; so that v will always be supposed to denote a positive quantity, while u, v', V and V′, may be either positive or negative, according to circumstances. After this, it is evident that the bodies are in the same state, as if they met each other with the velocities $v - u$ and $u - v'$, while the right line BD, on which the centres move, were itself in motion, with the velocity u; the motion of this right line will obviously have no influence on the impact of the two bodies.

Let us suppose the velocity u, which we have seen may be assumed at pleasure, to be taken such that

$$m\,(v - u) = m'\,(u - v');$$

this gives

$$u = \frac{m\,v + m'\,v'}{m + m'}.$$

Now we have seen that when two bodies meet with velocities, such, that the products of each mass into its respective velocity, are equal to each other; the bodies will rebound with the same velocities as those with which they met; the body A, therefore, abstracting from the supposed motion of the line BD, will rebound with a velocity equal to $v - u$, and the body A′ with the velocity $u - v'$; and consequently, their actual velocity, introducing again the velocity u, will be for A, after the stroke u diminished by $v - u$; and that of A′ equal to u augmented by $u - v'$; that is, we shall have

$$V = u - (v - u) = 2u - v,$$
$$V' = u + (u - v') = 2u - v';$$

or $$V = 2\frac{m\,v + m'\,v'}{m + m'} - v;$$

and
$$V' = 2\,\frac{m\,v + m'\,v'}{m + m'} - v';$$
which are equivalent to
$$\left.\begin{aligned} V &= \frac{(m - m')\,v + 2\,m'\,v'}{m + m'}, \\ V' &= \frac{(m' - m)\,v' + 2\,m\,v}{m + m'}. \end{aligned}\right\} \quad (4)$$

Motion after impact.

151. It will be observed, that in these formulæ, the values of v' and v are both supposed to be estimated in the same direction, viz. in the direction of the motion of the mass m; and therefore, when the bodies meet each other, v' must be accounted negative; or if we suppose it positive, we must change the above into the two following, viz.
$$\left.\begin{aligned} V &= \frac{(m - m')\,v - 2\,m'\,v'}{m + m'}, \\ V' &= \frac{(m - m')\,v' + 2\,m\,v}{m + m'}. \end{aligned}\right\} \quad (5)$$

And if one of the bodies, as for example A′, or the mass m', were at rest before impact, then we should have $v' = o$, and the formulæ would become
$$\left.\begin{aligned} V &= \frac{(m - m')\,v}{m + m'}, \\ V' &= \frac{2\,m\,v}{m + m'}. \end{aligned}\right\} \quad (6)$$

Particular deductions.

These two cases, however, are obviously both involved in equation (4), by giving to v' its requisite value; that is, considering it as positive, negative, or zero, as the case may require.

We may further observe, that, as an immediate consequence of our formulæ N° (4), it follows, that when $m = m'$, the values of V and V′, become $V = v'$ and $V' = v$; that is, in the particular case in which the masses are equal, each body will rebound back after impact, with the velocity which the other had before it.

And, therefore, if one of the bodies, as for example A′, were at rest before the impact, that is, if we had $v' = o$, we should then have $V = o$ after impact, and $V' = v$; the body A will therefore remain at rest after the stroke, and the other body A′ will be urged forward with the velocity A had before impact.

It follows also from our first determined values of V and V′, viz.
$$V = 2\,u - v, \text{ and } V' = 2\,u - v';$$
that
$$V - V' = v' - v = -(v - v').$$

Whence it appears that, in all cases, the relative velocity of the two bodies, after impact, is equal (but affected with a contrary sign) to the relative velocity before the impact; understanding, in general, by *relative velocity*, the difference of the absolute velocities of the two bodies estimated in the same direction; or it may otherwise be defined, that velocity with which the bodies approach or recede from each other.

152. Let us illustrate the preceding results by a few numerical examples.

Examples.

Exam. 1. The masses of two perfectly elastic balls, A and A′, are as 6 to 5, and their velocities in the same direction 7 and 4; to determine the velocity of each after impact:

Here $m = 6$, $m' = 5$, $v = 7$, $v' = 4$; whence our equations (4), viz.
$$V = \frac{(m - m')\,v + 2\,m'\,v'}{m + m'},$$
$$V' = \frac{(m' - m)\,v' + 2\,m\,v}{m + m'};$$
become, when expressed numerically,
$$V = \frac{(6 - 5)\,7 + 2\cdot5\cdot4}{6 + 5} = \tfrac{47}{11} = 4\tfrac{3}{11},$$
$$V' = \frac{(5 - 6)\,4 + 2\cdot6\cdot7}{6 + 5} = \tfrac{80}{11} = 7\tfrac{3}{11}.$$
That is, A will move after impact with a velocity of $4\frac{3}{11}$, and A′ with a velocity $7\frac{3}{4}$ in the same direction.

Exam. 2. Let now two bodies, A and A′, whose masses are as 3 to 5, *meet* each other with velocities which are as 4 to 7; to determine their motion after impact:

Here $m = 3$, $m' = 5$, $v = 4$, and $v' = -7$; whence our formulæ in numbers become
$$V = \frac{(3 - 5)\,4 - 2\cdot7\cdot5}{3 + 5} = \frac{-8 - 70}{8} = -9\tfrac{3}{4};$$
$$V' = \frac{-(5 - 3)7 + 2\cdot3\cdot4}{3 + 5} = \frac{-14 + 24}{8} = 1\tfrac{1}{4}.$$

Exam. 3. Suppose two bodies, A, A′, the masses being as $1 : r$; of which the latter is at rest, and the other moving with the velocity a; to determine the motion of both after impact:

Here we have $v = a$, $v' = o$, $m = m$ and $m' = r\,m$; whence
$$V = \frac{(1 - r)\,m\,a}{(1 + r)\,m} = \frac{1 - r}{1 + r}\,a;$$
$$V' = \frac{2\,m\,a}{(1 + r)\,m} = \frac{2}{1 + r}\,a.$$

The velocity of each body therefore, after impact, will have a determined ratio to the velocity of the first body; and hence, if we conceive a series of balls, all in geometrical progression, and at rest, to be struck by another, forming the last ball of the series, moving with the velocity a, we shall have, for the successive velocities of the several balls, after being each struck by the preceding one,
$$\frac{2\,a}{1 + r},\ \frac{2^2 a}{(1 + r)^2},\ \frac{2^3 a}{(1 + r)^3},\ \frac{2^4 a}{(1 + r)^4},\ \&\text{c. } \frac{2^n a}{(1 + r)^n};$$
the last term answering to any number (n) of quiescent balls; or if the whole number of balls, including the first projected ball, be called n, then the velocity of the last ball will be
$$\frac{2^{n-1} a}{(1 + r)^{n-1}}.$$
It is obvious from this expression, that if $r = 1$, or the balls are all equal, that we shall have
$$\frac{2^{n-1} a}{(1 + r)^{n-1}} = a;$$
or the velocity of the last ball will be the same as the first.

If r be greater than unity, that is, if the balls form an increasing series, then the successive velocities will diminish; the denominator of the fraction being greater than its numerator; but if r be less than unity, or if the balls decrease from the beginning, then the successive velocities all increase; and by continuing the series, the last ball may be propelled with any velocity at pleasure.

153. We have hitherto principally considered the circumstances attending the impact of bodies, either perfectly elastic, or perfectly non-elastic: let us now bestow a few paragraphs on that class which are imperfectly elastic.

If we denote the ratio of perfect to imperfect elasticity, by $1 : r$, and if we suppose, as in (art. 150), two bodies A and A′ of this class to meet each other, having their masses such, that if they were perfectly non-elastic, they would remain at rest; that is, (retaining our former notation), such that $m v = m' v'$.

With these values of the two products, we have seen, in the article above referred to, that supposing the bodies perfectly elastic, each would rebound with a velocity equal to that they met with; but in our case, from what has been stated (art. 149), it appears, that the bodies, instead of rebounding with the velocities v and v', will only acquire the velocities $r v$ and $r v'$, r being a fraction from the nature of the above substitution.

In the case where the velocities are unequal, we may, as in (art. 150), conceive our two velocities, v and v', to be composed each of two velocities, viz.

$$v, \text{ of } u \text{ and } v - u,$$
$$v', \text{ of } u \text{ and } v' - u, \text{ or } -(u - v'),$$

and determine u such, that

$$m(v - u) = m'(u - v'); \qquad (a)$$

then, considering the bodies as meeting with these two velocities, while the line in which their centres of gravity move is carried along uniformly with the velocity u, we shall find that the bodies will, independently of this last motion, rebound with a velocity of

$$r(v - u), \text{ and } (u - v')r.$$

Whence the velocity V and V′, of the two bodies A and A′, will be

$$V = u - r(v - u) = (1 + r)u - r v,$$
$$V' = u + r(u - v') = (1 + r)u - r v'.$$

And since equation (a) gives

$$u = \frac{m v + m' v'}{m + m'},$$

we have

$$V = (1 + r)\frac{m v + m' v'}{m + m'} - r v,$$
$$V' = (1 + r)\frac{m v + m' v'}{m + m'} - r v';$$

which, when reduced, give the following results:

$$\left.\begin{aligned} V &= \frac{(m - m' r)v + (1 + r)m' v'}{m + m'}, \\ V' &= \frac{(m' - m r)v' + (1 + r)m v}{m + m'}. \end{aligned}\right\} \qquad (7)$$

These two formulæ, if we suppose $r = 1$, that is, when the bodies are perfectly elastic, coincide exactly with our equations (N° 4), as they ought to do.

As a numerical example, let the two bodies A, A′, be imperfectly elastic, the ratio to perfect elasticity being as 3 to 4, their masses as 4 to 5, and their velocities 6 and 7, also suppose the bodies to meet each other; to determine their velocities after impact:

Here,

$$r = \tfrac{3}{4}\; m = 4,\; m' = 5,\; v = 6 \text{ and } v' = -7;$$

whence

$$V = \frac{(4 - \frac{15}{4})6 + (1 + \frac{3}{4})5. - 7}{4 + 5} = -6\tfrac{23}{36},$$
$$V' = \frac{-(5 - 3).7 + (1 + \frac{3}{4})4.6}{4 + 5} = 3\tfrac{1}{3}.$$

Oblique impact.

154. In the preceding articles we have only considered the case in which the bodies impinge directly on each other; it remains, therefore, now for us to investigate the motion arising from oblique impact.

Let A, A′, (fig. 92) denote two elastic bodies, which impinge on each other in the point R, after approaching in the oblique directions BA and DA′; to determine their velocities after the stroke:

Let LM be the plane tangent common to the two balls at their point of contact; then denoting their velocities and masses as before, and calling the angle BAE $= \alpha$, and DAE $= \beta$, we may resolve the velocity v into the two velocities $v \sin\alpha$, $v \cos\alpha$; and v' into the two $v' \sin\beta$ and $v' \cos\beta$. The velocities $v \cos\alpha$ and $v' \cos\beta$, being parallel to the plane tangent LM, will not be affected by the impact, but will continue the same after the stroke as before it; and we must consider the impact of the bodies as if they impinged on each other directly with the velocities $v \sin\alpha$, $v' \sin\beta$; if, therefore, these values be substituted in our general formulæ (N° 4), or the still more general equations (N° 7), (the latter involving the case of imperfect elasticity) we may compute the motion of A and A′, in the directions AP, A′P′; that is, as resulting from the impact; after which, combining these velocities respectively with the velocities $\cos\alpha$ and $\cos\beta$, we shall have the velocities required.

That is, retaining the above notation, and denoting the masses as before by m and m', and the ratio of perfect to imperfect elasticity by $1 : r$; we shall have, for the velocities of the bodies as arising from the impact,

$$\left.\begin{aligned} V &= \frac{(m - m'r)v \sin\alpha + (1 + r)m' v' \sin\beta}{m + m'}, \\ V' &= \frac{(m' - m r)v' \sin\beta + (1 + r)m v \sin\alpha}{m + m'}. \end{aligned}\right\} (8)$$

Denoting now the actual velocities after impact by U and U′, we have

$$\left.\begin{aligned} U &= \surd(V^2 + v^2 \cos^2\alpha), \\ U' &= \surd(V'^2 + v'^2 \cos^2\beta). \end{aligned}\right\} \qquad (9)$$

The above equations general.

These equations (8) and (9) will be found to involve in them all the preceding equations relative to the impact of bodies, whether they be non-elastic, or perfectly, or imperfectly elastic; or whether the impact be direct or oblique.

Suppose, for instance, that the bodies were perfectly non-elastic, and that they impinged upon each other directly. In this case $r = o$, the angles α and β are each $= 90°$; therefore $\sin\alpha = 1$, $\sin\beta = 1$, $\cos\alpha = o$, and $\cos\beta = o$; these substitutions give

$$V = \frac{m v + m' v'}{m + m}$$
$$V' = \frac{m v + m' v'}{m + m'};$$

which are equal, and agree with our equation (1); and this, as we have shewn (art. 147), involves, under certain relations of m, m', v and v', equations (2) and (3).

Let now the bodies be perfectly elastic, and impinge on each other directly; then we shall have as before $\sin\alpha = 1$, $\sin\beta = 1$, and in this case $r = 1$. These substitutions being made, give

$$V = \frac{(m - m')v + 2 m' v'}{m + m'},$$
$$V' = \frac{(m' - m)v' + 2 m v}{m + m'}.$$

These are our equations N° (4), and which we have shown involve, under certain relations of the several quantities, equations (5) and (6).

If the impact be supposed direct, but the bodies imperfectly elastic, then since $\sin\alpha = 1$ and $\sin\beta = 1$, we shall have

$$V = \frac{(m - m'r)v + (1 + r)m'v'}{m + m'},$$

$$V' = \frac{(m' - mr)v' + (1 + r)mv}{m + m'};$$

which are our equations (N° 7).

Therefore all the circumstances of the impact of homogeneous spheres, whether non-elastic, perfectly, or imperfectly elastic, or whether the impact be direct or oblique, are involved in equations (8) and (9).

We might have extended our investigations relative to the impact of bodies, to a much greater length, by considering the case of bodies of various forms, and of indefinite number; but as we conceive these to be rather subjects of curiosity than utility, we do not introduce them: the reader, however, who is desirous of examining these cases, will find them developed at considerable length in Poisson's "Traité de Mecanique," tom. ii.

Principle of living forces.

155. The principles of the *conservation of the motion of the centre of gravity*, and *the conservation of living forces*, as it relates to the impact of bodies, may be illustrated in this place.

When we have occasion to value the effect of a machine, or to compare the effect of different machines with each other, it is frequently necessary to consider the product of the mass by the square of its velocity; which product is what is called the *force vive*, *vis viva*, or *living force*.

It is a general principle, when the motion of a system of bodies experiences an abrupt change, that there necessarily results a diminution in the sum of the *living forces* of all the bodies.

Carnot has investigated the circumstances of such a motion, and has deduced the following general theorem, viz. That the diminution is equivalent to the sum of the living forces due to the velocities lost or gained by the bodies.

Loss of, in non-elastic bodies.

In order to illustrate this, in the case of the impact of two non-elastic bodies, we may refer to our equations (1) and (2), which may be reduced to one, viz.

$$(m + m')\,u = m\,v \pm m'\,v'.$$

Now multiplying the two members of this equation by $2\,u$, we obtain

$$(m + m')\,2\,u^2 = (m\,v \pm m'\,v')\,2\,u.$$

Substracting this from the identical equation

$$m v^2 + m' v'^2 + (m + m')u^2 = m^2 v^2 + m' v'^2 + (m + m')\,u^2,$$

there remains

$$m\,v^2 + m'\,v'^2 - (m - m')\,u^2 = m^2\,v^2 + m'\,v'^2 - 2\,u\,(m\,v \pm m'\,v) + (m + m')\,u^2;$$

or, which is the same thing,

$$m\,v^2 + m'\,v'^2 - m\,u^2 - m'\,u^2 = m\,(v - u)^2 + m'\,(u \mp v')^2;$$

this equation obviously agrees with the enunciation of the theorem. For $m\,v^2 + m'\,v'^2$ is the sum of the living forces before the stroke, and $m\,u^2 + m'\,u^2$; the sum after the stroke u being the common velocity: consequently,

$$m\,v^2 + m'\,v'^2 - (m\,u^2 + m'\,u^2)$$

will be the living force lost in the impact, which we have shown to be equal to

$$m\,(v - u)^2 + m'(u \mp v)^2;$$

and this is obviously the living force due to the velocities $(v - u)$ and $(u \mp v)$ lost or gained.

Consei tion of, elastic bodies

156. In elastic bodies there is no loss of the living forces, these being the same both before and after impact.

We have seen (art. 150) that

$$V = 2\,u - v,$$

$$V' = 2\,u - v';$$

whence

$$mV^2 + m'V'^2 = 4\,u^2(m + m') - 4\,u(m\,v + m'v') + m^2v^2 + m'^2v'^2.$$

Now we have, in the same article, determined

$$u = \frac{m\,v + m\,v'}{m + m'};$$

therefore the two first members of the second side of our equation destroy each other, and we have

$$m\,V^2 + m'\,V'^2 = m\,v^2 + m'\,v'^2;$$

that is, the sum of the living forces in elastic bodies is the same both before and after impact.

We have already stated, that we know of no bodies that are perfectly elastic, nor of any that are entirely divested of that property: for this reason, there will always be a loss of the living force in the impact of two bodies; but this loss will always be less than that determined in the case of perfectly non-elastic bodies in the preceding article.

Conse tion o motio the ce of gra

157. The impact of non-elastic and elastic bodies have one common property, which is only a particular case of a general principle in Mechanics, known under the denomination of the *conservation of the motion of the centre of gravity*. This principle consists in this, that the reciprocal action of the bodies of the same system which act the one on the other, in any manner whatever, has no effect, or produces no alteration in the motion of the centre of gravity of the entire system. This conservation, as it relates to the impact of bodies, may be shown as follows:

Preserving still the same notation as in the preceding articles, and, moreover, denoting by s and s' the variable distances of the centres of the bodies from the fixed point B (fig. 89), chosen at pleasure on the line which they describe, and by x, the distance of the centre of gravity of the two bodies from the same point;

Then, observing that the weights are proportional to the masses, we shall have, in order to determine x,

$$(m + m')\,x = m\,s + m'\,s'. \qquad (1)$$

Taking the fluxions of this equation, introducing the time, which we call t, we shall have

$$(m + m')\frac{\dot{x}}{\dot{t}} = m\,\frac{\dot{s}}{\dot{t}} + m'\,\frac{\dot{s}'}{\dot{t}} \qquad (2)$$

Here $\frac{\dot{s}}{\dot{t}}$ and $\frac{\dot{s}'}{\dot{t}}$ express the velocity of the two bodies at the end of the time t, and $\frac{\dot{x}}{\dot{t}}$ expresses the corresponding velocity of their centre of gravity; we may obtain therefore this velocity before the stroke, by replacing $\frac{\dot{s}}{\dot{t}}$ and $\frac{\dot{s}'}{\dot{t}}$ with v and v' in equation (2), and the velocity after impact, by introducing into the same equation V and V'.

Let therefore y and y' be the two motions of the centre of gravity before and after impact, and we shall have

$$y = \frac{m v + m' v'}{m + m'}\ y' = \frac{m \mathrm{V} + m' \mathrm{V}'}{m + m'}.$$

This value of y is obviously the same as we have hitherto denoted by u, whence we have $y = u$.

Now in non-elastic bodies we have also $\mathrm{V} = \mathrm{V}' = u$;

whence $$y' = \frac{m u + m' u}{m + m'} = u;$$

therefore in this case $y' = u$, and consequently $y = y'$.

In elastic bodies $\mathrm{V} = 2u - v$, or $2y - v$, and $\mathrm{V}' = 2u - v'$, or $2y - v'$.

Whence

$$y' = \frac{2my - mv + 2m'y - m'v'}{m + m'}, \text{ or}$$

$$y' = \frac{(m + m') 2u - (mv + m'v')}{m + m'} = 2y - y;$$

that is, $y' = 2y - y$, or $y' = y$.

Our results would be exactly the same if we employed equations (N° 7) which appertain to bodies imperfectly elastic; whence we conclude, that when any two bodies impinge on each other, whether they be perfectly elastic or non-elastic, or imperfectly elastic, the motion of their centre of gravity will be the same before as after impact; and this is, as we have stated above, a general principle in Mechanics, in whatever manner a system of bodies act upon each other; but it would carry us too far to attempt the general demonstration in this place.

§ XXVI. *Solution of miscellaneous dynamic problems.*

Problem I.

158. Suppose a thread ACn C′A′ (fig. 93), having two equal weights A, A′, suspended at its extremities, to be hung over two fixed points or pulleys, C, C′, in the same horizontal line; and that to the middle point of the thread n, equally distant from C, C′, another given weight B is fixed, which is allowed to descend by its own gravity, so as to cause the other two at the same time to ascend; it is proposed to find the law of the velocity by which those weights ascend and descend, abstracting from the resistance of the air, the weight of the thread, and the friction over the fixed points.

Draw nD perpendicular to the horizontal line CC′, then it is obvious that CD = C′D. Let $\mathrm{CD} = a$, $\mathrm{D}n = x$, and $\mathrm{C}n = y$; $g = 16\frac{1}{12}$ feet. Put $\dot{t}$ to denote the indefinitely small interval, or the fluxion of the time which is supposed to flow uniformly; then the velocity of the ascending weight will be $\frac{\dot{y}}{\dot{t}}$; and the variation of the velocity or the accelerating force $= \frac{\ddot{y}}{\dot{t}^2}$.

But the weight A cannot ascend at all, unless its gravity be counteracted, and an additional force impressed upon it equivalent to produce the actual velocity in ascent; therefore the force constantly acting on each particle of A, necessary to cause A to ascend with the velocity $\frac{\dot{y}}{\dot{t}}$, is measured by

$$2g + \frac{\ddot{y}}{\dot{t}^2};$$

and the whole motive force acting on A is, therefore,

$$\left(2g + \frac{\ddot{y}}{\dot{t}^2}\right) + \mathrm{A};$$

an expression which, for the sake of brevity, we shall denote by F.

Now the motive force F, by which A is urged upwards, being derived from the action of the weight B, B itself will be re-acted upon in the direction n C, by an equal motive force F. But the force in the direction n C, may be resolved into two others; viz.

$$\frac{\mathrm{CD}}{\mathrm{C}n} \times \mathrm{F} = \frac{a}{y} \times \mathrm{F}, \text{ parallel to DC,}$$

and

$$\frac{n\mathrm{D}}{n\mathrm{C}} \times \mathrm{F} = \frac{x}{y} \times \mathrm{F}, \text{ in the direction } n\mathrm{D};$$

the latter, therefore, retarding the descent of B.

Because A = A′, and that each of these weights acts in a similar manner on the weight B, therefore the effect of the weight A′ is, in like manner, resolvable into two forces acting on B; one in the direction DC′, and precisely equal to the former force in the direction DC, and therefore destroying its effect; and another force in the direction n D, likewise equal to the former force in that direction, and conspiring with it to retard the descent of B.

Therefore, the whole effect of the two weights A and A′ on the weight B is equivalent to a motive force =

$$2 \times \frac{x}{y} \times \mathrm{F} = \frac{x}{y}\left(2g + \frac{\ddot{y}}{\dot{t}^2}\right) \times 2\,\mathrm{A},$$

directly opposing the descent of B.

Again, the velocity with which B actually descends is $\frac{\dot{x}}{\dot{t}}$, and the variation of that velocity, or the accelerating force with which B actually descends, $= \frac{\ddot{x}}{\dot{t}^2}$; whence the accelerating force constantly destroyed by the connection of B with the weights A, A′, is

$$2g - \frac{\dot{x}}{\dot{t}^2};$$

and the whole motive force constantly lost,

$$\left(2g - \frac{\ddot{x}}{\dot{t}^2}\right) \times \mathrm{B}.$$

Now as the motive force constantly lost by B, must be equal to the motive force constantly retarding its descent, we shall have

$$\frac{x}{y} \times \left(2g + \frac{\ddot{y}}{\dot{t}^2}\right) \times 2\,\mathrm{A} = \left(2g - \frac{\ddot{x}}{\dot{t}^2}\right) \times \mathrm{B},$$

where it will be observed that $y^2 = x^2 + a^2$; consequently, $y\dot{y} = x\dot{x}$; therefore, $\frac{x}{y} = \frac{\dot{y}}{\dot{x}}$.

Let $\frac{\mathrm{B}}{2\,\mathrm{A}} = m$; then by substitution, and properly reducing the terms, we obtain

$$\frac{m(\ddot{x}\dot{x} + \ddot{y}\dot{y})}{\dot{t}^2} = 2g(m\dot{x} - \dot{y});$$

and taking the fluents, considering $\dot{t}$ as constant,

$$\frac{m(\dot{x}^2 + \dot{y}^2)}{2\dot{t}^2} = 2g\,(mx - y) + \mathrm{C},$$

C being a constant quantity or correction.

In the last equation, substitute $\frac{x^2\dot{x}^2}{y^2}$ for its equal $\dot{y}^2$, and make $v = \frac{\dot{x}}{\dot{t}}$, the velocity of B corresponding to n D, or x; and we shall have

$$v = y + \frac{\sqrt{\{4g(mx - y) + 2C\}}}{\sqrt{(mx^2 + y^2)}};$$

the velocity sought.

This question is proposed, and answered by T. Simpson, in his Select Exercises, and his result corresponds with the above; it was afterwards solved by Atwood, in his Treatise on Rectilinear Motion, and a different answer obtained: by the above solution, due to Mr. Ivory (having been published by that gentleman in Leybourn's Mathematical Repository), it appears that Atwood committed an error in his solution, which is traced to its source in the second volume of the same work, by Mr. Berry.

PROBLEM II.

Problems.

159. Let AFC, BGC (fig. 94), be two curves, the planes of them being vertical; and let a line ACB be stretched over a fixed pulley C, by two given weights A and B, of which B preponderates against A, and descends along the curve CGB: it is required to assign the velocity of B, when it has descended through a given perpendicular altitude, and A has ascended through a given altitude in the same time.

Let BL be the perpendicular altitude through which B has descended, and AH the perpendicular altitude through which A has ascended in the same time; let BL $= y$, and AH $= q$.

Conceive the weight B to describe in its descent the evanescent arc *b* B, and during the same time let the other weight A rise through the arc *a* A. Through *b* and A, draw *b o* and A *l* parallel to the horizon; also through *b*, draw *b n* perpendicular to CB, and A *m* perpendicular to C *a*; and let CB $= x$, B *n* $= \dot{x}$ B *o* $= \dot{y}$, B *b* $= \dot{s}$, A *a* $= \dot{r}$, and *a l* $= \dot{q}$.

Now the whole force by which B is urged in the direction *b* B, is $\frac{B\dot{y} - A\dot{q}}{\dot{s}}$.

For the force which impels B in the direction *b* B $= \frac{B\dot{y}}{\dot{s}}$; and since A acts in opposition to this force, its effect when referred to the direction *b* B, must be taken from $\frac{B}{\dot{s}}$, in order to obtain the force, which, upon the whole, impels B in the direction *b* B; now the force of A, in the direction BC $= \frac{A\dot{q}}{\dot{x}}$; which being resolved into the direction *b* B, becomes $\frac{A\dot{q}}{\dot{s}}$; whence we have, as above stated, $\frac{B\dot{y} - A\dot{q}}{\dot{s}}$ for the whole force by which B is urged in the direction *b* B.

In a similar way it may be shown, that B impels A in the direction *a* A, with a force equal to $\frac{B\dot{y} - A\dot{q}}{\dot{r}}$.

Let us next endeavour to ascertain what is the mass moved by B, or the inertia that is opposed to the communication of motion to B. The inertia of B opposed by its own mass is B; but as the bodies A and B move with different velocities, the inertia opposed by A will not be equivalent to the mass A; therefore, in order to determine an equivalent mass, which being accumulated in B (when A is removed), will resist the communication of motion in the same manner as the mass A, ascending by the force of B: let B be first supposed without inertia, and to possess weight only; then since the force which impels A, in the direction *a* A, is $\frac{B\dot{y} - A\dot{q}}{\dot{r}}$, and the mass moved is A (B being supposed void of inertia), the force which accelerates A will be denoted by $\frac{B\dot{y} - A\dot{q}}{\dot{r}A}$, because A is now the only mass of matter moved. But A describes the space $\dot{r}$ in the same particle of time in which B describes $\dot{s}$; and since when the fluxion of the time is given, the force which accelerates bodies describing different spaces, will be as the second fluxions of those spaces, we have the force which accelerates the body B equal to

$$\frac{B\dot{y} - A\dot{q}}{\dot{r}A} \times \frac{\ddot{s}}{\ddot{r}}.$$

Hence we shall obtain the mass, which, if placed at B, would exert a resistance or inertia equivalent to A; for since, in this case, the moving force and inertia are each applied at the same point, and referred to the same direction, the quantity of matter moved will be equal to the moving force divided by the accelerating force; and since the force which impels B is $\frac{B\dot{y} - A\dot{q}}{\dot{s}}$, and the force which accelerates B $= \frac{B\dot{y} - A\dot{q}}{\dot{r}} \times \frac{\ddot{s}}{\ddot{r}}$, we have the mass moved at B, or the inertia equivalent to that of A $= \frac{A\dot{r}\ddot{r}}{\dot{s}\ddot{s}}$.

Now let $v =$ the velocity of the descending weight B, $w =$ the velocity of the ascending weight A, $t =$ the corresponding time, and $g = 16\frac{1}{12}$ feet. Then we have $v = \frac{\dot{s}}{\dot{t}}$, and $w = \frac{\dot{r}}{\dot{t}}$; and since the evanescent arcs *a* A, *b* B, are described in the same time, $\dot{v} = \frac{\ddot{s}}{\dot{t}}$, and $\dot{w} = \frac{\ddot{r}}{\dot{t}}$; whence, by substitution,

$$\frac{A\dot{r}\ddot{r}}{\dot{s}\ddot{s}} = \frac{Aw\dot{w}}{v\dot{v}};$$

consequently the force which accelerates B is

$$\phi = g\frac{B\dot{y} - A\dot{q}}{\dot{s}} \div \left(B + \frac{Aw\dot{w}}{v\dot{v}}\right) = \frac{g(B\dot{y} - A\dot{q})v\dot{v}}{\dot{s}(Bv\dot{v} + Aw\dot{w})};$$

multiply by $2\dot{s}$, and we have

$$\phi\, 2\dot{s} = \frac{2\dot{s}\ddot{s}}{\dot{t}^2} = \frac{2g\dot{s}(B\dot{y} - A\dot{q})v\dot{v}}{\dot{s}(Bv\dot{v} + Aw\dot{w})}.$$

But $\frac{2\dot{s}\ddot{s}}{\dot{t}^2} = \text{flux. } v^2 = 2v\dot{v}$; whence

$$2v\dot{v} = \frac{g(B\dot{y} - A\dot{q})2v\dot{v}}{(Bv\dot{v} + Aw\dot{w})},$$

or $$\frac{g(\mathrm{B}\dot{y} - \mathrm{A}\dot{q})}{\mathrm{B}v\dot{v} + \mathrm{A}w\dot{w}} = 1;$$

whence $g(\mathrm{B}\dot{y} - \mathrm{A}\dot{q}) = \mathrm{B}v\dot{v} + \mathrm{A}w\dot{w}$;

and taking the fluents, we have

$$g(\mathrm{B}y - \mathrm{A}q) = \tfrac{1}{2}\mathrm{B}v^2 + \tfrac{1}{2}\mathrm{A}w^2 + \mathrm{C},$$

C being a constant arbitrary or correction. Hence we have

$$v^2 = 2gy - \frac{2g\mathrm{A}q}{\mathrm{B}} - \frac{\mathrm{A}w^2}{\mathrm{B}} - 2\mathrm{C};$$

which equation is general, whether the bodies move in right lines or curves.

Since $$v : w :: \dot{s} : \dot{r};$$

we have hence $$w^2 = \frac{v^2\dot{r}^2}{\dot{s}^2}.$$

Substituting, therefore, this value of w^2, we obtain

$$v^2 = 2gy - \frac{2g\mathrm{A}q}{\mathrm{B}} - \frac{\mathrm{A}v^2\dot{r}^2}{\dot{s}^2\mathrm{B}}\, 2\mathrm{C};$$

and $$v = \dot{r}\sqrt{\frac{2g(\mathrm{B}y - \mathrm{A}q) - 2\mathrm{C}\mathrm{B}}{\mathrm{B}\dot{s}^2 + \mathrm{A}\dot{r}^2}}.$$

It is easy to apply this general equation to the preceding problem, from which the same result may be deduced as that obtained in the preceding article.

Problem III.

160. A perfectly flexible chain is wound round a cylinder, supported with its axis parallel to the horizon. Then if the weight and dimensions of the cylinder be given, and also the weight and length of the chain; it is required to determine the time in which the chain, impelled by the force of gravity, will unwind itself; a given length being unwound at the commencement of the motion.

We have seen, in treating of the centre of gyration, that that centre in a circle or cylinder is distance from the axis by a quantity $= r\sqrt{\tfrac{1}{2}}$; therefore the moment of inertia will be the same, if we suppose half the weight of the wheel collected into an indefinitely thin rim at its circumference; we may hence consider the problem as if the cylinder were hollow, and as having only half the weight of the given cylinder, while its diameter must be supposed the same as in the problem.

Let 2 B denote the weight of the cylinder, l the length of the chain, and w its weight; and let a denote the length of the part of the chain depending from the cylinder at the commencement of the motion, and CP (fig. 95) $= x$, the variable length of the same, at any time of its motion; and let v denote the corresponding velocity of the point P.

Then we shall obviously have

$$l : x : w : \frac{wx}{l} = \text{weight of part } x;$$

and therefore $\frac{2gwx}{l}$ expresses the motive force urging the system when the length x of the chain hangs from the cylinder.

But it appears from the preceding part of the problem, that $\mathrm{B} + w$ expresses the whole inertia of the system; therefore the accelerating force

$$\phi = \frac{\ddot{x}}{\dot{t}^2} = \frac{2gwx}{l(\mathrm{B} + w)};$$

whence, multiplying by $2\dot{x}$, we have

$$\frac{2\dot{x}\ddot{x}}{\dot{t}^2} = \frac{4gwx\dot{x}}{l(\mathrm{B} + w)};$$

and taking the fluents,

$$v^2 = \frac{2gwx^2}{l(\mathrm{B} + w)} + \mathrm{C}.$$

To find C we have $v = o$ when $x = a$; whence

$$o = \frac{2gwa^2}{l(\mathrm{B} + w)} + \mathrm{C};$$

consequently, by introducing the value of C,

$$v^2 = \frac{2gw(x^2 - a^2)}{l(\mathrm{B} + w)};$$

and

$$v = \frac{\sqrt{2gw}\sqrt{(x^2 - a^2)}}{\sqrt{(\mathrm{B}l + wl)}}.$$

Since $\frac{\dot{x}}{\dot{t}} = v$, we have by substitution,

$$\dot{t} = \frac{\sqrt{(\mathrm{B}l + wl)}}{\sqrt{2gw}} \times \frac{\dot{x}}{x^2 - a^2}.$$

Taking the fluents and correcting, we have

$$t = \frac{\sqrt{(\mathrm{B}l + wl)}}{\sqrt{2gw}} \times h.\log.\frac{x + \sqrt{(x^2 - a^2)}}{a};$$

which, when $x = l$, becomes

$$t = \frac{\sqrt{(\mathrm{B}l + wl)}}{\sqrt{2gw}} \times h.\log.\frac{l + \sqrt{(l^2 - a^2)}}{a},$$

the time of unwinding the chain.

Problem IV.

161. Two equal bodies, connected by a flexible string, are placed on a smooth horizontal plane; the string being stretched out in a direction parallel to the edge of the plane; to the middle of this string there is fixed another string, at the extremity of which there is a given weight that hangs over the edge of the plane in a vertical direction, and by its descent communicates motion to the other bodies. Now supposing the length of the connecting string to be given, it is required to determine the circumstances of the motion of the three bodies.

Suppose that the bodies A and B are at A′ and B′ (fig. 96) when the motion commences, and let D′ be the middle of the string. Then the point D′ will be drawn along the straight line D′I, perpendicular to A′B′, by the action of gravity on the descending weight W; and the equal bodies A and B will describe equal portions of the similar curves AA′ and BB′, with the same velocities. At the end of any indefinite time t, let the bodies on the plane be at A and B, and the middle of the string at D; then the distance D′D will be equal to the space through which the weight W has descended in that time.

Let the velocities of the bodies in the curves AA′, BB′ $= u$, and their mass or weight $= w$; also the velocity of the descending weight $= v$, its weight being W. Draw AE perpendicular to D′D, and make AD $=$ AD′ $= a =$ half the length of the string, D′D $= s$, D′E $= x$, EA $= y$, the arc AA′ $= r$, ED $= z$, and $2g = 32\tfrac{1}{6}$ feet, the accelerative force of gravity.

Then, at the end of the time t, when the bodies on the plane have acquired the velocity u, if we suppose the W to be disengaged from the string, so that the bodies may be at liberty to move freely, it is evident that the bodies A and B would proceed with the uniform velocity u; and in the succeeding instant $\dot{t}$, the descending body W would acquire the velocity $v+2g\dot{t}$, and the quantity of motion would then be

$$wu,\ wu \text{ and } W(v+2g\dot{t}),$$

or $w(u+\dot{u}-\dot{u})$, $w(u+\dot{u}-\dot{u})$ and $W(v+\dot{v}+2g\dot{t}-\dot{v})$. But the motions which the bodies actually take in that time, are

$$w(u+\dot{u}),\ w(u+\dot{u}) \text{ and } W(v+\dot{v});$$

therefore, according to the principle of D'Alembert, (art. 89) the motions

$$w(-\dot{u}),\ w(-\dot{u}) \text{ and } W(2g\dot{t}-\dot{v})$$

must be such that, if the bodies were impressed with these only, the system would be in equilibrio; that is,

$$-2w\dot{u} = \frac{v}{u}W(2g\dot{t}-\dot{v}),$$

or $$2Wg\dot{t} = W\dot{v} + 2w\frac{u\dot{u}}{v}.$$

But $$\dot{t} = \frac{\dot{s}}{v}; \text{ therefore}$$

$$2gW\dot{s} = Wv\dot{v} + 2wu\dot{u};$$

and, taking the fluents, we have

$$4gWs = Wv^2 + 2wu^2.$$

It is evident that the string AD is a tangent to the curve at A; therefore

$$\dot{r} : \dot{x} :: a : z;$$

whence $$\dot{r} = \frac{a\dot{x}}{z};$$

also $$\dot{x} : -\dot{y} :: z : y, \text{ or } -\dot{y} = \frac{y\dot{x}}{z};$$

and since $$a^2 - y^2 = z^2,\ -\dot{y} = \frac{z\dot{z}}{y};$$

whence $$\frac{y\dot{x}}{z} = \frac{z\dot{z}}{y},\ \dot{z} = \frac{y^2\dot{x}}{z^2}, \text{ and } \dot{x} = \frac{z^2\dot{z}}{a^2-z^2}:$$

consequently $$\dot{s} = \dot{x} + \dot{z} = \dot{x} + \frac{y^2\dot{x}}{z^2} = \frac{a^2\dot{x}}{z^2} = \frac{a^2\dot{z}}{a^2-z^2},$$

and $$s = \int\frac{a^2\dot{z}}{a^2-z^2} = h.\log\frac{a+z}{a-z};$$

also $$\frac{v}{u} = \frac{\dot{s}}{\dot{r}} = \frac{z}{a}, \text{ or } u^2 = \frac{a^2v^2}{z^2}.$$

This value of u^2 being substituted in our equation

$$4gWs = Wv^2 + 2wu^2,$$

gives $$4gWs = Wv^2 + 2w\frac{a^2v^2}{z^4};$$

whence we deduce

$$v^2 = \frac{4gWsz^2}{Wz^2+2wa^2} = \frac{4gW\,h.\log\frac{a+z}{a-z}}{Wz^2+2wa^2};$$

and $$t = \int\dot{s}\sqrt{\left\{\frac{Wz^2+2wa^2}{4gWsz^2}\right\}}, \text{ or}$$

$$t = \int\frac{a^2\dot{z}}{a^2-z^2}\times\sqrt{\left\{\frac{W^2z^2+2wa^2}{4gWz^2h.\log\frac{a+z}{a-z}}\right\}}.$$

In the particular case where $z = a$, we have

$$v^2 = \frac{4gsW}{W+2w} \text{ and } t = s\sqrt{\left(\frac{W+2w}{4gW}\right)},$$

as they ought to be.

To find the equation of the curve described by the bodies, we have

$$\dot{x} = \frac{z^2\dot{z}}{a^2-z^2} = \frac{\dot{y}}{y}\sqrt{(a^2-y^2)}, \text{ and}$$

$$x = a\times h.\log\frac{a+\sqrt{(a^2-y^2)}}{y} - \sqrt{(a^2-y^2)}.$$

PROBLEM V.

162. There is a solid wheel and axle, made of oak (specific gravity 925); the diameter of the wheel being 3 feet, and of the axle 6 inches; the thickness of the wheel 3 inches, and the length of the axle 1 foot 9 inches: now, if the power acting at the circumference of the wheel, be equivalent to 1 cwt. what must be the weight raised by a cord, folding round the axle so that its momentum may be a maximum? And if the uniform thickness of the wheel were 6 inches, and the axle as before, then what would be the weight raised in the case of the maximum effect? In both instances regarding the difference caused by the power having inertia, and being divested of it.

Let 112 lbs. $= P$, the given power, $x =$ the weight whose momentum is to be a maximum. The radius of the wheel CB (fig. 97) $= r$, and CA the radius of the axle $= r'$; W the weight of the wheel, and W′ that of the projecting part of the axle.

Then the alsolute force at B is P; in opposition to which, the weight x is acting at A, which transferred to the distance CB or CD, will be

$$\frac{x\times CA}{CB} = \frac{r'x}{r};$$

whence the motive force, or that force which gives motion to the system, will be

$$P - \frac{r'x}{r} = \frac{rP - r'x}{r}.$$

Now the distance of the centre of gyration of the wheel from the mathematical axis being $r\sqrt{\frac{1}{2}}$, and of the axle $r'\sqrt{\frac{1}{2}}$; we shall have for the moment of inertia of the machine $\frac{1}{2}r^2W + \frac{1}{2}r'^2W'$; and for the moment of the power and weight, $r^2P + r'^2x$.

Multiplying the motive force by r^2, and dividing the product by the sum of the inertia above determined, we shall have

$$\frac{r^2P - rr'x}{\frac{1}{2}r^2W + \frac{1}{2}r'^2W' + r^2P + r'^2x}$$

for the accelerating force of the point B; and consequently

$$\frac{rr'P - r'^2x}{\frac{1}{2}r^2W + \frac{1}{2}r'^2W' + r^2P + r'^2x}$$

for the accelerating force of the point C, or of the weight x; and it is this accelerating force multiplied by the weight x, that is to be a maximum. If, therefore, for the sake of abridging, we make

$$\frac{1}{2}r^2W + \frac{1}{2}r'^2W' + r^2P = S,$$

we shall have to find

$$\frac{rr'Px - r'^2x^2}{S + r'^2x} = \text{a maximum};$$

which, being put into fluxions and reduced, gives

$$x^2 + \frac{2S}{r'^2}x = \frac{r}{r'^3}PS;$$

whence

$$x = -\frac{1}{r'^2}S\ \frac{1}{r'^2}\sqrt{(S^2 + r'rPS)},$$

the weight required.

This result is equally applicable to any proposed dimensions or specific gravities of the wheel and axle.

In our example, we find $W = {\cdot}9121708$ cwt. and $W' = {\cdot}1520285$ cwt. when the thickness of the wheel is 3 inches; and when that thickness is 6 inches, then $W = 1{\cdot}8243416$ and $W' = {\cdot}1266904$; also, in the first case, $S = 13{\cdot}123772$; and in the second, $S = 17{\cdot}225373$.

Whence we readily obtain

$x = 2{\cdot}9188{\cdot}$ cwt. when the thickness $= 3$ inches,

$x = 2{\cdot}9373$ cwt. when the thickness $= 6$ inches.

The power in both these results being supposed to be possessed of inertia: if we suppose it to be divested of it; then $r^2 P$ will vanish out of the denominator of our general maximum equation, and we shall, therefore, have

$S = \frac{1}{2} r^2 W + \frac{1}{2} r'^2 W' = 4{\cdot}1237$ in the first case, and $8{\cdot}2253$ in the second.

Which give, for the corresponding values of x,

$x = 2{\cdot}7677$ cwt. for the thickness 3 inches,

$x' = 2{\cdot}8744$ cwt. for the thickness 6 inches.

If the inertia of the wheel and axle be rejected, then we shall have $S = r r' P$, and

$$x = -\frac{r^2}{r'^2} P \frac{r P}{r'^2} \sqrt{(r^2 + r r')} = 2{\cdot}8844 \text{ cwt.}$$

If, in the last theorem $r = r'$, we shall have

$$x = P(\sqrt{2} - 1) = {\cdot}414214 \times P;$$

so that the common solution to this problem can only be true in the extreme case, when the wheel and axle are converted into a single pulley.

PROBLEM VI.

163. In a system of pulleys, in which the same string goes round all of them contained in two blocks, having the power P and the weight W raised by it, together with the number of the pulleys, and the weight and figure of each, it is required to assign the force which accelerates the descent of P, the weight of the lower block being included in the weight W.

Let the number of pulleys be n, and conceive them to be cylindrical, and that the weight of each $= q$, then the absolute moving force which acts on P, is equal to its own weight $= P$. But because W acts at P in a contrary direction with a force $\frac{W}{n}$, the force which upon the whole impels P in its descent, is

$$P - \frac{W}{n} = \frac{nP - W}{n}.$$

Now the inertia of P is its own weight, and that of W, whose velocity is $\frac{1}{n}$th of the velocity of P, will be $\frac{W}{n^2}$; and because the circumference of the lowest pulley moves with the same velocity as the weight W, and its inertia is the same as if $\frac{1}{2} q$ were collected into its circumference; the inertia by which it resists the descent of P, will be $\frac{q}{2n^2}$, the circumference of the next pulley, revolving twice as fast as the former, its inertia will be four times as great, and will therefore be $= \frac{4q}{2n^2}$, and so on; the sum of the inertia of the pulleys being

$$\frac{q}{2n^2}(1 + 2^2 + 3^2 + 4^2 + \&c.\ n^2) =$$

$$\frac{q}{2n^2} \times \frac{2n^3 + 3n^2 + n}{6}.$$

The whole inertia, therefore, which resists the communication of motion to P, will be

$$P + \frac{W}{n^2} + q \times \frac{2n^3 + 3n^2 + n}{12 n^2} =$$

$$\frac{12 n^2 P + 12 W + q(2n^3 + 3n^2 + n)}{12 n^2};$$

and consequently the accelerating force arising from dividing the motive force by the inertia, will be

$$\frac{12 n^2 P - 12 n W}{12 n^2 P + 12 W + q(2n^3 + 3n^2 + n)}.$$

PROBLEM VII.

164. Let ABCD (fig. 98) represent a system of pulleys, in which the string that goes round each pulley is fixed to the weight: having given the weight W, and the power P which raises it, together with the weight and number of the equal cylindric pulleys A, B, C, D, &c. it is required to assign the force which accelerates the descent of the power P, the weights of the pulleys B, D, C, being included in P.

Let the weight of each pulley be q, and let the number of them $= n$, then the absolute force that urges P is its own weight; but since W acts in opposition to it, and because the power at $W = \frac{1}{2^n - 1}$ of P; the force with which P has a tendency to descend, will be

$$P - \frac{W}{2^n - 1} = \frac{P(2^n - 1) - W}{2^n - 1}.$$

The inertia of W, with reference to the motion of P, will be

$$\frac{W}{(2^n - 1)^2};$$

because the velocity of W is to that of P as $1 : (2^n - 1)$; and since the velocity of the circumference of the pulley A is equal to that with which W ascends, the inertia of this pulley referred to P's motion, will be

$$\frac{q}{2(2^n - 1)^2}.$$

And because the angular velocities of the pulleys A, B, C, D, &c. are as

$$2 - 1,\ 2^2 - 1,\ 2^3 - 1,\ 2^4 - 1, \&c. \text{ or as}$$
$$1,\ 3,\ 7,\ 15, \&c.$$

$$\text{The inertia of B} = \frac{3^2 . q}{2(2^n - 1)^2};$$

$$\text{of C} = \frac{7^2 q}{2(2^n - 1)^2};$$

$$\text{of D} = \frac{15^2 q}{2(2^n - 1)^2};$$

&c. &c.

and therefore the inertia of all the pulleys which arises from their rotation, will be

$$\frac{q}{2(2^n - 1)^2} \times \{1^2 + 3^2 + 7^2 + \&c.\ (2^n - 1)^2\}.$$

The sum of the series in the parenthesis is

$$\frac{2^{2n+2} - 3{\cdot}2^{n+2} + 3n + 8}{3};$$

whence the above becomes

$$\frac{q}{6(2^n - 1)^2} \times (2^{2n+2} - 3{\cdot}2^{n+2} + 3n + 8).$$

Moreover, the inertia which arises from the motion of the centres of gravity of the several pulleys as they descend, will be found, by the same reasoning, to be

$$\frac{q}{3(2^n-1)^2}\times(2^{2n}-3\cdot 2^{n+1}+3n+5);$$

which being added to the inertia before found, will give, for the whole inertia,

$$\frac{q}{6(2^n-1)^2}\times(2^{2n+2}+2^{2n+1}-6\cdot 2^{n+2}+9n+18).$$

The whole inertia, therefore, which resists the communication of motion to P during its descent =

$$\mathrm{P}+\frac{\mathrm{W}}{(2^n-1)^2}+\frac{q}{6(2^n-1)^2}\times(2^{2n+2}+2^{2n+1}-6\cdot 2^{n+2}+9n+18)=$$

$$\frac{6\mathrm{P}(2^n-1)+6\mathrm{W}+q(2^{2n+2}+2^{2n+1}-6\cdot 2^{n+2}+9n+18)}{6(2^n-1)^2}$$

And since we have found the motive force to be

$$\frac{\mathrm{P}(2^n-1)-\mathrm{W}}{2^n-1},$$

we shall have, by dividing the latter expression by the former, for the force which accelerates P,

$$\frac{6\mathrm{P}(2^n-1)-6\mathrm{W}(2^n-1)}{6\mathrm{P}(2^n-1)^2+6\mathrm{W}+q(2^{2n+2}+2^{2n+1}-6\cdot 2^{n+2}+9n+18)},$$

and for the force which accelerates W in its ascent,

$$\frac{6\mathrm{P}(2^n-1)-6\mathrm{W}}{6\mathrm{P}(2^n-1)^2+6\mathrm{W}+q(2^{2n+2}+2^{2n+1}-6\cdot 2^{n+2}+9n+18)}.$$

In order to render the solutions of the two latter problems less complicated, the weights of the moveable pulleys have been included, either in the weight W, or the power P. In the latter, there is this particular advantage; viz. that the weights of the pulleys constitute a part of the moving force; and consequently facilitate the elevation of any weight to be raised by it; whereas the weights of the pulleys in the other case, act in opposition to the moving force.

The weights of the pulleys in the above problem, have been supposed to be included in the power P; but as they are not equal in their effect on the raised weight, it may not be amiss to indicate the manner their forces, when referred to the power P, are estimated.

It is manifest that the pulley A, being fixed, contributes nothing to the moving power: also the weight of each power being q, the pulley B acts on W with a force equal to q; the pulley C with a force $= 3q$, and D with a force $7q$; that is to say, the successive co-efficients of q, will be $2-1$, 2^2-1, 2^3-1, &c. If, therefore, the number of pulleys be n, the moving force exerted by them on the weight to be raised $= q(1+3+7+$, &c.), continued to $n-1$ terms $= (2^n-1-n)$, and this weight would be balanced by

$$\frac{q(2^n-1-n)}{2^n-1},$$

acting at P; wherefore, if a power G be applied at P, exclusive of the weights of the pulleys, the whole moving force which acts on P will be

$$\mathrm{G}+\frac{q(2^n-1-n)}{2^n-1}-\frac{\mathrm{W}}{2^n-1}.$$

If this value for the moving forces be substituted for P in the preceding solution; that is, as far as the body is concerned in contributing to the moving force, we shall have a general expression for the acceleration of the descending weight, every circumstance being taken into the account, except those of friction, the weight of the line, and the resistance of the air; the two latter of which are too small to have any sensible effect; and the former may be so diminished, that the real motion of the descending weight shall scarcely deviate from that deduced from the investigation.

Let, therefore, G be any moving force applied to raise the weight W, by means of the system of pulleys described in the last problem; every thing else remaining the same; we shall then have the force which accelerates the descending weight G, equal to that part of gravity expressed by the fraction

$$\frac{6\mathrm{G}(2^n-1)+6q(2^n-1)-6\mathrm{W}(2^n-1)}{6\mathrm{G}(2^n-1)+6\mathrm{W}+q\mathrm{S}};$$

where we take S to denote the quantity

$$2^{2n+2}+2^{2n+1}-6\cdot 2^{n+2}+9n+18;$$

and on similar principles, the motion generated in any of the simple mechanical powers may be investigated.

HYDRODYNAMICS,

Is that branch of general Mechanics which considers the equilibrium and motion of fluids. The terms *Hydrostatics* and *Hydrodynamics* have corresponding significations to the *Statics* and *Dynamics* in the Mechanics of solid bodies; viz. Hydrostatics is that division of the science which treats of the equilibrium of fluids, and Hydrodynamics that which relates to their forces and motions. It is, however, very usual to include the whole doctrine of the mechanics of fluids under the general term Hydrodynamics, and to denote the divisions relative to their equilibrium and motion by the terms *Hydrostatics* and *Hydraulics*. We propose to adopt the latter division in the following treatise.

Historical sketch of the rise and progress of Hydrodynamics.

We have seen, in the historical chapter prefixed to our treatise on Mechanics, that we are indebted to Archimedes for the first correct theoretical notions of the doctrine of Statics; and it was the same celebrated philosopher who first established the fundamental laws of Hydrostatics, or that branch of Hydrodynamics which relates to the equilibrium of fluids.

With regard to the theory of the motion of bodies, whether solid or fluid, or the sciences of Dynamics and Hydraulics, they have had their birth wholly amongst the moderns; the former of these we have already noticed in the chapter above alluded to, and the latter will be introduced in its proper place in the present article.

According to some authors, the work which Archimedes composed on Hydrostatics, we owe, as it now exists, to a translation from the Arabic; while others maintain that we have derived it from an immediate translation of the original Greek text. This work is entitled "De Humido insidentibus," and is divided into two books. The basis on which this author founds his theory is this: that every particle of a fluid being supposed equal, and equally heavy, will remain in the place in which it is found; or that the whole mass will be in equilibrio when each particular particle is equally pressed in every direction. This equality of pressure, on which the state of equilibrium is made to depend, is demonstrated by experiment. The author afterwards examines the conditions which ought to obtain, in order that a solid homogeneous body, floating on a fluid, may take and preserve the situation of equilibrium: he shows that the centre of gravity of the body, and that of the part immersed, must be situated in the same vertical right line; that the weight of the body is equal to the portion of fluid displaced by it; that the body will be entirely immersed when its specific gravity is equal to, or exceeds that of the fluid; and other principles of the science of Hydrostatics, which constitute the basis of the theory of present times. It appears likewise, from his investigations, that two bodies of equal magnitude, both heavier than the fluid in which they are immersed, will lose equal parts of their weights; and that reciprocally, when the weights lost in the same fluid are equal, the bodies are of equal magnitudes. The solution of the well known problem of Archimedes, relative to the crown of Hiero, king of Syracuse, depends on the above principles.

History.

Screw of Archimedes.

Besides the theoretical principles of Hydrostatics, we owe also to this philosopher, according to some authors, an ingenious hydraulic engine, called, from the name of its supposed inventor, the *screw of Archimedes.* It is employed in elevating water to small heights; and is very simple in its construction, and commodious in its application.

Diodorus asserts, that Archimedes invented this machine in his voyage to Egypt; and that the Egyptians afterwards employed it for the purpose of draining the marshes of that country; but Vitruvius, a contemporary of Diodorus, does not enumerate it amongst the discoveries of Archimedes, of whom he was nevertheless a great admirer; and Claudius Perrault, the translator and commentator of Vitruvius, adds, that the use Diodorus gives to this machine, namely, that it was employed to render Egypt habitable, by draining off the waters with which it was formerly inundated, makes it highly probable that the engine is of much earlier date than the time of the Syracusan philosopher. If this conjecture have any foundation, let us not mix with the legitimate claims of Archimedes, an invention which may be contested with him: he is too rich in other respects to render important the sacrifice of an equivocal right.

Ctesibius and Hero. Invention of the pump and siphon.

About a century after Archimedes, two mathematicians of the Alexandrian school, viz. Ctesibius, and Hero, his disciple, invented the *pump*, the *siphon*, and the *fountain of compression;* the latter of which is to this day known under the appellation of *Hero's fountain.* We owe more especially to Ctesibius, a machine of the same kind, composed of a sucking and a forcing pump, so combined, that by their alternate action, the water is drawn and forced into a tube placed between them. The effects produced by these machines are in truth highly curious and interesting, and doubtless must have appeared very extraordinary to their original inventors, who not knowing to what principle to attribute them, had recourse to their grand scheme of occult qualities, so commodious for explaining all the phenomena of nature. The water rose in the pumps, according to these philosophers, because nature abhorred a vacuum, and consequently the place abandoned by the piston was immediately supplied by the water: we know not whether at that time philosophers were aware of the limit to which the elevation of the water was confined; but we do know, that when this was pointed out to the great Galileo, the father of modern physics, he could only explain it by stating, that nature's abhorrence of a vacuum only extended to about 33 or 34 feet! Such were the illustrations of the ancients: their whole science consisted in the operations of secret and occult powers; they transferred from the moral to the physical world, the ideas of affection and hatred; both celestial and terrestrial bodies had their sympathies and antipathies; and philosophers considered that they had explained a phenomenon, when they had, after one manner or another, brought it under the influence of these chimerical agents.

Clepsydra.

The *Clepsydra*, or *water clock*, may be considered as an hydraulic machine, of which the invention is attributable to the Egyptians. This instrument indicated the hours by the successive elevations of the water which entered into a vessel, in quantities, regulated according to the proposed divisions of time, or by means of a hand, which the falling water caused to revolve on a graduated face or dial plate. Ctesibius, and even some moderns, as Tycho Brahe, Dudley, and others, have not disdained to turn their attention to the improvement of this machine; the great perfection, however, that has been attained in the construction of clocks and watches, renders the clepsydra, in the present day, a mere matter of curiosity.

Water-mills.

Wind-mills.

Water-mills, which must be classed amongst the most valuable hydraulic engines, were also an ancient invention, of the date of which we are ignorant. An epigram of the Greek Anthology seems to indicate that water-mills were first invented in the time of Augustus; but Vitruvius, who flourished under this prince, in his descriptions, does not speak of them as a recent invention; it is, therefore, highly probable that they were known long before that period. As to windmills, they were not employed in Europe till long after water-mills; some authors pretend, that the former were first invented by the French in the sixth century of the christian era; while others assert that we owe them to the crusaders, who brought them from the east, where they were even then very ancient; and generally preferred to water-mills, in consequence of the sources of the rivers being much more rare and uncertain in those countries than in Europe. But whether they are the invention, or merely the adoption of Europeans, this is certain—that the progress of their improvement was very slow, and that we generally prefer the use of water-mills as more commodious and regular in their operation.

Bossut, when speaking of these ancient and important constructions, observes, "In viewing so many labours, so many monuments of human genius, the man, alive to gratitude, asks, to whom do we owe all these useful and sublime discoveries? What honours, what recompences, have these benefactors of man received of their country, or of the world at large? history commonly answers nothing to these enquiries: while great pains are taken to transmit the names and the exploits of conquerors, who have ravaged the earth, and left traces of misery and destruction in all their steps."

It is, however, only the *construction* of certain hydraulic engines that we owe to the ancients; for they were wholly ignorant of any theoretical hydraulic principle: we may, therefore, easily conceive, that their first attempts were very rude and imperfect, and the defects of one machine were their only lessons for the construction of others less imperfect; and it was thus, by successive attempts, and reiterated experiments and failures, that they were led by degrees to that state of perfection to which they ultimately attained.

To Sextus Julius Frontinus, is commonly attributed the first theoretic notions of the motion of fluids. This author was inspector of the public fountains of Rome, under the emperors Nerva and Trajan, and he left, on this subject, a work entitled "De aquæductibus urbis Romæ commentarius."

In this treatise, the author first describes the aqueducts of Rome, cites the names of those which the Romans had constructed, and the dates of their constructions; he then fixes and compares with each other the measures of capacity which he employed at Rome for measuring the products of the adjutages. Thence he passes to a description of the means of distributing the waters of an aqueduct, or of a fountain. On these subjects he made several correct observations; for example, he showed that the quantity of water issuing from an adjutage, did not wholly depend upon its magnitude or superficies, but that the height of the reservoir above it must also be considered; a very obvious fact, but yet such an one as some more recent constructors have neglected to introduce into their investigations. He knew, also, that the tube designed to carry off part of the water of an aqueduct, ought to have, according to circumstances, a position more or less oblique with respect to the course of the fluid, &c. Notwithstanding all this, however, he did not exhibit a mathematical precision on this subject; for he did not know the correct law which obtained between the velocity of the adjutage, and the height of the reservoir.

No other ancient author approximated in any manner towards a theoretical view of the principle of hydraulics; we are therefore completely justified in claiming the honour of the discovery of this science as wholly due to the moderns.

Progress of Hydrodynamics with the moderns.

We have seen that Archimedes had established the first grand principles of the theory of Hydrostatics; that is to say, the principle of equal pressure, which is sufficient, with the assistance of geometry and analysis, for the solution of every problem which depends upon the equilibrium of fluids. We shall, therefore, pass slightly over this branch of the general science of Hydrodynamics, and turn our attention to the other principal division of it, which treats of the motion of fluids, a subject still involved in considerable difficulties: for, notwithstanding all the efforts that have been made, mathematicians are not able, even now, to exhibit formulæ, which unite at the same time the advantages of accuracy of principle with facility of application; an union highly requisite in all scientific researches.

From the time of Frontinus, to that of Torricelli, we are not aware of any step that was made in the progress of Hydraulics, which requires any particular notice. To the latter philosopher we are indebted for racter and sagacity of a mathematical philosopher, attentive to the progress of nature, and accustomed to follow her steps. Calculation was always to him an instrument of necessity, and not a vain exhibition of analytical formulæ, difficult to follow, and still more difficult to comprehend and apply.

John Bernoulli. Maclaurin.

Notwithstanding the success of this work, John Bernoulli, the father of Daniel, and Maclaurin, judging that the principle of the conservation of the living forces, however true in itself, ought not to be immediately employed in the determination of the motion of fluids, each resolved the same problem by new methods (and each differently from the other), which they regarded as more direct and more intimately connected with the pure principles of Mechanics. Their results were in most respects conformable to those of Daniel Bernoulli; and it is but justice to state, that their solutions display great talent and ingenuity: at the same time, we cannot deny that they are involved in some obscurity, and depend upon many doubtful and precarious suppositions. The Hydraulique of John Bernoulli, is printed in vol. iv. of his works, and in the Memoirs of the Academy of St. Petersburgh for the years 1737 and 1738; and the theory of Maclaurin forms a part of his Treatise of Fluxions.

D'Alembert.

D'Alembert, after having given an almost entirely new character to the science of Dynamics, by means of the principle of which Newton and James Bernoulli had given the first ideas, applied it, with the same success, to the motion of fluids. His work on this subject, "Traité de l'Equilibre et du Mouvement des Fluides," was published in 1744. Here, in the problem relative to the spouting of fluids, and the discharge of vessels, he makes the same suppositions as we stated in speaking of Daniel Bernoulli; but this is all that is common to the two methods. D'Alembert considers the motion of each lamina as composed, at every instant, of two motions; viz. that which it had the moment preceding, and the motion lost; and thus establishes with simplicity and elegance the conditions of equilibrium between the motions lost; and, consequently, the resulting equations make known the motion that is preserved, and all the circumstances attending the flow from the orifice. He likewise solved various problems that had never been before attempted.

Notwithstanding the facilities that the author thus introduced into the science of Hydraulics, he was still not satisfied with the hypothesis on which he had rested the basis of his calculus, and he therefore endeavoured to establish his equations on other principles, derivable immediately from the first laws of Hydrostatics; viz. 1st. that a rectangular canal, taken at pleasure, in a fluid mass, in equilibrio, is itself also in equilibrio; and, 2d. that a portion of fluid, in passing from one place to another, preserves its volume, when the fluid is incompressible, or dilates itself according to a given law, when the fluid is elastic; so that in both cases the mass remains the same. This new solution was published in his "Essai sur la Resistance des Fluides," in 1753; and he afterwards entered into a farther developement of his principles in several volumes of his "Opuscules Mathematiques."

Euler.

We have as yet said nothing of the works of the celebrated Euler, who, during the same period, was employing himself in reducing the science of Hydraulics to general formulæ and equations; and who ultimately produced on this subject one of those elegant analytical exhibitions of result which have always distinguished the works of this great master. His first solution, or rather his general theory, was published in the Memoirs of Berlin, and he afterwards extended and perfected it, in several papers which appeared in the *Acta Petropolitana*. The science of Hydrostatics is there presented under an entirely new form, with its application to various novel and interesting problems. The theory of the motion of fluids is all comprised in two differential equations of the second order; the author applies the general principles to the discharge of water through orifices of any magnitude, to the ascent of water in pumps, to its course in the tubes of a conduit, both of constant and variable diameters, &c. He also treats of the motion of elastic fluids; and that of the air in particular gives rise to certain formulæ relative to the propagation of sound, and to the manner in which sound is produced in the tubes of an organ, in the flute, &c.

Lagrange gave also a learned memoir on the motion of fluids, in which he proposed to diminish the difficulties that have retarded the progress of this theory, and which have obliged mathematicians to employ, even in the most simple problems, indirect methods of doubtful and uncertain suppositions. It must be acknowledged, however, that the author failed of accomplishing what he had attempted, at least if we regard the practical application of his results.

Hitherto we have considered simply the circumstances attending the motion of a fluid; it remains for us to offer a few observations relative to the percussion of a fluid in motion against a solid body, or the resistance which a solid body experiences when moving in such a medium. A very simple idea, and in part correct, which mathematicians first formed, was to regard a fluid in motion as an infinite number of parallel filaments, which gave each its respective blow to the solid body, without being affected by the adjacent filaments. Hence they found, 1st, that in the perpendicular impact of a fluid against a plane, or of a plane against a fluid, the percussion or the resistance, is as the product of the plane by the density of the fluid, and by the square of the velocity with which the percussion is made; 2nd, that in an oblique impact, the percussion which results from it, perpendicular to the plane, is as the product of the plane, by the density of the fluid, the square of the sine of the angle of incidence, and the square of velocity. Nothing can be desired more simple than this theory; and it answers very nearly to the result of experiment as far as regards the direct impact; but it becomes more and more defective as the angle of incidence is diminished, so as to fail almost entirely when that angle is less than 45°.

This defect has led many mathematicians to attempt the introduction of other theories, or to modify that of which we have been speaking, so as to render *it* applicable to those cases where we have seen it is insufficient.

Newton, in his *Principia*, treated the question of the resistance of fluids under different points of view, according to their nature, or the medium in which the bodies moved. He supposed at first a rare medium, composed of equal particles, kept at equal distances from each other, so that each particle might be supposed to give its respective blow without impeding, or being impeded by the others, which is in fact similar to the preceding theory. He then found that the resistance of a globe, is only half that which is due to its circumscribing cylinder, moving perpendicularly to its end; and he afterwards sought the absolute resistance of the sphere, supposing, either that the particles of the fluid are perfectly elastic, or that they are entirely divested of their elasticity. In the first case, he found that the resistance of the globe is to the force with which its total motion might be produced, or destroyed, in the time that it would employ to pass over two-thirds of its diameter with an uniform velocity, as the density of the medium to the density of the globe; in the second case, the resistance is half the above. This theory, however, is evidently not applicable to continuous and dense fluids, such as water, mercury, oil, &c.; the author therefore gave another theory for fluids of this kind, in which the globe does not immediately strike all the resisting particles of the fluid, but merely communicate to the particles in contact with itself, and those near them, a pressure which is transmitted by degrees to those more remote. According to this new theory, founded on various propositions, which we cannot enter upon in this place, the resistance of the globe is the same as that of its circumscribing cylinder, a result which is contradicted by experiment; and whence we may conclude, that this manner of considering continuous fluids, is not conformable to their physical constitution.

Daniel Bernoulli, in the eighth volume of the *Acta Petropolitana*, proposes a method for determining the perpendicular force of impact of a fluid vein spouting from an orifice against a plane. He observes, that supposing the plane of a certain extent, the threads of which the vein is composed will terminate by curving themselves, following directions parallel to the same plane; and he regards the curve described by each thread, as a canal, in which a body moves, and which consequently experiences at each point the action of a centrifugal force; the author moreover supposes these submitted to a tangential force, variable according to any law; he calculates the effect of all these forces, and finds that there result from them, parallel to the axis of the vein, or perpendicular to the plane, an impulsion equal to the weight of the cylinder of the fluid, which has for its base the section of the vein before the threads began to curve, and for its height, double the height due to the velocity of the fluid; which is very nearly conformable to experiment. It must be observed, however, that this method is rather distinguishable for its ingenuity than for its utility, being extremely limited in its application.

D'Alembert, in his " Esai sur la resistance des fluides," which appeared in 1752, determined the law of the resistance of fluids, by those of their equilibrium. He supposed at first, a body kept at rest from some exterior cause, in the midst of a fluid which is about to impinge. The threads at the meeting of the body curve themselves in different directions; and the portion of the fluid which covers the anterior part of the body is, to a certain degree, in a state of stagnation. The author then observes, that the pressure offered by the body, or the resistance that it opposes to the motion of the particles, is produced by the loss of velocity which these particles have experienced, for one body only acts on another so far as it communicates or tends to communicate to it a part of its motion: and he thence shows that the question is reduced to finding, in the first place, the velocity of the fluid immediately along the surface of the body, which he determines by two different methods. This velocity being found, we have the formulæ for the pressure or the resistance; but the calculation is very complicated, and of no use in its general state. In modifying it, and relaxing a little from mathematical rigour, D'Alembert found that the perpendicular action of the fluid against the plane, is a little less than the weight of the cylinder which has for its base the area of the vein, and for its height double the altitude due to the velocity with which it impinges. This answers very nearly to the result of experiments, for water confined in its course; but not for the resistance of fluids of indefinite breadth and depth. Euler also attempted an investigation of this problem, and admitted, with D'Alembert, that the resistance of the fluid is due to its pressure: his solution is much more concise than D'Alembert's, but it is, like that, insufficient to meet the general question.

Bossut.

All the theories to which we have hitherto adverted, can only be considered as so many ingenious analytical speculations, founded on hypotheses more or less admissible; but it is obvious that the only sure grounds, on which to rest a practical theory, or such an one as may be employed with confidence by engineers, are to be obtained by means of certain experimental data: a well conducted course of experiments was accordingly began by Bossut in 1766, and continued by him afterwards for many years. This labour, which embraces very nearly every part of practical Hydraulics, is given in considerable detail in the different editions of that author's " Traité d'Hydrodynamique," to which we shall have frequent occasion to advert in the following treatise. Du Buat also, in 1786, published an excellent work on the " Principes Hydrauliques," in which we find a great number of experiments and theoretic views of the various parts of this science.

When we contemplate the great diversity of subjects to which the theory of Hydrodynamics becomes applicable, it will be readily admitted that it is impossible, within the limits of this historical sketch, to enter upon even a mere enumeration of all that has been written in connection with the application of this science. We must, therefore, select such as appear to be of the greatest importance, and this will immediately direct us to the action of fluids on a ship floating on the ocean; a subject of universal utility, and one where human ingenuity stands in continual need of the rules and assistance of science.

Equilibrium of floating bodies.

So early as in the year 1689, M. Renau undertook to submit the motion of a ship to calculation, in a work entitled " Théoric de la Manœuvre du Vaisseau." One of his principal propositions in this work was, that if a ship be urged at the same time by the action of two sails perpendicular to each other, and that we represent these forces by the two contiguous sides of a parallelogram, constructed on their directions; the ship will experience from the water, a resistance represented by the diagonal of the figure. Huygens observed, that this proposition would be true, if the resistance of the water were simply as the velocity, but that it was false on the hypothesis that the resistance is as the square of the velocity, which is, either nearly or exactly, con-racter and sagacity of a mathematical philosopher, attentive to the progress of nature, and accustomed to follow her steps. Calculation was always to him an instrument of necessity, and not a vain exhibition of analytical formulæ, difficult to follow, and still more difficult to comprehend and apply.

John Bernoulli. **Maclaurin.** Notwithstanding the success of this work, John Bernoulli, the father of Daniel, and Maclaurin, judging that the principle of the conservation of the living forces, however true in itself, ought not to be immediately employed in the determination of the motion of fluids, each resolved the same problem by new methods (and each differently from the other), which they regarded as more direct and more intimately connected with the pure principles of Mechanics. Their results were in most respects conformable to those of Daniel Bernoulli; and it is but justice to state, that their solutions display great talent and ingenuity: at the same time, we cannot deny that they are involved in some obscurity, and depend upon many doubtful and precarious suppositions. The Hydraulique of John Bernoulli, is printed in vol. iv. of his works, and in the Memoirs of the Academy of St. Petersburgh for the years 1737 and 1738; and the theory of Maclaurin forms a part of his Treatise of Fluxions.

D'Alembert. D'Alembert, after having given an almost entirely new character to the science of Dynamics, by means of the principle of which Newton and James Bernoulli had given the first ideas, applied it, with the same success, to the motion of fluids. His work on this subject, "Traité de l'Equilibre et du Mouvement des Fluides," was published in 1744. Here, in the problem relative to the spouting of fluids, and the discharge of vessels, he makes the same suppositions as we stated in speaking of Daniel Bernoulli; but this is all that is common to the two methods. D'Alembert considers the motion of each lamina as composed, at every instant, of two motions; viz. that which it had the moment preceding, and the motion lost; and thus establishes with simplicity and elegance the conditions of equilibrium between the motions lost; and, consequently, the resulting equations make known the motion that is preserved, and all the circumstances attending the flow from the orifice. He likewise solved various problems that had never been before attempted.

Notwithstanding the facilities that the author thus introduced into the science of Hydraulics, he was still not satisfied with the hypothesis on which he had rested the basis of his calculus, and he therefore endeavoured to establish his equations on other principles, derivable immediately from the first laws of Hydrostatics; viz. 1st. that a rectangular canal, taken at pleasure, in a fluid mass, in equilibrio, is itself also in equilibrio; and, 2d. that a portion of fluid, in passing from one place to another, preserves its volume, when the fluid is incompressible, or dilates itself according to a given law, when the fluid is elastic; so that in both cases the mass remains the same. This new solution was published in his "Essai sur la Resistance des Fluides," in 1753; and he afterwards entered into a farther developement of his principles in several volumes of his "Opuscules Mathematiques."

Euler. We have as yet said nothing of the works of the celebrated Euler, who, during the same period, was employing himself in reducing the science of Hydraulics to general formulæ and equations; and who ultimately produced on this subject one of those elegant analytical exhibitions of result which have always distinguished the works of this great master. His first solution, or rather his general theory, was published in the Memoirs of Berlin, and he afterwards extended and perfected it, in several papers which appeared in the *Acta Petropolitana*. The science of Hydrostatics is there presented under an entirely new form, with its application to various novel and interesting problems. The theory of the motion of fluids is all comprised in two differential equations of the second order; the author applies the general principles to the discharge of water through orifices of any magnitude, to the ascent of water in pumps, to its course in the tubes of a conduit, both of constant and variable diameters, &c. He also treats of the motion of elastic fluids; and that of the air in particular gives rise to certain formulæ relative to the propagation of sound, and to the manner in which sound is produced in the tubes of an organ, in the flute, &c.

Lagra Lagrange gave also a learned memoir on the motion of fluids, in which he proposed to diminish the difficulties that have retarded the progress of this theory, and which have obliged mathematicians to employ, even in the most simple problems, indirect methods of doubtful and uncertain suppositions. It must be acknowledged, however, that the author failed of accomplishing what he had attempted, at least if we regard the practical application of his results.

Perc of flui Hitherto we have considered simply the circumstances attending the motion of a fluid; it remains for us to offer a few observations relative to the percussion of a fluid in motion against a solid body, or the resistance which a solid body experiences when moving in such a medium. A very simple idea, and in part correct, which mathematicians first formed, was to regard a fluid in motion as an infinite number of parallel filaments, which gave each its respective blow to the solid body, without being affected by the adjacent filaments. Hence they found, 1st, that in the perpendicular impact of a fluid against a plane, or of a plane against a fluid, the percussion or the resistance, is as the product of the plane by the density of the fluid, and by the square of the velocity with which the percussion is made; 2nd, that in an oblique impact, the percussion which results from it, perpendicular to the plane, is as the product of the plane, by the density of the fluid, the square of the sine of the angle of incidence, and the square of velocity. Nothing can be desired more simple than this theory; and it answers very nearly to the result of experiment as far as regards the direct impact; but it becomes more and more defective as the angle of incidence is diminished, so as to fail almost entirely when that angle is less than 45°.

This defect has led many mathematicians to attempt the introduction of other theories, or to modify that of which we have been speaking, so as to render it applicable to those cases where we have seen it is insufficient.

Newt Newton, in his *Principia*, treated the question of the resistance of fluids under different points of view, according to their nature, or the media in which the bodies moved. He supposed at first a rare medium, composed of equal particles, kept at equal distances from each other, so that each particle might be supposed to give its respective blow without impeding, or being impeded by the others, which is in fact similar to the preceding theory. He then found that the resistance of a globe, is only half that which is due to its circumscribing cylinder, moving perpendicularly to its end; and he afterwards sought the absolute resistance of the sphere, supposing, either that the particles of the fluid are perfectly elastic, or that they are entirely divested of their elasticity. In the first case, he found that the resistance of the globe is to the force with which its total motion might be produced, or destroyed, in the time that it would employ to pass over two-thirds of its diameter with an uniform velocity, as the density of the medium to the density of the globe; in the second case, the resistance is half the above. This theory, however, is evidently not applicable to continuous and dense fluids, such as water, mercury, oil, &c.; the author therefore gave another theory for fluids of this kind, in which the globe does not immediately strike all the resisting particles of the fluid, but merely communicate to the particles in contact with itself, and those near them, a pressure which is transmitted by degrees to those more remote. According to this new theory, founded on various propositions, which we cannot enter upon in this place, the resistance of the globe is the same as that of its circumscribing cylinder, a result which is contradicted by experiment; and whence we may conclude, that this manner of considering continuous fluids, is not conformable to their physical constitution.

Daniel Bernoulli, in the eighth volume of the *Acta Petropolitana*, proposes a method for determining the perpendicular force of impact of a fluid vein spouting from an orifice against a plane. He observes, that supposing the plane of a certain extent, the threads of which the vein is composed will terminate by curving themselves, following directions parallel to the same plane; and he regards the curve described by each thread, as a canal, in which a body moves, and which consequently experiences at each point the action of a centrifugal force; the author moreover supposes these submitted to a tangential force, variable according to any law; he calculates the effect of all these forces, and finds that there result from them, parallel to the axis of the vein, or perpendicular to the plane, an impulsion equal to the weight of the cylinder of the fluid, which has for its base the section of the vein before the threads began to curve, and for its height, double the height due to the velocity of the fluid; which is very nearly conformable to experiment. It must be observed, however, that this method is rather distinguishable for its ingenuity than for its utility, being extremely limited in its application.

D'Alembert, in his "Esai sur la resistance des fluides," which appeared in 1752, determined the law of the resistance of fluids, by those of their equilibrium. He supposed at first, a body kept at rest from some exterior cause, in the midst of a fluid which is about to impinge. The threads at the meeting of the body curve themselves in different directions; and the portion of the fluid which covers the anterior part of the body is, to a certain degree, in a state of stagnation. The author then observes, that the pressure offered by the body, or the resistance that it opposes to the motion of the particles, is produced by the loss of velocity which these particles have experienced, for one body only acts on another so far as it communicates or tends to communicate to it a part of its motion: and he thence shows that the question is reduced to finding, in the first place, the velocity of the fluid immediately along the surface of the body, which he determines by two different methods. This velocity being found, we have the formulæ for the pressure or the resistance; but the calculation is very complicated, and of no use in its general state. In modifying it, and relaxing a little from mathematical rigour, D'Alembert found that the perpendicular action of the fluid against the plane, is a little less than the weight of the cylinder which has for its base the area of the vein, and for its height double the altitude due to the velocity with which it impinges. This answers very nearly to the result of experiments, for water confined in its course; but not for the resistance of fluids of indefinite breadth and depth. Euler also attempted an investigation of this problem, and admitted, with D'Alembert, that the resistance of the fluid is due to its pressure: his solution is much more concise than D'Alembert's, but it is, like that, insufficient to meet the general question.

Bossut.

All the theories to which we have hitherto adverted, can only be considered as so many ingenious analytical speculations, founded on hypotheses more or less admissible; but it is obvious that the only sure grounds, on which to rest a practical theory, or such an one as may be employed with confidence by engineers, are to be obtained by means of certain experimental data: a well conducted course of experiments was accordingly began by Bossut in 1766, and continued by him afterwards for many years. This labour, which embraces very nearly every part of practical Hydraulics, is given in considerable detail in the different editions of that author's "Traité d'Hydrodynamique," to which we shall have frequent occasion to advert in the following treatise. Du Buat also, in 1786, published an excellent work on the "Principes Hydrauliques," in which we find a great number of experiments and theoretic views of the various parts of this science.

When we contemplate the great diversity of subjects to which the theory of Hydrodynamics becomes applicable, it will be readily admitted that it is impossible, within the limits of this historical sketch, to enter upon even a mere enumeration of all that has been written in connection with the application of this science. We must, therefore, select such as appear to be of the greatest importance, and this will immediately direct us to the action of fluids on a ship floating on the ocean; a subject of universal utility, and one where human ingenuity stands in continual need of the rules and assistance of science.

Equilibrium of floating bodies.

So early as in the year 1689, M. Renau undertook to submit the motion of a ship to calculation, in a work entitled "Théorie de la Manœuvre du Vaisseau." One of his principal propositions in this work was, that if a ship be urged at the same time by the action of two sails perpendicular to each other, and that we represent these forces by the two contiguous sides of a parallelogram, constructed on their directions; the ship will experience from the water, a resistance represented by the diagonal of the figure. Huygens observed, that this proposition would be true, if the resistance of the water were simply as the velocity, but that it was false on the hypothesis that the resistance is as the square of the velocity, which is, either nearly or exactly, con-

Floating bodies.

formable to the operations of nature. In fact, according to this hypothesis, the resistance which the water opposes to the action of the two sails, being as the square of the velocity, it is necessary, first, to construct a parallelogram to represent the two velocities which the two sails impress separately on the vessel, and then to construct a second parallelogram, such, that its sides having still the same direction as those of the first, may be proportional to their squares: then the diagonal of this second parallelogram will be that which expresses the compound resistance; and the velocity of the ship, directed according to this diagonal, will be proportional to its square root. Renau, however, would not be convinced by this argument, but persisted in his error, till John Bernoulli published his " Essai sur la Manœuvre des Vaisseaux," in 1714; this put the question beyond the reach of argumentation by completely exposing the inaccuracies to which we have alluded, and others of a much more important nature. But in the first attempts at the solution of this problem, mathematicians had too little considered the essential conditions of the equilibrum of vessels; and more recent authors have therefore been obliged to retrace the steps of their predecessors, and in some measure to establish a new foundation of naval science.

It has been known, since the time of Archimedes, that for a solid body, floating on a fluid, to remain in equilibrio, it is necessary, 1st, that its absolute weight, and that of the fluid which it displaces, be equal to each other; 2nd, that the centre of gravity of the body, and that of the volume of fluid displaced by it, be in the same vertical line. But this is not sufficient to constitute a solid and permanent equilibrium. Daniel Bernoulli shows, that in having regard to the different situations which the centres of gravity may have on the vertical line, there exists several states of equilibrium more or less permanent. When the centre of gravity of the system of all the materials which compose the entire weight of the vessel, is placed below the centre of gravity of the immersed part, the equilibrium is always firm or permanent, and will, if deranged by any external force, have a tendency to re-establish itself; and the ship will return with so much the more energy to its first situation, according as its centre of gravity is placed lower; but when the two centres of gravity coincide, or when that of the vessel is higher than that of the displaced fluid, the equilibrium is unstable, and more and more so as the distance between the two centres augments. The same author also gave formulæ for valuing the stability of the ship in all cases. It appears that Euler had, on his part, at the same time come to similar results, which are illustrated and demonstrated in his elegant *Scientia Navalis*, 1749.

In 1746, Bouguer published his "Traité du Navire," and explained the same theory in a new and very simple manner. He shows, under the appellation of the *meta-centre*, the limits below which the centre of gravity of the whole weight of the vessel, and its load, ought to be placed; he examines the best position for the masts; the extent that ought to be given to the sails, and the divers motions of the *seeling* of the ship, &c. which may take place in consequence of a change in the position of the *centre velique*, that is to say, of the point in which we may conceive to be applied the resultant of all the efforts of the wind on the different points of the sail, and the vertical drawn through the centre of gravity of the total load of the vessel.

Another work, by the same author, in which he treats this subject at length, " La Manœuvre des Vaisseaux," was published in 1757; but there is unfortunately in his researches this radical defect, that they are for the greater part founded upon the common theory of the resistance of fluids; and therefore can only be employed under certain restrictions and limitations.

Dup
new

It remains for us to add to the above sketch of the progress of this science, some account of a paper lately presented to the Institute of France, by M. C. Dupin, author of a valuable work, entitled " Développements de Geometrie," as also of various memoirs which have been presented to, and approved of by the above learned body. The memoir to which we now refer is entitled " De la Stabilité des Corps Flottans," and is intended to form a leading article of the second volume of the work above referred to. We know nothing of this theory, except through the report of M. M. Sané and Poinsot; but they have entered sufficiently at length into its merits, and into an illustration of its object and of the author's mode of operation, to enable us to judge of its importance, and to lead us to regret that it has not yet been presented to the public.

M. Dupin appears, by this report, to have taken an entire new view of this interesting branch of Hydrostatics; or at least he has (as Lagrange formerly observed with respect to another production of this author) " found the secret of making new and interesting observations on a subject which seemed to be exhausted." His method of investigation is entirely different from those of Bouguer and Euler; the geometry that he employs was unknown in their time; and this new instrument has led him to new and important results. He considers, under one general point of view, all the positions that a body can take, while floating on the same fluid; its weight and form being supposed constant. That part of a floating body which is immersed in the fluid, is termed by French authors the *caréne*; and as we are not aware of any equivalent English word, we shall adopt the above term in the following brief illustration of the principles of our author.

We know, that in order that a floating body may be in equilibrio, it is necessary that its centre of gravity should be in the same vertical as the centre of volume of its caréne; this caréne being terminated at the level of the fluid by a horizontal plane, which is called the *plane of floatation*. But the weight of the body being supposed constant, the mass or volume of the caréne is so likewise. If, therefore, by changes in the interior of the body, we cause its centre of gravity to take every possible position, without in any respect changing its exterior form; we shall find, for the different states of the same body, an infinite number of planes of floatation; each of these carénes has its centre of volume in a particular point; and consequently there are an infinite number of centres of caréne; these points together form a surface, which the author denominates the *surface of the centres of caréne*. All the planes of floatation are tangents to another surface, which, with reference to these planes, is of the same kind as those which M. Monge, in his Descriptive Geometry (the parent of the theory of M. Dupin), denominates *enve-*

loppes, and which therefore the latter calls *la surface enveloppe des flottaisons*.

Preceding authors have not generalized their views to the same extent: these surfaces have never presented themselves to their imagination; and it is this new view of the subject that has led M. Dupin to general theorems, comprising all that was before known on this subject, with many new and highly interesting results. We hope to see the theory, at no very distant period, presented to the public by the author, who will not, we should imagine, wait for its appearance in the volumes of "Memoires des Etranges Savans," where it is ordered to be published by the committee of the Institute.

In the above historical sketch, we have confined our remarks principally to those authors who have made some decided change in the theory of Hydrodynamics; but, besides these, many other writers are deserving our attention, either for the new light that they have thrown upon the most abstruse parts of this science, or for their experiments, from which many valuable data have been derived. Of these, Atwood, to whom we owe an ingenious paper on the stability of floating bodies, published in the Philosophical Transactions for 1769, is worthy of particular notice. Dr. T. Young has also given several interesting particulars relative to different branches of Hydraulics, both in his Natural Philosophy and in the Philosophical Transactions; where also, in 1795, was published an elaborate paper by professor Vince: but as we shall have frequent occasion, in the course of the following treatise, to refer to the works of these authors, it would be superfluous to enter minutely into a description of them in this place.

M. Hachette, lately one of the professors in the Polytechnic School, and author of a Treatise on Mechanics, has recently presented to the French National Institute a memoir, detailing the results of a very interesting course of experiments on the spouting of fluids, and the discharge of vessels; the principal objects of which were, first, to measure the contraction of the vein of fluid proceeding from a narrow aperture; secondly, to examine the singular phenomena which take place when small cylindrical or concave pipes are added; and, thirdly, to examine and describe the figure of the fluid vein, and the variations occasioned by the different forms of the orifice. The second part of this memoir relates to a class of very curious phenomena attending the discharge of fluids, viz. the greater or less quantities yielded in the same time through tubes of different figures, the apertures in all having the same dimensions. Bossut has shown that a short additional cylindric tube will yield a greater quantity of fluid in a given time than a simple orifice of the same diameter, in the ratio of 13 to 10; other experimentalists have shown that tubes of a conical figure will give more than a cylinder of the same exterior orifice; and Venturi shows that the discharge may be increased in the ratio of 24 to 10. To these M. Hachette has added many other curious and interesting results. To this author we shall also have occasion to refer, in the experimental part of our treatise. We may add to the above, the names of Dr. Matthew Young, who has an interesting article on this subject in vol. vii. of the Transactions of the Irish Academy; Mr. Eytelwein Handbuch, der Mechanik und der Hydraulik; Ventruri, on the lateral Communication of Motion in Fluids; George Juan, Examen Maritimo, a Spanish work, published in 1771; Bossut, "Traité Théorique et Experimental d'Hydrodynamique:" Poisson, "Traités Hydrostatique et Hydrodynamique," contained in that author's "Traité de Mecanique:" Prony, "Architecture Hydraulique:" Gregory's "Treatise of Mechanics," and most of the other works to which we haye referred under the article Mechanics.

HYDROSTATICS.

§ I. *Definitions.*

Definitions.

1. *Hydrostatics*, is that branch of Hydrodynamics which relates to the equilibrium and pressure of fluids.

A fluid is a mass of material points which yield to the least effort that is made to separate them, the one from the other. The fluids which nature presents to us, approach, more or less, to this state of perfect fluidity; but there are many of them also which possess a certain quality of adherence, commonly expressed by the term *viscosity* or *tenacity*; and the greater this quality, the less perfect the state of fluidity; because this opposes itself to the separation of the particles from each other: but, in what follows, we shall make abstraction of this adherence, and consider only perfect fluids.

Amongst what are called perfect fluids, we distinguish two distinct species: the one *compressible*, as water, and all liquids. These may take an infinite number of different figures, but under all these forms they preserve always the same space; at least their compressibility is so small under the greatest pressure, that we may consider it as nothing, without giving rise to any sensible error. Fluids of the second species comprehend the air, gasses and vapours; these are *compressible*, and possess a perfect elasticity, so that they may change at the same time both their form and volume by compression, but will return again to their original state as soon as that compression is removed. Vapours differ from the air, and the permanent gasses, in this, that they lose their character of elastic fluids, and are reduced into liquids, when compressed to a certain degree, or when their temperature is diminished; while the air and the gasses always preserve this characteristic, whatever may be the temperature and the pressure to which they are submitted.

No fluid perfectly incompressible.

2. We have observed above, that water is incompressible, or, at least, that it possesses this property in a very high degree: it may not, however, be amiss to enter a little more particularly into this question, and to explain the nature of the experiments by which it has been shown to yield to a superincumbent pressure.

The idea of the total incompressibility of water was founded chiefly, if not entirely, upon a gross and inadequate experiment made by the *Academi del Cimento*, at Florence, commonly denominated the *Florentine experiment*; in which, water, when violently pressed, made its way through the fine pores of a globe of gold, rather than yield to compression: but it is obvious that such an experiment was wholly insufficient to demonstrate the absolute incompressibility of water; it only proved that fluid, if at all compressible, was so in a very slight degree, because there were no means of ascertaining whether the quantity of water forced into and through the pores of the gold, was exactly equal to the change which had taken place in the internal dimensions of the globe.

Canton's experiments.

This experiment, however, gross as it was, had generally been considered as conclusive, till Mr. Canton, upon an attentive consideration of it, found that it was not sufficiently accurate to justify the conclusion that had always been drawn from it. In order, therefore, to bring the matter to a more accurate and decisive trial, he procured a small glass tube, about two feet long, with a ball at one end, of an inch and a quarter diameter. Having filled the ball, and part of the tube, with mercury, and brought it exactly to the temperature of 50° by the Fahrenheit's thermometer, he marked the place where the mercury stood in the tube, which was about 6½ inches above the ball; he raised the mercury by heat to the top of the tube, and then sealed it hermetically; and, upon reducing the mercury to the same temperature as before, it stood in the tube ·32, or nearly one-third of an inch higher than the mark. The same experiment was repeated on water exhausted of air, instead of the mercury, and the water stood in the tube ·43 of an inch above the mark. Since the weight of the atmosphere on the outside will compress the ball, without any counterbalance from within, and equally raise both the mercury and water, it appears that water will expand about ·11 of an inch more than the mercury, by removing the weight of the atmosphere.

Having thus shown that water is compressible, he proceeded to estimate the degree of compression, corresponding to any given weight; with which view he prepared another ball, with a tube joined to it; and finding that the mercury in ·23 of an inch of the tube was the hundred thousandth part of that contained in the ball, he divided the tube accordingly. He then filled the ball, and part of the tube, with water, exhausted of air; and leaving the tube open, placed his apparatus under the receiver of an air pump, and observed the degree of expansion of the water, answering to any degree of rarefaction of the air; and again, by putting it into the glass receiver of a condensing machine, he noted the degree of compression of the water, corresponding to any degree of condensation of the air. He thus found, by repeated trials, that in a temperature of 50°, and when the mercury was at its mean heighth in the barometer, the water expands one part in 21740, and is as much compressed with the weight of an additional atmosphere; or the compression of water, by twice the weight of the atmosphere, is one part in 10870 of its whole bulk. Seeing that it might be objected, that the compression of the water was owing to the air which it might be supposed to contain, he observes, that more air would render it more compressible; and he therefore let into the ball, a bubble of air, but found that the water was not more compressed with the same weight than before.

In some farther experiments of the same kind, he found, that water is more compressible in winter, than in summer; but he observed the contrary in spirits of wine and oil of olive.

The following table was formed when the barometer was at 29½ inches, and the thermometer at 50 degrees.

Compression of	*Millionth parts.*	*Spec. grav.*
Spirits of wine	66	846
Oil of olive	48	918
Rain water	46	1000
Sea water	40	1028
Mercury	3	13595

See Philosophical Transactions for 1762 and 1764.

It appears, therefore, that the distinction that is made between incompressible and compressible fluids, is not, strictly speaking, correct. Nevertheless, since the compression of the liquids, as appears from the preceding table, is *very small*, compared with their mass, it may be neglected in all practical cases, without the danger of committing any sensible error.

§ II. *Characteristic property of fluids.*

3. The fundamental and characteristic property of fluids, which distinguishes them from solids, and which serves as the basis of the theory of their equilibrium and motion, is the faculty that they possess of transmitting equally and in all directions any pressures exerted on their surface. We admit this property, as a fact established by experiment, and universally assented to by philosophers; and it is the only hypothesis which is necessary to introduce, in order to establish the entire theory of Hydrostatics, or that branch of Hydrodynamics, which treats of the equilibrium of fluids.

In order to illustrate this hypothesis, let us take the case of perfectly incompressible fluids. Let BE (fig. 1, HYDROSTATICS) represent a prismatic vessel, fixed, with its base horizontal, and filled to a certain height BC, with any liquid, as water, for example; and conceive this water to be covered by a piston CD, which exactly fits the vessel; and, for the sake of simplification, let us make abstraction of the weight of the water, that is, let us conceive that the water itself exercises no pressure on the sides of the vessel; and, finally, let us suppose the weight of the piston, or the force with which it is urged, to be denoted by a given weight P. It is obvious that the horizontal base of the prism is pressed in the same manner as if the weight P was applied immediately to that base and distributed uniformly over its surface: every point of the base will experience vertical pressures equal to each other; and, therefore, the pressure on any given portion a of the base will be proportional to a, and will be equivalent to a vertical force applied to the centre of gravity of the area a, and equal to $\frac{Pa}{A}$, A being the area of the entire base.

Now the principle of *the equality of pressure in all directions* consists in this, that the pressure which the weight P exercises at the upper surface of the water, is transmitted by the intermediate fluid, not only to the base of the vessel, but also to every point in its other faces, in a direction perpendicular to those faces; so that any area a, taken in one of the lateral sides of the prism, will experience the same pressure, $\frac{Pa}{A}$, as if it formed part of its horizontal base.

Generally, let us suppose that the vessel have the form of any polyëdron, as (fig. 2), which we shall suppose closed on all sides, and exactly filled with a liquid, without weight. Now, let us remove one of the faces, as ACD, of the polyëdron, and replace it by a piston; now conceive a given force P to be applied to this piston perpendicularly to the surface of the adjacent liquid. The fluid will not be put in motion by this force; and, according to our principle, the pressure which it exercises on the surface where it is in contact, is transmitted, through the medium of the fluid, to all the faces of the vessel. Every point of the polyëdron, comprising amongst them the face of the piston, will be equally pressed from the inside to the outside, in directions perpendicular to each respective face; and if we suppose an area a, taken in any one of those sides, or in the surface of the piston, its pressure will be a force perpendicular to its face, applied at its centre of gravity, and equal to $\frac{Pa}{A}$, A being the entire area of the piston.

This result is easily extended to the case where the sides of the vessels are curve surfaces, continued or discontinued; it is sufficient, in such cases, to decompose these surfaces into indefinitely small elements, which may be considered as the plane faces of a polyëdron of an infinite number of faces; and if we denote the area of one of these areas by W, we shall have $\frac{PW}{A}$ for the pressure which it experiences; A being always taken to denote the area of the piston, and P the perpendicular force applied to it.

If we denote by p the constant pressure which an area of the plane has to support, that is considered as the unit of surface, it is evident that we shall have $\frac{P}{A}=p$; then $p\,\omega$ will be the pressure on the element ω, and $p\,a$ the pressure on the surface a.

4. When the liquid contained in the vessel is supposed to possess weight, it transmits the pressures applied at its surface in the same manner as when void of gravity; but in this case it exercises also on the sides of the vessel another pressure, namely, that due to its own weight. It is the same with regard to a fluid in equilibrio, of which the particles are solicited by given accelerating forces, and which is contained in a vessel shut on all sides. If the faces of this vessel are necessarily in equilibrio, so that we cannot make an opening without the liquid escaping, we must conclude that these faces experience at each point a perpendicular pressure, within and without, in the direction of the normal to the surface of the vessel: for it is only in this direction that a surface can receive a pressure, and destroy it by its resistance.

It follows, from what has been stated above, relative to the replacing of one of the sides of the vessel (fig. 2) by a piston, that if we make a second opening, and to this apply another piston, which is acted upon by a force Q, there will be an equilibrium when the force Q is to the force P, as the area of the second piston is to that of the first; for the pressure of each point of the piston P will be transmited to each point of the piston Q, and reciprocally each point of the piston Q will be transmitted to each point of the piston P; whence these partial pressures, which are opposite to each other, will be in equilibrio when they are equal; that is, when the pressure on each piston is proportional to the area of the fluid with which it comes in contact.

It will be the same for any greater number of openings. Whatever may be their number, if we apply to each of them a piston which is urged by a power proportional to its surface, all these powers will mutually counterbalance each other, so that no one of them will be able to preponderate and raise another.

Proposition. Fig. 3.

5. If we imagine the surface of a fluid mass EFHO (fig. 3), not heavy, and perfectly free, to be parted into an infinite number of elements A, B, C, D, and that to each of these elements pistons were applied perpendicularly to the surface, and acted upon by powers proportional to their bases; all these powers will be in equilibrio; so that the particles of the fluid will remain at rest, and the entire mass will have no motion, either of rotation or translation.

In order to demonstrate this proposition in a complete and rigorous manner, we must first show, that

Lemma 1. Fig. 4.

If the sides of an inflexible polygon ABCDE (fig. 4) be acted upon perpendicularly at their middle points by the powers P, Q, R, &c. proportional to these sides, and all directed, either from the inside to the outside, or from the outside to the inside; these several powers will all be in equilibrio.

Draw the diagonals AC, AD; it is demonstrated (art. 25, *Statics*) that when two forces meet in a point, their resultant must necessarily pass through the same point; and the three forces may be represented by the sides of a triangle perpendicular, each respectively to the three proposed forces. Thus the two forces, P and Q, being perpendicular, and proportional to the sides AB, BC of the triangle ABC, have for their resultant a force (which we may call X) perpendicular and proportional to the side AC of the same triangle. Moreover, the force X is perpendicular to the middle of AC, for it necessarily passes through the point of meeting a, of the two component forces P and Q; which point is evidently the centre of the circle circumscribing the triangle ABC; whence it follows, that the force X is perpendicular to, and at the middle of the side AC. We may demonstrate in the same manner, that the two forces X and R, meeting at the point b, have for their resultant a force Y, proportional to AD, and perpendicular to its middle point; that the two forces Y and S, meeting in the point c, have for their resultant a force Z, proportional to AE, and perpendicular to its middle point, and so on, if the polygon had a greater number of sides.

Therefore the powers P, Q, R, S, have for their resultant a force Z, equal and directly opposed to the last force T, consequently, the system of the forces P, Q, R, S, T, is in equilibrio.

The same demonstration has place, whatever may be the number of sides of the polygon, and consequently it also obtains, when the number of sides become infinite: whence we see, that if we have any inflexible curve, which we divide into an infinite number of elements, and that at the middle of each of these elements we apply, perpendicularly, forces which are proportional to them; these forces will be in equilibrio.

Let us now consider the polygon ABCDE' as a section arising from cutting a right prism through the middle of its height, and parallel to its two opposite bases; then it is obvious that the middle of the sides AB, BC, &c. are the same, or coincide with the centres of gravity of the rectangular faces of the prism; and if to these points we apply powers perpendicular and proportional to the faces above mentioned, these powers will have the same ratios with each other, as the powers P, Q, R, S, T. Now the powers P, Q, R, S, T, are in equilibrio; therefore the powers perpendicular to the centres of gravity of the rectangular faces of a right prism, and proportional to these faces, are also in equilibrio.

The same result will be obtained, when the section and the opposite bases of the prism are curves, and when all the points of the rectangular faces of the body are acted upon perpendicularly by equal forces.

Lemma 2. Fig. 5. 6. If at the point G (fig 5), the middle of the mean breadth EF of a trapezium ACDB, there be applied a force P, perpendicular and proportional to its surface; this force may be decomposed into two others; the one perpendicular and proportional to the orthogonal projection A *c d* B of the trapezium, and the other perpendicular and proportional to the rectangle LEMNFK, which is perpendicular to the two parallel planes A *c d* B, MCDN, by which it is terminated.

Draw through the direction of the power P, and perpendicularly to the parallel right lines AB, CD, *c d*, LK, EF, MN, the plane STR, which will consequently be perpendicular to the four planes ACDB, A *c d* B, CD *d c*, LMNK; resolve the force P into two others, V and H, directed in the plane SRT; the first V, perpendicular to ST, and the second H, perpendicular to RT. This being done, the three forces P, V, and H, being perpendiculars to the three sides of the triangle R, S, T, we shall have

$$P : V : H :: SR : ST : RT \text{ or } ML;$$

or, multiplying the several consequents by the equal lines EF, LK, we have

$$P : V : H :: SR \times FE : ST \times LK : ML \times LK,$$
$$:: ACDB : A\,c\,d\,B : LMNK.$$

Now, the direction of the force V is perpendicular to A *c d* B, and that of the force H is perpendicular to LMNK, because they are in a plane SRT, perpendicular to the two planes A *c d* B, LMNK; and, moreover, the first is perpendicular to ST, and the second to RT, which is parallel to the right lines LM, KN; whence the truth of the proposition is manifest; that is, we have resolved the force P into two other forces, the one perpendicular and proportional to the projection A *c d* B, and the other perpendicular and proportional to the plane LKMN.

We may remark here, that if the height SR of the trapezium ACDB be infinitely small, the point G will be its centre of gravity; the point *g* situated on the line GV, will be the centre of gravity of the projection A *c d* B; and as the point G is always the centre of gravity of the rectangle LMNK, it follows, that if to the centre of gravity of the infinitely small trapezium, a perpendicular force be applied proportional to its surface, that force may be decomposed into two others, of which the first is perpendicular to the centre of gravity, and proportional to the surface of the trapezium of projection; and the second, perpendicular to the centre of gravity, and proportional to the surface of a rectangle, which has its base equal to the mean breadth of the trapezium, and for its height the distance comprised between the two parallel planes drawn through the parallel and opposite bases, and on one of which is formed the orthogonal projection of the said trapezium.

The same hypothesis subsisting, let us imagine an infinitely thin solid lamina, parallel to the bases; and the convex surface of which is composed of an infinite number of trapeziums, such as ACDB, acted upon perpendicularly by forces P, all directed from the inside to the outside; or the contrary, and conceive orthogonal projections of all these trapeziums to be made on one of the bases of the lamina. Then substituting, instead of the forces P, the forces H and V, we shall find, first (art. 5), that the forces H will produce an equilibrium about the contour of the lamina; and, secondly, that the forces V, being perpendicular to the centres of gravity of the trapeziums of projections, and proportional to these trapeziums, will have for their resultant a force represented by the sum of the same trapeziums, and passing perpendicularly through the centre of gravity of their system.

7. We may now proceed to the demonstration of our principal preposition (art. 5); viz.

If we imagine the surface of a fluid mass, as EFHO (fig. 3), not heavy, and perfectly free, to be parted into an infinite number of elements A, B, C, D; and that to each of these elements pistons be applied perpendicularly to the surface, and urged with forces proportional to their bases, all these powers will be in equilibrio; so that the particles of the fluid will remain at rest, and the entire mass will have no motion either of rotation or translation.

Let us imagine, first, that the fluid mass EFHO is entirely shut in on all sides, in a vessel pierced with an indefinite number of holes A, B, C, D, &c. to which pistons are applied proportional to the area of these holes. We have seen (art. 4) that all these forces will mutually balance each other, and that no one or more of the pistons can preponderate so as to raise the others; whence it follows, that the particles of the fluid mass will remain absolutely immoveable, and consequently, with regard to the entire mass, it may be considered the same as if it formed a solid body.

In order to demonstrate that, as a solid body, it will have no motion of translation; let us in the first place conceive the body divided into two parts by a plane, which we shall suppose to be horizontal, and the section which it forms we will call the *principal section.* Let us farther suppose that each of the parts is divided into an infinite number of lamina, by planes parallel to the principal section; and finally, that the convex surfaces of all these lamina are divided each into an infinite number of trapeziums, by vertical planes. The forces perpendicular to the centres of gravity of the lateral trapeziums, all directed from the outside to the inside, and proportional to these trapeziums being supposed to be resolved each into two others, the one horizontal and the other vertical, we see from the preceding article, 1st. That for all the lamina which compose the entire body, the horizontal forces are in equilibrio. 2d. It is also evident relative to the two parts of the body, that the resultant of the vertical forces for the lower part is proportional to the area of the principal section, that it passes perpendicularly through its centre of gravity, acting from below upwards; and that on the same principle the resultant of the vertical forces for the upper part is proportional to the area of the same principal section; that it passes perpendicularly through its centre of gravity, and acts downward from above: whence it follows that these two resultants being equal, and directly opposed, will mutually destroy each other. And we have shown that the horizontal forces are in equilibrio; consequently the whole, as a mass, will remain absolutely immoveable. The particles of the fluid we have also seen, are at rest with respect to each other; consequently

The particles of the fluid will remain at rest, and the whole, as a mass, will neither take a motion of translation nor of rotation, since all the forces, both vertical and horizontal, are in perfect equilibrio.

§ III. *On the equilibrium and pressure of fluids arising from their own weights.*

8. We have already, in our preceding article, made a slight reference to the pressure of heavy fluids against the bottom and sides of the vessels in which they are contained; but as our principal object has hitherto been, to illustrate the manner in which forces are transmitted from one part of a fluid to another, we have, generally speaking, found it most convenient to consider a fluid merely as the agent of such transmission; and have therefore, for the sake of simplifying our investigations, supposed it divested of its own natural weight: it is obvious, however, that in all practical cases, this is one of the most important data we have to consider; and we propose, therefore, in the present section, to investigate the nature and conditions of the equilibrium of a fluid, submitted to the action of gravity; and the pressure with which it acts against the sides and bottom of the vessel where it is contained.

We may observe here, as we have already done in considering the action of gravity on solid bodies, that we shall suppose it a constant force, and proportional to the mass, for all distances that differ but little from each other with regard to their respective distances from the centre of the earth; and that the directions of gravity in different parts of the same body, which is not of great extent, are parallel to each other. When any other hypotheses of gravity become requisite, they will be specifically stated.

9. As we have restricted ourselves to one hypothesis only; viz. *the equality of pressure in all directions*, it will be necessary for us to demonstrate, what might otherwise be admitted as an experimental fact; namely, that the surface of a heavy fluid, perfectly free, when in equilibrio in any vessel, arranges itself in such a manner as to be perpendicular at each of its points to the direction of gravity.

Fig. 6.

Let A *m d* (fig. 6) be the free surface of the fluid; any particle *m* is acted upon by gravity, according to the vertical direction *m n*; let this force be denoted by the line *m n*, and resolve it into two other forces *m p*, *m q*, in the directions of the two elements of the curve A *m* D contiguous to *m*.

Now, in order that the particle *m* may remain in equilibrio, the forces *m p*, *m q*, must be each equal to those which the adjacent particles exercise against them in the opposite directions *p m*, *q m*.

These last forces, from the principle of equal pressure, must be equal to each other; therefore the forces *m p*, *m q* must also be equal to each other; consequently, the direction of the force *m n* will divide the angle *p m q* into two equal parts, and will incline no more to the element *m p* than to *n q*, or which is the same, it is, at the point *m*, perpendicular to the curve A *m* D; and as this perpendicularity of gravity will equally obtain in all the other points of the curve A *m* D, we may conclude reciprocally, that the surface of the fluid is perpendicular in each of those points to the direction of gravity.

It follows hence, that if all the directions of gravity meet in a given point, the curve A *m* D will be the arc of a circle, or the surface of the fluid will form part of a spherical surface, of which the centre is the common point of tendency of all the particles. When the dimensions of the surface of the fluid is very small, with respect to the radius of the earth, that surface may be regarded as a plane, the directions of gravity being throughout such extent, parallel as to sense, agreeably to the supposition (art. 8).

Siphon.
Fig. 7 & 8.

10. If a siphon KMNO (fig. 7) of any figure, and of which the branches may be equal or unequal, contain water, or any other heavy fluid; the surfaces AB, DE, of this fluid in the two branches will be on a level, or at the same distance from any common horizontal plane.

Let us imagine, in a given vessel or reservoir FGHK (fig. 8), a siphon KMNO, perfectly equal to that proposed; and conceive also, that the water of the reservoir, with the exception of the part ABMDEN, which answers to the fictitious siphon, becomes solid, or hard, without changing either its place or volume; then it is clear that the portion of the water remaining in its fluid state, will be in the same state of compression and of stagnation, as it was in before the other portion of it became hard; and consequently, the surfaces AB, DE, will also remain horizontal. Now if we consider the simple siphon (fig. 7), we shall find it differ in no respect from the supposititious one (fig. 8); and therefore, we obtain the same conclusion in both cases.

It may be proper to observe here, that the above proposition falls under a certain restriction in the natural and physical state of fluids; for, in order that the fluid may become exactly level in the two branches of the siphon, it is requisite that they should both exceed a certain dimension, although it is not necessary for them to be of equal magnitude, or similar in form. But when one of the branches is very small, as for example, such that its internal diameter does not exceed $\frac{1}{8}$th of an inch, or thereabout, while that of the other is more considerable; then the fluid will not rise to the same level in the two branches. The greater number of fluids, as wine, water, oil, spirit of wine, &c. rise highest in the small branch (which is called a capillary tube, from capillus, *a hair*) than in the larger one; while, on the contrary, mercury will sustain itself lower in the capillary than in the greater tube. These phenomena are due to a particular cause, which will be examined and illustrated in its proper place: as far as we shall have occasion to consider this instrument, we shall make abstraction of this case, and consider the fluid as rising to the same level in both branches.

Uses of the siphon.

The siphon is an useful instrument in various branches of arts and manufactures; the property of it also, above investigated, serves to explain many natural phenomena. Thus, in sinking a well in the vicinity of the sea, a river, &c. the water is frequently found to rise in the well to the same level with the water in the river or sea; and when these latter are affected by the tides, the water in the well is also frequently known to ebb and flow because the well and the river, or principal

Hydrodynamics.

reservoir, may be regarded as the two branches of a siphon, which communicate with each other by means of fissures and openings in the interior of the earth. In the same manner, the water that we draw from one point to another by a long tube, as for example, in forming a public fountain, &c. naturally tends to rise to the same level at both extremeties of the pipe; and it would rise to the same height, and remain at rest, if the tube at the fountain were carried to the same level as at the reservoir; but the former being always made shorter than the latter, the water discharges itself in the form of a fountain; and with so much the greater force and effect, as the difference of level is greater between the two extremities of the conduit pipe.

Pressure at different depths. Fig. 9.

11. When the water contained in any vessel ABCD (fig. 9), at rest, is submitted only to the action of gravity, any particle m is equally pressed in all directions, with a force equal to the weight of the corresponding column om, situated vertically above it.

In the first place, we know from our general principle, that the particle being supposed in equilibrio, it is equally pressed in all directions; and in order to estimate the quantity of that pressure, let us suppose all the water in the vessel, except the column om, to become hard, without in any respect changing either its form or volume; then it is obvious that the particle m would still be in precisely the same state as when all the liquid of the vessel was in its fluid state. But in this case it is obvious that the pressure which the particle has to support, is equal to the weight of the column above it, om; consequently, when the whole is fluid, the particle is equally pressed in all directions, by a force equal to the weight of the vertical column above it.

Fig. 10.

This result is wholly independent of the form of the vessel; for imagine in the same vessel, represented in fig. 10, any curve F m QSH, and conceive all the water beyond this curve to become hard, without changing its volume or form; then, according to the principle which we have all along assumed (viz. that the state of equilibrium is not disturbed by supposing any part of the system or mass to become fixed), the particle m will still be subject to the same pressure as before; that is, it will be equally pressed in all directions with a force equivalent to a vertical column, whose height is equal to the perpendicular depth of the particle below the surface of the fluid; and it will obviously be the same with regard to a vessel of the form FQSH.

Therefore, whatever may be the form of a vessel, every particle at the same perpendicular depth is subject to the same pressure, in all directions; viz. a pressure equivalent to a column of the fluid, equal in height to the depth of the particle below the horizontal surface of the fluid.

As the vertical pressure of any column is as the number of particles contained in it; and as this number is as the height of the column and the specific gravity of the fluid, the base being supposed indefinitely small; therefore, the pressure in all directions varies in the same proportion; viz. as the infinitely small base, the vertical depth, and the specific gravity of the fluid.

Pressure on finite bases.

12. We have, at present, only considered the pressure on any elementary particle of the fluid: it remains now for us to connect the foregoing principles with the pressure of fluids on given bases; in order to which, it will be requisite to demonstrate the following proposition.

Hy sta

Fig. 1

The liquid contained in any vessel ABCD (fig. 11) being at rest, and submitted to the action of gravity only, the perpendicular pressure which the sum of all the elementary particles of any part fnr of the bottom or the sides of the vessel has to sustain, is equal to the absolute weight of any column which has for its base the surface fnr of that base; and for its height, the vertical distance GO of the centre of gravity G of the same surface, from the surface AD of the fluid.

Conceive the surface fnr, to be parted into an indefinite number of elements fg, gx, xy, &c. and draw the verticals ft, gu, xz, &c. terminated by the surface of the fluid. Now the perpendicular pressure, which the elements fg, gx, xy, &c. have to sustain, are, by the last article, equal to $p \cdot fg \cdot ft$, $p \cdot gx \cdot gu$, $p \cdot xy \cdot xz$, &c.; p being assumed to denote the specific gravity of the fluid, and fg, gx, xy, &c. the indefinitely small bases of the columns.

The whole pressure, therefore, will be represented by

$$p \times \{fg \cdot ft + gx \cdot gu + xy \cdot xz + \&c.\}$$

Now if we consider these products as the moments of so many small weights, with reference to the surface of the fluid, we know that

$$fg \cdot ft + gx \cdot gu + xy \cdot xz + \&c. =$$

the sum of $fg + gx + xy + \&c. \times \mathrm{GO}$; G denoting their centre of gravity; consequently, since

$$fg + gx + xy + \&c. = \text{base } fnr,$$

we shall have for the amount of the whole pressure,

$$p \times fnr \times \mathrm{GO}.$$

Hence, if the base BC (figs. 12, 13, 14) of a vessel of any form, be horizontal, the pressure which the bottom has to sustain is expressed by

$$p \times \mathrm{BC} \times \mathrm{GO};$$

p being the specific gravity of the fluid; GO the vertical drawn through the centre of gravity of the base BC, and terminated in the surface of the fluid produced, if necessary.

Figs. 14, 1

Consequently, if the bases of vessels be equal, and the perpendicular depths of the fluids contained in them equal, as also the specific gravities of the fluids, the pressure upon their bases will be the same, whatever may be their forms or capacities. This is pleasingly illustrated experimentally, by means of the apparatus shown in figs. 12, 13, 14, 15, in which the three former figs. being successively applied to the base plate BC (fig. 15), and water being then introduced into each by the pump MN, it will, when it rises to the same height in each, press down the bottom, which turns on a joint at Q, by elevating the weight P in the scale S; which weight, acting through the medium of the lever, is exactly equal to the weight of the water contained in the cylindric column BC mn.

It is obvious, that in consequence of this property, we may construct vessels so that the pressure upon their bases shall exceed in any proportion the weight of the fluid which they contain.

We see, also, the difference which there may be between the power necessary to support a vessel, with the fluid contained in it, and that with which it presses on its own base. If the base BC, of the vessel, fig 14, were detached from its sides, we must, in order to prevent the escape of the fluid, apply a force equal to the weight of the column B mn C; whereas it would only require, to support the vessel, a weight equal to those of the fluid and vessel: in the one case we have to support the re-action of the fluid against the upper surface of the vessel; whereas, in the other this pressure is sustained between the two opposite actions of the bottom and sides.

Thus, when the vessel is narrowest at bottom, it will require more force to support the vessel, than to keep its bottom from falling; while, if the vessel is broadest at bottom, it may be supported with a less effort than would be necessary to prevent the bottom from separating from the sides of the vessel. But the pressure of the fluid on the bottom of an upright prismatic vessel, is equal to its weight.

It follows, from the above considerations, that the pressure against the side of any vessel is wholly independent of the magnitude of it; so that the walls of a reservoir, of moderate extent, have no greater pressure to sustain, at the same depth, than the bank of a large river, or even an embankment against the sea itself, the water being supposed in a quiescent state; which pressure, from what has been above demonstrated, is obviously equal to the weight of a column, whose base is equal to the surface pressed, and its altitude equal to half the depth of the fluid, or the depth of the centre of gravity of the surface pressed below the surface of the fluid. From the same principles we also readily derive the following properties:

1. The entire lateral pressure of a vessel, whose side is perpendicular to its base, is equal to the weight of the fluid contained in a rectangular prism, whose altitude is that of the fluid, and its base a parallelogram, one side of which is equal to the altitude of the fluid, and the other to the semiperimeter of the vessel.

2. The pressure against one side of a cubical vessel, filled with a fluid, is equal to half the pressure against the bottom; and the whole pressure against the sides and bottom is equal to three times the weight of the fluid.

13. *Of the hydrostatic paradox.* Upon the two principles that fluids press equally in all directions, and in proportion to their perpendicular depths, depends the explanation of a circumstance known by the appellation of the *hydrostatic paradox;* which is this: any quantity of water, or other fluid, however small, may be made to balance and support any quantity or any weight, however great; a circumstance which has been converted to a very useful purpose in the construction of some machines, particularly in that of the Bramah press. A well known contrivance to illustrate this principle, is the *hydrostatic* bellows; it consists of two thick boards, EF, CD (fig 16), about sixteen or eighteen inches diameter, covered or connected firmly with pliable leather round the edges, to open and shut like common bellows, but without valves; there is a pipe AB, about three feet high, fixed into the bellows at B. Now let the water be poured into the pipe at A, and it will run into the bellows, gradually separating the boards by raising the upper one; then, if several weights, three hundred weights for instance, be laid upon the upper board, the water being poured into the pipe till it is full, will sustain all the weights, though the water in the pipe may not weigh a quarter of a pound; for the narrower the pipe the better, provided only that it exceeds certain limits, and that we make it long enough; the proportion being always this:

As the area of the orifice, or section of the pipe,
To the area of the bellows board EF;
So is the weight of water in pipe AG,
To the weight it will sustain on the board.

Hydrostatics.

For the fluid at B, the bottom of the tube, is pressed with a force varying as its altitude AB; and this pressure is communicated horizontally to all particles in the space FE; and then distributed equally throughout the fluid in the bellows: consequently the pressure upwards at FE, is equal to the weight of a cylinder of the fluid, whose base is FE, and height AB; whilst the actual weight of water borne up is only that of the cylinder, whose base is FE, and height BG: and hence no weights laid on CD, that do not exceed the weight of a cylinder of the fluid, whose base is EF, and altitude AG, will disturb the equilibrium.

Centre of pressure.

14. *The centre of pressure* is that point of a surface pressed by any fluid, through which the resultant of all the individual pressures passes; or to which, if a force equal to the whole pressure were applied in a contrary direction, it would keep the surface at rest.

There is a remarkable analogy between the centre of pressure of fluids, and the centre of percussion; which may be stated as follows:

If a plane surface, not horizontal, which is pressed by a fluid, be produced to the horizontal surface, and their common intersection be made the axis of suspension, the centre of percussion will be the same, or coincide with, the centre of pressure.

Fig. 17.

Let ABCD (fig. 17) be the horizontal surface of the fluid which presses upon the plane EIF; produce this plane till it meets the surface of the fluid in the line MN, and let O be the centre of pressure.

From any point p, of the surface pressed, draw the vertical $p\,m$, meeting the horizontal surface in m; and in the plane BC, draw from m the line m M, perpendicular to MN. The pressure upon p (art. 12), is as $p \,.\, p\,m$. and its effect to turn the plane about MN, is as $p \,.\, p\,m \,.\, p\,\mathrm{M}$, by the nature of the lever; also its effect to turn the plane about NI, is as $p \,.\, p\,m \,.\, \mathrm{MN}$. In like manner, if the plane EIF be supposed to revolve about the axis MN, and to strike an obstacle at O, the percussive force of the particle p, by which it endeavours to move the plane about MN, will be as $p \,.\, p\,m \,.\, p\,\mathrm{M}$; and its force to turn the plane about NI, will be as $p \,.\, p\,\mathrm{M} \,.\, \mathrm{NM}$; or as $p \,.\, p\,m \,.\, \mathrm{MN}$. And the like correspondence between the percussive, and the pressive forces, of any other particles in the plane EF, may be shown in the same manner; consequently, the percussive forces of the whole of the particles, whereby they endeavour to move the plane in the two directions, have the same relation as the forces of pressure; and therefore, the centres of pressure and percussion are coincident.

Hence the theorem given for the centre of percussion in (art. 112) may be applied to the determination of the centre of pressure: hence also appears the mistake of those who assert, that the centres of percussion, and of pressure, do not coincide; they are the centres of *oscillation* and of *pressure*, that are not universally coincident.

Application of the above principle.

15. In order to adapt the general formulæ for the centre of percussion to the instance of the centre of pressure, it will be proper to make a slight change in the notation; let it be the distance from MN of any particle, or of any horizontal lamina of the fluid, in contact with the plane EIF; let l be the length of such lamina, and d' its depth (being considered as

evanescent), then will $l\,\dot{d}$ be its area; also let δ be the distance of the centre of gravity of the plane EIF, from the line MN, where that plane intersects the surface; and let the horizontal distance of $l\,\dot{d}$, from the line NI, be denoted by h; then, with respect to the line MN, the formula for the centre of percussion (art. 112) will become

$$\frac{\int l\,d^2\,\dot{d}}{\int l\,d\,\dot{d}}, \text{ or } \frac{\int l\,d^2\,\dot{d}}{\text{EIF} \times \delta};$$

and with respect to the line NI, the formula (art. 113) will be

$$\frac{\int l\,\mathrm{H}\,d\,\dot{d}}{\int l\,d\,\dot{d}}, \text{ or } \frac{\int l\,\mathrm{H}\,d\,\dot{d}}{\text{EIF} \times \delta}.$$

16. We shall add a few examples to illustrate this formula.

Examples.

1. Let a reservoir which contains water, or any other fluid, (its specific gravity being s), have one of its sides plane and vertical: if we imagine a right line drawn vertically from this plane, its length being λ, and the distance of its centre of gravity from the surface of the fluid δ, the pressure exerted upon this line will be $s\,\delta\,\lambda$ (art. 12). Let the distance of the superior extremity of the line λ from the surface of the fluid be a, and make $a + \lambda = e$, so shall e be the distance of the lower extremity of the line λ, from the horizontal surface of the fluid.

Then, to find the centre of pressure of the line λ, we take the complete fluent of the expression

$$\frac{\int l\,d^2\,\dot{d}}{\int l\,d\,\dot{d}} = \frac{2\,(d^3 - a^3)}{3\,(d^2 - a^2)},$$

l being constant; and when the variable d becomes e, the latter is then

$$\frac{2\,(e^3 - a^3)}{3\,(e^2 - a^2)};$$

which is the distance of the centre of pressure from the horizontal surface of the fluid.

When one extremity of the line λ coincides with the surface of the fluid, we have $a = o$, and $e = \lambda$, and the distance of the centre pressure becomes $\frac{2}{3}\,\lambda$.

2. If upon the vertical line λ we construct a rectangle, of which the horizontal base is h, the whole pressure upon it will be $s\,h\,\delta\,\lambda$, and the distance of the centre of pressure from the surface of the fluid will be $\frac{2\,(e^3 - a^3)}{3\,(e^2 - a^2)}$, the same as we have found it above; and this centre must evidently be found upon the vertical line which divides the parallelogram into two equal parts.

Fig. 18.

If the upper horizontal side of the parallelogram coincide with the surface of the fluid (as the side AB of the parallelogram ABCD, fig. 18), its tendency to turn about its base will be

$$\tfrac{1}{2}\,s\,h\,\lambda^2 \times \tfrac{1}{3}\,\lambda = \tfrac{1}{6}\,s\,h\,\lambda^3,$$

and its tendency to turn about one of its vertical sides will be

$$\tfrac{1}{2}\,s\,h\,\lambda^2 \times \tfrac{1}{2}\,h = \tfrac{1}{4}\,s\,h^2\,\lambda^2;$$

whence, the first of these efforts will be to the second as $\frac{1}{6}\,s\,h\,\lambda^3 : \frac{1}{4}\,s\,h^2\,\lambda^2$, or as $2\,\lambda : 3\,h$; which reduces to $2 : 3$, when the rectangle becomes a square.

Centre of pressure of a triangle.

3. To determine the centre of pressure of the triangle ABC (fig. 19), whose side AB is horizontal, and which is placed vertically in a fluid, whose horizontal surface is SS′:

Fig.

Through C, the summit of the triangle, draw the vertical line SCP, also the line CQ bisecting the base AB, and any line TR parallel to AB. Make CP $= \lambda$; AB $= h$, CS $= a$, SP $= a + \lambda = e$, the distance of the horizontal line, in which lies the centre of gravity of the triangle from S $= \delta$ (that is, if C$g = \frac{2}{3}$ CQ, the distance between SS′, and $g = \delta$), the angle PCQ $= k$, SM $= d$, TR $= l$, CM $= c = d - a$.

The whole pressure upon this triangle will be represented by

$$s \times \tfrac{1}{2}\,h\,\lambda \times \delta = \tfrac{1}{2}\,s\,\delta\,\lambda\,h;$$

and to find the depth of the centre of pressure below SS′, we must find the fluent of the expression

$$\frac{\int l\,d^2\,\dot{d}}{\text{ABC} \times \delta}, \text{ or } \frac{\int l\,d^2\,\dot{d}}{\frac{1}{2}\,\delta\,\lambda\,h}.$$

In order to this, we have

$$\text{CP} : \text{AB} :: \text{CM} : \text{TR},$$

or,
$$\lambda : h :: c : l = \frac{h}{\lambda}\,(d - a);$$

which, substituted for l in $\int l\,d^2\,\dot{d}$, gives

$$\frac{h}{\lambda}\int (d^3 - a\,d^2)\,\dot{d} = \frac{h}{\lambda}\,(\tfrac{1}{4}\,d^4 - \tfrac{1}{3}\,a\,d^3) + \text{C};$$

C being a constant arbitrary of corection, which is determined by considering that when $l = 0$, or when $d = a$, the fluent vanishes, which gives

$$\frac{h}{\lambda}\,(\tfrac{1}{4}\,a^4 - \tfrac{1}{3}\,a^4) + \text{C} = 0, \text{ when C} = \frac{h}{12\,\lambda}\,a^4;$$

and consequently the correct fluent is

$$\frac{h}{\lambda}\,(\tfrac{1}{4}\,d^4 - \tfrac{1}{3}\,a\,d^3 + \tfrac{1}{12}\,a^4).$$

Hence then for the whole triangle, in which SM $=$ SP, or $d = e$, if we substitute e for d, in the fluent, and divide by the denominator $\frac{1}{2}\,\delta\,\lambda\,h$ of the general expression; we shall have, for the depth of the centre of pressure of the whole triangle,

$$\frac{\tfrac{1}{4}\,e^4 - \tfrac{1}{3}\,a\,e^3 + \tfrac{1}{12}\,a^4}{\tfrac{1}{2}\,\delta\,\lambda^2};$$

which is evidently independent of the base h of the triangle.

When the vertex of the triangle is at the horizontal surface of the fluid $e = \lambda$, $a = 0$, $\delta = \frac{2}{3}\,\lambda$, and the expression becomes $\frac{3}{4}\,\lambda$.

4. To determine the distance of the centre of pressure from the vertical line CP:

Here we must take the formulæ

$$\frac{\int l\,\mathrm{H}\,d\,\dot{d}}{\int l\,d\,\dot{d}};$$

in which H represents MN $= c \tan k = (d - a).\tan k$.

Substituting this value of H, and $\frac{h}{\lambda}\,(d - a)$ for l, its value found above, we have

$$\int l\,\mathrm{H}\,d\,\dot{d} = \frac{h \tan k}{\lambda}.\int (d^3 - 2\,a\,d^2 + a^2\,d)\,\dot{d} =$$
$$\frac{h \tan k}{\lambda}\,(\tfrac{1}{4}\,d^4 - \tfrac{2}{3}\,a\,d^3 + \tfrac{1}{2}\,a^2\,d^2) + \text{C}.$$

In this case also the fluent vanishes, when $d = a$; whence we have

$$\frac{h\,.\tan k}{12\,\lambda}\,a^4 + \text{C} = o, \text{ and C} = -\frac{h \tan k}{12\,\lambda}\,a^4;$$

consequently, the correct fluent is

$$\frac{h \tan k}{\lambda}\left(\tfrac{1}{4} d^4 - \tfrac{2}{3} a d^3 + \tfrac{1}{2} a^2 d^2 - \tfrac{1}{12} a^4\right).$$

Whence, making $d = e$, and dividing by

$$\int l d \dot{d} = \tfrac{1}{2} \delta \lambda h,$$

we have

$$\tan k \times \frac{\tfrac{1}{4} e^4 - \tfrac{2}{3} a e^3 + \tfrac{1}{2} a^2 e^2 - \tfrac{1}{12} e^4}{\tfrac{1}{2} \delta \lambda^2},$$

for the distance sought.

From this general result we readily deduce the following particular cases; as,

1. When the triangle is isosceles, the angle $k = 0$, and then the preceding value vanishes, as it obviously ought to do; for in that case the triangle is symmetrical with respect to CP, and the pressures are in equilibrio about that line.

2. In the case where the triangle is right angled, and has its base or horizontal side AB $= h$, one of the sides about the right angle; that is, when CA coincides with CP, then $h = 2 \lambda \tan k$, or $\tan k = \frac{h}{2 \lambda}$, which changes the expression for the distance of the centre of pressure from SP, into

$$\frac{h}{\delta \lambda^2}\left(\tfrac{1}{4} e^4 - \tfrac{2}{3} a e^3 + \tfrac{1}{2} a^2 e^2 - \tfrac{1}{12} a^4\right).$$

3. Again, when the vertex of the triangle is at the surface SS′ of the fluid, then we have

$$a = 0, e = \lambda, \delta = \tfrac{2}{3} \lambda;$$

and the expression becomes $\frac{3}{4} \lambda \tan k$, which for the right-angled triangle reduces to $\frac{3}{8} h$.

4. Lastly, when the triangle has its vertex at the horizontal surface of the fluid, the tendency to turn about the base, is to that to turn about the perpendicular, let fall from the vertex upon the base, as $1 : 3 \tan k$; and in case of the right-angled triangle, as $2 \lambda : 3 h$; which, when the legs of the triangle are equal, reduces to the ratio of 2 : 3, as in the case of the rectangle and square.

When C and AB lie on different sides of SS′, that is, when part of the triangle is out of the fluid, no other alteration will be necessary in the preceding expressions than a change of the signs of the terms which contain uneven powers of a. So that this simple transformation will accommodate the preceding general theorems to the case of trapezoids.

It will be found, by a similar investigation to the above, that if the radius of a circle be r, and the depth of its centre below the surface of the fluid δ, the distance of the centre of pressure from the upper surface in the same plane, is expressed by $\delta + \frac{r^2}{4\delta}$, which, when the upper part of the circle just touches the surface, becomes $\frac{5}{4} r$.

17. When two fluids of unequal densities are included in a bent tube, and balance each other, their perpendicular altitudes, estimated from a horizontal plane drawn through the common surface where they are in contact, will be reciprocally as their specific gravities or densities.

For, let ABCD (fig. 20) be such a bent tube, its form and dimensions being arbitrary; and let the common surface of the fluids be GH; one fluid occupying the space EFHG, and the other the space GHBCKI. Let the specific gravity of the fluid in the former be s, and that of the other S: also through the surface GH draw the horizontal plane GHLOM; then it is manifest that the part GHBCML is naturally in equilibrio: in order, therefore, that the equilibrium may exist in the whole, the pressure exerted upon GH by the fluids contained in EFHG, IKML, must be equal.

Now we have seen that the former of these pressures is denoted by GH × FH × s,
and the latter by GH × NO × S;
we must, therefore, have

$$\text{FH} \times s = \text{NO} \times \text{S, or}$$
$$\text{FH} : \text{NO} :: \text{S} : s;$$

that is, the altitudes are reciprocally as the specific gravities of the respective fluids.

Suppose, for example, the fluids were mercury and water, the densities of which are as 14 : 1, or $s = 1$, and $\text{S} = 14$; and let it be required to determine the column of mercury that will balance a column of water of 32 feet:

Here we have

14 : 1 :: 32 feet : 28 inches *nearly*.

Consequently, if a fluid, of which the specific gravity is S, press under an altitude h, we may substitute, instead of that pressure, the pressure of another fluid, whose specific gravity is s, by giving to the latter an altitude equal $\frac{\text{S} \times h}{s}$.

From the above we readily deduce the method for determining the pressure on any base; as, for example, MN (fig. 21) of a vessel containing horizontal beds or strata of different fluids, as MNKD, DKGC, &c. For, let ML, LP, PQ, &c. be the heights or thicknesses of the different strata; s, s', s'', &c. their specific gravities: the pressure on the bottom MN, will be the same as if, in the place of the superior beds, we substituted others, all similar to the lower one MNKD, and of which the respective heights should be

Fig. 21.

$$\frac{s' \times \text{LP}}{s}, \frac{s'' \times \text{PQ}}{s}, \frac{s''' \times \text{QR}}{s}, \&\text{c.}$$

Consequently, the pressure on MN $=$

$$s \times \text{MN} \times \left\{ \text{ML} + \frac{s'.\text{LP}}{s} + \frac{s''.\text{PQ}}{s} + \frac{s'''.\text{QR}}{s} + \&\text{c.}\right\} =$$
$$\text{MN} \times \{s.\text{ML} + s'.\text{LP} + s''.\text{PQ} + s'''.\text{QR} + \&\text{c.}\};$$

that is to say, in order to have the pressure on the bottom MN, we must multiply this base by the sum of the products of the heights, and the specific gravities of the fluid strata contained in the vessel.

Density variable according to any law.
Fig. 22.

18. Let now the fluid ABC (fig. 22) be heavy, and suppose it to be composed of an infinite number of strata of which the densities vary according to any given law; to determine the pressure that it exerts against any point in the bottom or sides of the vessel:

In the first place, we may observe, in order that there may be an equilibrium in the mass, the several beds or strata ought to be horizontal or perpendicular to the action of gravity. This, therefore, being supposed, through the lowest point B of the fluid draw the vertical BH, and it is obvious that the lowest point B will have to bear the entire weight of the column or line BH, or the sum of the weights of the particles B b, b c, c d, &c.

Now, if we denote by s, s', s'', &c. the densities or specific gravities of the lamina MNB, MNQP, PQRS, &c.; the weight of the lines B $b = s.$ B b, of b $c = s'.$ b c, of $c\,d = s''.\ c\,d$, &c. and consequently, the entire pressure on the point B will be expressed by

$$s.\,B\,b + s'.\,b\,c + s''.\,c\,d + \&c.;$$

and this value being the same as

$$s \times \left(B\,b + \frac{s'.\,b\,c}{s} + \frac{s''.\,c\,d}{s} + \&c.\right), \text{ or}$$

$$s' \times \left(b\,c + \frac{s.\,B\,b}{s'} + \frac{s''.\,c\,d}{s'} + \&c.\right), \text{ or}$$

$$s'' + \left(c\,d + \frac{s.\,B\,b}{s''} + \frac{s'.\,b\,c}{s''} + \&c.\right), \text{ or}$$

$$\&c. \qquad \&c.;$$

it follows, that we may substitute instead of the proposed fluid, another of which the density is the same throughout the whole height BH.

Draw through any two points F, f indefinitely near, the two horizontal planes GT, $g\,t$, which determine the limits of the two elements on the sides of the vessel; and it follows, that each point of Gg, or of Ft, will experience a perpendicular pressure equal to the pressure on the point F; now the latter pressure is the same as the absolute weight of the vertical line HF; and as by making HF $= x$, and the density or specific gravity of the fluid in F $= \phi$, the weight of the line HF is evidently $\int \phi\,\dot{x}$, it follows that the perpendicular pressure excited in Gg is

$$G g \times \int \phi\,\dot{x};$$

and that the sum of the pressures against AG is

$$\int.\,G g \times \int \phi\,\dot{x}.$$

Example. 19. Let us illustrate these results by the application of our formulæ to an example.

1. Required the pressure requisite to support the horizontal base of a vessel, or the pressure it has to sustain; the densities of the lamina being to each other respectively as their altitudes:

Let the whole altitude $= a$, the density of the fluid at any point $= s$, x any variable altitude, and ϕ the corresponding density; we shall have

$$\phi = \frac{s\,x}{a}, \text{ and hence}$$

$$\int \phi\,\dot{x} = \int \frac{s\,x\,\dot{x}}{a} = \frac{s\,x^2}{2\,a};$$

which, when $x = a$, becomes $= \frac{s\,a}{2}$;

whence the pressure on the base $= \frac{s\,a}{2} \times$ *base.*

The same result is obtainable from the most obvious principles of geometry, by considering that the densities may be supposed to form an arithmetical progression, of which the first term, which answers to the point H, is zero, the last s, and the number of terms $=$ HB.

2. Let a vessel be cylindrical, or rectangular, and let it be required to determine the pressure which any given part of the side of the vessel has to sustain (as VY, fig. 23), the densities being proportional to the heights, as in the preceding example. Fig. 23.

Let the whole height of the vessel HB (fig. 23) $= a$, the density in KD $= s$, x and variable depth, and ϕ the corresponding density; and we shall have

$$\phi = \frac{s\,x}{a};$$

consequently,

$$\int \phi\,\dot{x} = \int \frac{s\,x\,\dot{x}}{a} = \frac{s\,x^2}{2\,a};$$

$$\int.\,G\,g.\int \phi\,\dot{x} = \int \frac{s\,x^2\,\dot{x}}{2\,a} = \frac{s\,x^3}{6\,a} + C.$$

In order to find C, we may observe that the fluent ought to be such that it vanishes when $x =$ AV; this gives

$$C = \frac{s\,.\,AV^3}{6\,H\,B};$$

making, afterwards, $x =$ AY, we shall have

$$\frac{s\,.\,AY^3 - s\,.\,AV^3}{6\,H\,B},$$

for the total value of the pressure on VY.

This result, however, like the former, may be obtained independent of the fluxional calculus, as the reader will immediately perceive. The above is given for the sake of its greater generality.

20. The preceding deductions find an important application in proportioning the thicknesses of water pipes, intended to elevate water, or any other fluid, to different heights; for we have seen, that in any vessel containing a heavy fluid, the parts that are deepest below the surface, sustain a proportional greater pressure: we may, therefore, run into a superfluous expence, by giving the same thickness to the material in every part. For if the substance be uniformly thick, and the lower parts are sufficiently strong, the upper parts are, of consequence, much thicker than necessary; and the same may be observed in the construction of sluice gates, dams, banks, &c. In all such cases, it is advisable to determine, first, the adequate strength to resist the pressure at the greatest depth; and to proportion the dimensions of the upper parts to the pressure they will have to sustain. Thick of wa pipes.

For the latter purposes, the computations are more direct than in the former; indeed, they are so obvious, by referring to our chapter on the *resistance of solids*, that we deem it unnecessary to say more on the subject in this place; but with respect to the thickness of tubes, the calculation is not so obvious: we shall, therefore, enter a little upon this subject before concluding the present section.

In order to examine this question, we must investigate the relation which obtains between the resistance of a circular ring of metal or other matter, and the power tending to produce a rupture in it, the latter force emanating from the centre, and acting all round against its internal surface.

21. It will, however, be best to begin with the case of a regular polygon, which we shall suppose flexible, as ABCDEF (fig. 24), and let us conceive that to the several angles of it, there are applied the powers or forces P, Q, R, &c. directed from the centre O to the circumference; to determine the conditions of the exciting forces: Flexib polyg Fig. 2

Draw through the angle where any one of the powers is applied, as, for example, the angle A, the line PO, and take any part of it, as Ay, to represent the force P, applied at that point, and construct the parallelogram Axyz, of which the sides Ax, Az, fall on the sides AB, AF, of the polygon. Now the power P, the tension of the side AB, and that of the side AF, being in equilibrio, they are proportional to the three lines Ay, Az, xy, or (because the two isosceles triangles Axy, AOB, are similar) to the three lines AB, OB, OA: in the same manner the power Q, the tension of the sides BC and BA, are proportional to the three lines BC, OC, and OB; and so on with all the other forces.

Now in all this series of proportionals, the same ratio evidently obtains; for since the whole is supposed in equilibrio, each side of the polygon has an equal tension. Therefore, calling x, g, h, k, l, z, the tensions of the sides AB, BC, CD, &c. we shall have
$P : Q : R, \&c. : x : g : h, \&c. :: AB : BC : CD, \&c. : OB : OC : OD, \&c.$;
whence we conclude that, since
$AB = BC = CD =$, &c. we shall have
$P = Q = R =$, &c.
Also, because
$OB = OC = OD =$, &c. we find
$x = g = h =$, &c.
And, lastly, by a composition of ratios, we have
$P + Q + R +, \&c. : x :: AB + BC + CD +, \&c. : OB$;
that is, when a polygon, under the circumstance we have supposed, is in equilibrio, it appears,

1. *That all the powers must be equal.*
2. *That all the sides of the figure have to sustain equal tensions.*
3. *That the sum of all the forces, is to the tension of any of the sides, as the perimeter of the polygon is to the radius of the circumscribing circle.*

ble ders 22. The above principles being established, it will be easy to investigate the ratio of the tensions which the lower circumferences of different cylinders have to sustain, in proportion to their radii and altitudes, and the specific gravity of the fluids.

Let C and c denote the area of the bases of two cylinders, which we shall suppose to be horizontal, or at least, that every point of the base of each cylinder is at equal depths below the upper surface of the fluid.

We have assumed C and c to denote the area of the two bases; let also the radii be R and r, the altitudes H and h, the densities S and s, and the tension of the circumferences in the direction of their tangents T and t.

It is evident that the sum of the pressures acting, from the inside to the outside, according to the direction of the radii, on all the points of the two circumferences, are proportional to the products C.H.S and $c.h.s$; and, by the preceding article, we have the two following ratios

$$C.H.S : T :: C : R,$$
$$c.h.s : t :: c : r.$$

But $C : c :: R : r$;

consequently, from the equality of the ratios

$$C.H.S : c.h.s :: T : t;$$

or, substituting R and r, for C and c (the ratio being the same), we obtain

$$R.H.S : r.h.s :: T : t;$$

that is, the tension at every point of the circumference of the two bases, will be directly as the product of the radius of the base, the altitude, and the specific gravities of the fluid.

PROBLEM.

blem. 25 and 23. To determine the ratio of the thicknesses that we ought to give to two cylinders, composed of flexible rings, in order to resist the pressure of the fluids contained in them.

Let us conceive the two cylinders to be cut through, parallel to their bases, the sections of which are shown in fig. 25, and fig. 26. And conceive the circular rings BSE, &c. $b\,s\,e$, &c. to be composed of an indefinite number of lamina, of which XYVZ, and $x\,y\,v\,z$, represent one in each; the resistances which these oppose to rupture, are evidently in the ratio of the number of such lamina, or to the thicknesses BS, $b\,s$, and the tanacity of the materials of which the tubes or cylinders are composed.

If therefore we denote the resistances which it is necessary for the metal to oppose, by F and f, the thicknesses by E and e, the tenacity of the metal by U and u; we shall have $F : f :: EU : e\,u$.

Now, in order that there may be an equilibrium between the resistance of the fluid and the strength of the tube, or between the tension and resistance, we must have $T = R = EU$, and $t = r = e\,u$; T and t denoting the tensions as determined in the preceding article; but in that article we found $T : t :: R\cdot H : r.\,h$, (supposing the densities in this case the same); therefore

$$EU : e\,u :: R\cdot H : r.\,h,$$

and
$$E : e :: \frac{R\cdot H}{U} : \frac{r.\,h}{u};$$

that is, the thickness of the metal ought to be proportional to the radii and heights directly, and as the tenacity of the metal reciprocally.

Consequently, when the question relates only to the proportionate dimension of the metal in the same tube, the bore being the same throughout, we see that the thickness in such cases ought to be proportional to the depth below the surface of the fluid in the general reservoir.

Practical application. 24. The above results obviously apply only to the proportional thickness and resistance of tubes; in order to connect them with the determination of the actual thickness, according to the nature of the metal, and the maximum of pressure to which tubes and cylinders are exposed, we must have some experiments on which to found our computation. In our section of Mechanics, we have detailed some experiments, performed by Mr. Barlow, on the strength of various kinds of iron, and which we believe is the only series of this kind carried on upon a sufficiently large scale to inspire us with any confidence in the results. The dreadful consequences that may attend a want of proper strength, have been but too often indicated by the fatal accidents that have taken place in the explosion of the boilers of certain steam vessels, particularly that which happened at Norwich, in 1817, where, we believe, *thirteen* persons lost their lives, and others were dreadfully wounded and mangled. It is, therefore, highly important for engineers engaged in the construction of such machines, to examine with attention the nature of the pressure, and to ascertain the thickness of metal necessary to resist it; the investigations given above, will furnish them with the proportional dimensions, and the section to which we have referred, will supply them with well established data for their actual thickness for any given pressure.

With respect to the pipes of water works, &c. the same fatal consequences are not to be apprehended, as in the cases to which we have above referred; but in these it is highly desirable to have data, on which we may with confidence, compute the requisite dimensions, so that, while we are under no fear of the explosion or the rupture of any of the vessels, we may avoid increasing the expences of the construction, by using a greater quantity of metal than is actually requisite to ensure safety.

§ IV. *Of the determination of the specific gravity of bodies.*

General principles.

25. In the twelfth section of our article STATICS, we have had occasion to illustrate the term *specific gravity*, and to show in what respects it differs from the *density* of a body: it will, therefore, be unnecessary in this place to enter again upon the subject; but we shall, in the present article, endeavour to explain the principles on which the different methods of ascertaining the specific gravity of solids and fluids are founded, and to give some account of the best of those methods: but previous to this, it may be proper to make a few observations which flow immediately from the definition of specific gravity, and the nature of bodies in general.

1. The magnitudes of two bodies being equal, their specific gravities will have the same proportion as their weights; and when the weights of bodies are equal, their specific gravities will be inversely as their magnitudes; consequently, we may say generally, that the weights of different bodies are to each other in the compound ratio of their magnitudes and specific gravities.

Let W and w denote the weights of two bodies;
S and s their respective specific gravities;
M and m their magnitudes;

and let c be the constant co-efficient which converts the preceding proportionalities into equalities, and we shall have

$$W = c\,SM,\ w = c\,sm;$$

whence

$$\frac{W}{w} = \frac{M}{m} \times \frac{S}{s}.$$

Assumed units.

In estimating the weights, magnitudes, and specific gravity of substances, some standard quantities are always assumed to which other bodies are referred; the letters w, m, s, may represent the standards, and each of them might be assumed equal to 1; but such assumption would not correspond with our present weights and measures. It will, therefore, be more eligible that m should represent some known magnitude which may be assumed as the unit of capacity, as a cubic inch, or cubic foot, &c. and s and w any convenient numbers.

We have shown in the article above referred to, that, by a most singular coincidence, the cubic foot of distilled water, at the temperature of 39° of Fahrenheit's thermometer (the temperature at which the density of this fluid is at its maximum), weighs very nearly 1000 oz. avoirdupoise, differing from it by a quantity scarcely perceptible. If, therefore, we make $w = 1000$, $m = 1$, and $s = 1000$, our preceding equation will be converted into

$$W = MS;$$

that is, *the weight of any body in avoirdupoise ounces, will be expressed by the product of the number of cubic feet into the specific of the body taken from that scale, in which the specific gravity of distilled water, at the temperature of* 39°, *is* 1000. See STATICS (art. 114).

Various systems of weights.

As we have unfortunately in this country a variety of weights, it frequently becomes necessary to convert the weight of a body from one denomination to another, and in particular, from avoirdupoise to troy weight, and the contrary. Now a troy ounce is to an ounce avoirdupoise in the ratio of 960 to 875; consequently $\frac{1000}{1728} \times \frac{875}{960} = \cdot 52746$ of an ounce troy, or 253·181[*] grains, the weight of a cube inch of water.

Hence the magnitude of a solid, estimated in cubic inches, is $= \frac{W}{253\cdot 181 \times S}$, when the weight W is given in grains, or $= \frac{W}{\cdot 52746 \times S}$, when the weight is in ounces troy weight; and, conversely, the weight $W = M \times S \times 253\cdot 181$, when the magnitude is expressed in inches, and the weight in grains troy; or $W = M \times S \times \cdot 52746$, when the magnitude is given in inches, and the weight in troy ounces. In all these cases, the specific gravity being supposed to be taken from that scale of relation in which the specific gravity of water is 1000.

From these equations we may readily determine, and with great accuracy, the diameter D of any small sphere, whose specific gravity is S′, its weight W being known in grains troy.

For the content of a sphere whose diameter is 1, being ·523598, we have

$$1 : \cdot 523598 :: 253\cdot 181 \text{ grains} :$$

$\cdot 523598 \times 253\cdot 181 = 132\cdot 5648$ grains, the weight of the sphere.

Therefore, since the weights are as the magnitudes and specific gravities conjointly; and the magnitudes of spheres are as the cubes of their diameters, we have for the weight W of the sphere, whose diameter is D,

$$W = 132\cdot 5648 \cdot D^3 . \frac{S}{1000};$$

and, consequently,

$$D = 1\cdot 961208 \sqrt[3]{\frac{W}{S}}.$$

In a similar manner, various other useful rules and theorems, applicable to the measurement of bodies either regular or irregular, may be easily deduced. The above, however, will, it is presumed, be deemed a sufficient indication of the method; at the same time that it shows the importance of the subject, and the necessity of a correct knowledge of the principles on which the theory is founded.

PROPOSITION.

Prop

26. *A body, when immersed in a fluid, will, when left to itself, sink, if the specific gravity be greater than that of the fluid; it will rise to the surface and float there, if its specific gravity be less than that of the fluid; but it will remain at rest in any position when the specific gravities of the body and fluid are equal.*

1. For the body has a tendency to descend by its own weight, and is supported or resisted by a force equivalent to the weight of an equal bulk of the fluid, or of as much fluid as will fill the space occupied by the body. If, therefore, the body be heavier than the fluid, bulk for bulk, its weight will be greater than the pressure upwards, caused by the action or resistance of the fluid; and consequently the latter pressure is not sufficient to prevent the body from sinking.

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[*] This exceeds the actual weight, as determined from experiment, by ·212 of a grain. See note, page 33.

2. If the body be specifically lighter than the fluid, its pressure downwards will be less than the upward pressure of the water at the same depth; consequently, as the greater force necessarily overcomes the less, and the upward pressure is the greater, the body will rise.

3. When the body and the fluid have the same specific gravity, equal masses of each are of the same weight; and consequently the force with which the body endeavours to descend, and the force which opposes it, are equal to each other; and as they act in contrary directions, the body will rest between them, so as neither to sink by its own weight, nor to ascend by the upward pressure of the fluid.

From the above we readily draw the following deductions; viz.

1. If by any contrivance the specific gravity of a body can be so varied, as to be at one time *greater*, and at another *less*, and then *equal* to the specific gravity of the fluid wherein it is immersed, the body will *sink* or *rise*, or remain at *rest*, according to the variations caused in its specific gravity.

This is the case in the experiment with those little glass images which some philosophers exhibit in their lectures, these being made to ascend or descend, or remain suspended, at pleasure.

2. If a solid, specifically heavier than a fluid, be immersed to a depth, which is to its thickness as the specific gravity of the solid to that of the fluid, and the pressure of the fluid from above be removed, the body will be sustained in the fluid: for the pressure from above being removed, the body is in the same state with respect to the contrary pressure, as if the same weight filled the whole space to the surface of the fluid; that is, as if its specific gravity and that of the fluid were equal.

This serves for the explanation of the common experiment of making lead swim, in consequence of being fitted to the bottom of a glass tube.

3. Hence, also, we see the meaning of the proposition, that all bodies, when immersed in a fluid, lose the weight of an equal volume of that fluid. The weight is not otherwise lost than as it is sustained by the action of a contrary force; and it therefore becomes obvious, why the weight of a bucket of water is not perceived while it is in the water; it is not because that weight is destroyed, but because it is supported; not because fluids do not gravitate, when they are in fluids of the same sort; but because there is a pressure in a contrary direction, which is precisely equal to their gravity. The weights thus lost by immerging the same body in different fluids, are as the specific gravities of the fluids; hence, bodies of equal weight, but of different volume, lose in the same fluid, weights which are reciprocally as the specific gravities of the bodies, or directly as their volumes.

4. The whole weight of a body which will float in a fluid, is equal to the weight of as much of the fluid as the immersed part of the body displaces.

5. Hence, the magnitude of the whole body, is to that of the part immersed, as the specific gravity of the fluid to that of the body. And if the body be any prism, with its base kept horizontally, the altitude of the prism will be to the depth immersed, as the specific gravity of the fluid to that of the body.

6. And because when the weight of a body taken in a fluid is subtracted from its weight out of the fluid, the difference is the weight of an equal volume of the fluid; this, therefore, is to its weight in the air, as the specific gravity of the fluid is to that of the body.

Consequently, if

W be the weight of a body in air,
W′ its weight in water, or any other fluid,
S the specific gravity of the body,
s the specific gravity of the fluid,

we shall have

$$W - W' : W :: s : S;$$

whence

$S = \frac{W}{W-W'}\, s =$ the specific gravity of the body, and

$s = \frac{W-W'}{W}.\, S =$ the specific gravity of the fluid;

so that *the specific gravities of bodies are as their weights in the air directly, and their loss in one and the same fluid inversely.*

To determine the specific gravity of bodies.

27. Hence, for two bodies connected together, or mixed together into one compound of different specific gravities, we may (supposing there is no penetration of dimensions) easily deduce the necessary equations.

Let the respective weights and specific gravities be denoted thus:

$H =$ weight of the heavier body in air;
$H' =$ weight of the same in water;
and S its specific gravity;
$L =$ weight of the lighter body in air;
$L' =$ weight of the same in water;
and S′ its specific gravity;
$C =$ the weight of the compound in air;
$C' =$ the weight of the same in water;
$\sigma =$ its specific gravity; and
$s =$ the specific gravity of water.

Then

$$(1.)\quad (H - H')\,S = H s.$$

$$(2.)\quad (L - L')\,S' = L s.$$

$$(3.)\quad (C - C')\,\sigma = C s.$$

$$(4.)\quad H + L = C.$$

$$(5.)\quad H' + L' = C'.$$

$$(6.)\quad \frac{H}{S} + \frac{L}{S'} = \frac{C}{\sigma}.$$

From which equation any of the above quantities may be found in terms of the others.

When the lighter body L is of less specific gravity than the water, L′ must be considered as negative; and to find its specific gravity, we must have recourse to a compound mass, as C; thus, because from equations 4 and 5,

$$L - L' = (C - C') - (H - H');$$

and from equation 2, $S' = \frac{L s}{L - L'}$, we shall have, by substituting for the latter denominator,

$$(7.)\quad S' = \frac{L s}{(C - C') - (H - H')};$$

or, if we deduce the value of S′ from the latter equation, we have

$$(8.)\quad S' = \frac{S \sigma L}{CS - H \sigma}.$$

Two fluids.

28. If a vessel contain two fluids of unequal density (such as water and mercury), and a solid of some intermediate specific gravity, be immersed under the surface of the lighter fluid and float on the heavier; the part of the solid immersed in the heavier fluid, is to the

whole solid, as the difference between the specific gravities of the solid and the lighter fluid, to the difference between the specific gravities of the two fluids.

The truth of this proposition or deduction will appear immediately from the following considerations. Let the specific gravity of the heavier fluid be s, and that of the lighter fluid s'; the part of the body immersed in the former $=$ B, and the part immersed in the latter $=$ B′; also the specific gravity of the body itself $=$ S.

Assume the area of the horizontal section of the solid coinciding with the contiguous surface of the two fluids $=$ A, and its perpendicular distance from the upper surface of the lighter fluid $= d$. Now the pressure on this area, as arising from the lighter fluid, would be $\mathrm{A}\,d\,s'$, if no body were immersed; but in consequence of the part B′ of the body, this pressure is reduced to $(\mathrm{A}\,d - \mathrm{B}')\,s'$; and the pressure arising from the body itself $= (\mathrm{B} + \mathrm{B}')\,\mathrm{S}$; which, added to the former, will give the whole force by which the section of the solid is urged downwards; and the pressure upwards against the same section is $\mathrm{A}\,d\,s' + \mathrm{B}\,s$.

But by the supposition the body is sustained in equilibrio by the opposite action of these forces, consequently they are equal to each other, and we have

$$\mathrm{A}\,d\,s' - \mathrm{B}'\,s' + (\mathrm{B} + \mathrm{B}')\,\mathrm{S} = \mathrm{A}\,d\,s' + \mathrm{B}\,s;$$

whence we find

$$\mathrm{BS} + \mathrm{B'S} = \mathrm{B}'\,s' + \mathrm{B}\,s;$$

or $$\mathrm{B} : \mathrm{B}' :: \mathrm{S} - s' : s - \mathrm{S};$$

or, lastly, $$\mathrm{B} : \mathrm{B} + \mathrm{B}' :: \mathrm{S} - s' : s - s'.$$

The latter analogy agrees with the enunciation of the proposition; and the one preceding it; viz.

$$\mathrm{B} : \mathrm{B}' :: \mathrm{S} - s' : s - \mathrm{S};$$

may be expressed in words, thus:

As the part of the solid within the heavier fluid, to the part within the lighter; so is the difference between the specific gravities of the solid and lighter fluid, to the difference between the specific gravities of the heavier fluid and the solid.

When the specific gravity of the lighter fluid is very small compared with that of the heavier or, that of the solid, then we may, without any sensible error, consider s' as zero; and the analogy becomes

$$\mathrm{B} : \mathrm{B} + \mathrm{B}' :: \mathrm{S} : s.$$

Specific gravity of air.

The specific gravity of air is very small, in comparison with that of water; and hence, it is very commonly neglected when we attempt to ascertain the specific gravity of a fluid, and that of a lighter solid; the rule generally given having simply a reference to the ratio between the whole volume and the part immersed. This is obviously inaccurate, because the air being a heavy fluid, every solid floating on a solid, and in air, is, in fact, a solid of intermediate specific gravity between those of the two fluids; and therefore strictly falls under the case above investigated. But, as we have before observed, the specific gravity of air, is so small, in comparison with most fluids we have to consider, that it may (except in cases where the utmost accuracy is required) be omitted without producing any important error; and it may be rendered perfectly accurate by subtracting the number expressing the specific gravity of the air, from the two numbers expressing the specific gravities of the solid, and that of the fluid on which it floats. The remainders will express the actual ratio between those specific gravities, and will then coincide with our preceding analogy, viz. we shall have

$$\mathrm{B} : \mathrm{B} + \mathrm{B}' :: \mathrm{S} - s' : s - s'.$$

29. Having laid down the preceding principles, it now only remains to illustrate the method of employing them, in order to find, practically, the specific gravity.

In order to this practical determination, it is necessary to make use of instruments of great delicacy: that is to say, such as will ascertain, with the utmost accuracy, the quantity and variation in the weights of bodies. The most improved instrument of this kind, the *hydrostatic balance*, will be described under the general term BALANCE. In the present article we shall suppose the weights ascertained, and thence show the method of finding the specific gravity of the bodies. The general problem may be divided into three cases.

1. *When the body is heavier than water.*

Find the weight of the body, both in the water and out of the water; and the difference of these will express the weight lost in water.

Then assuming B to denote the weight of the body out of water, and B′ its weight in water, S its specific gravity, and s that of the water; then our first equation (art. 27) becomes, by changing H into B,

$$(\mathrm{B} - \mathrm{B}')\,\mathrm{S} = \mathrm{B}\,s;$$

whence

$$\mathrm{S} = \frac{\mathrm{B}\,s}{\mathrm{B} - \mathrm{B}'}:$$

or, converting it into an analogy,

$$\mathrm{B} - \mathrm{B}' : \mathrm{B} :: s : \mathrm{S};$$

that is,

As the weight lost in water
Is to the absolute weight,
So is the specific gravity of water
To the specific gravity of the body.

2. *When the body is lighter than water.*

In this case, attach to it another body heavier than water, and of such magnitude that the mass compounded of the two may sink together. Weigh the denser and the compound body separately, both out of the water and in it, and find how much each loses in water, by subtracting its weight in water from its weight in air; and subtract the less of these remainders from the greater; then retaining the notation in art. 27, our equation (7) gives

$$\mathrm{S}' = \frac{\mathrm{L}\,s}{(\mathrm{C} - \mathrm{C}') - (\mathrm{H} - \mathrm{H}')};$$

or, converting this into an analogy,

$$(\mathrm{C} - \mathrm{C}') - (\mathrm{H} - \mathrm{H}') : \mathrm{L} :: s : \mathrm{S}';$$

that is,

As the last remainder above
Is to the weight of the light body in air,
So is the specific gravity of water
To the specific gravity of the body.

3. *When the specific gravity of the fluid is required.*

Take any body of known specific gravity, weigh it both in the fluid and out of it; and find the loss of weight by taking the difference of those two: then our equation (1) of the article above referred to becomes (by changing H into B),

$$(\mathrm{B} - \mathrm{B}')\,\mathrm{S} = \mathrm{B}\,s;$$

or, $$s = \frac{(\mathrm{B} - \mathrm{B}')\,\mathrm{S}}{\mathrm{B}}.$$

This equation is immediately transformed into the following proportion:

$$B : B - B' :: S : s;$$

that is,

As the weight of the body in air
Is to the weight lost in water,
So is the specific gravity of the body
To the specific gravity of the fluid.

The specific gravities of fluids, and even of solid bodies, may be also determined by means of an instrument called an hydrometer, of which there are various constructions, described under that article in our alphabetical part.

30. We have had occasion, in our historical sketch of the science of Hydrostatics, to refer to a celebrated ancient problem, solved by Archimedes, respecting the crown of Hiero, king of Syracuse; we may now enter more minutely upon this subject, which in its most general sense may be thus enunciated.

To find the respective weights of two known ingredients, in a given compound.

Adopting the notation of art. 27, and employing our 4th and 6th equations, viz.

$$H + L = C$$
$$\frac{H}{S} + \frac{H'}{S'} = \frac{C}{\sigma},$$

we shall hence find

$$H = \frac{(\sigma - S')\,S}{(S - S')\,\sigma}\,C$$
$$L = \frac{(S - \sigma)\,S'}{(S - S')\,\sigma};$$

which obviously express the quantity of the two ingredients, in terms of the quantities that are supposed by the question to be known.

These formulæ may be reduced into the form of a theorem, or rule, as follows:

Take the three differences of every pair of the three specific gravities, viz. of the compound, and of each ingredient; and multiply each specific gravity by the difference of the other two; then say

As the greatest product
Is to the whole weight of the compound,
So is each of the other two products
To the respective weights of the two ingredients.

But if, instead of finding the weights, we were to find the magnitudes M and M′ of the two ingredients, the specific gravities being as above; we should have the weight of M equal to MS, and the weight of M′ equal to M′S′; while the weight of the compound would be $(M + M')\sigma$.

Hence we should have

$$SM + S'M' = \sigma M + \sigma M',$$

or

$$SM - \sigma M = \sigma M' - S'M';$$

whence

$$M : M' :: \sigma - S' : S - \sigma,$$

and

$$M : M + M' :: \sigma - S' : S - S'.$$

It may be proper here to remark, that this deduction will not always obtain, because it is founded upon a supposition that the magnitude of the compound is equal to the sum of the magnitudes of the two ingredients. But it very frequently happens, that the magnitude of the mixture is less than this sum; a circumstance that is probably occasioned by two causes; the different magnitudes of the constituent particles of the bodies or ingredients, and their chemical affinity. But this rule is notwithstanding of use in determining the quantity of penetration or rarefaction, by comparing the computed magnitudes or densities, with those discovered from observation.

Of the construction of tables of specific gravities.

Tables of specific gravities.

31. We have already remarked, that in forming tables of specific gravities, any solid or fluid may be made use of for forming a standard by which other bodies may be compared; but by the universal assent of philosophers pure water has been at all times preferred, on account of its being more plentiful, and because it is found to be less anomalous than any other fluid whatever; being always the same in all parts of the earth at the same temperature.

With respect to temperature this will affect the specific gravity of all bodies, as far, at least, as observation has yet extended. Water, when heated from 60° to 100°, increases in its volumn nearly $\frac{1}{167}$th part of its own bulk; mercury $\frac{1}{243}$d part, and others more or less than these numbers. It follows, therefore, that an experiment determines the specific gravity only in that temperature at which the bodies are examined.

It will consequently be proper, in the more delicate kind of experiments, to note the temperature at which they are made; and as far as possible to choose some convenient degree, such as about 60° of Fahrenheit's thermometer, which may be generally procured in these climates. A temperature of 39° or 40° would be in many respects advantageous, particularly as this answers to the greatest degree of condensation of water: but it is not always to be obtained.

It will frequently be useful to know the variation which heat produces on pure water; and this will be seen by the results in the following table, which are extracted from Mr. Gilpin's results, published in the Philosophical Transactions for 1792.

Temperature of Water.	Bulk of Water.	Specific Gravity.
35°	99910	1·00090
40	99070	1·00094
45	99914	1·00086
50	99932	1·00068
55	99962	1·00038
60	1·00000	1·00000
65	1·00050	·99950
70	1·00106	·99894
75	1·00171	·99830
80	1·00242	·99759
85	1·00320	·99681
90	1·00404	·99598
95	1·00501	·99502
100	1·00602	·99402

TABLE OF SPECIFIC GRAVITY,

32. *Extracted from Davies, Lavoisier, Young, and other authentic sources.*

Note.—Water at 60° is assumed 1000 specific gravity.

MINERAL PRODUCTIONS.

Table of specific gravity.

Platina, purified	19500
hammered	20336
Platina wire	21041
Platina, laminated	22069
Pure gold, cast	19258
hammered	19361

Hydrodynamics.

Minerals.

Gold, 22 carats fine, of the standards of London and of Paris, cast	17486
hammered	17589
Gold, French coin, $21\frac{1}{16}$ carats, fine, cast	17402
coined	17647
trinket, 20 carats, fine, cast	15709
hammered	15774
Mercury	13568
Lead, cast	11352
Litharge	6300
Silver, pure, cast	10474
hammered	10510
Silver, Parisian, 11 den. 10 gr. fine, cast	10175
hammered	10376
Silver, French coin, 10 den. 21 gr. fine, cast	10047
hammered	10407
Bismuth, cast	9822
Copper, cast	8788
wire	8878
Brass, cast	8395
wire	8544
Cobalt, cast	7812
Nickel, cast	7807
Iron, cast	7207
bar	7788
Steel, hard, not screwed	7816
screwed	7818
soft, not screwed	7833
screwed	7840
Loadstone	4800
Haematite	4200
Tin, cast	7291
screwed	7299
Zinc, cast	7190
Antimony, cast	6702
Antimony, glass of	4916
Antimony, crude	4064
Tungstein	6066
Arsenic, cast	5763
Molybdena	4738
Spar, ponderous	4430
Jargon of Ceylon	4416
Ruby, oriental	4283
spinelle	3760
ballas	3645
Brasilian	3531
Pseudotopas	4270
Garnet, Bohemian	4188
Syrian	4000
Sapphire of Puy	4076
oriental	3994
Brasil	3130
Topas, oriental	4010
Saxon	3564
Brasilian	3536
Emery	4000
Hyacinth	3687
Beryl, or oriental aquamarine	3548
occidental aquamarine	2722
Diamond, rose coloured	3531
white	3521
lightest	3501
Manganese, crude	3530
Schorl, black, crystallized	3385
amorphous	2922
Glass, flint	3329

Glass, white	2892
bottle	2732
green	2642
of St. Gobin	2488
Fluor, red	3191
green	3181
violet	3175
blue	3168
white	3155
Hone, black and white	3131
white	2876
Granitello	3062
Serpentine, green, of Dauphine	2988
green	2896
Ophite	2972
Jade, green	2966
white	2950
Mica, black	2900
Basaltes, from the Giants' Causeway	2864
from Auvergne	2415
Marble, white Parian	2837
green	2741
red	2725
white, of Carrara	2716
Jasponyx	2816
Chrysolith	2782
of Brasil	2692
Emerald, Peruvian	2775
Porphyry, red	2765
Jasper, grey	2764
violet	2711
yellow	2710
brown	2691
red	2661
Alabaster, white, antique	2730
Calcarious spar, rhombic	2715
pyramidal	2714
Slate	2671
Pitch stone, red	2669
blackish	2319
yellow	2086
black	2049
Onyx, pebble	2664
Chalcedony, transparent	2664
Granite, Egyptian, red	2654
Rock crystal, pure	2653
Amorphous quartz	2647
Agate, onyx	2637
Carnelian	2613
Sardonyx	2602
Purbeck stone	2601
Flint, white	2594
blackish	2581
Agate, oriental	2590
Prase	2580
Portland stone	2570
Whetstone of Auvergne	2563
Zeolith, red	2486
crystallized	2083
Mill-stone	2483
Paving-stone	2415
Touchstone	2415
Porcelain, Chinese	2384
of Limoge	2341
of Seves	2145
Lapis obsidianus	2348

Selenite	2322
Sulphate of potash	2250
of soda	2200
Grindstone	2142
Salt	2130
Sulphur, native	2033
melted	1990
transparent	1950
Nitre	2000
Brick	2000
Alabasta	1874
Plumbago	1860
Sulphate of zinc	1850
Alum	1720
Borax	1718
Sulphate of iron	1700
Asphaltum	1400
Coal, Scots	1300
Newcastle	1270
Staffordshire	1240
Jet	1238
Ice, *probably*	930
Pumice-stone	914

LIQUIDS.

Sulphuric acid	1840
Ph. London	1850
Nitrous acid, Ph. London	1550
Nitric acid	1217
Salt, solution of	1244
Water 27, Salt 10	1240
Water 3, Salt 1	1217
Water 12, Salt 1	1060
Water of the Dead Sea	1240
Sea water	1026
Solution of caustic soda	1200
Muriatic acid	1194
Water of the Seine, filtered	1001
Naphtha	708

Substances partly Minerals.

SOLIDS.

Acetite of lead	2700
Tartrite of antimony	2100
Muriate of ammonia	1400

LIQUIDS.

Æther, sulphuric	7340
nitric	9088
muriatic	7298

ELASTIC FLUIDS.

	Kirwan. Barometer, 30.	*Lavoisier.* Thermometer 52°.
Sulphureous acid gas	2·265	—
Carbonic acid gas	1·500	·00176
Nitrous gas	1·194	—
Hepatic gas	1·106	—
Oxygen gas	1·103	·00137
Atmospheric air	1·000	·00128
Nitrogen gas	·985	·00120
Ammoniacal gas	·600	—
Hydrogen gas.	·084	·000096

VEGETABLE PRODUCTIONS.

Vegetable productions.

Crystal of tartar	1850
Liquorice, extract of	1723
Opopanax	1622
Sugar, white	1606
Potash, solution of	1570
Gum Arabic	1452
Honey	1450
Catechu	1398
Aloes, socotrine	1379
hepatic	1358
Bdellium	1371
Myrrh	1360
Cocoa shell	1345
Opium	1336
Assafœtida	1327
Tragacanth	1316
Ivy gum	1294
Scammony, from Smyrna	1274
from Aleppo	1235
Sarcocolla	1268
Myrrh	1250
Guaiacum	1229
Gamboge	1221
Resin of jalep	1218
Galbanum	1212
Gum ammoniac	1207
Dragon's blood	1204
Sagapenum	1201
Lignum nephriticum	1200
Olibanum	1173
Pitch	1150
Copal, opaque	1140
transparent	1045
Euphorbium	1124
Storax	1110
Oil of sassafras	1094
Benzoin	1092
Sandarac	1092
Yellow amber	1078
Mastic	1074
Frankincense	1071
Acetic acid	1062
Oil of cinnamon	1044
Anime, occidental	1043
oriental	1028
Malmsey, Madeira	1038
Oil of cloves	1036
Gall nuts	1034
Elemi	1018
Cider	1018
Vinegar, distilled	1009
Water at 60°	1000
Bourdeaux wine	994
Burgundy wine	991
Turpentine liquid	991
Camphor	988
Oil of mint	975
Oil of nutmeg	948
Linseed oil	940
Oil of carraway	940
of marjoram	940
of spike	936
of rosemary	934
Elastic gum	933
Oil of poppy seed	929
of beech mast	917
of almonds	917
of rape	913

Oil of lavender	893
of oranges	888
essential, of turpentine	870
Balsam of Tolu	896
Acetic ether	866
Indigo	769

ANIMAL SUBSTANCES.

Animal substances.

Pearl	2750
Coral	2680
Sheep's bone, recent	2222
Oyster shell	2092
Ivory	1917
Stag's horn	1875
Ox's horn	1840
Blade bone of an ox	1656
Lac	1139
Isinglass	1111
Egg of a hen	1090
Human blood	1053
dº buffcoat	1056
serum	1028
red globules	1126
Milk, ewe's	1041
ass's	1035
mare's	1034
goat's	1034
cow's	1032
woman's	1020
Whey, of cow's milk	1019
Wax, white	968
yellow	965
Lard	948
Spermaceti	943
Butter	942
Tallow	942
Fat of hogs	937
veal	934
mutton	923
beef	923
Ambergrease	926
Lamp oil	923
Solution of pure ammonia	897

WOODS.

Woods.

Pomegranate tree	1354
Lignum vitæ	1333
Box, Dutch	1328
French	912
Letter wood, Surinam	1286
Ebony	1177
Heart of oak, 60 years felled	1170
Oak, English, just felled	1113
the same, seasoned	743
usually stated at	925
of Canada	872
Dantzic	756
Adriatic	996
Bog oak, of Ireland	1046
Teak, of the East Indies	from 745 to 657
Poon, of the East Indies	from 771 to 579
Mahogany	from 1063 to 637
Pear tree trunk	646
Yellow wood, Cape of Good Hope	657
Stink wood, ditto	681
Medlar tree	944
Olive wood	927
Logwood	931
Beech	852
Ash	from 845 to 600
Yew, Spanish	807
Dutch	788
Alder	800
Elm	from 800 to 600
Apple tree	793
Plum tree	755
Maple	755
Cherry tree	715
Quince tree	705
Orange tree	705
Walnut	671
Pitch pine	660
Red pine	657
Yellow pine	529
White pine	420
Fir, of New England	553
of Riga	753
of Mar Forest, Scotland	696
yellow	657
white	569
male	550
female	498
Cypress	644
Lime tree	604
Filbert wood	600
Willow	585
Cedar	560
Juniper	556
Poplar, white Spanish	529
common	383
Sassafras wood	482
Larch, of Scotland	530
Cork	240

33. With respect to the above table of the specific gravity of different woods, the results are selected from what are generally conceived to be the best authorities; yet they do not agree with some accurate experiments that we have made on several of the species above enumerated: but the fact seems to be, that there is so much difference in the weight of this substance, according to the soil in which the tree grows, the time after it is felled before the observations are made, and the part of the tree which is employed in the experiment, that little agreement can be expected in such determinations. It appears, from numerous experiments performed by Du Hamel, the detail of which are found in his treatise "Sur l'Exploitation des Bois," that the specific gravity of different trees of the same species, will vary as much as in the ratio of 5 : 7, according to the soil which produced them; and from other experiments, reported in Barlow's "Essay on the Strength of Timber," &c. it appears, that in the seasoning, English oak will lose ⅓d of its original weight, or specific gravity; and that the specific gravities of the top and butt of the same tree, vary as much as in the ratio of 4 to 5, and even more.

As we conceive the results above referred to, are highly interesting, in relation to our present enquiry, we shall make a short abstract of them, from the work above referred to.

34. **RESULTS of Experiments on eight pieces of English Oak, from 10⅜ inches to 3 inches in thickness, in order to ascertain the loss of weight, monthly, &c. in seasoning. By Mr. BENJAMIN COUCH, of his Majesty's Dock-yard, Plymouth.**

These pieces were laid on the beams of a smith's shop, and placed at such a distance from the forges that the fire might only operate to keep the air dry.

They were converted from trees just received from the forest, and weighed every month, from February, 1810, to August, 1812; at which latter period, observing the thickest pieces lost but little of their weight, the weighing of them monthly was discontinued; after which it was only performed annually, as shown in the following Table.

Order.	Length.	Breadth.	Thickness.	Feb. 1810. Weight.	March.	April.	May.	June.	July.	August.	Sept.	Oct.	Nov.	Dec.	January, 1811.	Feb. * Very much rain since last weighed.	March.	April. Rained several days previous to weighing	May.	June.	July. Constant rain for two days previous to weighing
	Inchs.	Inchs.	Inchs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.
1	24¾	16⅜	10⅜	163½	154½	149½	144¾	140½	137¾	135¼	133	131¾	130¼	129¾	129	130¼	127¼	127¾	125¼	124¼	124½
2	25⅝	14½	9¼	133	123¼	118	113½	109¼	106¾	104½	102¾	101½	100¼	99½	99	100¼	98	97	96½	95¼	96
3	30⅝	16⅛	8⅛	164	155¼	151¼	147	143½	141	139¼	137¼	136	134¾	134¼	133¾	135	132	132¾	130	129¼	129½
4	31½	12⅝	7⅛	104½	99	96	92¾	90¼	88¼	87	85¼	84½	84	83¼	82¾	84	81¾	83	80¼	79½	80¾
5	39¾	16⅜	6	163½	148¾	142½	135¾	130¼	127	123¾	121	119¼	117¾	117½	116¾	118	115	116	113¼	112¼	112½
6	30½	12¾	5⅛	77¼	71½	68¾	66¼	64	62	61	59¾	58¾	58¼	58	57¾	57¾	57	58½	56¼	55¾	55½
7	37⅞	14⅛	4	92	82¼	78	75	71¼	69½	67¾	66	65	64½	63¾	63½	65	62¾	64	61¾	61	62¼
8	38¾	14¼	3	74½	65½	60¾	56¾	53¼	51½	50½	49¼	48¼	47¾	47¾	47½	47¼	47	46½	46¼	46¼	46
			Total...	972¼																	

Order.	August.	Sept.	Oct.	Nov.	Dec.	January, 1812.	Feb.	March.	April.	May.	June.	July.	August, 1812.		August, 1813.	August, 1814.	August, 1815.
	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.		lbs.	lbs.	lbs.
1	121¾	122	119¾	121	119¾	119	118¾	118½	117½	117¼	116¾	116	115½	...	111¼	108½	106¼
2	93¼	92¼	92	93¾	91¾	91¼	91¼	91	90¾	90½	89¼	89¼	89	...	85¾	85	84½
3	127	127½	125½	126½	125	124½	124	124	123	122¾	122	121½	121	...	116¾	114¼	112½
4	78½	79¼	77½	77	77¼	76½	76¾	76¾	76¼	75¾	75½	75¼	74¾	...	72½	71¼	70½
5	109¾	109	108¼	110	108½	107¼	107½	107¾	106¾	106½	106	105¾	105¾	...	103¼	103	102¾
6	54½	56	54	56	54	53¾	53½	53½	53¼	53	52¾	52¼	52½	...	51½	51	51¼
7	60¼	59½	59½	59¼	59½	59	59	59¼	58¾	58¼	58	58¼	58½	...	57¼	57⅛	57¼
8	45½	45½	45½	45¾	45¾	45¾	45¾	45¾	45½	45½	45½	45½	45¼	...	45	45⅛	45
																	630½

	Weight. lbs.
February, 1810.	972¼
August, 1816 .	630½
Loss of Weight.	341¾

Lost in seasoning little more than one-third.

* The state of the weather is noticed, in order to account for the *increase* of weight in some of the pieces.

35. *Experiments made in order to determine the weight of a cubic foot, of different kinds of wood; the foreign when first imported, and those of English growth, as soon as felled. Also the weight of each when fully seasoned; from different parts of the same tree.* By Mr. B. Couch.

Species (in the language of commerce.)	Country where produced.	What part of the tree.	Lateral dimensions.		Weight of a cubic foot in ounces avoirdupois, or specific gravity.	
			When first planed.	When seasoned.	When first planed.	When seasoned.
			Inches.	Inches.		
Riga Fir, superior	Russia .	*Butt .	18 diameter . .	$17\frac{3}{4}$ diameter † . .	672	644
		Top .	11 by 11 . .	$10\frac{13}{16}$ by $10\frac{3}{4}$ . .	546	552 ‡
Ditto, inferior	Russia .	Butt .	12 diameter . .	$11\frac{7}{8}$ diameter . .	577	494
		Top .	$8\frac{3}{4}$ diameter .	$8\frac{1}{2}$ diameter . .	464	464
Pitch Pine .	Baltimore, America .	Butt .	10 by 10 . . .	$9\frac{3}{4}$ by $9\frac{3}{4}$ . . .	755	741
		Top .	$7\frac{1}{2}$ by $7\frac{1}{2}$ . .	$7\frac{3}{8}$ by $7\frac{3}{8}$ . . .	518	524
Ditto . . .	Virginia, America .	Butt .	$18\frac{3}{8}$ diameter .	$18\frac{1}{4}$ diameter . .	628	597
		Top .	10 by 8 . . .	$9\frac{15}{16}$ by $7\frac{15}{16}$ . . .	540	529
Yellow Pine .	Canada .	Butt .	18 by 18 . . .	$17\frac{7}{8}$ by $17\frac{7}{8}$ . . .	683	461
		Top .	16 by 16 . . .	$15\frac{7}{8}$ by $15\frac{7}{8}$ . . .	495	420
White Pine .	American States .	Butt .	12 by 12 . . .	$11\frac{7}{8}$ by $11\frac{3}{4}$ . . .	555	405
		Top .	12 by 12 . . .	$11\frac{7}{8}$ by $11\frac{7}{8}$ . . .	633	448
Northern Pine	New York	Butt .	$17\frac{5}{8}$ diameter .	$17\frac{5}{8}$ diameter . .	658	549
		Top .	$7\frac{3}{4}$ by $7\frac{1}{2}$ . . .	$7\frac{11}{16}$ by $7\frac{7}{16}$ . . .	432	416
White Pine .	New Brunswick .	Butt .	12 by 12 . . .	$11\frac{15}{16}$ by $11\frac{7}{8}$ . .	679	436
		Top .	8 by 9 . . .	$7\frac{15}{16}$ by $8\frac{15}{16}$ . . .	411	368
Red Pine . .	Canada .	Butt .	12 by 12 . . .	$11\frac{3}{4}$ by $11\frac{3}{4}$ . . .	672	569
		Top .	11 by 9 . . .	$10\frac{5}{8}$ by $8\frac{3}{4}$ . . .	570	503
Spruce Spar .	Halifax .	Butt .	$8\frac{1}{2}$ diameter . .	$8\frac{3}{8}$ diameter . . .	587	580
		Top .	$5\frac{1}{2}$ diameter . .	$5\frac{3}{8}$ diameter . . .	541	554
Ditto . . .	Canada .	Butt .	7 diameter . .	$6\frac{15}{16}$ diameter . .	528	524
		Top .	$4\frac{1}{2}$ diameter . .	$4\frac{3}{8}$ diameter . .	485	512
Poon . . .	East Indies	Butt .	17 diameter . .	$16\frac{3}{4}$ diameter . .	651	576
		Top .	9 by 9 . . .	9 by $8\frac{7}{8}$	771	695
Teak . . .	East Indies	Butt .	12 diameter . .	12 diameter . . .	662	657
		Top .	$6\frac{1}{2}$ by $6\frac{1}{2}$ . .	$6\frac{1}{2}$ by $6\frac{1}{2}$ . . .	688	675
Yellow Wood .	Cape of Good Hope	Butt .	$5\frac{1}{2}$ diameter .	$5\frac{7}{16}$ diameter . .	661	657
		Top .	5 diameter . .	5 diameter . . .	632	630
Stink Wood .	Ditto . .	Uncertain	9 by 4 . . .	9 by 4	700	681
Letter Wood .	Surinam .	Butt .	$4\frac{1}{4}$ by 4 . . .	$4\frac{1}{4}$ by 4	1286	1286
Cedar . . .	Spanish America	Root .	$5\frac{7}{8}$ by 6 . . .	$5\frac{7}{8}$ by 6	722	681
		Trunk .	12 by $5\frac{1}{2}$ . . .	12 by $5\frac{1}{2}$ . . .	457	453
Ditto . . .	Canada .	Uncertain	14 by 14 . . .	$13\frac{1}{4}$ by $13\frac{1}{4}$ . . .	909	753
Oak	England .	Butt .	12 by $11\frac{3}{4}$ . .	$11\frac{5}{8}$ by $11\frac{3}{8}$ . . .	1113	743
		Top .	$6\frac{1}{2}$ by 6 . . .	$6\frac{3}{16}$ by $5\frac{3}{4}$ . . .	1071	777
Elm . . .	Ditto . .	Uncertain	11 by 11 . . .	$10\frac{3}{4}$ by $10\frac{3}{4}$ . . .	940	588
Bog Oak . .	Ireland .	Ditto .	3 by $1\frac{1}{2}$ . . .	3 by $1\frac{1}{2}$	1046	1046

* The butts and tops were cut from the same tree.

† When diameter is expressed, the pieces were cylindrical; the others were parallelopipedons.

‡ It happens in some cases, that the specific gravity is greater after seasoning than before; this arises from the pieces havin more in dimensions, than in weight.

§ V. *Of the equilibrium of floating bodies.*

Preliminary remarks.

36. In order that a heavy body may preserve itself in equilibrio on the surface of a fluid at rest, it is in the first place necessary that its weight be less than that of a volume of the fluid equal to its own. There are some exceptions to this rule, in the case of a very small body, the matter of which exercises no attraction on the particles of the fluid, or where its action is much less than that of the fluid on itself. Thus, for example, every one knows the experiment by which a needle is made to float on the surface of wa in a bason or other vessel: this arises from the v small weight of the body not being sufficient to o come the natural cohesion or attraction of the part i of the water towards each other; a consideration w will not be attended to in this article: we shall sup the bodies, whose equilibrium is to be investigated, sufficient magnitude to overcome every resistanc the kind above alluded to.

The condition that the weight of the body is than an equal volume of the fluid being fulfilled, the body will be in part immersed in the fluid; that is, to such a depth, that the weight of the fluid displaced is equal to that of the body; and when these two weights are equal, the body will remain in equilibrio, if its centre of gravity, and that of the displaced fluid, be in the same vertical line; for the body is kept in equilibrio by the opposite action of two systems of forces, viz. the vertical action of gravity on the several particles of the body, by which it tends downwards, and the opposite vertical pressure of the fluid upwards, which counteracts the former. Now, since both these systems of forces are parallel to themselves, and to each other, their resultants are also parallel, and consequently both vertical; and since their opposite action preserves the body in equilibrio, they are necessarily found in the same vertical line, which, from what has been demonstrated of parallel forces, passes through their respective centres of gravity. In homogeneous bodies, the centre of gravity of the immersed part of the body, coincides with the centre of gravity of the fluid displaced; and since the weight of the displaced fluid is equal to that of the body, it follows, that the densities are in the inverse ratio of the volumes, or that the volume of the immersed part is to that of the whole body as the density of the latter is to that of the fluid, as we have already seen in the preceding section. Consequently the research of the position of equilibrium of a heavy body on a fluid of given density, greater than itself, is reducible to a problem of pure geometry, viz.

It is required to cut any proposed body by a plane, so that the volume of one of the segments, may be to that of the whole body in a given ratio; and such that the centre of gravity of the whole body, and that of one of its segments, may be both found in a line perpendicular to the cutting plane.

In order to the complete solution of this problem, it is necessary, in each particular case, to express the two conditions of equilibrium, by means of equations, the solutions of which will make known all the directions that can be given to the cutting plane, and whence necessarily result all the possible positions of equilibrium of the body.

Note. The cutting plane above referred to, is usually denominated the *plane of floatation*, and its contour the *line of floatation*; also the vertical line which we have seen passes through the two centres of gravity, is called the *line of support*; and the intersection of this line, with the axis passing through the centre of gravity (about which the body revolves through an indefinitely small angle), is the *meta-centre*.

37. Let us first, as one of the most simple cases, investigate the conditions of equilibrium of a right triangular prism, placed in a fluid with its edges horizontal, and consequently so that its plane of floatation is parallel to them.

It is obvious here (and it is the same with all prismatic or cylindric bodies, that have their bases perpendicular to the sides or edges, and which are placed horizontally) that the positions of equilibrium are independent of the length of the solid, that is, of the distance of the bases; we may, therefore, abstract from this consideration, by determining only the line of intersection of the plane of floatation with one of the two bases, or with any vertical section.

Let, then, ABC (fig. 27) be a vertical section of the given prism. It may happen that this prism is placed in the fluid, so that one of its edges only is immersed, or it may have two, which present two cases that are best considered separately.

One edge immersed.

Fig. 27.

In the first place, let us suppose that the summit or angle C only is plunged into the fluid, and that $m\,n$ is the intersection of the cutting plane with the section or base, which intersection obviously represents the line of level of the fluid.

Call a, b, c, the given sides of the triangle ABC, a denoting the side opposite the angle A, b the side opposite the angle B, and c that opposite the angle C: also let x and y be the unknown sides C n, C m, of the triangle C $n\,m$; so that we may have

$BC = a$, $AC = b$, $AB = c$, $Cn = x$, and $Cm = y$;

and since the surface of a triangle is equal to the product of any two of its sides with half the sine of the included angle, we shall have

$$ABC = \tfrac{1}{2}\, a\, b \sin C,\ C\,m\,n = \tfrac{1}{2}\, x\, y \sin C.$$

Now the entire prism, and the part or prism immersed in the fluid, being to each other as their bases ABC and $m\,n$ C, the lengths being equal, we must have

$$m\,n\,C : ABC :: r : 1,$$

r being a quantity less than unity, or the ratio which denotes the density or specific gravity of the body, that of the fluid being unity. Substituting, therefore, for $m\,n$ C and ABC, their values $\frac{1}{2}\, x\, y \sin C$ and $\frac{1}{2}\, a\, b \sin C$, and suppressing the common factor sin C, the above analogy becomes

$$x\,y : a\,b :: r : 1;$$

whence $$x\,y = r\,a\,b. \qquad (1)$$

Let D represent the middle of the base AB, and draw the line CD; on this right line take $DG = \frac{1}{3}\,DC$, and the point G will be the centre of gravity G of the triangle ABC (art. 125, STATICS).

In the same manner, k being the middle of $m\,n$, if we take $k\,g$ equal to one-third of k C, the point g will be the centre of gravity of the triangle C $m\,n$. Now the line G g must be perpendicular to the line $m\,n$; but because the lines CD, C k, are cut proportionally at the points G and g, the right lines D k and G g are parallel; therefore, the right line D k which joins the middle of the two bases AB and $m\,n$, is also perpendicular to $m\,n$; consequently the two right lines D m and D n are equal.

Reciprocally, if we have $Dm = Dn$, the right line D k will be perpendicular to $m\,n$, as also its parallel G g; hence, in order that the line G g, which joins the two centres of gravity G and g, may be perpendicular to the section $m\,n$ of the cutting plane; it is necessary, and it is sufficient, that the values of the two quantities D m and D n be equal to each other.

In order to determine these values, make $CD = h$, and denote by α and β the two parts DC m, and DC n into which the angle C is divided; then with reference to the triangle CD m, we shall have

$$(D\,m)^2 = (CD)^2 + (C\,m)^2 - 2\,CD \cdot C\,m .\ \cos DC\,m,$$

or $$(D\,m)^2 = h^2 + y^2 - 2\,h\,y .\ \cos\beta;$$

the triangle CD m will, in like manner, give

$$(D\,n)^2 = h^2 + x^2 - 2\,h\,x \cos\alpha;$$

and equating the values of D m and D n, we have

$$y^2 - 2\,h\,y .\ \cos\beta = x^2 - 2\,h\,x .\ \cos\alpha. \qquad (2)$$

This equation, and the preceding one, viz.

$$x\,y = r\,a\,b,$$

express the two conditions of the problem, and are sufficient for determining the two unknown quantities x and y.

Our second equation gives

$$y = \frac{r\,a\,b}{x};$$

which substituted in the first, it becomes

$$\frac{r^2\,a^2\,b^2}{x^2} - \frac{2\,h\,r\,a\,b}{x}\cos\beta = x^2 - 2\,h\,x\cdot\cos\alpha;$$

and this reduces immediately to

$$x^4 - 2\,h.\cos\alpha.\,x^3 + 2\,h\,r\,a\,b.\cos\beta.\,x - r^2\,a^2\,b^2 = 0. \quad (3)$$

The value of x is, therefore, determined by means of an equation of the fourth degree, and the value of y is then immediately found from the equation

$$y = \frac{r\,a\,b}{x}.$$

We know from the principles of algebra that every equation of an even degree, of which the last term is negative, has two real roots, the one positive, and the other negative; our equation (3) has, therefore, necessarily two such roots, but the other two may be either real or imaginary; if they are real, we know from the property of equations that they must be both positive. The body, therefore, that we have been considering, immersed with one edge only in the fluid, may have three different positions of equilibrium, but it cannot have more than three.

One base immersed.

If we suppose the angle or vertex C out of the fluid, or the two angles A and B below the level $m\,n$, the problem will still be of the same kind as the preceding, with this difference only, that we change the density r, in our equations (1) and (3), into $1 - r$. In fact, because the centres of gravity of the triangles ABC, and of its two parts C$m\,n$ and $m\,n$ AB, are in the same right line, the triangles C$m\,n$ and CAB have their centres of gravity in the right line perpendicular to $m\,n$; so that we shall have, in the first place, equation (2), the same in both cases, also the proportion

$$m\,n\,\mathrm{AB} : \mathrm{CAB} :: r : 1$$

is readily converted into

$$\mathrm{C}\,m\,n : \mathrm{CAB} :: 1 - r : 1;$$

whence we draw an equation analogous to equation (1); viz.

$$x\,y = (1 - r)\,a\,b;$$

and if we make use of this equation in order to eliminate y out of our second equation, we arrive immediately at the following expression; viz.

$$x^4 - 2\,h.\cos\alpha.\,x^3 + 2\,h(1 - r)\,a\,b\cos\alpha.\,x - (1 - r)^2\,a^2\,b^2 = 0; \quad (4)$$

which, as we have stated, is precisely analogous to our equation (3); and, therefore, the same in this case as in that, we conclude that the body may have three positions of equilibrium, or at least that it cannot have a greater number than three.

We arrive at the very same results which ever of the three edges of the prism we conceive to be immersed in the fluid, as also for every two edges, as in the second case above; these all together form six cases, and in each there *may* be three positions of equilibrium; and consequently the entire prism *may* have *eighteen* such positions, but it cannot have more.

Isosceles triangular prism.

Fig. 28.

38. When the triangle ABC is isosceles, as in (fig. 28), the calculation of the position of equilibrium becomes a little more simple; for supposing AC $=$ BC, and the angle C immersed in the fluid, we may dispense with equation (3), and resolve directly equations (1) and (2). In this case the right line CD is perpendicular to the middle of AB, and the right angled triangle ADC gives

$$h^2 = a^2 - \tfrac{1}{4}\,c^2\,a.\cos\alpha = h;$$

substituting these values, and making also $a = b$, and $\beta = \alpha$, in equations (1) and (2), they become

$$x\,y = r\,a^2,$$

$$y^2 - x^2 - \left(\frac{4\,a^2 - c^2}{2\,a}\right)(y - x) = 0;$$

the latter is satisfied by making $x = y$; whence the first becomes

$$x^2 = y^2 = r\,a^2, \text{ or}$$

$$x = y = a\surd r.$$

According to this mode of solution, we find but one position of equilibrium, which is that where the base of the triangle is horizontal and parallel to the surface of the fluid; but we must not thence conclude that only one position is possible. We obviously only obtained a partial solution of our second equation, by supposing $x = y$; let us, therefore, now find the solution of the two equations on more general principles.

With this view we may divide the second equation by $y - x$, and we obtain

$$y + x = \frac{4\,a^2 - c^2}{2\,a};$$

squaring this

$$y^2 + 2\,x\,y + x^2 = \left(\frac{4\,a^2 - c^2}{2\,a}\right)^2;$$

subtract $\quad 4\,x\,y \quad = 4\,r\,a^2,$

and we have

$$y^2 - 2\,x\,y + x^2 = \left(\frac{4\,a^2 - c^2}{2\,a}\right)^2 - 4\,r\,a^2;$$

$$\text{whence } y - x = \surd\left\{\left(\frac{4\,a^2 - c^2}{2\,a}\right)^2 - 4\,r^2\,a^2\right\}$$

$$y + x = \frac{4\,a^2 - c^2}{2\,a};$$

by addition and subtraction,

$$y \text{ or } x = \frac{4\,a^2 - c^2 \pm \surd\{(4\,a^2 - c^2) - 16\,r\,a^4\}}{4\,a}. \quad (5)$$

We may take each of the two roots successively, the one for x, and the other for y; and when these roots are real and positive, and less than a, there will result two new positions of equilibrium, in which the base of the prism is inclined to the surface of the fluid. If the radical is zero, the two roots become equal, and we shall then have but one position of equilibrium, which will be the same as that we have already found; for in order that the radical may disappear, we must have

$$4\,a^2 - c^2 = 4\,a^2\surd r,$$

which is the same as our preceding case; viz.

$$y = x = a\surd r.$$

In the same manner as, in the preceding article, we determined the positions of equilibrium when the base was immersed in the fluid, from the case previously found for the vertex, so also in this, we shall determine the same by merely substituting $1 - r$ instead of r.

In all cases, therefore, there will be one position of equilibrium with the base immersed, and which will be given by the equation

$$y = x = a\surd(1 - r);$$

and in this the base will be horizontal.

And as, in the former case, two other positions of equilibrium may sometimes be found from our equation (5); so also the analogous equation

$$x \text{ or } y = \frac{4a^2 - c^2 \pm \surd\{(4a^2 - c^2)^2 - 16(1-r)a^4\}}{4a}$$

will give two in this. It is obvious, however, that as the conditions of the roots being real depend upon the radical part of the expression, it may happen that in one case there are three positions of equilibrio, while in the other there is only one; for it is obvious that we may have

$$(4a^2 - c^2)^2 > 16ra^4; \text{ but } < 16(1-r)a^4,$$

or $(4a^2 - c^2)^2 > 16(1-r)a^4$; but $< 16ra^4$.

39. Let us next consider the case in which the triangle is equilateral: here we may make $a = b = c$ in all the formulæ where these quantities enter. Consequently our preceding determination for the equal values of x and y for the isosceles triangle will not be changed, but will remain as before; viz.

$$x = y = a \surd r.$$

And with regard to equation (5), expressing the unequal values of these quantities, we have only to write a for c, and it becomes

$$x \text{ or } y = \frac{3a^2 \pm \surd(9a^4 - 16ra^4)}{4a},$$

$$x \text{ or } y = \frac{a(3 \pm \surd(9 - 16r)}{4},$$

for the case in which only one vertex is immersed;

and $$x \text{ or } y = \frac{a(2 \pm \surd(16r - 7)}{4}$$

when two of its edges are in the fluid.

It follows, therefore, that if we have $r < \frac{9}{16}$ and $> \frac{8}{16}$ or $\frac{1}{2}$, the first of the above formulæ will be real, and its two values will be both less than a; but beyond these limits the expression will either be imaginary, or one of its two roots, viz. that which answers to the upper sign, will be greater than a, and will therefore be inadmissible.

In the same manner, in order that the second formulæ may give real and admissible values, it is necessary that $r < \frac{1}{2}$ and $> \frac{7}{16}$. We see, therefore, that from $r = \frac{9}{16}$, to $r = \frac{1}{2}$, the first formula alone will give admissible results; and that, on the contrary, the second will only obtain from $r = \frac{1}{2}$ to $r = \frac{7}{16}$. When $r > \frac{9}{16}$ or $< \frac{7}{16}$, they fail in both cases.

It follows, therefore, that when there are three positions of equilibrium with the vertex immersed, there can be but one with the vertex upwards; and when there are three in the latter case, there can be but one in the former; consequently, it being the same with each of its three edges, there may be 4 × 3 = 12 positions of horizontal equilibrium: but in order that this may obtain, the density of the body must be comprised between the fractions $\frac{9}{16}$ and $\frac{7}{16}$, that of the fluid being 1; and there will be but 2 × 3 = 6 positions, when the density of the body falls beyond these limits. When there are 12 positions, 9 of them will be with the vertex down, and 3 with it up; or 9 with the vertex up, and 3 with it downwards; but when there are only 6, there will be an equal number of each sort.

We haye stated, at the conclusion of art. 37, that the greatest number of horizontal positions of equilibrium of any triangular prism is *eighteen;* but from what is shown above, it is highly probable, that if we were to stop to examine the conditions of the limits of the equation by which they are expressed, we should find that several of those conditions would be inconsistent with each other, and that the greatest number of such positions would fall far within what we have stated as their greatest limit. This, however is an enquiry that we must leave to the curious reader.

Vertical equilibrium.

40. Besides their horizontal positions of equilibrium, prismatic and cylindric bodies, terminated with bases perpendicular to their sides or edges, have also vertical positions, in which their bases are parallel to the horizontal surface of the fluid; and there are two such positions for each of these bodies, because either base may be immersed. It is, therefore, only requisite to ascertain the depth of the body in the fluid; and for this we may take on one of the edges of the prism, measuring from the immersed base, a part which shall be to the whole length, as the density of the body to that of the fluid; for it is obvious that the immersed part will then have to the whole mass the same ratio.

All solids of revolution have also two positions of equilibrium, in which the fixed axis of the figure is vertical, and which are, therefore, easily determined for each particular figure. If, for example, the solid be a right cone, it will remain in equilibrio, both when the vertex is immersed in the fluid, and when the base is so immersed. In the first place, the cone below the surface of the fluid is similar to the whole, and their volumes will be to each other as the cubes of their lengths or heights; therefore, calling h the height of the whole cone, and x the height of that which is immersed, also as before, r the ratio of the density of the body to that of the fluid, we must have

$$x^3 : h^3 :: r : 1;$$

whence we deduce

$$x = h \sqrt[3]{r}.$$

In the second case we find

$$h - y = h \sqrt[3]{r}, \text{ or } y = h(1 - \sqrt[3]{r}),$$

where y denotes the height of the immersed conic frustrum.

Symmetrical bodies.

Generally, all bodies that are symmetrical with reference to any axis or right line, have the same properties as prisms and solids of revolution. Understanding here by the term symmetrical, any body homogeneous or heterogenous, composed of a series of lamina perpendicular to a right line, indefinitely thin, and whose centres of gravity all fall on that right line, which may thus be denominated the *axis* of the body.

Every plane perpendicular to this axis will cut bodies of this kind into two parts, of which each has its centre of gravity on this axis, as well as the centre of gravity of the whole body; consequently, keeping the axis vertical, and immersing a part of the body in the fluid, we shall be certain that the centre of gravity of this part, and that of the entire solid, are in the same vertical, which we have seen is sufficient for establishing a position of equilibrium, providing only that, besides this condition, the weight of the displaced fluid be equal to that of the whole body. This latter equality is always possible, if the density of the body when it is homogeneous, or its mean density when it is heterogeneous, be less than that of the fluid; whence we may conclude, that in either of these cases the body will have two positions of equilibrio, the inverse of each other, and in both of which the *axis* is vertical.

Of the various positions of equilibrium.

41. Having thus indicated the method of determining the different positions of equilibrium that a body may assume, it will be proper now to make a few remarks with reference to their different qualities. It

may be easily imagined that in some cases the equilibrium is more permanent than in others; viz. in some cases it may be such, that if the position of the body be changed, it will have a tendency to regain its primitive position; while in others, if its state be at all changed, the body, instead of tending to regain its first position, will have a tendency to revolve farther from it, while in others, it will remain in any position in which it is placed. These three different states are denominated respectively, an equilibrium of *stability*, of *instability*, and of *indifference* or *neutrality*.

Something similar to this has place in solid bodies: if the centre of gravity of a solid body be placed in the same vertical line as the fixed axis about which it is free to revolve, it will be in equilibrium; and this may happen either when the centre of gravity is above or below the fixed axis, or when it coincides with it: in the first case, if the body be slightly put out of its position, it will have a tendency to restore itself again, as soon as the external disturbing cause is removed; but in the second, if the position is in any respect altered, the body will have a tendency to revolve; and in the third, it will obviously be indifferent to every position.

Remarkable property.

The several positions of equilibrium of the same solid body floating on a fluid, have the following remarkable property, which may be demonstrated entirely independent of any calculation; viz. If we make a body turn about a moveable axis, which always remains parallel to a fixed horizontal right line; and that in this manner we make it pass successively through all its positions of equilibrium, in which the axis has the same direction; then will the positions of equilibrium of stability and instability succeed each other alternately, so that if the first be one of stability, the second will be one of instability, the third one of stability, &c.

In fact, while the body is still very near its position of stability, it will, as we have seen, tend to return to it again: but as we recline the body more and more, this tendency will gradually diminish, till finally it will no longer have any tendency to return to its first position; but on the contrary will revolve farther from it.

Now before this change of sign of its position (to speak analytically) takes place, the body must pass through some position, in which that tendency is zero, or where it will neither tend to, nor from its first position; this, therefore, will be its second position of equilibrium, and before the body arrives at it, it will tend to its first position, and consequently from its second; and after it has passed the second, its tendency is to revolve from the first position towards the third: consequently, this second position is one of instability, because on either side of it the body has a tendency to revolve more and more from it. With respect to the third position, it is obviously one of stability, because we have seen that the body tends towards, as soon as it has passed its second position; the third position being thus one of stability, we may prove in the same manner that its fourth is an equilibrium of instability, and so on.

Thus, when the body has returned again to its primitive position, it must necessarily have passed through some even number of positions of equilibrium, alternately of stability and instability. For example, if we suppose the equilateral prism, whose equilibrium we have investigated in (art. 39), to be made to turn about such an axis, it will pass either through six or twelve positions of equilibrium, which we have determined. Now the half of these will answer to the state of stability, and the other to a state of instability, and the equilibrium of both species will succeed each other alternately.

42. It is very important to be able to distinguish the equilibrium of stability of a floating body, from that which is not staple, for which a general rule may be investigated, deduced from the principle of living forces; but let us, in the first place, consider a particular case which frequently presents itself, where the stability depends only on a simple condition, which is easily verified.

This case is that of a body, which is divided by a vertical plane CED (fig. 29) into two parts, perfectly equal both in form and density. We shall suppose also that the body is displaced from its position of equilibrium in such a manner that this section shall still remain vertical; and that having thus deranged its position, it is left to itself, without impressing upon it any initial velocity. According to this supposition, the section CED will remain in one and the same vertical plane during the whole motion of the body; for every thing being the same on each side of this section, we can see no reason for its deviating from the vertical plane in which it is originally placed. For the same reason, the centre of gravity of the displaced fluid will be in some point in the section CED, as well as the centre of gravity of the whole body.

Let G represent this last centre; let also O, in the position of equilibrio, be the centre of gravity of the displaced fluid, and AB the intersection of the level of the fluid, with the plane CED. In this position the right line GO, which join the two centres, is vertical, and consequently perpendicular to the right line AB; but it will be inclined at a different angle as soon as the body is displaced from its position of equilibrium; and at the same time the centre of gravity of the immersed portion, and the line of intersection of the level with the plane CED, will change their positions in this plane. In this situation of the body, let O be the centre of gravity of the part immersed, and AB the line of level; the point G still remaining the same. Now the forces which tend to put the body in motion are its own weight, directed according to the vertical GH, drawn through the centre of gravity G, and the resultant of the vertical pressures of the fluid on the surface of the body: this resultant may be denominated the *thrust* of the fluid: it is equal to the weight of the displaced fluid, and acts at the point O, in a direction opposed to that of gravity; that is, in the vertical line O′H′. This vertical and the right line GO being in the same plane, they will cut each other in some point *m*, which, as we have already stated (art. 36), is called the *meta-centre*; and it is, as will be shown, the position of this point with respect to that of the centre of gravity G, on which depends the degree of stability of the body's equilibrium.

As we may always conceive a force to be applied at any point in the line of its direction (Statics, art. 15), we shall suppose *m* to be the point of application of the thrust of the fluid, acting in the direction *m* H′, and the body will then obviously be solicited by two parallel and contrary forces, applied at the extremities of the right line GO; and it remains to examine the circumstances of the motion which the body will in consequence take; that is, whether these forces will tend to cause the body to regain the position of equilibrium from which it has been displaced, or whether they will cause it to revolve farther from that position of equilibrium towards some other.

In the first place, they must necessarily produce a motion of oscillation about the point G; for the motion of this point will be the same as if the two forces were applied immediately to it; as we have seen in treating of parallel forces; therefore, because by the hypothesis the initial velocity is zero, its motion will be rectilineal, vertical, and only equal to the excess of the greater of these two forces above the less. If, at the origin of the motion, the weight of the body exceed the thrust of the fluid, the point G will begin to descend, and its motion will be at first accelerated; but as the body is thus plunged deeper into the fluid it will displace a greater volume of it, and the thrust of the latter will be increased; till, finally, it will arrive at that point at which the latter will be equal to the weight of the body; then the acceleration will cease, but the point G will still continue to move in the same direction, in consequence of the acquired velocity, till the thrust of the fluid, prevailing over the weight of the body, its motion is first retarded, and ultimately destroyed. From this point, the centre G having lost all its velocity, it will again return towards its original position, and will thus continue to oscillate till the resistance of the fluid has entirely destroyed the motion. The magnitude or length of these oscillations will be so much the less, as the difference between the weight of the body and that of the displaced fluid, is less in comparison with the first of these quantities. If the body have been only very slightly deranged from its position of equilibrium, this difference will itself be very small, and consequently also the length of the oscillations; and these oscillations, being thus inconsiderable, may be supposed to have no influence on the stability of equilibrium of the body.

During these oscillations of the point G, the body will turn about this centre as if it were fixed, and its motion of rotation will therefore be wholly due to the force acting at the point m, according to the direction mH'; and its state of equilibrium will be one of stability, or of instability, according as the right line GO, by this motion, returns towards its vertical position, or is carried farther from it. Now, from a simple inspection of the figure, it is obvious that the thrust of the fluid will tend to bring the line OG towards the vertical position, whenever that point is found above G; and, on the contrary, if the meta-centre be placed below G, as, for example, at m'; the thrust of the fluid, which acts according to the vertical m'H'', will tend to carry the line GO still farther from the vertical, and to upset the floating body.

Therefore, when the meta-centre is below the centre of gravity of the floating body, such as we are considering, the equilibrium is not staple; and, on the contrary, when that point is found above the centre of gravity of the body, the equilibrium is one of stability, at least while we only consider the derangement to be such that the plane CED remains vertical: and if in any particular case the meta-centre coincide with the centre of gravity, there will be no tendency to rotation, or the line GO will preserve any inclination that is given to it.

43. When the form of the floating body is known, it will always be easy, by putting it into a position very near its position of equilibrium, to determine the place of the meta-centre, and to ascertain whether this point is above or below the centre of gravity of the body.

Example.

Suppose, for example, the proposed body to be a horizontal cylinder, with homogeneous elliptic bases, and its density equal to half that of the fluid.

Figs. 30, 31.

Let AEBD (fig. 30 and 31) represent a vertical section, made at equal distances from the two bases. Now, in the position of equilibrium, one of the two axes of this ellipse will be vertical; and as half the volume will be immersed in the fluid, it follows that the other axis will be found to coincide with the surface of the fluid, and will represent its level. In fig. 30, we suppose the longest axis to be vertical; and in fig. 31, the shorter; and it is proposed to determine the position of the meta-centre in these two cases.

Let us draw through the point G in each figure, a right line E'D', making a very small angle with the axis ED, and conceive that we afterwards incline the latter axis till E'D' becomes vertical; and at the same time let us suppose that we raise or depress the centre of gravity G, till the level of the fluid coincide with the line A'B', perpendicular to the right line E'D', at the point G'. In this position, the part A'E'B'G' of the ellipse AEDB, will be immersed in the fluid; and this part is divided into two unequal parts A'E'G' and B'E'G', by the right line G'E'. Now it is evident, that the centre of gravity of the displaced fluid is situated in some point in the greatest of those parts, as in the point O'. Whence it is obvious, in examining the two figures, that O'm parallel to E'D', will meet the right line ED below the centre G in fig. 30, and above it in fig. 31. And since by the definition m is the meta-centre, it follows, that in the former position the equilibrium is one of instability, and in the second it is staple. And if we suppose the body to turn about the horizontal right line which joins the centre of the two bases, the body will pass through four positions of equilibrium, alternately, of stability and of instability, conformably with our deductions (art. 41.)

Body of any form.

44. It would carry too far to enter into the question of the equilibrum of a body of any form, and of heterogeneous composition; we shall not therefore attempt the question under this general character, but shall confine ourselves to the consideration of a floating body, which is symmetrical with respect to a vertical plane passing through its centre of gravity, when at rest, and of such kind that we need only resolve the problem for the area of the plane section of the body. Again, we shall suppose the derangement of the body, from its position of equilibrium, to be very small, as we have already done in the preceding cases.

Fig. 32.

Let $a\,b$ (fig. 32) be the plane of floatation, G the centre of gravity of the floating body DFE, and O the centre of gravity of AFB, or of the displaced fluid when the line AB coincides with the plane of floatation; the inclination of the body will then be the angle GOV, formed by GO, and the vertical line OV, passing through the centre of a F b; and suppose this evanescent angle, or the arc which measures it, equal to A.

Then, disregarding the vertical motion of the centre of gravity as indefinitely small, we shall suppose that the new position to be consistent with the equality of the weights of the body, and of the displaced fluid. Thus will the portion AC a, which is moved out of the fluid, be equal to the part BC b, which has entered it in consequence of the motion.

These suppositions are not precisely what the general solution of the problem requires; it is obvious, however, that they differ so little from the truth, in the case of very small oscillations, as to produce no sensible error, at the same time they throw great facilities into the solution. Granting therefore these suppositions, the equal areas AC a and BC b, may be regarded as isosceles triangles, since we may suppose

$$CA = Ca, \text{ and } CB = Cb;$$

whence it follows, that $a\,C = C\,b$, that is, the intersection C of the two lines of floatations, is in the middle of AB.

Now the pressure of the fluid upwards on $a\,F\,b$, is equal to a weight of a lamina of that fluid, of an equal magnitude; and this vertical force acts at the centre of gravity of $a\,F\,b$. So that, as it is necessary to know this centre, we shall first endeavour to assign its position.

For this purpose we may observe, that

$$a\,F\,b = AFB + CB\,b - AC\,a;$$

and if we conceive the laminæ of the fluid equal to these areas, it is evident, that the moment of the thrust of the fluid, taken with relation to any vertical whatever, is equal to the sum of the fluid laminæ, of which the volumes are AFB, CB b, *minus* the moment of the laminæ AC a: let us therefore estimate these moments with respect to the vertical G i. First, the moment of the weight of the fluid laminæ AFB $=$ W·GV; W denoting the weight of the body, or that of the displaced fluid.

Secondly. If we represent by w (CB b) the weight of a column of fluid equal to CBb, we have for the moment of this weight, its product by the distance $q\,i$; the point q being the foot of the perpendicular demitted upon C b, from the centre of gravity CB b: we find therefore the point q, by taking $C\,q = \frac{2}{3}\,C\,b$. Thus the momentum of CB b, is $w\,(CB\,b) \times q\,i$.

Lastly. The moment of AC a, is in like manner, $w\,(AC\,a) \times p\,i$; making $p\,C = \frac{2}{3}\,a\,C$. Where, as this latter force tends to produce a motion in the contrary direction to the former, it must be taken negatively; and moreover, since it ought to be subtracted from the two others, it becomes positive, and the moment M of the pressure, or thrust of the fluid upon $a\,F\,b$, is expressed by $M = W \times GV + w(CB\,b)\,p\,q$, (because $CB\,b = AC\,a$.)

Now since the fluid is supposed homogeneous, the areas AFB, B$c\,b$, are proportional to the weights of the quantities of fluid to which they are respectively equal; that is to say, if we put the surface AFB $=$ S, since $CB = c\,b$, and $B\,c = CB \sin A = CB{\cdot}A$ (because $\sin A = A$, the angle being indefinitely small), we have the triangle $CB\,b = \frac{1}{2}\,A \times CB{\cdot}CB$: therefore

$$S : W :: \tfrac{1}{2}\,A{\cdot}BC^2 : w\,(CB\,b) = \frac{W}{2S}\,A{\cdot}BC^2.$$

But $p\,q = \frac{2}{3}\,a\,b = \frac{2}{3}\,AB$ and $GV = A{\cdot}GO$; if, therefore, we put $GO = a$, and $AB = b = 2\,BC$, there results

$$M = A{\cdot}W\left(a + \frac{b^3}{12\,S}\right). \qquad (a)$$

Measure of the pressure.

45. Since we consider the body as if it were retained by a fixed axis passing through G, the preceding value of M will measure the pressure of the fluid. So that if we would compare these pressures upon different bodies, we must consider them with reference to the same angle of inclination A, and the relative stabilities will then be measured by

$$W\left(\frac{b^3}{12\,S} \pm a\right); \qquad (b)$$

and if the body be an irregular solid, the stability will be as the sum of all the

$$W\left(\frac{b^3}{12\,S} \pm a\right).$$

By stability is here to be understood the resistance which a body opposes to its change of position; that is, the effort made by the fluid to retain the body in its position of equilibrium, or to carry it farther from it; and the ambiguous sign prefixed to a, must be taken, positive or negative, according as the centre of gravity of the whole body is below or above the centre of gravity of the volume of the displaced fluid. If the body be homogeneous, then the centre of gravity of the displaced fluid is the same as that of the part of the body immersed, and, consequently, it is then lower than that of the whole body; but when the body is heterogeneous, it may happen, and frequently does (except in ships of war), that the centre of gravity of the body is below that of the displaced fluid, it being customary to dispose about the lower parts substance specifically heavier than the materials of the floating body, in order to accomplish this object.

We have seen, in the case of bodies perfectly uniform, that the stability may be either positive, negative, or nothing, and our equation or expression (b) may, it is obvious, comply with all these conditions. When the centre of gravity of the body is below that of the displaced fluid, then a is positive, and consequently also the stability, as is otherwise obvious. On the contrary, when the centre of gravity of the body is above that of the displaced fluid, then a is negative; but the stability may still be positive, provided $\frac{A\,b^3}{12\,S}\ 7\ a$; but if $a\ 7\ \frac{A\,b^3}{12\,S}$, then the stability is negative; and if $\frac{A\,b^3}{12\,S} = a$, it is then obviously nothing, or zero.

We see therefore the advantage of forming the body, and disposing the materials of which it is composed, in such a manner as to bring its centre of gravity as low as possible, as also of constructing it so that it may be narrower towards the bottom than towards the plane of floatation.

46. If the formula (b) be divided by W, the weight of the displaced fluid, the quotient will express the distance from the vertical G i, to the direction of the resultant of the upward pressure of the fluid; that distance being taken equal to G n, we shall have

$$G\,n = \frac{A\,b^3}{12\,S} \pm GV = A\left(\frac{b^3}{12\,S} \pm a\right). \qquad (c)$$

If a be positive, the vertical G i faces to the left of the centre of gravity of the displaced fluid; in this case G n must be taken to the right of G i, and the same thing obtains in the contrary case if

$$\frac{A\,b^3}{12\,S}\ 7\ a, \text{ or GV};$$

the vertical $g\,m$ intersecting the right line GO, in a point m more elevated than G. But, if in the latter case (that is, if GV be negative) we have $\mathrm{GV} = \frac{\mathrm{A}\,b^3}{12\,\mathrm{S}}$; then $\mathrm{G}\,n = o$, and the point m and G coincide. Finally, if $\mathrm{GV}\ 7\ \frac{\mathrm{A}\,b^3}{12\,\mathrm{S}}$, and G consequently higher that O, the line $\mathrm{G}\,n$ must be taken to the left of $\mathrm{G}\,i$, and the point m will fall below G.

To find the distance $\mathrm{G}\,m$, it may be observed, that $\mathrm{G}m.\sin\mathrm{A} = \mathrm{G}\,n$, or because $\sin\mathrm{A} = \mathrm{A}$, $\mathrm{A}\times\mathrm{G}m = \mathrm{G}n$; wherefore $\mathrm{G}\,m = \frac{b^3}{12\mathrm{S}} \pm a$. (d)

As the point m obviously corresponds here with what we have defined the metacentre we come to the sames conclusion in this case as before, viz.

The stability of a body or vessel is positive, nothing, or negative, according as the metacentre is above, coincides with, or is below the centre of gravity of the body.

The following example will, in some measure, illustrate the preceding deductions:

Let the floating body be a homogeneous rectangular parallelopiped, whose altitude is perpendicular to the surface of the fluid. Suppose the height of the solid $= h$, the breadth of the base b, the specific gravity of the body s, then will the cube of the line coinciding with the line of floatation $= b^3$, as in the formula: the height of the immersed part $= s\,h$, and S the space proportional to the part immersed $s\,b\,h$. The distance of the centres of gravity of the whole body and of the part immersed will be

$$\mathrm{GO} = \tfrac{1}{2}\,h - \tfrac{1}{2}\,s\,h = \frac{h - s\,h}{2};$$

consequently, the stability which varies as

$$\frac{b^3}{12\ \mathrm{S}} \pm \mathrm{GO},$$

will vary as

$$\frac{b^3}{12\,s\,b\,h} - \frac{h - s\,h}{2} = \frac{b^2}{6} - h^2\,(s - s^2).$$

That is, the stability of such a solid will be proportional to the difference between the sixth part of the square of the breadth of the base, and the product of the square of the altitude, into the difference between the specific gravity of the body and its square, that of the fluid being unity.

As an example, suppose the height of the parallelopiped equal to the breadth of the base, or $h = b$, and let it be required to determine the specific gravity of the solid when its stability is nothing.

In this case, writing h for b, we have

$$\frac{h^2}{6} - h^2\,(s - s^2) = o, \text{ or}$$

$$s^2 - s = -\tfrac{1}{6};$$

whence $s = \frac{1}{2} \pm \frac{1}{6}\sqrt{3} = {\cdot}78868$, or ${\cdot}21132$ *nearly*.

These specific gravities agree very nearly with those of dry seasoned oak and cork, that of water being 1; and therefore a solid of the form proposed of either of these substances, would float in water with the equilibrium of neutrality or indifference.

2. If the body be a homogeneous cylinder, whose axis is vertical, and the circumstances of its equilibrium were required, we should find, by an investigation similar to the above, that its stability is as the quotient of the square of the radius of the base divided by four times the height of the part immersed, diminished by the distance between the centres of gravity of the whole solid and the part immersed; that is as

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$$\frac{\mathrm{CG}^2}{4\,d} - \mathrm{OG},$$

where CG is the radius of the base, at the depth of the part immersed, and OG the distance between the centres of gravity of the immersed part, and of the whole solid. We leave this investigation, and the corollaries that result from it, as exercises for the reader.

Oscillations of a floating body.

47. If we consider the fluid as non-resisting, the floating body will oscillate incessantly about a horizontal axis, passing through the centre of gravity, according to laws analogous to those of bodies in a vacuum; and we propose here to examine the particular nature of these oscillations, regarding them as indefinitely small.

Let us suppose that the vertical line passing through the centre of gravity of the body, in its equilibrated position, has been inclined by the quantity f, then denoting by A the inclination of the body at the end of the time t, and by α, the arc described by the point which is placed at the unit of distance from G (fig. 32), we shall have $\mathrm{A} = f - \alpha$. Now the angular accelerating force, as we have seen in Dynamics, is the quotient of the sum of the momenta of the moving forces divided by the moment of inertia. The dividend being the total vertical pressure of the fluid is readily obtained, knowing the mass of the body, by means of our preceding equation (a); and the divisor is $\mathrm{M}\,k^2$, M being the mass of the body, and k^2 the quotient of the momentum of inertia (with respect to any axis passing through the centre of gravity) divided by the mass.

Fig. 32.

Hence, as the weight W of the body is equivalent to $2\,g$ M, $2\,g$ denoting the force of gravity, we have

$$\frac{\ddot{\alpha}}{\dot{t}^2} = \frac{2g}{k^2}\cdot\mathrm{A}\left(a + \frac{b^3}{12\ \mathrm{S}}\right) = \frac{2g}{k^2}\left(a + \frac{b^3}{12\ \mathrm{S}}\right)(f - \alpha).$$

Or if we put $l = \left(a + \frac{b^3}{12\,\mathrm{S}}\right) \div k^2$, there will arise

$$\frac{\ddot{\alpha}}{\dot{t}^2} = \frac{2g}{l}(f - \alpha).$$

Multiplying this by $2\,\dot{\alpha}$, it becomes

$$\frac{2\,\dot{\alpha}\,\ddot{\alpha}}{\dot{t}^2} = \frac{4\,g}{l}(f\,\dot{\alpha} - \alpha\,\dot{\alpha});$$

and taking the fluents

$$\frac{\dot{\alpha}^2}{\dot{t}^2} = \frac{2g}{l}(2f\alpha - \alpha^2) + \mathrm{C}.$$

Here $\frac{\dot{\alpha}}{\dot{t}}$ is the angular velocity; so that when

$\frac{\dot{\alpha}}{\dot{t}} = o$, $\alpha = o$, and consequently $\mathrm{C} = o$;

whence $\dot{t} = \sqrt{\frac{l}{2g}} \times \frac{\dot{\alpha}}{\sqrt{(2\,\alpha f - \alpha^2)}}$,

taking the fluent we obtain

$$t = \sqrt{\frac{2g}{l}} \times \text{arc}\left(\cos = \frac{f - \alpha}{f}\right) + \mathrm{C}'.$$

Here, in order to determine C′, we have, when $t = o$, $\alpha = o$, consequently $\mathrm{C}' = o$;

therefore $t = \sqrt{\frac{2g}{l}} \times \text{arc}\left(\cos = \frac{f - a}{f}\right)$ is the complete fluent; and we hence find

$$\frac{f - a}{f} = \cos\left(\text{arc} = t\sqrt{\frac{2g}{l}}\right), \text{ or}$$

$$a = f\left(1 - \cos\left(\text{arc} = t\sqrt{\frac{2g}{l}}\right)\right).$$

The variables comprised in this equation are the arcs a and $t\sqrt{\frac{2g}{l}}$; where, if we attribute successively to $t\sqrt{\frac{2g}{l}}$ the values o, $\frac{1}{2}\pi$, π, $1\frac{1}{2}\pi$, 2π, &c. we find o, f, $2f$, $3f$, $4f$, &c. for the corresponding values of a; which shows that the moving body after describing the descending arc f, will describe an ascending arc f in the same time; then return and rise the contrary way, and so on. It is obvious also, that the time of making a complete oscillation is given by taking $\pi = t\sqrt{\frac{2g}{l}}$, since that hypothesis gives $a = 2f$, as it ought to do.

Hence there results $t = \pi\sqrt{\frac{l}{2g}}$.

This expression for the time does not comprise the quantity f; whence it follows that the vibrations are isochronal; and by comparing the above with the value of t, for the simple pendulum vibrating in indefinitely small arcs (art. 82), it will appear that they exactly agree; consequently, the principle circumstances traced there, apply immediately to the present case.

Now if we restore the value of l, we shall have $l = \frac{12\,k^2\,S}{12\,a\,S + b^3}$; an expression which shows the length of the synchronal simple pendulum. And for the time of oscillation, $t = \pi\,k\sqrt{\frac{12\,S}{2g(12\,a\,S + b^3)}}$.

Application.

In order to make an application of these principles, let us take the instance of a prismatic body, such that the transverse section of the immersed part in the position of equilibrium is an isosceles triangle, of which the vertices of the equal angles are in the plane of floatation; the upper part of the section may have almost any variety of form. Let b represent the half base of the isosceles triangle, which is the transverse section of the part immersed; h the height of this triangle, and at the distance of the upper surface of the fluid from the centre of gravity of the body; the distance of that surface from the centre of gravity of the triangle immersed will be $\frac{1}{3}h$; consequently, the distance between the centres of gravity of the body and of the fluid displaced will be $a = d - \frac{1}{3}h$; the area immersed will be $S = b\,h$; and the value of the stability (formula b, art. 45) will be

$$\frac{W}{3h}(3\,d\,h - h^2 + 2\,b^2),$$

which may be either positive, nothing, or negative, as in the various cases before specified; and the length of the synchronal pendulum will be

$$L = \frac{3\,h\,k^2}{3\,d\,h - h^2 + 2\,b^2}.$$

It may not be amiss to observe, that these results are obtained only on a supposition of the oscillations being indefinitely small, and that they will not apply to any other case. See Gregory's Mechanics, vol. i.

48. We cannot enter at much length upon the application of our preceding deductions to the construction of ships and vessels generally, as that would lead us upon a subject sufficient of itself to form a considerable volume, we therefore beg to refer the reader for such application to the works to which we have alluded in our historical chapter, as also to an ingenious paper on the same subject, published by Mr. Atwood in the Philosophical Transactions for 1796. All we can attempt in this place is to examine a few of the most simple cases, such as river and canal boats, in which we shall avail ourselves of an article published by Mr. English, in Tilloch's Philosophical Magazine.

Vessels of this kind are generally of the same transverse section throughout their whole length, except a small part in prow and stern, formed by segments of circles or other simple curves; therefore such a length may be easily assigned, that any of the transverse sections being multiplied thereby, the product will be equal to the whole solidity of the vessel. The form of the transverse section is generally either rectangular, as in fig. 33; trapezoidal, as in fig. 34; or mixtilineal, as in fig. 35; in all which MM represents the line of floatation when upright; and EF the same when inclined at any angle MXE; also G represents the centre of gravity of the whole vessel, and R that of the part immersed.

If the vessel be loaded quite up to the line AB, and the specific gravity of the boat and burden be the same, then the point G is simply the centre of gravity of the section ABCD; but if not, the centres of gravity of the boat and burden must be found separately, and reduced to one, by the usual method, by dividing the sum of the momenta by the sum of the weights or areas, which in this case are as the weights.

The point R is always the centre of gravity of the section MMCD, which, if consisting of different figures, must also be found by dividing the sum of the momenta by the sum of the weights as usual. These two points being found, the next thing is to determine the area of the two equal triangles MXE, MXF, their centres of gravity o, o, and the perpendicular projected distance $n\,n$ of these points on the water line EF.

This being done, through R, and parallel to EF, draw RT a fourth proportional to the whole area MMCD, either triangle MXE, or MXF, and the distance $n\,n$: through T, and at right angles to RT or EF, draw TS meeting the vertical axis of the vessel in S, the metacentre; also through the points G, B, and parallel to ST, draw NGW and BV; and again, through S and parallel to EF, draw WSV, meeting the two former in V and W; then SW is as the stability of the vessel, which will be positive, nothing, or negative, according as the point S is above, coinciden with, or below the point G.

If now we assume W to represent the weight of the whole vessel and burden (which will be equal to the section MMCD multiplied by the length of the vessel) and P to denote the required weight to be applied at the gunwale B, to sustain the vessel at the given angle of inclination, we shall have this proportion

$$VS : SW :: W : P;$$

which proportion is general, whether SW be positive o negative; it must only, in the latter case, be supposed to act upward to prevent an overturn.

In rectangular vessels of given weight and dimensions, the whole process is so evident, that any farther explanation would be unnecessary.

In trapezoidal vessels, after having found the points G and R, let AD, BC, be produced till they meet in K; then, since the two sections, MMCD, EFDC, are equal, the two triangles, MMK, EFK, are also equal; and therefore the rectangle KE × KF = KM × KM = KM^2; and since the angle of inclination is supposed to be known, the angle at E and F are given; consequently, if a mean proportional be found between the sines of the angles E and F, we shall have the following proportions.

As the mean proportional thus found : sin E :: KM : KF; and as the same mean proportional : sin F :: KM : KE: therefore ME, MF, become known; whence the area of either triangle MXE, MXF, the distance *n n*, and all the other requisite quantities, may be found.

In the mixtilineal section, let AB = 9 feet = 108 inches, the whole depth = 6 feet = 72 inches, the altitude of MM the line of floatation = 4 feet, or 48 inches; and let the two curvilineal parts be circular quadrants of 2 feet, or 24 inches radius each. Then the area of the two quadrants = 904·7808 square inches, and the distance of their centres of gravity from the bottom = 13·8177 inches nearly; also the area of the included rectangle *a b i e* = 1440 square inches, and the altitude of its centre of gravity = 12 inches. In like manner, the area of the rectangle AB*cd*, will be found = 5184 square inches, and the altitude of its centre of gravity = 48 inches; we shall therefore have

$$\begin{array}{lllll}
\left.\begin{array}{r}\text{Momentum of the}\\ \text{two quadrants}\end{array}\right\} & =904{\cdot}7808 & \times\, 13{\cdot}817 & =12501{\cdot}989; \\
\left.\begin{array}{r}\text{Momentum of the}\\ \text{rectangle } abie\end{array}\right\} & =1440 & \times\, 12 & =17280; \\
\left.\begin{array}{r}\text{Momentum of the}\\ \text{rectangle AB}cd\end{array}\right\} & =5184 & \times\, 48 & =248832; \\
\hline
 & 7528{\cdot}7808 & & 278613{\cdot}989.
\end{array}$$

Now the sum of the momenta, divided by the sum of the areas, will give

$$\frac{278613{\cdot}989}{7528{\cdot}7808} = 37{\cdot}006 \text{ inches,}$$

which is the altitude of G, the centre of gravity of the section ABCD, above the bottom.

In like manner R, the centre of gravity of the section MMCD, will be found to be equal to

$$\frac{123093{\cdot}989}{4936{\cdot}7808} = 24{\cdot}934 \text{ inches;}$$

consequently, their difference, or the value GR = 12·072 inches.

Now, suppose the vessel to *heel* 15°, then we shall have the following proportion:

radius : tan 15° :: MX = 54 : ME or MF = 14·469 inches; whence the area of either triangle MXE or MXF = 390·663 inches. Therefore

4936·7808 : 390·663 :: 72 = *n n* = $\frac{2}{3}$ AB : 5·6975 inches = RT; and again,

radius : sin 15° :: 12·072 = GR : 3·1245 = SW, the stability required.

Again, as sin 15° : rad :: 5·6975 = TR : 22·013 = RS; to which, adding 24, the altitude of the point R, we shall have 46·947 for the height of the metacentre; which taken from 72, the whole altitude, there remains 25·053; from which, and the half width 54 inches, the distance RS is found = 59·529 inches nearly; and the angle SRV = 80° 6′ 42″; whence SV = 58·645.

Let us next suppose the mean length of the vessel to be 40 feet, or 480 inches, and we shall have the weight of the whole vessel equal to the area of the section MMCD = 4936·780, multiplied by 480, or 4936·780 × 480 = 2369654 cubic inches of water, equal in weight to 85708 lbs. avoirdupois.

Lastly, as SV : SW, or as 58·645 : 2·573 :: 85708 : 3760 +, the weight of the gunwale, which will sustain the vessel at the given inclination. Therefore, a vessel of the above dimensions, and weighing 38 tons 5 cwts. 28 lbs. will require a weight of 1 ton 13 cwts. 64 lbs. to cause her to incline 15°.

In the above example, the deflecting power has been supposed to act perpendicularly to the gunwale at B; but if the vessel be navigated by sails, the *centre velique* (or that point in a single equivalent sail, where, if the resultant of all the forces of the wind were applied, the same effect would be produced) must be found; with which, and the angle of deflection, the projected distance of the same, on the line SV, may be obtained; and then the power calculated as above, necessary to be applied at the projected point, will be that part of the force of the wind which causes the vessel to heel; and conversely, if the weight and dimensions of the vessel, the area and altitude of the sails, and the direction and the velocity of the wind, be given, the angle of direction may be determined.

We cannot pursue this enquiry to a greater length in this place; we beg, therefore, to refer the reader, for a full investigation, as it relates to the stability of ships, &c. to the papers published by Atwood, Phil. Trans. for 1796.

§ VI. *On the figure of the earth.*

Figure of the earth.

49. The figure of the earth was for a long time supposed to be that of a perfect sphere; but when the doctrine of centrifugal and centripetal forces began to be better understood, its exact spherical figure was doubted; but who it was that first suggested this doubt does not appear to be known. Picard, in his measurement of the earth, published in 1671, speaks of a conjecture proposed to the Academy, that supposing the diurnal motion of the earth, heavy bodies would descend with less force at the equator than at the poles, and observes, that for the same reason there would be a difference in the length of the pendulum vibrating seconds in different latitudes. The same year Richer was sent to Cayenne, and among other objects of his voyage, he was charged by the Academy to observe the length of the pendulum vibrating seconds. It will be sufficient to observe, in this place, that the conjecture was verified; the seconds pendulum at Paris having been found to be a line and a quarter shorter than that vibrating at the same rate in Cayenne.

This may be said to be the first direct proof of the diurnal motion; and Huygens was then led to suspect that the same cause might produce a protuberance of the equatorial parts of the earth, and a consequent depression of the poles; whether this philosopher were, or not, the first person who offered this suggestion, he was undoubtedly the first who attempted the solution of the problem; viz. the determination of the figure of the earth. Newton soon after undertook the like investigation, but on a different hypothesis, and they have been since followed by numerous other celebrated mathematicians, through whose labours and ingenuity the problem has finally received every degree of elucidation of which it appears to be susceptible. We cannot in this place attempt to develope the views that different authors have taken in the solution of this celebrated problem; at the same time as it is obviously a question intimately connected with the principles of Hydrostatics; that is to say, the determination of the equilibrium of a fluid under certain circumstances; we cannot close this treatise without bestowing a few pages, by way of illustrating some of the most usual modes of solution.

Huygens assumes, as the base of his solution, that "if a fluid heavy mass have all its points in equilibrio, its surface ought to cut perpendicularly the directions of the gravity of the particles of which it is composed."

Newton's hypothesis was, "that if a fluid heavy mass be in equilibrio, any two columns of the fluid directed to one and the same fixed point, considered as a centre, will mutually counterbalance each other, independent of the rest of the mass."

Before attempting the general solution on the principles which we have assumed for the bases of our theory, viz. *the principle of equal pressure*, it will not be uninteresting to the reader to see an exposition of the solution of the above two celebrated philosophers.

Solution on Huygens' hypothesis.

Huygens' solution.

50. *The earth being supposed to have been originally fluid, having the form of a solid of revolution; and all its particles being supposed submitted to the action of a given gravitating and centrifugal force; it is required to determine the figure it will assume.*

Fig. 36.

Let C (fig. 36) represent the centre of the earth, DAEB one of its meridians, or the section of the body made by a plane passing through its axis of revolution DE. Suppose the gravity of the point M to be in the direction MO, and let this force be denoted by MF, which decompose into two others MH, MK, perpendicular to the axes DE, AB of the meridian. The centrifugal force of the point M, which revolves about DE, is proportional to the ordinate MP; let us represent this force, which acts in a contrary direction to MH, by HI. Then it is obvious that the point M is solicited by the action of the two forces MI, MK; which by the hypothesis are perpendicular to the element M m of the meridian.

Assume CP $= x$, PM $= y$; the force MF $= \phi$, the angle OMP $= z$; and call f the given centrifugal force for any known distance k. Then we shall have

$$\text{MH} = \phi \cos z; \ \text{M}k = \phi \sin z;$$

$$\text{HI} = \frac{f y}{k} \text{ and } \text{MI} = \phi \cos z - \frac{f y}{k}.$$

Now drawing the ordinate $m\,p$, the similar right angled triangles M r m, VKM, will give

$$\text{M}r : rm :: \text{KV} : \text{MK};$$

or, $$-\dot{x} : \dot{y} :: \phi \cos z - \frac{f y}{k} : \phi \sin z;$$

whence we have for the equation of the meridian

$$\left(\phi \cos z - \frac{f y}{k}\right) \dot{y} = - \phi z \dot{x}.$$

Solution on Newton's principle.

51. Let DAE (fig 37) be the meridian, DE the axis, C the centre of the earth, towards which the polar column of the fluid DC, and any other column MC tend (which by the hypothesis mutually counterbalance each other). Take on CM any point R, the gravity of which is in the direction RO, and which may be denoted by R t; resolve this force into two others R h, R i; the one in the direction MC, the other perpendicular to the same line. Let us represent by R z, the centrifugal force of the point R, which acts in the direction QR perpendicular to the axis of revolution DE, and resolve this into two forces R s, R u, the one in the direction CM, and the other perpendicular to it.

Now it is evident, that the forces R i, R u, contribute nothing to the column or canal RC in the direction MC, and that we have only to consider the forces R h, R s.

Let the absciss CP $= x$, the ordinate MP $= y$, the angle CRO $= p$, CR $= r$, QR $= u$.

The force R $t = \phi$, the centrifugal force $= f$, at a given distance k, then we shall have

$$\text{R}\,h = \phi \cos p, \text{R}\,z = \frac{f u}{k}, \text{ and } \text{R}\,s = \frac{f u^2}{k r};$$

therefore the pressure of the canal RC on the point C is equal to

$$\int \dot{r} \left(\phi \cos p - \frac{f u^2}{k r}\right) = \int \dot{r} \left\{\phi \cos p - \frac{f y^2 r}{k (x^2 + y^2)}\right\},$$

these fluents must be taken with x and y constant: and afterwards, in order to have the weight of the canal MC, we must make

$$r = \text{CM} = \sqrt{(x^2 + y^2)};$$

which will give an expression that we may then equate with the given weight of the column DC.

By way of illustration, let us suppose that gravity is constant, and that its direction is towards the centre of the earth C (figs. 36 and 37). Then in the former figure we have

$$\sin z = \frac{x}{\sqrt{(x^2 + y^2)}},$$

$$\cos z = \frac{y}{\sqrt{(x^2 + y^2)}};$$

and the equation of the meridian as depending upon the supposition of Huygens, will become

$$\frac{\phi y \dot{y}}{\sqrt{(x^2 + y^2)}} - \frac{f y \dot{y}}{k} = \frac{- \phi x \dot{x}}{\sqrt{(x^2 + y^2)}};$$

or $$\frac{\phi (x \dot{x} + y \dot{y})}{\sqrt{(x^2 + y^2)}} - \frac{f y \dot{y}}{k} = o;$$

of which the fluent is

$$\phi \sqrt{(x^2 + y^2)} - \frac{f y^2}{2 k} + \text{C} = o,$$

where the constant C must be so taken, that when $y = o$, we may have $x =$ CD a given quantity.

In fig. 37, we have $p = o$, and the quantity

$$\int \dot{r} \left(\phi \cos p - \frac{f y^2 r}{k (x^2 + y^2)}\right)$$

becomes at once

$$\phi r - \frac{f y^2 r^2}{2 k (x^2 + y^2)};$$

making $CM = r = \sqrt{(x^2 + y^2)}$, we have for the weight of the column CM

$$\phi \sqrt{(x^2 + y^2)} - \frac{f y^2}{2 k},$$

which ought to be equal to the weight of the column DC; whence again we obtain

$$\phi \sqrt{(x^2 + y^2)} - \frac{f y^2}{2 k} + C = o$$

for the equation of the meridian.

52. The principles employed by Huygens and Newton, it appears, from the above investigations, give on the hypothesis assumed in the preceding article the same results, and the same has place on some other hypotheses relative to the action of gravity; while in others the equations obtained are very different. The principle employed by Huygens establishes the equilibrium only on the surface of the fluid; while that of Newton has reference to the interior of the mass with regard to the centre; but these two conditions are not sufficient separately, nor even in certain cases conjointly, to establish the equilibrium in every component particle of the mass, as is shown by Clairaut in his "Traité de la Figure de la Terre."

The equilibrium will only have place in every particle of the mass, when, taking any point whatever (and not a determinate and fixed point), we find this to be equally pressed in all directions. This general view of the problem was first clearly developed by Maclaurin, and from it we derive the following results:

1. That when any fluid mass OAN (fig. 38) is submitted to any forces, and in equilibrio, the angular canal OMN, terminated on both extremities by the surface, is separately in equilibrio. For since the whole mass is in equilibrio, the particle M is equally pressed in every direction, and this equality of pressure will remain the same, if we conceive the canal OMN to remain fluid while all the other part of the mass become solid.

2. That the triangular canal OMR, of which one angle O is at the surface of the fluid, is also in equilibrio. For if MR be produced to the surface of the fluid, the two angular canals OMN, ORN, are each separately in equilibrio; whence denoting the pressure of a column by P, we have

$$P{\cdot}OM = P{\cdot}NM \text{ and } P{\cdot}OR = P{\cdot}NR, \text{ or}$$
$$P{\cdot}NM = P{\cdot}NR + P{\cdot}RM = P{\cdot}OR + P{\cdot}RM,$$

therefore $P{\cdot}OM = P{\cdot}OR + P{\cdot}RM$;

that is, the point M sustains the same pressure from the column MN, as from the two columns OR and RM, which act on this point in the same manner as if they were placed in a right line. There will be, therefore, no motion in the canal OMN, either in the direction OMR, or ORM, consequently the canal OMN will remain in equilibrio as far as relates to these forces.

3. It follows also that the triangular canal MRQ, taken in the interior of the mass, is in equilibrio. For if we produce MR, MQ, QR to the surface of the fluid, the three canals OMN, OQZ, NRZ will be in equilibrio, as we have seen above. Therefore, P denoting the pressure as before,

$$P{\cdot}OM = P{\cdot}NM, \text{ or } P{\cdot}OQ = P{\cdot}QM = P{\cdot}NR + P{\cdot}RM,$$

and
$$P{\cdot}OQ = P{\cdot}RQ + P{\cdot}ZR, \text{ or}$$
$$P{\cdot}OQ = P{\cdot}RQ + P{\cdot}NR.$$

Substituting this value of P·OQ in the preceding equation, it will become

$$P{\cdot}QM + P{\cdot}RQ = P{\cdot}RM.$$

Thus, the sum of the pressures of the columns QM, RQ on the point M, is equal to the pressure of the column RM on the same point; and consequently the canal QMR is in equilibrio.

Fig. 39.

4. Again, the canal OMRQN (fig. 39) of any form terminating at each extremity at the surface of the fluid is in equilibrio. For if we draw the diagonals OR, OQ, all the canals OMR, ORQ, OQN, will be in equilibrio. Therefore

$$P{\cdot}OM = P{\cdot}RM + P{\cdot}OR;\ P{\cdot}OR = P{\cdot}QR + P{\cdot}OQ;$$
$$P{\cdot}OQ = P{\cdot}NQ. \text{ Thus } P{\cdot}OM = P{\cdot}RM = P{\cdot}QR + P.NQ;$$

that is to say, the pressure of the column OM on the point M, is equal to the sum of the pressure of the columns RM, QR, NQ on the same point. Consequently the canal OMRQN is in equilibrio.

Fig. 4

5. Lastly, the canal MRTQV (fig. 40) of any figure returning on itself, taken in the interior of the fluid, is in equilibrio. For if the diagonals MT, MQ, be drawn, it will be seen that all the triangular canals MRQ, MQT, MTV, being in equilibrio, the pressure of the column RM on the point M, is equal to the sum of the pressures of the columns RQ, QT, TV, VM, on the same point; consequently, the canal MRQTV, whatever may be the number of its sides, will be in equilibrio.

Newton's and Huygens' particular cases.

53. It appears, therefore, that the principle assumed by Newton, that is, the equilibrium of the two central columns, is only a particular case of (N° 1) in the preceding article; and that the hypothesis or principle of Huygens, that is, the perpendicularity of gravity to the surface of the fluid, will only obtain when (N° 4) of the same article is verified; viz. when a canal of any figure, terminated at each extremity by the surface of the fluid, shall be in equilibrio; for we may conceive that this canal is immediately situated at the surface of the fluid, as for example OKM (fig. 39), and then the state of equilibrium necessarily requires that gravity to be perpendicular in every point of the canal.

Mathematicians, and particularly Maclaurin and Clairaut, have made great use of these supposititious canals, in order to determine the figure of the earth according to different hypotheses of the action of gravity.

When gravity is supposed primitively to be directed towards a fixed point, whatever may be the law which it observes, the equation of the terrestrial meridian, considered as a solid of revolution, may be found immediately from our general principle, viz. of equal pressure, as will appear from the following solution.

Solution on the principle of equal pressure. Fig. 41.

54. *Suppose the fluid mass* ADBE (fig. 41) *to revolve about the fixed axis* DE, *each of its particles being submitted to the action of a centrifugal force, and to the action of gravity directed towards a fixed centre* C, *and proportional to any given function of the distance from this point. It is required to determine its figure, on the principle of equal pressures.*

Let N be any point in the body, and conceive the locus of all the points, whose pressure is equal to N, to be the curve OTKH: draw from the centre C the right line CNM, and from the points N and M, the perpendiculars NQ, MP, to the axis of revolution DE.

Solution on the principle of equal pressure.

Assume CQ $= s$, QN $= z$, CN $= u$, the central force of the point N $= \phi$, ϕ being some function of s and z; the centrifugal force due to any distance $k = f$; the pressure in N $= p$ also some function of s and z, such that

$$\dot{p} = \mathrm{P}\,\dot{s} + \mathrm{Q}\,\dot{z},$$

P and Q being such functions of s and z, that $\mathrm{P}\,\dot{s} + \mathrm{Q}\,\dot{z}$ may be a complete fluxion or differential; for otherwise p would be an imaginary quantity, and N could not be in equilibrio.

Let us consider the portion of the fluid situated in N as a small rectangle $\mathrm{N}nrq$, of which $\mathrm{N}n = \dot{s}$, and $\mathrm{N}q = \dot{z}$; then it is evident that $\mathrm{P}\,\dot{s}$, supposing only s to be variable, will express the element of the pressure on each point of nr; and that $\mathrm{Q}\,\dot{z}$, supposing only $\dot{z}$ variable, will express the element of the pressure on each point of qr; therefore the elementary pressure against $nr = \mathrm{P}\,\dot{s}\,.\,\dot{z}$, and the elementary pressure against $qr = \mathrm{Q}\,\dot{z}.\dot{s}$.

Now resolve the central force ϕ into two others; the one in the direction $\mathrm{N}n$, and the other in the direction $\mathrm{N}q$; the first of which $= -\frac{\phi s}{u}$, and the second $= -\frac{\phi z}{u}$; to the latter adding the centrifugal force of the point N $= \frac{fz}{k}$; we have for the absolute force, soliciting the element $\mathrm{N}nrq$ in the direction $\mathrm{N}n$, $\frac{-\phi s}{u}\dot{s}\,\dot{z}$,

and $$\left(\frac{fz}{k} - \frac{\phi z}{u}\right)\dot{s}\,\dot{z},$$

for the absolute force acting in the direction $\mathrm{N}q$.

Now, in order that there may be an equilibrum, it is requisite that these two forces be equal, each respectively, to the corresponding and contrary pressures.

Whence $-\frac{\phi s}{u}\dot{s}\,\dot{z} = \mathrm{P}\,\dot{s}\,\dot{z}$, or $\frac{\phi s}{u} = \mathrm{P}$;

and $\left(\frac{fz}{k} - \frac{\phi z}{u}\right)\dot{s}\,\dot{z} = \mathrm{Q}\,\dot{s}\,\dot{z}$, or $\frac{fz}{k} - \frac{\phi z}{u} = \mathrm{Q}$;

$$\text{also } \dot{p} = -\frac{\phi s \dot{s}}{u} + \left(\frac{fz}{k} - \frac{\phi z}{u}\right)\dot{z},$$

$$= -\frac{\phi(s\dot{s} + z\dot{z})}{u} + \frac{fz\dot{z}}{k},$$

$$= -\phi\dot{u} + \frac{fz\dot{z}}{k};$$

taking the fluents $p = \mathrm{C} - \int \phi\dot{u} + \frac{fz^2}{2k}$,

which expression ought to be constant for every point of the same bed or lamina of the fluid. This condition gives us the nature of the extreme curve ADBE, as well as of the interior curve OTKH.

All things remaining the same, let CP $= x$, PM $= y$; CM $= r$; the central force in M $=$ F. Then the general value of p becomes for the point M,

$$p = \mathrm{C} - \int \mathrm{F}\,r + \frac{fy^2}{2k}.$$

Now, it is evident, that throughout the whole extent of the upper bed ADBE, the pressure ought to be zero; whence the equation of the curve is

$$\mathrm{C} - \int \mathrm{F}\,\dot{r} + \frac{fy^2}{2k} = 0.$$

This equation appertains to every point of the curve ADBE, and in order to determine the correction C, we must suppose a fixed point through which the curve ought to pass. Let B be this point, and assume CB $= b$, and for the same point $y = b$, also $\int \mathrm{F}\,\dot{r} = \mathrm{D}$, a known quantity; then we shall have

$$\mathrm{C} - \mathrm{D} + \frac{fb^2}{2k} = 0;$$

$$\text{or,} \qquad \mathrm{C} = \mathrm{D} - \frac{fb^2}{2k};$$

substituting for C, the equation of the curve between r and y, will be

$$\mathrm{D} - \frac{fb^2}{2k} - \int \mathrm{F}\,\dot{r} + \frac{fy^2}{2k} = 0. \qquad (1)$$

In order to determine the equation of the curve OTKH, let us take again the formula

$$p = \mathrm{C} - \int \phi\dot{u} + \frac{fz^2}{2k};$$

$$\text{or,} \qquad p = \mathrm{D} - \frac{fb^2}{2k} - \int \phi\dot{u} + \frac{fu^2}{2k}.$$

Suppose now the point K to be given, and assume CK $= c$, and for the same point $z = c$, also $\int \phi\dot{u} = \mathrm{E}$ a known quantity;

$$\text{whence} \qquad p = \mathrm{D} - \frac{fb^2}{2k} - \mathrm{E} + \frac{fc^2}{2k}.$$

Thus the equation of the curve between u and z is

$$\mathrm{D} - \frac{fb^2}{2k} - \mathrm{E} + \frac{fc^2}{2k} = \mathrm{D} - \frac{fb^2}{2k} - \int \phi\dot{u} + \frac{fz^2}{2k};$$

that is, by reduction,

$$\mathrm{E} - \frac{fc^2}{2k} - \int \phi\dot{u} + \frac{fz^2}{2k} = 0. \qquad (2)$$

This general problem, and others of a similar kind, the reader will find treated in a very elegant manner by Euler, in the Memoirs of Berlin for the year 1755.

55. If now, according to the assumption of Huygens, we suppose the central force ϕ to be constantly directed towards the fixed point C, and, moreover, that it is constant in its intensity; and if also we denote this constant force by $2g$, according to the notation employed in Dynamics, we shall have

$$\mathrm{F} = 2g; \int \mathrm{F}\,\dot{r} = 2gr, \ \mathrm{D} = 2gb; \int \phi\dot{u} = 2gu;$$

whence the equation (1) of the curve ADBE, of the preceding article, will be

$$2gb - \frac{fb^2}{2k} - 2gr + \frac{fy^2}{2k} = 0;$$

and equation (2) of the curve OTKH, will be

$$2gc - \frac{fc^2}{2k} - 2gu + \frac{fz^2}{2k} = 0.$$

It still remains to determine the ratio between t radius CB of the equator, and the semi-polar axis C or the axis of revolution.

In order to this determination, we may observe, th we have for CB, $r = y = b$, and for CD, $y = o$, wh give CD $= r = b - \frac{fb^2}{4kg}$; therefore

$$\text{CB} : \text{CD} :: b : b - \frac{f b^2}{4 k g} :: 1 : 1 - \frac{f b^2}{4 k g}.$$

Assuming now the arbitrary distance $k = b$, there results

$$\text{CB} : \text{CD} : 1 : 1 - \frac{f}{4g};$$

and it only remains to find the value of the fraction $\frac{f}{2g}$; the principle of which determination has been already illustrated in § 14. DYNAMICS.

Huygens, from the measurements and experiments made in his time, found the fraction $\frac{f}{2g} = \frac{1}{291 \cdot 47}$ nearly; and hence again

$$\text{CB} : \text{CD} :: 1 : 1 - \frac{1}{582 \cdot 94};$$

or

$$\text{CB} : \text{CD} :: 582 \cdot 94 : 581 \cdot 94,$$

for the ratio of the equatorial and polar radii.

Newton, after having established the law of universal gravitation, supposed the earth originally a fluid of a spherical form, and that it was changed by the action of a centrifugal force into an oblate spheroid, or a solid, produced by the revolution of a semi-ellipse about its shorter axis; and calculated the sum of the attractions on all the particles of two columns; the one terminating at the equator, and the other at the pole, meeting and counterbalancing each other at the centre of the earth. These two sums expressed the weights of the two columns; then, from the weight of the equatorial column he subtracted the sum of all the centrifugal forces on the same; which latter being made equal to the difference of weights of the polar column, gave him the ratio of the equatorial and polar axes. Supposing the earth homogeneous, this computation gives for the ratio in question 230 to 229.

Having said thus much with reference to the figure of the earth, as it depends upon the laws of the equilibrium of fluids, we shall conclude by showing in what manner the same may be drawn from the actual measurement of terrestrial arcs in different latitudes.

Figure of the earth determined from measured arcs.

PROBLEM.

56. *To determine, by means of measured arcs, the ratio of the earth's axes, supposing it to be an elliptic spheroid, differing but little from a sphere.*

Let ADBE (fig. 42) represent a meridian of the earth, DE its axis; AB a diameter of the equator; MN, *mn*, two arcs of the same number of degrees, of which the lengths are supposed to be known from actual measurement, their magnitude being such, that they may be considered, without sensible error, to be parts of circular arcs. MO, NO, *m o*, *n o*, the radii of curvature, corresponding to the points M, *m*, N, *n*, and M*p*, *m p*, ordinates perpendicular to the diameter AB. Assume $\text{CD} = a$, $\text{CB} = b$, $b^2 - a^2 = c^2$; $\text{CP} = x$; $\text{C}p = u$; $\text{MO} = \text{R}$, $m o = r$; radius $= 1$; the given angle of the latitude BSM of the point $\text{M} = p$, and the given angle B *s m* of the latitude of the point $m = q$; the length of the arc $\text{MN} = \text{M}$, that of the arc $m n = m$, and we shall have (in consequence of the similar sectors MON, *m o n*)

$$\text{R} : r :: \text{M} : m;$$

whence

$$m \text{R} = r \text{M}. \qquad (1)$$

The property of the ellipse gives

$$\text{PM} = \frac{a}{b} \sqrt{(b^2 - x^2)}; \ \text{SP} = \frac{a^2 x}{b^2};$$

$$p m = \frac{a}{b} \sqrt{(b^2 - u^2)}; \ \text{SP} = \frac{a^2 u}{b^2};$$

$$\text{R} = \frac{(b^4 - c^2 x^2)^{\frac{3}{2}}}{a^4 b}; \ r = \frac{(b^4 - c^2 u^2)^{\frac{3}{2}}}{a^4 b}.$$

Again we have

$$\text{SP} : \text{PM} :: \cos p : \sin p,$$

which gives by substitution, and reduction,

$$x^2 = \frac{(b^4 - c^2 x^2)^{\frac{3}{2}}}{b^2 - c^2 \sin^2 p},$$

$$u^2 = \frac{(b^4 - c^2 u^2)^{\frac{3}{2}}}{b^2 - c^2 \sin^2 q}.$$

Substituting now in equation (1) for R and r their values as found above, and introducing into these also the values of u^2 and x^2, and farther observing, that

$$\sin^2 p + \cos^2 p = 1, \text{ and } \sin^2 q + \cos^2 q = 1,$$

we obtain

$$\frac{m}{(b^2 - c^2 \sin^2 p)^{\frac{3}{2}}} = \frac{\text{M}}{(b^2 - c^2 \sin^2 q)^{\frac{3}{2}}};$$

or, $m\,(b^2 - c^2 \sin^2 q)^{\frac{3}{2}} = \text{M}\,(b^2 - c^2 \sin^2 p)$;

or, which is still the same,

$$m^{\frac{2}{3}}\,(b^2 - c^2 \sin^2 q) = \text{M}^{\frac{2}{3}}\,(b^2 - c^2 \sin^2 p);$$

whence

$$c^2 = \frac{b^2\,(\text{M}^{\frac{2}{3}} - m^{\frac{2}{3}})}{\text{M}^{\frac{2}{3}} \sin^2 p - m^{\frac{2}{3}} \sin^2 q};$$

or, substituting for c^2 its value $b^2 - a^2$

$$b^2 - a^2 = \frac{b^2\,(\text{M}^{\frac{2}{3}} - m^{\frac{2}{3}})}{\text{M}^{\frac{2}{3}} \sin^2 p - m^{\frac{2}{3}} \sin^2 q};$$

wherefore

$$\frac{b^2}{a^2} = \frac{\text{M}^{\frac{2}{3}} \sin^2 p - m^{\frac{2}{3}} \sin^2 q}{m^{\frac{2}{3}} \cos^2 q - \text{M}^{\frac{2}{3}} \cos^2 p};$$

which last expression is reducible to

$$\frac{(\text{M}^{\frac{1}{3}} \sin p + m^{\frac{1}{3}} \sin q) \cdot (\text{M}^{\frac{1}{3}} \sin p - m^{\frac{1}{3}} \sin q)}{(m^{\frac{1}{3}} \cos q + \text{M}^{\frac{1}{3}} \cos p)\ (m^{\frac{1}{3}} \cos q - \text{M}^{\frac{1}{3}} \cos p)};$$

whence we have, as a final result,

$$\frac{b}{a} = \sqrt{\left\{ \frac{(\text{M}^{\frac{1}{3}} \sin p + m^{\frac{1}{3}} \sin q)\,(\text{M}^{\frac{1}{3}} \sin p - m^{\frac{1}{3}} \sin q)}{(m^{\frac{1}{3}} \cos q + \text{M}^{\frac{1}{3}} \cos p)\,(m^{\frac{1}{3}} \cos q - \text{M}^{\frac{1}{3}} \cos p)} \right\}}$$

in which all the quantities on the second side of the equation are supposed to be given.

57. The only violation of the strict principles of mathematical computation introduced into the preceding solution, is where we suppose the elliptic arc to be confounded with the circular arc at the place where the measurement is taken, which we may add is so nearly the case that little inaccuracy is to be apprehended from such an hypothesis.

As this question will be again examined in our treatise on Physical Astronomy, we shall merely in this place show the application of the above formula to a numerical example taken from the measurements of Bouguer at the equator, and that of Maupertuis in latitude 66½°. The former making a degree of latitude 56753 toises, and the latter 57437·9. We have therefore

$$\text{M} = 57437 \cdot 9, \ m = 56753; \ p = 66\tfrac{1}{2}^\circ; \ q = 0; \text{ or,}$$

$\sin p = \cdot 91706$, $\cos p = \cdot 39875$; $\sin q = 0$; $\cos q = 1$. Which numbers substituted in the above formula, give

$$\frac{b}{a} = \frac{35383}{35213};$$

or, $b : a :: 35383 : 35213$ or as $179 : 178$ *nearly*. Other measurements, as will be seen in the article above referred to, give very different ratios.

HYDRAULICS.

§ I. *Definitions and general principles.*

Definition.

58. *Hydraulics*, as we have already stated, is that branch of general mechanics which treats of, and by which we examine, the phenomena attending the motion of fluids.

One of the first, and most simple, cases connected with this science, is to determine the circumstances connected with the issuing of water, or any other fluid through given orifices, made in the bottom and sides of vessels.

General principles.

Let us consider a vessel filled to a certain height with water, its base being horizontal, having an orifice cut in the bottom; and endeavour to investigate the motion of the water which will necessarily issue through that orifice. In the first place, we may observe, that during this motion each particle will take a particular velocity, which is variable from one particle to another, as well in magnitude as in direction; but in order to simplify the solution of the problem, we shall suppose that all the particles that appertain to the same horizontal lamina, or stratum, have, at the same instant, the same velocity; so that each stratum will remain parallel to itself, and be composed of the same fluid particles during the whole continuance of the motion. Each lamina will thus take the place of that which is below it, and consequently its horizontal section will augment or diminish according as the section of the vessel is greater or less at different parts of its altitude; at the same time the thickness of each lamina, which we suppose always infinitely little, will vary inversely as the sections of the vessel. In order that this may obtain, it is obvious that the particles must, in a certain degree, have a horizontal motion, so that each particle, besides the vertical velocity common to the stratum to which it appertains, has also a proper horizontal velocity; but, if we were to introduce that consideration into our solution, it would become extremely complicated; we shall therefore only investigate the conditions of the vertical velocity. We may in fact conceive, and experiment proves, that in vessels which differ but little from a cylinder, or when the height of the water is very considerable with regard to the different horizontal sections of the vessels, the horizontal velocities of the particles are indefinitely small, with respect to the vertical velocities, which at the same time are very nearly equal in each particle of the same stratum; there are therefore a great number of cases in which the *parallelism of the strata*, if not exactly, at least approximatively obtains; and in all such cases, the calculation founded on the above hypothesis, will lead very nearly to the correct solution of the problem.

On the theoretic discharge of vessels.

Investigation of formulæ.

Fig. 1.

59. Let ABCD (fig. 1. pl. I. HYDRAULICS) represent the vertical section of a vessel, AB an orifice, O z a vertical axis, on which we estimate or measure the distances of the fluid strata from the fixed point O, or from the horizontal plane drawn through that point. Let the time the fluid has been in motion be denoted by t, v the corresponding velocity of any stratum, and z the distance of the same from the point O, so that v and z will be two unknown functions of the time t. Conceive $m\,q\,n$, $m'\,q'\,n'$ to represent the vertical section of the stratum in question; then we shall have $O\,q = z$, the thickness of the stratum $q\,q' = \dot z$, and its volume will be equal to this fluxion multiplied by the area of its horizontal section, or that of the vessel in the same place; which area is therefore some function of z, depending upon the equation of the generating curve of the vessel.

Hence, if we denote this section by y, the volume of the stratum we are considering will be $y\,\dot z$. Let us also, as we have hitherto done, represent the force of gravity by $2\,g$. During the instant $\dot t$, the velocity of the lamina $y\,\dot z$ would be increased by the quantity $2\,g\,\dot t$, if it were free and isolated; but $\dot v$ is the increase which it actually obtains, consequently $2\,g\,\dot t - \dot v$ is the infinitely small velocity lost at each instant; whence by the principle of D'Alembert, the fluid would remain in equilibrio, if all these strata were solicited by moving forces capable of impressing upon them the velocities which they lose at each instant: let us suppose, therefore these strata to be solicited by similar forces, and endeavour to determine according to this hypothesis, the conditions of their motion.

The accelerating force of any lamina $y\,\dot z$, will be expressed by $2\,g - \frac{\dot v}{\dot t}$, because it ought to produce the velocity $2\,g\,\dot t - \dot v$, in the instant $\dot t$. Its motive force will therefore be equal to the product

$$\left(2\,g - \frac{\dot v}{\dot t}\right) s\,y\,\dot z;$$

s denoting the density, or specific gravity of the fluid. Let us call p the pressure, on the unit of surface, which this stratum has to sustain on its upper face $m\,q\,n$; this pressure is transmitted through the medium of the fluid of which the stratum is composed, not only to the lower base $m'\,q'\,n'$, but also all round the sides of the vessel bounding its circumference. So that when the value of p is determined in terms of z and t, we shall thence be able to find the pressure exercised on every point of the vessel at every instant. But besides the pressure p, the lower base of the stratum will also, in the state of equilibrium we are considering, have to sustain a pressure due to the motive force of the stratum, viz. equal to

$$\left(2\,g - \frac{\dot v}{\dot t}\right) s\,\dot z, \text{ or } \left(2\,g - \frac{\dot v}{\dot t}\right) s\,y\,\dot z \div y.$$

If, therefore, the entire pressure on a unit of surface be denoted by p', we shall have

$$p' = p + \left(2\,g - \frac{\dot v}{\dot t}\right) s\,\dot z.$$

But the difference $p' - p$ is obviously the fluxion of p, as depending upon the variable z, we have therefore

$$\dot p = \left(2\,g - \frac{\dot v}{\dot t}\right) s\,\dot z,$$

an equation whence we may determine the value of the pressure p, when that of the velocity v is known.

60. As the fluid is supposed to be incompressible, there must necessarily pass in any given interval of

time, the same volume of fluid through all the horizontal sections of the vessel; whence we may conclude, that the velocity of any two strata ought to be every instant, in the inverse ratio of its horizontal section. If, therefore, we denote the velocity at the horizontal orifice AB by u, and the area of the orifice by a, we shall have, by comparing this velocity with that whose area is y, the equation $y\,v = a\,u$, which gives

$$v = \frac{a\,u}{y}. \qquad (1)$$

In this equation we may observe, that u is a function of t, and y a function of z; we may therefore take the fluxion of either with reference to t or z, as variable. The fluxion with reference to z, will express the difference between the velocity of two consecutive strata which have place at the same instant; and the fluxion with reference to t, will be the difference between the velocities of two strata of the fluid which answer successively to the same section of the vessel; but, in order to have the difference between the successive velocities of the same stratum, we must take the fluxion of v with reference to the two variables t and z, by considering the second as a function of the first: this gives us

$$\frac{\dot{v}}{\dot{t}} = \frac{a}{y} . \frac{\dot{u}}{\dot{t}} - \frac{a\,u}{y^2} . \frac{\dot{y}\,\dot{z}}{\dot{z}\,\dot{t}}.$$

This expression being now substituted instead of $\frac{\dot{v}}{\dot{t}}$, in the value of $\dot{p}$, and replacing also $\frac{\dot{z}}{\dot{t}}$, by the velocity v, or by its equal $\frac{a\,u}{y}$, we shall have

$$\dot{p} = 2\,s\,g\,\dot{z} - a\,s\,\frac{\dot{u}\,\dot{z}}{\dot{t}\,y} + a^2 s\,u^2\,\frac{\dot{y}}{y^3}.$$

Taking the fluent with reference to z variable, and observing, that in this case, u and $\frac{\dot{u}}{\dot{t}}$ must be considered as constant, it will be

$$p = 2\,g\,s\,z - a\,s\,\frac{\dot{u}}{\dot{t}} \int \frac{\dot{z}}{y} - \frac{a^2 s\,u^2}{2\,y^2} + \text{C}.$$

The correction C may be a function of t; its value depends on the pressure which the upper surface of the fluid has to support. Suppose, for example, this pressure to be that of the atmosphere, which let us denote by π. Let also EHF be the level of the water in the vessel; make OH $= z'$, that is to say, denote by z' the value of z, as referred to this level, and let y' represent the section of the vessel answering to the same level; so that y' becomes y when $z' = z$. Supposing, moreover, the fluent $\int \frac{\dot{z}}{y}$ to vanish, for this value of $z = z'$, we shall then have for the same value

$$\pi = 2\,g\,z' - \frac{a^2 s\,u^2}{2\,y'^2} + \text{C};$$

whence

$$\text{C} = \pi + \frac{a^2 s\,u^2}{2\,y'^2} - 2\,g\,s\,z'.$$

Substituting, in the preceding equation, this value of C, we have

$$p = \pi + 2gs(z - z') - a\,s\,\frac{\dot{u}}{\dot{t}} \int \frac{\dot{z}}{y} - \frac{a^2 s\,u^2}{2\,y^2} + \frac{a^2 s\,u^2}{2\,y^2}. \qquad (2)$$

This value of p belongs to every point of the vessel. Now the pressure at the orifice AB is given; it is equal to the pressure of the atmosphere, if the fluid run out of the vessel into air; and it is nothing if it run into a vacuum. Generally, we may represent it by $\pi - c$, c being the difference between the pressures at the level of the fluid and at the orifice.

Hydraulics.

Let now OA $= l$ be the value of z, which answers to the orifice AB, a value which is given and constant, h the height of the fluid above this orifice; viz. $h =$ HA $= l - z$. Assume also $\int \frac{\dot{z}}{y} = \text{N}$.

This fluent, being taken from $z = z'$ to $z = l$, N will in this case be a function of h, and will be determinable for each particular case, as depending upon the form of the vessel.

We shall have also, at the same time,

$$z - z' = h ; \int \frac{\dot{z}}{y} = \text{N}; \; y = a, \; p = \pi - c;$$

whence, from equation (2),

$$a\,s\,\text{N}\,\frac{\dot{u}}{\dot{t}}. = c + 2\,g\,s\,h - \left(1 - \frac{a^2}{y'^2}\right)\frac{s\,u^2}{2}. \qquad (3)$$

Two cases of the general problem.

61. The above investigation contains a complete solution of the problem we proposed to resolve; but it must be observed, that the motion of the water may take place under two cases very distinct from each other; viz. the one where the water constantly remains at the same height above the orifice, and the other, where for want of a sufficient supply it sinks as the fluid issues out.

When the height of the fluid is constant.

In the first case, h, N and y' are given and constant quantities; and we may therefore find the fluents in equation (3) without any difficulty, and from this will be determined the velocity u at the orifice of the vessel in terms of the time, and substituting this value in equations (1) and (2), we shall have the velocity v and the pressure p in functions of t and z.

When the height is variable.

In the second case, N and y' will be always given functions of h; but h will be variable, and in order to determine its value in functions of the time, we must have one equation more than in the first case. Now at the level of the water the velocity is $\frac{\dot{z}'}{\dot{t}}$; and it is equal to $\frac{a\,u}{y'}$ by equation (1); and farther, because $h = l - z'$, we have $\frac{\dot{h}}{\dot{t}} = -\frac{\dot{z}'}{\dot{t}}$, l being constant: therefore

$$\frac{\dot{h}}{\dot{t}} + \frac{a\,u}{y'} = 0. \qquad (4)$$

Combining this with equation (3), we have two fluxional equations of the first order, which are sufficient for determining h and u in functions of t; we shall know, therefore, by taking the fluxions, the depression of the surface, and the velocity of the fluid at the orifice for any time t; and then equations (1) and (2) will give, as in the first case, the velocity and pressure in any point of the vessel. But it is only in a few cases chosen for the express purpose that we have any chance of finding the fluents in equations (3) and (4) under a finite form. When this cannot be done we must have recourse to series, and then h and u, which depend upon these equations, can only be obtained by approximation.

When the orifice is very small.

When the orifice AB is very small in comparison with the other dimensions of the vessel, so that we may without any sensible error neglect the terms multiplied by a in equation (3), it reduces to

$$0 = c + 2gsh - \tfrac{1}{2}su^2;$$

and if, moreover, the pressure is the same at the orifice as at the upper surface or level of the fluid, in which case the quantity c becomes zero, the same equation gives $u = 2\sqrt{gh}$;
whence we draw the following important theorem:

The velocity of the water issuing from a small orifice in the bottom of any vessel, is equal to that acquired by a heavy body in falling through a height equal to the height of the fluid above the orifice.

When the pressure at the orifice and surface are unequal.

62. What we have stated above, is upon the supposition that the pressure at the orifice is the same as at the level of the fluid; but if the latter be greater than the former, the velocity u will be augmented with this excess of pressure, and the augmentation will be the same as if the level of the upper surface were so much higher as to make up the excess of pressure. Let h' denote the height of fluid that gives $2gsh' = c$, c being the difference of pressure, or the excess at the surface over that at the orifice, then we must substitute $h + h'$ instead of h, and we shall have

$$u = 2\sqrt{(gh + gh')};$$

that is, the velocity u would be the same as if the height of the fluid above the orifice were $h + h'$.

Hence, when water is projected into a vacuum, as the pressure of the atmosphere is equal to about 34 feet of water; we should have in this case $h' = 34$, and $u = 2\sqrt{(gh + 34g)}$.

Again, if in our equation (2) we reject all those terms that we multiplied by a, we shall have

$$p = \pi + 2gs(z - z');$$

whence we may conclude, that in this case, viz. when the orifice is very small, the pressure on any point of the vessel is equal to the pressure applied at the upper surface, plus the pressure $2gs(z - z')$, due to the height of the level of the water above that point, that is, *the pressure will be the same at every instant, as if the fluid had no motion in the vessel.*

This and the preceding theorem, relative to the pressure p and the velocity u, will equally obtain whether the water in the vessel be continually sinking by the discharge, or whether it be kept constantly full by a supply from above.

The velocity is progressive.

63. Before we conclude this part of our investigation, it may be proper to remark, that the velocity impressed upon the several particles of water, or other fluid, issuing through any orifice, is due wholly to the pressure upon them, which pressure being a motive force, we cannot suppose that this will produce instantaneously a finite velocity; it is therefore only after some finite time that the velocity u becomes equal to $2\sqrt{gh}$.

In order to demonstrate this, suppose the level of the water constant, and the quantity c equal to zero; then, taking the fluents of equation (3) on this hypothesis, and resolving it with reference to $\dot{t}$, it will become

$$\dot{t} = \frac{2aN\dot{u}}{4gh - u^2\left(1 - \frac{a^2}{y'^2}\right)},$$

which may be put under the form

$$\dot{t} = \frac{aN}{2\sqrt{.gh}} \times$$

$$\left\{\frac{\dot{u}}{2\sqrt{gh}+u\sqrt{\left(1-\frac{a^2}{y'^2}\right)}} + \frac{\dot{u}}{2\sqrt{gh}-u\sqrt{\left(1-\right.}}\right.$$

whence, by taking the fluents, we have

$$t = C + \frac{aN}{2\sqrt{gh} \times \sqrt{\left(1 - \frac{a^2}{y'^2}\right)}}$$

$$. \log. \left\{\frac{2\sqrt{gh} + u\sqrt{\left(1 - \frac{a^2}{y'^2}\right)}}{2\sqrt{gh} - u\sqrt{\left(1 - \frac{a^2}{y'^2}\right)}}\right\}.$$

If we reckon the time t from the origin of the n
the constant arbitrary or correction C ought to b
for the fluid leaving a state of rest, we shall have
same time $t = 0$, and $u = 0$, which gives $C = 0$.
pressing therefore this part of the expressio
exhibiting the logarithms under their exponential
we find

$$2\sqrt{gh} - u\sqrt{\left(1 - \frac{a^2}{y'^2}\right)} =$$

$$\left\{2\sqrt{gh} + u\sqrt{\left(1 - \frac{a^2}{y'^2}\right)}\right\} e^{-\lambda t.}$$

e, being the number whose hyp. log. $= 1$, and
suming for the sake of abridging the expression

$$\lambda = \frac{2\sqrt{gh} \times \sqrt{\left(1 - \frac{a^2}{y'^2}\right)}}{aN}.$$

Now when t is zero, the exponent λt is so lik
this exponent and the value of u encreasing with
as the factor a which is found in the denominato
is by the supposition very small, it follows th
factor λt will acquire a great value in a very sh
terval of time; therefore, the second member o
tion (5) which has for its factor the exponential
will become very nearly equal to zero; conse
we shall have

$$2\sqrt{gh} - u\sqrt{\left(1 - \frac{a^2}{y'^2}\right)} = o;$$

or simply, when we neglect the fraction $\frac{a^2}{y'^2}$, $2\sqrt{g}$

We see, therefore, that, strictly speaking, the
sion $2\sqrt{gh} = u$ is only a limit of the valu
which it can never attain, but to which, notwithst
it approaches indefinitely near, after a very sh
terval of time, this time being the less as the
is smaller.

On the same principle it follows that it is onl
a certain interval of time the pressure of th
against the sides of the vessel becomes a c
quantity at each point. When the fluid begins
this pressure is at first less than in the state of
brium, and while the motion has not attained
mity, or while the value of u is still variab
pressure also is variable with the time; its
value being determinable by substituting in e
(2) that of u as given by equation (4).

64. When the orifice is very small, as we have
supposed in the preceding article, all our d
nations will equally apply, whether they be
sides or in the bottom of the vessel; but whe

not the case, it is necessary that the orifice be horizontal, in order that the hypothesis of the parallelism of the lamina may obtain. If we make a very small opening in the bottom, or in any part of the side of the vessel, and the height of the water above it is great with reference to the horizontal section of the vessel, experiment shows that the motion of the fluid differs insensibly from the parallelism we have supposed, except it be very near the orifice, where the particles of the water take a velocity whose direction concur towards a point. The results, therefore, above obtained, will apply equally to such small apertures, whether they be horizontal, vertical, or any way inclined to the base. Thus the water issuing through an orifice, the plane of which has any inclination, will in an indefinitely short time attain the same velocity as a body would acquire in falling freely through the height of the level of the fluid above it; consequently, if the jet, by means of a tube, have a vertical direction, the particles of the water, which are projected with this velocity, will rise in a vacuum to the height of the level of the fluid. This result has been repeatedly verified by experiment. By knowing thus the velocity and angle of projection, the ranges or distances to which the ajutage will be thrown on any given plane, may be computed in the same manner as any other projectile.

It follows from this, that when the holes are made at different depths, as at C, D, E (fig. 2) of the vessel AA′ BB′, and are very small, the velocity with which the fluid issues will be as the square roots of the several depths, AC, AD, AE. And as the time of descending through CB, DB, EB, will be as the square root of these lines, the distances to which these several jets will be projected on the horizontal plane FB will vary as $\surd$ (AC × CB), $\surd$ (AD × DB), $\surd$ (AE × EB), &c. Consequently, if upon the height AB a semicircle AIB be described, the ranges of the several jets will be as the ordinates HC, ID, KE, these being, by the nature of the circle, mean proportionals between the corresponding abscisses. Again, the velocity through any hole C, is such as would carry the water horizontally through a space equal to 2 AC in the same time; therefore to find the actual distance it will fall from the bottom B, we have

$$\surd \text{AC} : \surd \text{CB} :: 2\,\text{AC} : 2 \surd (\text{AC} \times \text{CB}) = 2\,\text{HC}.$$

In the same manner we shall find the projected distance from the orifice D = 2 ID, and from E = 2 KE. The several distances of projection are therefore always equal to double the ordinate drawn from the orifice to meet a semicircle described upon the height of the vessel as a diameter.

Hence also it follows, that the greatest range is that proceeding from an orifice which bisects the height of the vessels, and all those equally above and below this middle point will have equal ranges.

We have here supposed the vessel standing on the horizontal plane; if it stand higher, then the distance from that plane from the surface of the fluid must be used instead of what we have above supposed to be the height of the vessel.

65. When we know the velocity of the fluid at the orifice, and the area of the latter, it is easy to compute the quantity of water that will issue through it in a given time. In fact, the quantity which passes during the indefinitely small portion of time $\dot{t}$, is equal to the product $a\,u\,\dot{t}$, a denoting the area of the orifice, and u the velocity of the spouting fluid: if, therefore, x represent the volume of water run out in the time t, we shall have

$$\dot{x} = a\,u\,\dot{t};$$

whence, by taking the fluents, the value of x in functions of t will be determined.

When the orifice is very small we have $u = 2 \surd g\,h$, therefore

$$\dot{x} = 2\,a\ (\surd g\,h)\,\dot{t},$$

and

$$x = 2\,a\,t\,(\surd g\,h),$$

provided the height h of the level of the fluid above the orifice be constant.

Let h' be the height through which a heavy body would fall in the time t, or $h' = g\,t^2$; then $g = \frac{h'}{t^2}$, which substituted for g, gives $x = 2\,a \surd\,h\,h'$.

Whence it follows, that the volume of water discharged in the case where the surface of the fluid is constant, is equal to double the volume of a cylinder, having for its base the orifice a, and for its altitude a mean proportional between h and h'.

When the level is variable.

When the level is variable, we must find h in terms of t. Now by making $u = 2 \surd\ g\,h$ in equation (4) we have

$$y' \frac{\dot{h}}{h^{\frac{1}{2}}} + 2\,a\,\dot{t} \surd\ g\,h = o,$$

an equation of which the fluents may be taken whenever the form of the vessel is given. We thus obtain the value of y' in terms of h and t, as also h in terms of t and y'; and substituting this last value in the equation $\dot{x} = 2\,a\,\dot{t} \surd g\,h$, we shall have by again taking the fluents, the quantity of water x in functions of t.

Contracted vein.

. 66. It should be observed, however, that the quantity of water discharged, computed on the above principles, does not agree with the result of experiments; the latter being always less than the former in the ratio of 1 : ·62; this ratio being constant, neither varying with the altitude of the fluid above the orifice, nor with the area of the latter; so that in fact we must always reduce the actual area by multiplying it by the decimal ·62.

This difference between the theory and practice, is attributable to the *contraction of the vein* of the fluid in passing through the orifice, in consequence of the different directions which the particles of the fluid take in their approach towards it. When they escape, they observe still to a certain extent the same directions, and this necessarily produces a contraction of the vein; the smallest transverse section of which must be considered as the area of the orifice, and which we have seen is equal to ·62 a, a denoting the actual aperture. It must be observed, however, that this coefficient is not universally admitted, different experimentalists giving different results, as we shall see in the following section: but we propose first to assume this number, and to illustrate our preceding results by a few examples.

Problems.

Problems reluting to the discharge of water through small apertures in the sides and bottoms of given vessels.

67. We have seen (art. 65) that x denoting the volume of the fluid which has run out in the time t, a the area of the orifice, or rather the contracted orifice; u the velocity at the latter, we have $\dot{x} = a\,u\,\dot{t}$.

We have seen also, that h denoting the height of the fluid above the orifice (the pressures at the surface of the fluid and at the orifice being equal), we have $u = 2gh$; and, farther, if we denote the area of the upper surface of the fluid by z, we shall have $\dot{x} = -z\dot{h}$;

whence $-z\dot{h} = 2a\dot{t}\sqrt{gh}$,

and $\dot{t} = \frac{-z\dot{h}}{2a\sqrt{gh}}$;

consequently, the resulting equation for the time is

$$t = -\int \frac{z\dot{h}}{2a\sqrt{gh}} + \text{C}.$$

Cylinder.

1. Let it be required to determine the time in which a given cylindric vessel will empty itself through an orifice in its base, the latter being small with regard to the dimensions of the vessel.

Let the area of the section be denoted by A, and the altitude of the water above the orifice, before the latter is opened by H; let h be any variable depth due to the time t; then the area z, being constant, and equal to A, the above expression becomes

$$t = \frac{-\text{A}}{2a\sqrt{g}} \times \int h^{-\frac{1}{2}}\dot{h} = \frac{-\text{A}}{a\sqrt{g}} \times h^{\frac{1}{2}} + \text{C}.$$

Now when $\text{H} = h$, $t = o$, whence

$$0 = \frac{-\text{A}}{a\sqrt{g}} \times \text{H}^{\frac{1}{2}} + \text{C};$$

wherefore the correct fluent is

$$t = \frac{\text{A}}{a\sqrt{g}} \times (\sqrt{\text{H}} - \sqrt{h}),$$

which, when $h = o$, becomes

$$t = \frac{\text{A}\sqrt{\text{H}}}{a\sqrt{g}},$$

the whole time of emptying.

If the vessel were to be kept constantly full, the velocity would be constant, and equal to $2\sqrt{g\text{H}}$; and therefore the quantity discharged in the time $t = \frac{\text{A}\sqrt{\text{H}}}{a\sqrt{g}}$, would be

$$2a\sqrt{g\text{H}} \times \frac{\text{A}\sqrt{\text{H}}}{a\sqrt{g}} = 2\,\text{AH};$$

that is, the quantity would be equal to double the content of the vessel.

As an example in numbers, let the area of the vessel be 100 inches, the altitude 3 feet, or 36 inches, the aperture $\frac{1}{10}$th of an inch, and, consequently, the area of the contracted vein ·062 inch, to find the time of emptying.

Here $\text{A} = 100$, $\sqrt{\text{H}} = \sqrt{36} = 6$, $a = {\cdot}062$; and assuming $g = 16\frac{1}{12}$ feet $= 193$ inches, we have

$$t = \frac{100 \times 6}{{\cdot}062\sqrt{193}} = 696\tfrac{1}{2}'' = 11^{m}\ 36\tfrac{1}{2}'',$$

the time required.

Hemisphere.

2. Let the vessel be a hemisphere, and the orifice in its vertex, to find the time of emptying.

Let the radius of the base or depth $= r$, any variable depth $= x$, then the radius at the surface will be $\sqrt{(2rx - x^2)}$, and $\pi(2rx - x^2) = z$ the area of the descending fluid, where $\pi = 3{\cdot}1456$, &c. and a the area of the contracted aperture, as before; then, substituting x for h in the general formula, we have

$$t = \int - \frac{\pi(2rx - x^2)\dot{x}}{2a\sqrt{gx}} + \text{C};$$

$$t = \frac{-\pi}{2a\sqrt{g}} \times \int (2rx - x^{\frac{3}{2}})\dot{x} + \text{C};$$

$$t = \frac{-\pi}{2a\sqrt{g}} \times \left(\frac{4}{3}rx^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right) + \text{C}.$$

Hyd

Hemi verte down

When $x = r$, $t = o$, wherefore

$$\text{C} = \frac{\pi}{2a\sqrt{g}}\left(\frac{4}{3}r^{\frac{5}{2}} - \frac{2}{5}r^{\frac{5}{2}}\right) = \frac{\pi}{2a\sqrt{g}} \times \frac{14}{15}r^{\frac{5}{2}}:$$

therefore the correct fluent is

$$t = \frac{\pi}{2a\sqrt{g}}\left\{\frac{14}{15}r^{\frac{5}{2}} - \left(\frac{4}{3}rx^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right)\right\},$$

which when $x = o$, gives $t = \frac{14\pi r^{\frac{5}{2}}}{30a\sqrt{g}}$ for the whole time of exhaustion.

3. Let us now suppose the hemisphere to have its orifice in the base, to determine the time of emptying. Hemi base wards

Here, employing the same notation, the equation of the descending surface is $z = \pi(r^2 - x^2)$;

whence $t = \frac{-\pi}{2a\sqrt{g}} \times \int \frac{(r^2 - x^2)}{\sqrt{x}}\dot{x} + \text{C}$;

or, $t = \frac{-\pi}{2a\sqrt{g}} \times \left(2r^2x^{\frac{1}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right) + \text{C}$.

Here when $x = r$, $t = o$,

therefore $\text{C} = \frac{\pi}{2a\sqrt{g}} \times \frac{8}{5}r^{\frac{5}{2}}$;

whence $t = \frac{\pi}{2a\sqrt{g}} \times \left\{\frac{8}{5}r^{\frac{5}{2}} - 2r^2x^{\frac{1}{2}} + \frac{2}{5}x^{\frac{5}{2}}\right\}$;

which, when $x = o$, becomes

$$t = \frac{8g\pi r^{\frac{5}{2}}}{10a\sqrt{g}}, \text{ the time required.}$$

The time of exhaustion in this case, therefore, is to the time in the former, as

$$\frac{8}{10} : \frac{14}{30}, \text{ or as } 12 : 7.$$

4. Let us now take a complete sphere, with an aperture at its lowest point, to find the time it will be in emptying itself. Spher

If we use here the same notation, we shall have the same general expression as in example 2, viz.

$$t = \frac{-\pi}{2a\sqrt{g}} \times \left(\frac{4}{3}rx^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right) + \text{C}.$$

But in order to determine C, we observe that when $x = 2r$, $t = o$, whence

$$\text{C} = \frac{\pi}{2a\sqrt{g}} \times \left(\frac{4 \,.\, 2^{\frac{3}{2}}}{3}r^{\frac{5}{2}} - \frac{2 \,.\, 2^{\frac{5}{2}}}{5}r^{\frac{5}{2}}\right);$$

or $t = \frac{\pi}{2a\sqrt{g}}\left(\frac{16\sqrt{2}}{15}r^{\frac{5}{2}} - \frac{4}{3}rx^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}}\right)$;

which, when $x = o$, becomes

$$t = \frac{8 \,.\, \pi r^{\frac{5}{2}}}{15a\sqrt{g}} \times \sqrt{2}.$$

If we make $x = r$, the above general expression gives

$$t = \frac{(8\sqrt{2} - 7)\pi r^{\frac{5}{2}}}{15a\sqrt{g}}$$

for the time of emptying the upper hemisphere.

The time of emptying the lower hemisphere has been already found $= \frac{7\pi r^{\frac{5}{2}}}{15a\sqrt{g}}$;

therefore, the times of exhausting the successive hemispheres are to each other $8\sqrt{2} - 7 : 7$.

5. If the vessel be a paraboloid, the radius of its base $= r$, and altitude h, we have

$$h : x : r^2 : \frac{x r^2}{h};$$

whence $\frac{\pi x r^2}{h} =$ descending area.

Our general formula, therefore, becomes

$$t = - \frac{-\pi r^2}{2 a h \sqrt{g}} \cdot \int x^{\frac{1}{2}} \dot{x} + C;$$

or $$t = \frac{-\pi r^2}{2 a h \sqrt{g}} \times \tfrac{2}{3} x^{\frac{3}{2}} + C.$$

When $x = h$, then $t = o$, wherefore

$$C = \frac{\pi r^2 h^{\frac{3}{2}}}{3 a h \sqrt{g}},$$

and $$t = \frac{\pi r^2}{3 a h \sqrt{g}} \times \left(h^{\frac{3}{2}} - x^{\frac{3}{2}}\right).$$

Consequently, when $x = o$, we have for the whole time

$$t = \frac{\pi r^2 h^{\frac{1}{2}}}{3 a \sqrt{g}} = \frac{\pi p}{3 a \sqrt{g}} \times h^{\frac{3}{2}},$$

where p denotes the parameter.

6. Let it now be proposed to find the time in which a given cylinder will empty itself into a vacuum, its upper surface being exposed to the pressure of the atmosphere.

Assuming h to denote the height of the vessel, A its area, h' the height of a column of water equal to the weight of the atmosphere, it appears from what has been stated, article 62, that we must now substitute $x + h'$ instead of x, the variable depth of the fluid, we have therefore

$$t = -\int \frac{z \dot{x}}{2 a \sqrt{g}(x + h')^{\frac{1}{2}}} = -\int \frac{A \dot{x}}{2 a \sqrt{g}(x + h')^{\frac{1}{2}}};$$

or $$t = \frac{-A}{2 a \sqrt{g}} \times \int \frac{\dot{x}}{(x + h')^{\frac{1}{2}}} + C;$$

taking the fluent, we obtain

$$t = \frac{-A}{a \sqrt{g}} \times (x + h')^{\frac{1}{2}} + C.$$

Now, in order to find C, we have, when $x = h$, $t = o$, whence

$$C = \frac{A}{a \sqrt{g}} (h + h')^{\frac{1}{2}};$$

therefore

$$t = \frac{A}{a \sqrt{g}} \left\{ (h + h')^{\frac{1}{2}} - (\dot{x} + h')^{\frac{1}{2}} \right\};$$

wherefore, when $x = o$, we obtain

$$t = \frac{A}{a \sqrt{g}} \left\{ (h + h')^{\frac{1}{2}} - h'^{\frac{1}{2}} \right\}$$

for the time of exhaustion.

68. In the preceding examples and investigations we have not considered the form of the orifice, because we have always supposed it small in comparison with the depth of the fluid; we have moreover generally confined ourselves to the issuing of fluids through horizontal apertures, in which case the form is of little importance; but when the opening is in the side of the vessel, and the depth of it bears any considerable ratio to the height of the water above it, we must then proceed on different principles, as the velocity will then be variable from the lower to the upper part of the orifice; we propose, therefore, in the present article, to investigate the relation between the time and the quantity of fluid discharged, through vertical orifices of different figures, adopting the principle illustrated in the preceding chapter, viz. that the velocity at every point is that due to a heavy body in falling through a space equal to the altitude of the fluid above it.

Hydraulics.

The depth of the fluid being constant.

Fig. 3.

Let XYVZ (fig. 3) be the vertical side of the vessel proposed, and AM$s t$M′ the vertical orifice, the contour of which is a plane curve; SB a vertical line passing through A the highest point of the orifice, and let MM′, $m m'$ be two horizontal lines indefinitely near the one to the other. Then the upper surface of the fluid being supposed to pass through S, and to continue during the whole discharge at the same height, it is proposed to ascertain the discharge due to any given time t.

Assume SA $= h'$, SB $= h$, AP $= x$, MM′ $= y$, and gravity $= 32\frac{1}{6} = 2g$.

The velocity of the fluid discharged through the elementary trapezoid MM′$m m'$, whose surface is $y \dot{x}$, may be considered as that due to the height SP $= h' + x$; consequently, according to the principle assumed, the velocity being uniform, we shall have

$$2 t y \dot{x} \surd (g(h' + x)) = 2 t y \dot{x} \surd g \times \surd (h' + x),$$

for the discharge through the element of the orifice in the time t.

Whence the discharge due to the whole aperture will be $2 t \surd g \times (\int y \dot{x} \surd (h' + x) + C)$. (1)

Since the nature of the curve which bounds the orifice is supposed known, we may thence find y in functions of x, and this value being substituted for y, the fluent, when it can be taken in finite terms, will give the discharge due to the time t.

Mean depth defined.

Again, since the velocity at the lowest point of the orifice is the greatest, and that it diminishes from this point to the upper part, where it is the least; there must necessarily be some intermediate point where the velocity is such that if the whole column of fluid issued with the intermediate velocity due to that point, the discharge would be the same in the same time as in the actual case: let the depth of this point below the surface of the fluid $= s$, then the velocity at that place will be $2 \surd g s$; consequently

$$2 t \surd g s \times (\int y \dot{x} + C')$$

will denote the discharge due to the time t, and this, as we have seen, will be equal to the former, whence

$$2 t \surd g s \times (\int y \dot{x} + C') = 2 t \surd g \times (\int y \dot{x} \surd (h' + x) + C);$$ or, dividing by $2 t \surd g$,

$$\surd s \times (\int y \dot{x} + C') = \int y \dot{x} \surd (h' + x) + C;$$

whence $$s = \frac{(\int y \dot{x} \surd (h' + x) + C)^2}{(\int y \dot{x} + C')^2}.$$

The quantity s is what is commonly called the *mean depth* or *mean height* of the fluid above the orifice.

69. Let us suppose the vessel to empty itself, and consequently the depth of the fluid variable, and investigate the quantity discharged under such circumstances in any given time.

This determination will evidently furnish, at the same time, the depression of the upper surface of the fluid in the vessel of which the interior form is known; because the portion of the capacity of the part exhausted is equal to the volume of the fluid expelled at the orifice. In order to solve this branch of our general proposition, let us suppose, that at the commencement of the mo-

The depth of the fluid variable.

tion the surface of the fluid is at S, and that at the end of the given time t, it has descended to K, making SK $= z$, we shall have to obtain an equation between z, t, g, x, h, h'. In order to this we shall conceive, at first, that the vessel is kept constantly full to the height K during the time t, and hence find from the preceding determination for the discharge in that time the value of

$$2t\sqrt{g} \times (\textstyle\int y\dot{x}\sqrt{(h'+x-z)} + C).$$

Here, since z is supposed constant, we shall easily determine the fluent of this expression; and correct it according to the required conditions, viz. that when $x = o$, the discharge $= o$, and, finally, by taking $x = h - h'$, we shall have the complete fluent.

The value of $2t\sqrt{g} \times (\int y\dot{x}\sqrt{(h'+x-z)} + C)$, determined as above, will contain only z and constant quantities.

Now, suppose that a portion of fluid is discharged during the element of time $\dot{t}$, and that the surface Q q is depressed to R r, through a distance K $k = \dot{z}$, also putting the section Q q of the vessel $=$ S, the quantity discharged will be S $\dot{z}$.

According to this hypothesis, the descent through K k may be imagined to take place with a uniform velocity, and the velocity at each point of the orifice to continue the same as when the surface was at K; therefore since when the velocities are equal the quantities discharged vary as the times, we have this analogy:

$$2t\sqrt{g}\{\textstyle\int y\dot{x}\sqrt{(h'+x-z)} + C)\} : S\dot{z} :: t : \dot{t};$$

whence

$$\dot{t} = \frac{S\dot{z}}{2\sqrt{g} \times (\int y\dot{x}\sqrt{(h'+x-z)} + C)};$$

consequently

$$t = \frac{1}{2\sqrt{g}} \times \int\frac{S\dot{z}}{\int y\dot{x}\sqrt{(h'+x-z)} + C)} + C''. \quad (3)$$

The form of the vessel being known, S is given in functions of z, and of constant quantities; on the other hand, when the integration indicated by the denominated is effected, it contains only z, and invariable quantities, so that the complete value of t may be found by the integration of an expression which contains only the variable z and constant quantities; and as to the value of C″, it will obviously be determined by observing that when $z = o$, $t = o$. The fluents in question may therefore always be referred to the quadrature of curves.

Let us now illustrate the preceding formulæ by a few examples:

Examples.
Rectangle.

1. Suppose the orifice to be a given rectangle to find the quantity discharged from it in a given time, the water remaining constantly at the same altitude.

Here the quantity indicated by y in the preceding formula is constant, viz. the horizontal side of the rectangle, which let us denote by b, its height being $a = h - h'$, substituting therefore b instead of y, our equation (1) becomes

$$q = 2tb\sqrt{g} \times (\textstyle\int\dot{x}\sqrt{(h'+x)} + C),$$

q denoting the quantity of the discharge.

Or, taking the fluent

$$q = 2tb\sqrt{g} \times (\tfrac{2}{3}(h'+x)^{\frac{3}{2}} + C).$$

But when $x = o$, $q = o$, wherefore

$$2tb\sqrt{g} \times \tfrac{2}{3}h'^{\frac{3}{2}} + 2tb\sqrt{g} \times \tfrac{2}{3}C = o;$$

consequently, $C = -h'^{\frac{3}{2}}$;

and introducing this value we have

$$q = \tfrac{4}{3}tb\sqrt{g} \times ((h'+x)^{\frac{3}{2}} - h')^{\frac{3}{2}}.$$

Now again assuming $x = h - h'$, we have for the total discharge through the orifice during the time t,

$$q = \tfrac{4}{3}bt\sqrt{g}(h^{\frac{3}{2}} - h'^{\frac{3}{2}}).$$

The area of the orifice being $h'b$, the quantity discharged in the time t, with the velocity due to the mean depth s, will be

$$2t\sqrt{g} \times ab = 2tb\sqrt{g}(h-h');$$

that is,

$$2t\sqrt{g} \times (\textstyle\int y\dot{x} + C') = 2tb\sqrt{g}(h-h');$$

whence we have for the mean depth,

$$s = \frac{4(h^{\frac{3}{2}} - h'^{\frac{3}{2}})^2}{9(h-h')^2}.$$

Trian

2. Let now the orifice be a triangle, whose vertex is uppermost and the base horizontal, the vessel being kept constantly full.

Denoting the ratio of the altitude to the base by $m : 1$, and calling x any variable base, we shall have $y = mx$, which being substituted for y in equation (1), that expression becomes

$$q = 2tm\sqrt{g} \times (\textstyle\int mx\dot{x}\sqrt{(h'+x)} + C);$$

or

$$q = 2tm\sqrt{g} \times \{\tfrac{2}{5}(h'+x)^{\frac{5}{2}} - \tfrac{2}{3}h'(h'+x)^{\frac{3}{2}} + C\}.$$

In order to determine C we have $q = o$, when $x = o$; wherefore

$$C = -\tfrac{2}{5}h'^{\frac{5}{2}} + \tfrac{2}{3}h'^{\frac{5}{2}} = \tfrac{4}{15}h'^{\frac{5}{2}}$$

the correct fluent therefore is

$$q = 2tm\sqrt{g} \times \{\tfrac{2}{5}(h'+x)^{\frac{5}{2}} - \tfrac{2}{3}h'(h'+x)^{\frac{3}{2}} + \tfrac{4}{15}h'^{\frac{5}{2}}\}.$$

Substituting now, in this equation, $h - h' = x$, we have for the total discharge in the time t,

$$q = \tfrac{4}{15}tm\sqrt{g}\{3h'^{\frac{5}{2}} + 2h'^{\frac{5}{2}} - 5hh'^{\frac{3}{2}}\}.$$

The area of the triangular orifice being $\frac{1}{2}m(h-h')^2 = \int y\dot{x} + C'$, we have for the value of the mean depth

$$s = \frac{16(3h^{\frac{5}{2}} + 2h'^{\frac{5}{2}} - 5hh'^{\frac{3}{2}})^2}{15^2(h-h')^4}.$$

3. Let now the triangular orifice be still the same, but with its vertex downwards, and its base horizontal, to determine the discharge due to the time t.

In this case $y = m(h - h' - x)$; then, by a similar investigation to the above, we shall find

$$q = t\tfrac{4}{15}m\sqrt{g}\{2h^{\frac{5}{2}} + 3h'^{\frac{5}{2}} - 5hh'^{\frac{3}{2}}\},$$

and $$s = \frac{16(2h^{\frac{5}{2}} + 3h'^{\frac{5}{2}} - 5hh'^{\frac{3}{2}})}{15^2(h-h')^4}.$$

Circ

4. Let the orifice be a circle, and the quantity discharged in a given time required.

As the accurate solution of this problem would require a long and rather difficult integration, we shall employ an approximative method, sufficiently correct for all practical cases.

Let the diameter of the circle $= d$, then from the property of that figure we have

$$\tfrac{1}{2}y = \sqrt{(dx - x^2)};$$

whence the above general expression for the discharge is $q = 4t\sqrt{g} \times (\int(dx - x^2)^{\frac{1}{2}}(h'+x)^{\frac{1}{2}}\dot{x} + C)$, making $\int(dx - x^2)^{\frac{1}{2}}(h'+x)^{\frac{1}{2}}\dot{x} = z\dot{x}$; the finding of the fluent is reduced to the determination of the area of a curve whose equation is

$$z = (d\,x - x^2)^{\frac{1}{2}}(h' + x)^{\frac{1}{2}};$$

its origin being at that point where $z = o$. We may approximate to the area of this curve by the method of equidistant ordinates. To this end suppose x divided into four equal parts, then we shall find the four ordinates corresponding to these points of division, by substituting in the preceding equation for z the values o, $\frac{1}{4}x$, $\frac{1}{2}x$, $\frac{3}{4}x$, and x; these values being found, we add to the sum of the first and fifth, four times the sum of the second and fourth, and double the third. One third of this latter sum, multiplied into the common distance between the ordinates will be an approximate value of the integral $\int z\,\dot{x}$ (see Hutton's Mensuration, Sec. 11, Part iv), which multiplied into $4\,t\,\surd\,g$ produces

$$q = \left\{\begin{array}{l} 2\,(\frac{1}{4}\,d\,x - \frac{1}{16}x^2)^{\frac{1}{2}}\,(h' + \frac{1}{4}\,x)^{\frac{1}{2}} \\ + \;(\frac{1}{2}\,d\,x - \frac{1}{4}x^2)^{\frac{1}{2}}\,(h' + \frac{1}{2}\,x)^{\frac{1}{2}} \\ +2\,(\frac{3}{4}\,d\,x - \frac{9}{16}\,x^2)^{\frac{1}{2}}\,(h' + \frac{3}{4}\,x)^{\frac{1}{2}} \\ +\frac{1}{2}\,(\,d\,x - x^2)^{\frac{1}{2}}\,(h' + x)^{\frac{1}{2}} \end{array}\right\}\frac{2}{3}\,t\,x\,\surd\,g.$$

If in this equation we make $x = d = h - h'$, and perform the requisite reductions, we shall have for the total discharge at the orifice $q = 2\,t\,d^2\,\surd\,g\,\times$

$$\left\{\frac{\surd\,(h + 3\,h') + \surd\,(h' + 3\,h)}{4\,\surd\,3} + \frac{1}{6}\,\frac{\surd\,(h + h')}{\surd\,2}\right\},$$

It is obvious that we should have had a more accurate approximation by assuming a greater number of ordinates, but the labour would have become very excessive; and had we attempted a rigorous integration, it would have led us into the summation of a series which would have been equally troublesome, and probably not more accurate, from the number of terms that must have been rejected. Of this, however, the reader may judge, the series in question being

$$q = 2\,\pi\,t\,r^3\,\surd\,g\,r\,n + \left\{1 - \frac{1}{32\,n^2} - \frac{5}{1024\,n^4} - \&c.\right\},$$

and

$$s = r\,n\left(1 \times \frac{1}{16\,n^2} - \frac{9}{1024\,n^4} - \&c.\right),$$

where $r =$ the radius of the circular orifice; n the quotient of the distance of its centre from the upper surface of the fluid, divided by the radius r, and $\pi = 3{\cdot}14159$, &c.; that is to say, the semicircumference of a circle whose radius is unity.

70. Let it be required to determine in what time t, the upper surface of the fluid in a prismatic vessel, will be depressed a vertical distance, z the orifice being supposed rectangular.

Here the section of the vessel S, and the breadth of the orifice y, are constant quantities. According to the principle illustrated under (art. 69), we must first determine the quantity of fluid that would escape through the orifice during the time t, on the supposition that the vessel were kept constantly full to the height of $h' - z$ above the upper part of the orifice. We have, therefore, to find the fluent of

$$y \int \dot{x}\,\surd(h' + x - z),$$

supposing z constant.

This fluent is

$$\tfrac{2}{3}\,y\,(h' + x - z)^{\frac{3}{2}} + \mathrm{C};$$

and when $x = o$, this expression vanishes, wherefore

$$\mathrm{C} = -\,\tfrac{2}{3}\,y\,(h' - z)^{\frac{3}{2}};$$

whence the correct fluent is

$$\tfrac{2}{3}\,y\,\{(h' + x - z)^{\frac{3}{2}} - (h' - z)^{\frac{3}{2}}\}.$$

Extending this to the entire orifice by making $x = h - h'$, we shall have

Depression of the surface.

$$\tfrac{2}{3}\,y\,\{(h - z)^{\frac{3}{2}} - (h' - z)^{\frac{3}{2}}\};$$

and if this value be substituted for

$$\int y\,\dot{x}\,\surd(h' + x - z) + \mathrm{C},$$

in our equation (3) it will become

$$t = \frac{3\,\mathrm{S}}{4\,y\,\surd g}\,.\int\frac{\dot{z}}{(h - z)^{\frac{3}{2}} - (h' - z)^{\frac{3}{2}}}.$$

The quantity which is found under the sign of integration in this equation is susceptible of being made rational; but the equations would be very complicated. The most easy method of obtaining the result is to square, by approximation, the curve, which for each value of δ has an ordinate equal to $\dfrac{1}{\delta^{\frac{3}{2}} - (\delta - a)^{\frac{3}{2}}}$, in the same manner as in the last example; observing that now, when $\delta = h$, $t = o$, the surface may be estimated between the limits $\delta = h$, and $\delta = a$; as in this case we should only have the terms to transcribe from the preceding example, it will be sufficient to have given this indication of the *modus operandi*, leaving the actual deduction as an exercise for the reader.

If the altitude of the orifice a be equal to the height h of the vessel, the time of exhaustion to any variable depth z, reckoned from the bottom, would be

$$t = \frac{3\,\mathrm{S}}{2\,b\,\surd g} \times \frac{h\,\surd z - z\,\surd h}{h\,z}.$$

When $z = o$, this expression is infinite; that is, the time of complete exhaustion is infinite.

We have been indebted to Gregory's Mechanics for some of the preceding problems; and the reader will find the subject still farther extended in "Bossut's Hydrodynamique."

§ II. *Experimental data relating to the discharge of vessels.*

Experiments.

71. Our preceding investigations and deductions are founded upon the hypothesis that all the fluid particles, issuing from a horizontal orifice in a vessel, are expelled with the velocity due to the height of the fluid in the reservoir above that orifice, no allowance having been made for the oblique direction in which the several particles approach the orifice; indeed this is a part of the subject which seems nearly to bid defiance to the powers of analysis, but it is fortunately of less consequence, as experiments in this case will supply what analysis is unable to effect.

If the orifice be horizontal, that filament of particles which answers to its centre will descend in a vertical line, and suffer no other resistance than that of the friction caused by the excess of its velocity above that of the colateral filaments, or by the retardation which arises from the attraction subsisting between them. The other filaments are soon compelled to turn from their vertical course, and to approach the orifice in spiral curves, and when they arrive at it, their directions become more or less horizontal according as they pass nearer to, or farther from, the edge of the orifice. The motion, therefore, may be decomposed into two directions; the one horizontal, which is destroyed by the equal and contrary resistance of the filaments diametrically opposite; the other vertical, in proportion to which the quantity of water discharged is estimated. Consequently the vertical velocity of the filaments will decrease from the centre of the orifice to the circumference; and the total discharge will therefore necessarily be less than if the filaments had issued vertically, as that filament does which answers to the centre of the aperture. It follows also, from the same circumstances, that the filaments which are nearer to the centre, moving faster than those contiguous to the circumference, supposing the orifice circular, will form a conic frustrum whose greater base is the aperture; that is to say, its diameter will diminish to a certain distance; because the exterior filaments are gradually drawn on, in consequence of the mutual attraction between them and the inferior filaments whose velocity is greater; and thus is formed the diminution in the diameter of the vein. This diminution of the transverse vertical section of the issuing fluid is called the *contraction* of the vein, and the vein itself, from the orifice to the greatest diminution, is called the *vena contracta*, or the contracted vein. Although it is very obvious that all these circumstances must necessarily take place from the physical constitution of fluids, it is also equally evident that it is impossible to introduce them into an analytical investigation of this problem; we must therefore supply this defect by actual experiments; and many have accordingly been undertaken by different philosophers; but of these the most important are those of Bossut, Du Buat, Venturi, Michelotti, and Polenus. It appears from the experiments of these authors, that when the orifice is circular and horizontal, the length of the contracted vein is nearly equal to the radius of the orifice. With respect to the ratio between the actual orifice and the transverse section of the flowing vein there is considerable difference. Polenus makes the ratio of the two diameters as 5½ to 6½; Bernoulli 5 to 7; Du Buat 6 to 9; Michelotti as 4 to 5; Venturi the same; and Bossut as 41 to 50; but it is obvious, that the most certain method of establishing this datum is to measure the quantity of the discharge in a given time, by which means the latter author found the ratio between the areas to be about 1 : ·62, as will be seen by the following detail of his experiments:

Bossut's Experiments.

Bossut's experiments.

On the quantity of water issuing from vessels kept constantly full, through horizontal apertures formed in thin brass plates.

72. In the following experiments, the head of water was kept constantly at 11 feet 8⅝ inches above the centre of the orifice; and the bottom of the vessel was a square, the side being 3 feet, or 9 feet area.

Horizontal apertures.

Experiment 1. In 50 seconds, a horizontal circular orifice of ¼ inch diameter, yielded 1926 cubic inches.

Exp. 2. In 90 seconds, a horizontal circular orifice 1 inch in diameter, yielded 13921 cubic inches.

Exp. 3. In 21 seconds, a horizontal circular orifice 2 inches in diameter, gave 13021 cubic inches.

Exp. 4. In 50 seconds, a horizontal rectangular orifice 1 inch long, and ¼ inch broad, gave 2444 cubic inches.

Exp. 5. In 71 seconds, a horizontal square orifice, whose side was 1 inch, yielded 13984 cubic inches.

Exp. 6. In 17 seconds, a horizontal square orifice, whose side was 2 inches, yielded 13419 cubic inches.

These being each reduced to the same time, as for example 1 minute, give the following results, viz.

TABLE I. *Of the reduced results from Bossut's Experiments.*

No	Constant height of water above the orifice 11 feet 8⅝ inch.	Inches discharged in one minute.	The same reduced to square inch orifices.
1	Circular orifice ¼ inch diameter	2311	11769
2	.... do. 1 diameter	9281	11817
3	.... do. 2 diameter	37203	11842
4	Rectangle 1 inch by ¼ inch	2933	11732
5	Square 1 inch side	11817	11817
6	Do. 2 inches side	47361	11840

The mean of all these results, gives for the discharge through an orifice of one square inch, in one minute 11803 cubic inches, or 196·7 cubic inches in one second.

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Note. The above are French feet and inches, and to make them agree with our English measures, we must assume such a height in English feet as will give the same nominal velocity; that is, we must reduce the altitude in exactly the same proportion as our number, by which we estimate the descent by gravity in a second, is greater than that of the French. For let h denote any height in French measure, and g the descent of a heavy body in a second in the same measure, then the velocity will be $2\sqrt{h\,g}$. Let h' and g' denote the similar quantities in English measure, then $v = 2\sqrt{h'\,g'}$; in order therefore that v may be the same in both, we must have $\sqrt{h\,g} = \sqrt{h'\,g}$, or $h\,g = h'\,g'$, that is,

$$g : g' :: h' : h, \text{ or } g' : g :: h : h'.$$

The proportion of $g' : g = 1{\cdot}06578 : 1$; whence $1{\cdot}06578 : 1 ::$ 11 feet 8⅝ inches : 10 feet 11¾ inches English. Substituting therefore 10 feet 11¾ inch for the height of the fluid; the above results will correspond with English measure. Or as we shall see in the subsequent articles, that the discharges or velocities are as the square roots of the altitudes of the fluid; if we suppose the tabular altitudes to be English measure, we must increase the discharges in the ratio of $\sqrt{1} : \sqrt{1{\cdot}06578}$, or in the ratio of $1 : 1{\cdot}03236$; that is, each tabular result must be increased by about $\frac{1}{30}$th part of itself.

Experiments on vertical apertures in thin brass plates.

Vertic
apertu

73. In the two following experiments the head of water was constantly nine feet above the centre of the aperture, the area of the bottom being still a square, whose side was three feet.

Exp. 1. In 55 seconds, a vertical circular orifice, ½ an inch in diameter, gave 1850 cubic inches.

Exp. 2. In 100 seconds, a vertical circular aperture, 1 inch in diameter, yielded 13558 cubic inches.

In experiments 3 and 4, the head of water was reduced to four feet.

Exp. 3. In 60 seconds a vertical circular orifice, ½ an inch in diameter, gave 1353 cubic inches.

Exp. 4. In 150 seconds a vertical circular orifice, 1 inch in diameter, discharged 13591 cubic inches.

Table II. *Of results reduced to the discharge in one minute, from Bossut's experiments.*

N°	Nature and dimensions of the orifices.	Height of the head of water.	Discharge in one minute.
1	Circular ½ inch diameter	9 feet	2018 cubic inches
2	Ditto 1 inch diameter	9 feet	8135 ditto.
3	Ditto ½ inch diameter	4 feet	1353 ditto.
4	Ditto 1 inch diameter	4 feet	5436 ditto.

Note.—To make these correspond to English measures, the above altitudes must be reduced to 8 feet 5¼ inches, and 3 feet 9 inches, or the results must be multiplied by $1\frac{1}{16}$.

If we compare the discharges from the same height, with the areas of the apertures, we shall find them nearly proportional to the apertures; viz.

$$\tfrac{1}{2}^2 : 1^2 \text{ or } 1 : 4 :: \begin{cases} 2018 : 8072 \\ 1353 : 5412. \end{cases}$$

The discrepance in the one case being $\frac{63}{8135}$, and in the other $\frac{6}{908}$.

Again, if we compare the discharges from the same apertures, under different altitudes, we shall find them to be nearly proportional to the square roots of those altitudes; for

$$\sqrt{9} : \sqrt{4} :: \begin{cases} 2018 : 1345 \\ 8135 : 5423. \end{cases}$$

The error in the first number being $\frac{8}{1353}$, and in the second $\frac{13}{5436}$.

74. From these and the preceding experiments, we therefore obtain the following practical conclusions.

1. That the discharges are nearly proportional to the area of the orifices, whatever their form may be, when placed horizontally; the height of the fluid being the same.

2. The discharges are nearly as the square roots of the heights of the fluid, the orifices being equal; our theoretical deductions are therefore so far verified by experiment.

Consequently, if we call Q and q the quantities of water discharged in the same time, from the two orifices A and a, under the same head of water in the reservoir; q and Q′ the quantities of water discharged during the same time by the same aperture A, under the different heads of water h and h', we have, from the above deductions,

$$Q : q :: A : a, \text{ and } q : Q' :: \sqrt{h} : \sqrt{h'};$$

whence $$q = \frac{aQ}{A} \text{ and } q = \frac{Q'\sqrt{h}}{\sqrt{h'}};$$

consequently $$\frac{aQ}{A} = \frac{Q'\sqrt{h}}{\sqrt{h'}},$$

and $$Q : Q' :: A\sqrt{h} : a\sqrt{h'}.$$

That is, generally, the quantities of water discharged in the same time by different orifices, and under different heads of water, are to each other in the ratio compounded of the area of the aperture and the square root of the heights of the fluid.

This general rule is sufficiently near for most practical cases, but the proportions above stated are obviously, from the above experiments, not perfectly correct.

The anomalies in question, Bossut attributes, in the first place, to friction, which is greater in proportion as the orifices are smaller, the parts in contact with the aperture being as the perimeters of the section, while the discharges are as the areas; the figures being similar, and the altitude the same.

Hydraulics.
Practical conclusions.

Again, of several orifices the friction will be, *cæteris paribus*, greatest when the perimeters are greatest; the circular orifice will give, therefore, a greater discharge than any other figure of equal perimeter.

The discharges are also less in proportion as the altitude increases, in consequence of the contraction being greater for greater altitudes.

These considerations must be introduced, in order to modify the rules laid down in this article, when great accuracy is required.

If we compute the velocity due to a heavy body in falling through 11 feet 8⅝ inches, French measure, or 10 feet 11¾ inches English, we shall find it to be 318·8 inches; and therefore the theoretic discharge due to this height of water from an aperture of 1 inch, ought to be (independent of the contraction of the vein) 318·8 inches per second; whereas, the practical discharge is found, by table I. to be 196·7 inches per second; therefore, if the velocity be what we have supposed, viz. that due to a heavy body falling through the height of the level of the fluid above the orifice, the vein must be contracted in the actual discharge, in the ratio of 318·8 : 196·7; or as 1 : ·616, which nearly corresponds with the number stated in article (71).

75. From the above and other experiments, which we cannot allow ourselves to transcribe, M. Bossut deduces the following results of the actual discharges under different altitudes, with the computed discharges, independent of the contraction of the vein; viz. by computing the velocities due to the several altitudes; the time of discharge being always one minute, and the circular aperture 1 inch in diameter.

Table III. *Of theoretic and practical discharges through circular orifices, 1 inch in diameter, in French feet and inches.*

Height of the fluid above the orifice.	Computed discharge, independent of contraction, per minute.	Actual experimental discharge, per minute.	Ratio of the theoretic to the actual discharge.
	Cubic inches.	Cubic inches.	
1 foot	4381	2722	1 : ·62133
2 feet	6196	3846	1 : ·62073
3 feet	7589	4710	1 : ·62064
4 feet	8763	5436	1 : ·62034
5 feet	9797	6075	1 : ·62010
6 feet	10732	6654	1 : ·62000
7 feet	11592	7183	1 : ·61965
8 feet	12392	7672	1 : ·61911
9 feet	13144	8135	1 : ·61892
10 feet	13855	8574	1 : ·61883
11 feet	14532	8990	1 : ·61873
12 feet	15180	9384	1 : ·61819
13 feet	15797	9764	1 : ·61810
14 feet	16393	10130	1 : ·61795
15 feet	16968	10472	1 : ·61716

The same remark applies here as in the preceding page, viz. that these being French feet and inches, the several altitudes must be reduced in the ratio of

1·06578 : 1, to make the results agree with English feet and inches; or, the discharges must be increased in the ratio of 1 : $1\frac{1}{30}$, if we consider the altitudes as English measures.

This reduction, for the convenience of the English reader, is made in the following table.

Table IV. *Of theoretic and practical discharges in English measures.*

Height of the fluid above the orifice.	Computed discharge, independent of contraction, per minute.	Actual experimental discharge, per minute.	Ratio of the theoretic to the actual discharge.
	Cubic inches.	Cubic inches.	
1 foot	4427	2812	1 : ·62133
2 feet	6402	3974	1 : ·62073
3 feet	7842	4867	1 : ·62064
4 feet	9055	5617	1 : ·62034
5 feet	10123	6277	1 : ·62010
6 feet	11089	6876	1 : ·62000
7 feet	11978	7422	1 : ·61965
8 feet	12805	7928	1 : ·61911
9 feet	13582	8406	1 : ·61892
10 feet	14317	8860	1 : ·61883
11 feet	15016	9289	1 : ·61873
12 feet	15686	9697	1 : ·61819
13 feet	16323	10089	1 : ·61810
14 feet	16939	10467	1 : ·61795
15 feet	17533	10821	1 : ·61716

Remarks on the preceding tables.

76. It may be proper to observe, that what are called in the above table the actual discharges, were not all of them absolutely determined experimentally; but the precautions employed in the computation were such (depending upon the preceding rules), that they may be considered as accurate, and may be employed, by means of the preceding formulæ, in the determination of any proposed case.

Let it be required, for example, to determine the quantity of water that would be discharged from an orifice of three inches diameter, under an altitude of thirty feet, in one minute, English measure.

Here, since the real discharges are in the ratio compounded of the area of the orifice and the square root of the altitude of the fluid; and since a height of 15 feet, through a circular orifice of 1 inch diameter, gives 10821 cubic inches, we have

$$10821 \times 3^2 \times \sqrt{2} = 137730 \text{ cubic inches,}$$

the discharge required.

Prony's formulæ.

M. Prony deduces from the preceding experiments the following formulæ for computing the discharge due to any altitude, and with any given orifice.

Let A denote the area in square feet, H the height of the head of water also in feet, T the time in seconds, g = the space fallen through in one second by the force of gravity, and Q the quantity of water in cubic feet, viz.

$$\left.\begin{array}{ll} 1 & Q = 0{\cdot}61938\ AT^2 \sqrt{g}\,H, \text{ or} \\ 2 & Q = \quad 4{\cdot}818\ AT \sqrt{H}; \end{array}\right\} \text{for an orifice of any form.}$$

$$3 \quad Q = 3{\cdot}7842\ d^2\,T \sqrt{H},$$ when the area is circular, d denoting its diameter.

And from equation (2) we readily deduce the following:

$$4 \quad A = \frac{Q}{4{\cdot}818\,T\sqrt{H}};$$

$$5 \quad T = \frac{Q}{4{\cdot}818\,A\sqrt{H}};$$

$$6 \quad H = \frac{Q^2}{(4{\cdot}818\ TA)^2}.$$

These formulæ are computed by assuming the mean coefficient 0·6194 for the area of contraction, and they will give results sufficiently correct for most practical cases; yet greater accuracy may be expected, if we use, instead of this mean coefficient, the tabular number answering to the height nearest to the given height of water in the reservoir; for example, if the head of water were 15 feet or thereabouts, we must take the coefficient ·617, whereas, if it were 3 feet we ought to take 6·20.

When the time T is one second, then our equation (2) becomes

$$Q = 4{\cdot}818\ A\sqrt{H};$$

and as the velocity per second, multiplied by the area, is obviously equal to the quantity discharged in a second, it follows conversely that the quantity divided by the area will give the velocity; calling, therefore, the velocity in feet V, we have

$$7 \quad V = \frac{Q}{A} = 4{\cdot}818\sqrt{H}.$$

The same may also be obtained by finding the velocity due to the height of the head of water in free space, and multiplying that velocity by the mean coefficient ·6194; or, where great accuracy is required, by the tabular co-efficient answering to the given altitude.

These formulæ, adjusted to English measures, are

$$1 \quad Q = {\cdot}61938\ AT^2\sqrt{H}\,g.$$

$$2 \quad Q = 4{\cdot}978\ AT\sqrt{H}.$$

$$3 \quad Q = 3{\cdot}9103\ d^2\,T\sqrt{H}.$$

$$4. \quad A = \frac{Q}{4{\cdot}978\,T\sqrt{H}}.$$

$$5 \quad T = \frac{Q}{4{\cdot}978\,A\sqrt{H}}.$$

$$6 \quad H = \frac{Q^2}{(3{\cdot}9103\,d^2\,T)^2}.$$

$$7 \quad V = \frac{Q}{A} = 4{\cdot}978\sqrt{H}.$$

77. Such are the principal experiments performed by M. Bossut on the discharges of water through simple orifices. We shall now conclude this article with the tabular results of Michelotti's experiments, which were conducted upon a grand scale, and must be esteemed highly valuable. The reservoir employed was 20 feet high, and 3 feet square within, having openings at different distances from the top. The water flowed into a cistern whose area was 289 square feet, and its figure uniform. The feet, inches, &c. are French measures.

TABLE V. *Containing the experiments of Michelotti on the quantities of water discharged by different vertical orifices, under different heads of water.*

Size and nature of the orifice.	Height of fluid above the centre of the orifice.			Time of running.	Cubic feet of water expended.		
	Feet.	in.	lines.	Minutes.	Feet.	in.	lines
Square of 3 inches.	6	7	$4\frac{1}{3}$	10	453	7	3
	6	10	$2\frac{2}{3}$	12	566	5	6
	11	8	$1\frac{1}{2}$	$8\frac{1}{2}$	516	9	5
	11	9	$9\frac{5}{6}$	10	612	1	5
	21	8	$3\frac{1}{2}$	5	415	5	3
	21	8	7	6	499	2	8
Square of 2 inches.	6	7	6	15	329	9	8
	11	5	$1\frac{1}{3}$	15	423	5	7
	21	5	$3\frac{7}{12}$	10	385	4	0
Square of 1 inch.	6	9	1	30	158	6	7
	11	10	$8\frac{1}{12}$	24	163	9	6
	21	6	1	60	562	11	4
Circular, 3 inches in diameter.	6	8	4	15	542	10	6
	11	7	1	12	570	11	8
	21	7	4	8	521	3	7
Circular, 2 inches in diameter.	6	9	5	30	488	8	3
	11	8	8	28	589	6	5
	21	10	10	20	575	5	10
Circular, 1 inch in diameter.	6	10	6	60	247	4	3
	11	8	11	60	324	1	5
	22	0	2	60	444	6	5

The mean coefficient deduced from these results is ·625, which agrees very nearly with that of Bossut, viz. ·619.

Similar experiments were performed by Brindley and Smeaton, which give ·630 for the mean ratio of contraction. The simple ratio of 8 : 5, agrees therefore very nearly with the mean of the three series, and may be employed with confidence in any practical cases.

§ III. *Experiments on the discharge of water through tubes of different dimensions.*

78. It was known even in the time of the Romans that the contraction of the vein, noticed in the foregoing experiments, depended upon the apertures being formed in thin plates, and therefore, if instead of such orifices, tubes of a certain length were made use of, the contraction might be avoided, and the discharge increased. M. Bossut undertook the examination of the circumstances attending such discharges through cylindrical tubes, and determined the ratio of the quantities yielded through different lengths of tubes, and under different altitudes of the fluid.

In the following experiments, the base of the vessel was a square whose side was three feet, the height of the waters above the horizontal orifice 11 feet $8\frac{5}{6}$ inches; and the length of the tubes as stated below.

Exp. 1. The tube being a cylinder, 1 inch in diameter, and 4 inches long, yielded in 66 seconds 13501 cubic inches of water.

Bossut's experiments.

Exp. 2. The tube being reduced to 2 inches in length, gave in the same time 13407 inches.

Exp. 3. The tube being farther reduced to $1\frac{1}{2}$ inch, yielded in the same time 13385 inches.

In the above experiments the issuing waters completely filled the tubes, but when these were farther reduced in length, the contraction took place in nearly the same manner as when the orifice was made in a thin plate; and by opening the aperture in a particular manner, these two effects, viz. a contraction, or no contraction, might be produced at pleasure in the shorter tubes, but not in the longer. This being tried upon the $1\frac{1}{2}$ inch tube, and a contraction produced, the quantity of water discharged in 91 seconds was 14078 cubic inches.

These several discharges being reduced to the discharges due to one minute of time, give the following results:

Constant height 11 *feet* 8·$\frac{5}{6}$ *inches.*

					Discharge in one minute.
Cylindric	tube inside	diam. 1 inch,	length 4	inches,	12274
	ditto	ditto	length 2	inches,	12188
	ditto	ditto	length $1\frac{1}{2}$	inch,	12168
The same	tube, vein	contracted			9282

It follows from the above experiments that the quantity of water discharged through vertical cylindrical tubes, are greater than the discharges through equal orifices made in a thin plate, and that the longest tubes give the greatest quantities in the same time; which quantities follow nearly the proportion of the square roots of the height of the water in the reservoir, above the lower extremity of the tube. In tubes, however, such as those above, whose lengths are inconsiderable with respect to the height of the water above them, the square roots differ but little from each other, or from that of the simple altitude of the vessel; the excess of discharge through the tube over that of the simple aperture, is, therefore, not attributable to this circumstance, but to the change which takes place in the motion of the fluid, which change we shall endeavour to account for in a subsequent article.

Ratio of theoretical and practical discharges.

With respect to the last experiment, the quantity discharged will be found to coincide very exactly with that produced by the simple circular orifice of 1 inch diameter (art. 75), from which in fact it differs in no respect, since the contraction obtained alike in both cases.

79. Comparing then together the computed or theoretical discharges, due to the altitude 11 feet $8\frac{5}{6}$ inches, the mean discharge through short tubes, as given in the first three of the above experiments, and the actual discharge through a simple orifice of the same diameter in a brass plate, we shall find them to each other very nearly as 16, 13, and 10.

Hence, the theoretic discharge, the discharge through short tubes, and that through a simple orifice, in the same time, the altitude of the water being the same, as also the diameter of the orifices, will be to each other as the numbers 16, 13, *and* 10

The same principles of solution may, therefore, be applied to the determination of the quantity of water discharged through tubes as through simple apertures, by observing to multiply the actual area by the fraction $\frac{13}{16}$, to get the equivalent contracted area, instead of the decimal ·62, or the fraction $\frac{5}{8}$, as employed in the latter case.

Velocity diminished by additional tubes.

80. It is, however, important to observe, that although we thus obtain what may be considered an equivalent contracted area, there is in reality no contraction, the tube emitting the water in a compact and full column; the area, therefore, in this case is not contracted, but the velocity is diminished, a fact which has been experimentally demonstrated, by showing that a *jet-d'eau*, proceeding vertically from a simple orifice, will rise to a greater heighth than the same stream conducted through a short tube. We ought, therefore, strictly, in computing discharges through additional tubes, to ascertain the altitude due to the diminished velocity, and to consider the issuing stream as a complete cylinder; but it obviously amounts to the same, if we consider the velocity to be that due to the whole height of the fluid, and make a corresponding contraction. We prefer the latter, as being more analogous to the discharge through simple orifices.

Physical examination.

81. Before detailing other experiments, connected with this enquiry, it may not be amiss to investigate the cause of the greater discharge through short tubes than through equal orifices in thin plates.

Fig. 4.

Let MOPN (fig. 4) represent a cylindric horizontal tube, attached to the reservoir ADCB; and conceive, in the first instance, prior to the discharge, the tube to be shut by a plug, or valve, at MN; if now we suppose this to be suddenly removed, each of the particles of the fluid which answers to the different points of the section NM, if they met no obstacle after issuing out, would describe, agreeably to the common principles of Dynamics, the parabolas Mmx, Nny. But, in reality, all the particles immediately on quitting the orifice, are united and mixed with each other, the parabolas are destroyed, and the fluid has a consequent tendency to contract. If there be no tube, or if the tube be so short, that in issuing, the point P falls between N and z, the vein of the fluid will experience a contraction, the same as in the simple orifice. But if the point P be found beyond z, as in our figure, the fluid will accumulate through the whole extent of z P, the vein will expand, and fill the tube, and will ultimately attain to, and continue to flow in a full stream. For the fluid being once supposed to fill the tube, the natural motion of the particles M, N, are altered, and will become less oblique to the plane of the orifice MN; whence it follows that the issuing stream will acquire a greater base, and continue to flow in a free stream, but the velocity, as we have stated above, is diminished. This diminution of velocity is, however, more than compensated for by the additional body of the issuing fluid, so that upon the whole the discharge in a given time, under these circumstances, will be greater than it is through an equal aperture in a thin plate, in the ratio above stated, viz. of 13 to 10.

Singular phenomenon.

It is a singular fact, that frequently, when the tube was short, the adherence of the water to the interior surface of the tube was so slight, that it might be destroyed by slightly striking the outside of the tube with a small key, or any such body. So that while the tube was discharging the water in a full stream, any little blows of the kind above stated would destroy the adhesion, the contraction of the vein would be formed, and the quantity of issuing fluid permanently diminished. We have an instance of two different discharges of this kind through the same tube in our third and fourth experiments (art. 78), and the following eight experiments are of the same kind.

TABLE VI. *Experiments on the discharge through additional tubes.*

N°. of Experiments.	Constant neight of the reservoir, in 12ths of inches.	Diameter of the tubes in 12ths of inches.	Length of the tubes.	Cubic inches discharged in 1 minute.
1	552	6 } full stream	2 inches	1689
2		10 }		4703
3	552	6 } contracted stream	do.	1293
4		10 }		3598
5	288	6 } full stream	2 inches	1222
6		10 }		3402
7	288	6 } contracted stream.	do.	935
8		10 }		2603

We may draw from these experiments the same conclusions as from the preceding, viz. that the discharges of water through additional short tubes when issuing with a full stream, are directly as the area of the interior section, and the square root of the height of the level of the fluid above the centre of the orifice. And that the actual discharge in this case, is to the theoretic discharge in a ratio, approaching to that of 13 to 16.

82. We shall conclude this part of our investigation by the following table of experimental and theoretic discharges for different heights, which may be very convenient to refer to in various practical cases.

TABLE VII. *Of theoretic and experimental discharges in one minute, through short cylindric tubes of one inch interior diameter, in French feet and inches.*

N°. of Experiments.	Height of the reservoir above the centre of the aperture.	Theoretic discharges in one minute.	Experimental discharge in the same time.	Ratio between the actual and computed discharges.
		Cubic inches.		
1	1 foot	4381	3539	0·817
2	2 feet	6169	5002	0·807
3	3 feet	7589	6126	0·807
4	4 feet	8763	7070	0·806
5	5 feet	9797	7900	0·806
6	6 feet	10732	8654	0·806
7	7 feet	11592	9340	0·805
8	8 feet	12392	9975	0·804
9	9 feet	13144	10579	0·804
10	10 feet	13855	11151	0·804
11	11 feet	14530	11693	0·804
12	12 feet	15180	12205	0·804
13	13 feet	15797	12699	0·803
14	14 feet	16393	13197	0·803
15	15 feet	16968	13620	0·802

TABLE VIII. *Of theoretic and experimental discharges in one minute, through short cylindric tubes of one inch interior diameter, in English measures.*

N°. of Experiments.	Height of the reservoir above the centre of the aperture.	Theoretic discharges in one minute.	Experimental discharge in the same time.	Ratio between the actual and computed discharges.
1	1 foot	4527	3657	0·817
2	2 feet	6374	5169	0·807
3	3 feet	7862	6330	0·807
4	4 feet	9055	7306	0·806
5	5 feet	10123	8163	0·806
6	6 feet	11087	8942	0·806
7	7 feet	11978	9651	0·805
8	8 feet	12805	10307	0·804
9	9 feet	13582	10932	0·804
10	10 feet	14317	11523	0.804
11	11 feet	15014	12082	0·804
12	12 feet	15686	12611	0·804
13	13 feet	16323	13122	0·803
14	14 feet	16939	13636	0·803
15	15 feet	17533	14074	0·802

83. Denoting here the same quantities by the letters A, Q, T, H, &c. as in article 77, the analagous equations in this case will be, for *French measures,*

$$1.\quad Q = 0{\cdot}81\,A\,T\,2\,\sqrt{H\,g}.$$

$$2.\quad Q = 4{\cdot}9438\,d^2\,T\,\sqrt{H}.$$

$$3.\quad d = \sqrt{\frac{Q}{4{\cdot}9438\,T\,\sqrt{H}}}.$$

$$4.\quad T = \frac{Q}{4{\cdot}9438\,d^2\sqrt{H}}.$$

$$5.\quad H = \frac{Q}{(4{\cdot}9438\,d^2\,T)^2}.$$

$$6.\quad V = \frac{Q}{{\cdot}7854\,d^2}.$$

The same reduced to *English measures*, become

$$1.\quad Q = 0{\cdot}837\,A\,T\,2\,\sqrt{H\,g}.$$

$$2.\quad Q = 5{\cdot}1086\,d^2\,T\,\sqrt{H}.$$

$$3.\quad d = \sqrt{\frac{Q}{5{\cdot}1086\,T\,\sqrt{H}}}.$$

$$4.\quad T = \frac{Q}{5{\cdot}1086\,d^2\sqrt{H}}.$$

$$5.\quad H = \frac{Q^2}{(5{\cdot}1086\,d^2\,T)^2}.$$

$$6.\quad V = \frac{Q}{{\cdot}7854\,d^2} = 6{\cdot}504\,\sqrt{H}.$$

Let us illustrate the latter formulæ by a few examples.

1. The head of water in a reservoir above the centre of a circular aperture, 3 inches in diameter, is 16 feet; required the number of cubic feet of water that will be discharged in an hour, through a short additional cylindrical tube, fitted to the aperture.

Here $T = 3600$, $H = 16$, and $d = 3$; whence equation (2) gives

$$Q = 5{\cdot}1086 \times 3^2 \times 3600 \times \sqrt{16}:$$

or $Q = 45{\cdot}9774 \times 14400 = 661574$ cubic feet.

2. A reservoir receives a constant supply of 9 cubic feet of water per second, required the diameter of a circular orifice, fitted with a short additional tube, which placed 9 feet below the level of the fluid will carry off the same quantity.

Here $T = 1$, $H = 9$, and $Q = 9$, whence equation (3) becomes

$$d = \sqrt{\frac{9}{5{\cdot}1086 \times \sqrt{9}}} = 3 \times \sqrt{\frac{1}{15{\cdot}3258}} = {\cdot}766$$

feet, or $9\frac{1}{6}$ inches.

3. Let the cylindric additional tube be 6 inches in diameter, and the required supply 6 cubic feet per second, required the depth of the orifice below the level of the fluid.

Here equation (5) gives

$$H = \frac{6^2}{(5{\cdot}1086 \times \frac{1}{4})^2} = \frac{36}{(1{\cdot}27715)^2} = 18\tfrac{1}{3}\ \text{feet}$$

nearly.

Investigations and experiments of Dr. M. Young, relative to the discharge through additional tubes.

84. In the seventh volume of the Transactions of the Royal Irish Academy, is given an interesting paper connected with the subject at present under consideration; that part which turns upon an illustration of the increased discharge through additional tubes, is, we think, highly deserving of the reader's attention.

Dr. Young's invetigations.

When a tube *m n r s*, fig. 5, is inserted into the vessel ABCD, it is found that the velocity is increased nearly in the subduplicate ratio of the length of the pipe, and that it approaches nearer to that ratio, according as the length of the pipe is increased.

Fig. 5.

To account for this increase of velocity has appeared a matter of some difficulty, since the water cannot issue at *r s* with a greater velocity than it enters at *m n*, and it does not appear how the velocity at *m n* can be increased by inserting a tube beneath it. In order to explain the cause of this effect, we are to consider that the whole force with which the plate *m n* is pressed down, is the weight of a column of water equal to *e m n f*, together with the weight of a column of air of the same base, reaching to the top of the atmosphere; and the whole force with which it is pressed up, is the weight of an equal column of air diminished by the weight of a column of water equal to *m n r s*; therefore the actual force with which the plate *m n* is pressed down, is the weight of a column of water equal to *e f r s*; the velocity, therefore, with which the plate *m n* will issue through the orifice *m n*, will be the same as through the orifice *r s* of the vessel A *b c* D, that is, equal to the velocity that a heavy body would acquire in falling through the altitude *e r*, and all the plates of water in the tube *m n r s* will descend with the same velocity; for they cannot descend faster, because otherwise there would be a vacuum in the tube, which is prevented by the upward pressure of the atmosphere. And the velocity of the effluent water will be the same, whatever be the pressure of the atmosphere, provided the weight of a column of air of the same base with *r s*, and whose height is equal to that of the atmosphere, be either equal to, or greater than the weight of the pillar of water *m n r s*. This might be proved experimentally by a vessel of water with a pipe inserted in the bottom, and placed under the exhausted receiver of an air pump: but as the operation of exhaustion is obstructed more by the evaporation of the water than of mercury, Dr. M. Young's illustration.

it will be better to use mercury in these experiments. Now if D be the defect of the guage from the standard altitude, it will measure the pressure of the air on the surface of the mercury in the vessel. Let A be the altitude of the mercury in the vessel above the upper orifice of the pipe, and P the length of the pipe; then the whole force pressing downwards, the plate of mercury, which is immediately in the upper orifice of the pipe will be equal to $D + A$, and the whole force pressing the same plate upwards will be $D - P$, and the difference between these forces will be the absolute force pressing the same plate of mercury downwards: while D is greater than P, this absolute force will consequently be equal to $A + P$; when $D = P$, $D - P$ vanishes, and the force pressing the plate downwards will be $D + A = P + A$; hence, therefore, no difference in the time of the efflux will be perceived, while the altitude of the mercury in the guage is equal to, or less than, the difference between the length of the pipe and the standard altitude. When D is less than P, the force upwards is also nothing; and therefore as before, the whole force pressing the plate downwards is $= D + A$, and A being given it decreases according as D decreases; and when D vanishes, that is, when the receiver is absolutely exhausted, the force becomes equal to A, and the time of the efflux will be the same as if the pipe had not been inserted in the bottom of the vessel.

"To try," says Dr. Young, "the truth of these deductions by experiment, I inserted a tube 7·8 inches long, in a cylindrical vessel, and, closing the orifice of the pipe, I filled the vessel with mercury to the height of 6 inches then placing the apparatus under the receiver of an air pump, when the barometer was at 30 inches, and the guage at 28·5 inches; the time of efflux was 26 seconds; when the experiment was repeated precisely in the same manner, but in the open air, the time of efflux was only 19 seconds. Now as the guage stood at 28·5 inches the defect D was $30 - 28{\cdot}5 = 1{\cdot}5$, and the pressure on the plate of mercury was $6 + 1\frac{1}{2} = 7\frac{1}{2}$; in the open air the pressure was $6 + 7{\cdot}8 = 13{\cdot}8$; therefore the ratio of the velocity of efflux in both cases, which is the same with the reciprocal ratio of the times, was as $\surd 7{\cdot}5 : \surd 13{\cdot}8$; or as 2·73 to 3·70; but 2·73 to 3·70 is very nearly as 19 to 26. No difference was observed in the time of efflux, when in the open air and in the exhausted receiver, unless the guage stood higher than 22½ inches; that is, unless the height of the mercury in the guage was greater than the difference between the length of the pipe and the standard altitude. In another experiment, the guage stood at 27·9 and the height of the barometer was 29·9, and the defect therefore $= 2$, and the pressure $= 8$. But $\surd 8 = 2{\cdot}828$, and $\surd 13{\cdot}8 = 3{\cdot}7$, now $2{\cdot}828 : 3{\cdot}7 :: 19 : 24$; and by the experiment the efflux appeared to be 23 seconds. When the efflux is made in vacuo it is evident to observation that the pipe is not filled during the efflux, as it is while the discharge is made in the open air."

85. "Since the column of water in the pipe *m n r s* adds to the pressure which the plate *m n* sustains, by diminishing the upward pressure of the air through the pipe, it appears that it produces this increase of pressure in the plate *m n* alone without producing any lateral pressure in the fluid which is on the level with *m n*; for it is manifest, that if an aperture were made in *m* B or *n* C, the velocity of the water issuing through it would not be effected by the insertion of the pipe; and consequently that the plate *m n*, which is immediately in the orifice of the pipe, is the only one, on the same level, whose tendency downwards is increased by the insertion of the pipe."

Hence, the particles of water at the edge of the aperture having their perpendicular pressure increased by the weight of the column *m n r s*, without any increase of their lateral pressure, they will issue through the orifice *m n* more perpendicularly; the sides also of the tube will obstruct the converging motion of the particles, and consequently, on both these accounts, the quantity of water discharged through a pipe thus inserted will exceed that discharged through a simple orifice, in a greater ratio than the sub-duplicate of the height of the water: and according as the length of the pipe increases, the ratio of the quantity of water actually discharged by the experiment, to that which would be discharged according to the theory, will increase; because the ratio of the perpendicular to the horizontal pressure increases, in the ratio of the sum of the depths of the vessel, and the length of the pipe to the depth of the vessel. It follows, therefore, that experiments made in this manner will approach nearer to coincidence with theory than when made with a simple orifice; except either when the tube is so long as that the friction of the fluid against the sides of the tubes shall produce a sensible effect, or when it is so short as not to be sufficient to give the particles a vertical direction, all which agrees very well with the experiments made by the ingenious Mr. Vince, of which he has given us an account in the Philosophical Transactions for the year 1795. Thus, he tells us, that having inserted a tube of ¼ of an inch in length, into a cylindrical vessel 12 inches deep, he found that the velocity did not sensibly differ from that through the orifice: the cause of which he discovered to be this, that the stream did not fill the pipe, but that the fluid was contracted as when it flowed through a simple orifice in a thin plate. When the pipe was half an inch long inserted into a vessel of the same depth as before, the velocity of the water from the pipe, and from the orifice, which ought by theory to have been as $\surd 12{\cdot}5$ to $\surd 12$, or as 49 : 48, was by experiment found to be nearly in the proportion of 4 to 3; now if the ratio of 49 : 48 be increased in the ratio of 7 to 6 (because this is the ratio of the diminution of the velocity on account of the contraction of the vein, and this contraction either nearly or entirely vanishes in the pipe), we shall have the ratio of 3·57 to 3. When the pipe was an inch long, the velocity from the pipe, and from the orifice, which, according to theory, ought to have been as $\surd 13 : \surd 12$, or as 26 to 25 appeared by the experiment to be nearly as 4 to 3; now, if the ratio of 26 to 25 be increased by the ratio of 7 to 6, we shall have the ratio of 3·64 to 3. When he made use of longer pipes, the velocity of the effluent water by experiment approached nearer to that which ought to have been discharged according to theory; so that in long pipes the difference between theory and experiment was not greater, he says, than what might be expected from the friction of the pipe and other causes, which may be supposed to retard the velocity.

When he inserted a pipe of the same diameter with the aperture, which terminated in a truncated cone

fixed in the orifice (see Philosophical Transactions for 1795), he expected that the quantity of water discharged, in a given time, would have been diminished, because the water issuing through the orifice $m\ n$, **would have room to form the** *vena contracta* **in the enlarging cone; but he found that the same quantity of water was discharged, as if the pipe had continued throughout of the same diameter and orifice. The reason of this is manifest from what has been said, for the pressure of the air will not suffer the truncated cone to remain partly empty, as it would be if the vena contracta were formed: it will, therefore, continue full, and consequently the water will pass through it in the same manner as if the water in the cone surrounding the pipe** $m\ a\ b\ n$ **were congealed.**

Mr. Vince likewise inserted into the bottom of the vessel a perpendicular pipe, in form of a truncated cone, the narrower part being fixed in the orifice, by which he found the efflux to be increased more than if he had inserted a cylindrical pipe of the same length, whose diameter was equal to that of the narrowest part of the conical pipe. This effect may be explained on the same principles by which we account for the augmentation of the diameter of a vertical vein of water through a simple orifice, when the velocity of efflux is considerable. For when a perpendicular pipe is inserted, the velocity of the discharge being considerably increased, the resistance from the air will be so likewise; and thus the diameter of the vein has a tendency to enlarge itself: now, in the widening cone, the pipe admits of this augmentation, at the same time that it increases the velocity; but the cylindrical pipe, though it equally increases the velocity, does not permit the vein to enlarge itself, and by thus confining it, the efflux is obstructed, and the quantity discharged in a given time is diminished. Accordingly, under the receiver of an air-pump, even in a moderate degree of exhaustion, there is no difference between the velocities with which a fluid is discharged through a conical or cylindrical pipe. (Irish Trans. vol. vii.)

This abstract will enable the reader to judge of the correctness of the objections made to some of these deductions by M. Hachette, given under a subsequent article.

Miscellaneous experiments of Venturi, Polenus, and Du Buat.

86. Venturi has made a variety of ingenious experiments on the discharge of vessels, and particularly on what he terms "the lateral communication of motion in fluids," which is the title of the English translation of this work by Nicholson. The principal deductions of this author are as follows:

In any fluid, those parts which are in motion carry along with them the lateral parts which are at rest.

To prove the truth of this proposition, Venturi introduced a horizontal current of water into a vessel filled with the same fluid at rest. The stream entering the vessel with a certain velocity, passes through a portion of the fluid, and is then received in an inclined channel, the bottom of which gradually rises, until it passes over the border or brim of the vessel itself; and the effect is found to be not only that the stream itself passes out of the vessel, through the channel, but that it carries along with it the fluid contained in the vessel; so that, after a short time, no more of the fluid remains than was originally below the aperture at which the stream enters. This fact is adopted as a primitive phenomenon, by the author, under the denomination of the lateral communication of the motion in fluids; and to this he refers many important hydraulic facts. He does not undertake to explain this phenomenon on physical principles, but shows that the mutual attraction of the particles of water is far from being a sufficient cause to account for it.

Hydraulics.

Conical tubes.

If that part of an additional tube which is nearest the side of the reservoir, be contracted according to the form of the contracted vein of fluid which issues through a hole of the same diameter in a thin plate, the expenditure will be the same as if the tube were not contracted at all; and the velocity of the fluid within this tube will be greater than that of a jet through a thin plate in the ratio of 121 to 100.

Polenus' experiments.

87. Bossut, in referring to experiments of this kind by Polenus, says, if water issue out of a conical tube, of which the greater base is next the reservoir, the form of the tube will facilitate the entry of the water into it; and, provided it be not too wide, it ought to give more water than the cylindric tube which has the same exterior orifice. Polenus having adapted horizontally to a reservoir, which was kept constantly to the height of $21\frac{1}{3}$ inches above the centre of the orifice, a conical tube $7\frac{2}{3}$ inches long, and having its greater base $2\frac{3}{4}$ inches in diameter, and its less $2\frac{1}{6}$ inches; then a cylindric tube of the same length and diameter $= 2\frac{1}{6}$ inches; he found that the former yielded 73035 cubic inches in 2 minutes 57″, and the latter the same quantity in 3 minutes 7″. He then made an orifice in a thin plate, the diameter being still the same as the above exterior ones, viz. $2\frac{1}{6}$ inches; he obtained the same discharge in 4 minutes 36″. The same author also compared the discharge through different conical tubes having the same common length, $7\frac{2}{3}$ inches, and the same exterior orifice, $2\frac{1}{6}$ inches; but with different orifices next the reservoir; the latter in four tubes, having their diameters respectively equal to $2\frac{3}{4}$ inches, $3\frac{1}{2}$ inches, 5 inches, $9\frac{5}{6}$ inches. The water in all these experiments being kept to the constant height of $21\frac{1}{3}$ inches. The times employed by these tubes in yielding each 73035 cubic inches were 2 minutes 57″, 2 minutes 58″, 3 minutes, and 3 minutes 5″.

If we reduce the results of these experiments to the discharges in one minute, they will be as follow:

		Cubic inches.
Orifice in a thin plate		15877
Cylindric tube, diameter	$2\frac{1}{6}$ inches	23434
Conical tube,	diameters $2\frac{3}{4}$ and $2\frac{1}{6}$	24758
Ditto	diameters $3\frac{1}{2}$ and $2\frac{1}{6}$	24619
Ditto	diameters 5 and $2\frac{1}{6}$	24345
Ditto	diameters $9\frac{5}{6}$ and $2\frac{1}{6}$	23687

It appears from these results, that the first conical tube gave the greatest discharge, and that even the last yielded a greater quantity than the cylindric tube.

Venturi.

88. To return again to Venturi's deductions. He endeavours to show that the pressure of the atmosphere is the agent which produces the greater discharge through a cylindric tube, than through a simple orifice of the same diameter; and that in a conical tube, the pressure of the atmosphere increases the expenditure in the proportion of the exterior section of the tube to the section of the contracted vein, whatever

may be the position of the tube, provided its internal figure throughout be adapted to the lateral communication of motion.

In cylindric pipes, the discharge is less than through conical pipes which diverge from the place of the contracted vein, and have the same exterior diameter.

For in the space between the inverted contracted vein, and the side of the cylinder, eddies, or circular whirls, are produced, as in a bason which receives water by a channel; and these retard the efflux of the stream, and produce a corresponding failure in the effect.

Discharge increased by certain adjutages.

89. By means of proper adjutages applied to a given cylindric pipe, placed horizontally, it is possible to increase the expenditure of water through that pipe in the ratio of 24 to 10, the charge on the height of the reservoir remaining the same.

Fig. 5-a.

For this purpose the inner extremity of the tube AD (fig. 5-*a*) must be fitted at AB with a conical piece of the form of the contracted vein; this increases the expenditure in the ratio of 12·1 to 10. Every other form will afford less. If the diameter at A be too great, the contraction must be made beyond B, and the section of the vein will be smaller than the section of the tube. Secondly, at the other extremitiy of the pipe BC, apply a truncated conical tube CD, of which let the length be nearly *nine* times the diameter C, and its external diameter D must be 1·8 C. This additional piece will increase the expenditure as 24 to 12·1. So that the quantity of effluent water will be increased by the two adjutages in the ratio of24 to 10. It is to be observed, however, that this is on the supposition that the pipe BC has no elbows, or sinuosities.

When a tube has any elbow, or is otherwise curved, the discharge is diminished. It appears from the experiments of this author, that the proportional quantities of fluid yielded by a straight pipe, another curved into a quadrantal arc, and a third bent at an angle of 90°, were as the numbers 70, 50, 45.

Decreased by friction, &c.

The internal roughness of a pipe diminishes the expenditure, though the friction of the water against these asperities does not form any considerable part of the cause. A right lined tube may have its internal surface highly polished throughout its whole length; it may every where have a diameter greater than that of the orifice to which it is applied; but still the expenditure will be greatly diminished if the pipe should have enlarged parts, or swellings. For by reason of these sudden changes in the interior dimensions of the pipe much of the motion will be consumed in eddies. This, as the author remarks, is a very interesting circumstance, to which perhaps sufficient attention has not been paid in the construction of hydraulic machines. It is not enough that contractions and elbows are avoided; for it may happen by an intermediate enlargement, that the whole advantage may be lost, which has been procured by the ingenious disposition of the other parts of the machine.

Experiments of Du Buat on the effects produced by bent tubes.

90. We have in the preceding account of Venturi's experiments referred to the effect produced by bent tubes, the following experiments by Du Buat relate to the same enquiry, and are pursued to a greater length.

TABLE IX. *Of Du Buat's results.*

Dimensions of the tubes.	Number of bendings.	Angle of the bending.	Velocity per second, both in straight and bent tubes.	Augmentation of the head of water which overcomes the resistance at the flexure.	Augmentation of the head, calculated from the formula $\frac{V^2 s^2 n}{m}$*
				Inches.	Inches.
Tube 1 inch in diameter, and 117 inches in length.	3	36° 0	84·945	2·490	2·490
	2	36 0	84·945	1·500	1·660
	4	24 34	84·945	1·500	1·660
	3	24 34	84·945	1·120	1·240
	1	36 0	84·945	0·750	0·830
	2	24 34	84·945	0·750	0·830
	1	24 34	84·945	0·370	0·410
	10	36 0	71·590	5·905	5·905
Tube 138 inches in length.	4	36 0	58·808	1·640	1·590
Tube 117 inches.	4	36 0	58·438	1·500	1·570
	2	36 0	58·438	0·750	0·780
	4	24 34	58·438	0·750	0·780
	1	36 0	58·438	0·370	0·390
Tube 138½ inches.	4	36 0	29·330	0·410	0·396
Tube 737 inches long.	4	36 0	28.657	0·390	0·378

Experiments of M. Hachette.

91. As the circumstances attending the flowing of water and the discharge of vessels are subjects of the greatest importance in the present advanced state of mechanical science, it is not surprising that so many writers have turned their attention to the determination of practical data, and the investigation of the causes of those discrepances which are still frequently found between the experimental and theoretical deductions.

M. Hachette is one of the latest mathematicians who have undertaken such a course of experiments; he has examined the results and deductions of earlier authors, and in many respects has benefited by their experiments, while in numerous cases he has far extended the views they had formed, and drawn several new and interesting results.

The first object that engaged his attention, was to determine whether the figure of the small orifice has any influence on the quantity of fluid discharged from it in a given time. It is generally admitted, that supposing the pressure the same, and the orifice of the same superficial area, the quantity which flows out is the same for all forms; and M. Hachette has determined the correctness of this principle when the orifice is circular, triangular, elliptical, or formed of an arc of a circle, and two straight lines; but he found the product very different, either in excess or defect, when the contour of the surface presented re-entering angles, these occasioning an important modification of the above general principle. He also found, that if the plane in which the aperture is pierced be not horizontal, the fluid vein forms a curve, which is a parabola corresponding to a certain initial velocity, a fact that the author has determined by direct measurement. Setting out from the place of greatest contraction, the thickness becomes constant for some considerable extent till the jet, by the resistance of the air, loses its shape; throughout this extent, all the fluid molecules describe the same curve, and the vein resembles a perfectly pure crystal, and immoveable line. We have frequently witnessed the above effect by means of the apparatus shown in (fig. 6), in which the principal vessel ABCD is of iron or copper; PM is a forcing tube, by means of which the air is brought to any state of condensation in the receiver ABCD, which is partly filled with water. QR denotes a tube moveable about a joint at Q, whereby it may be placed at any angle at pleasure, and S is a cock which opens, or cuts off, the communication of the tube QR, and the receiver ABCD.

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* In the formula $\frac{V^2 s^2 n}{m}$; V is the velocity, *s* the sine of the angle of the bending, *n* the number of bendings, and *m* a constant quantity, determined by experiment to be = 300.

Let us now conceive, that by working the piston H of the forcing tube PM, a quantity of air has been injected into the receiver, the cock S being shut so as to cut off the communication between the receiver and the moveable tube QR. In this state the air will remain in its state of condensation; but as soon as the cock S is turned, and the communication established between the receiver and tube, the increased elasticity of the air forces the water out of the principal vessel through the tube at R, from which it will continue to flow for a considerable time in a beautiful continued chrystal curve, exhibiting, as it were, a living projectile. The condensation of the air may be brought to any state consistent with the strength of the vessel, and a projection obtained equal to that due to a very considerable height of water; and by giving to the pipe different inclinations, all varieties of ranges may be obtained.

We have found by a comparison of these ranges of the fluid on planes differently posited, that the curve of the fluid differs almost insensibly from a parabola, and M. Hachette arrives at the same results. He has concluded likewise, from the principles of the parabolic theory of projectiles, the velocity of the fluid in a given point; for example, at the point of greatest contraction: he found also in a similar manner that the common velocity at all the points of the contracted vein is very nearly that due to the height of the surface above the orifice. Whence it appears that the common parabolic theory is accurate, or extremely nearly so, when applied to the velocity that obtains at this section of the vein; but it cannot be true at the same time with respect to the mean velocity of the particles which traverse the section of the orifice on account of the difference between the areas of these two sections.

92. The velocity at the contracted vein being known, the expence of the fluid in a given time will give the ratio of this section to that of the orifice, or with what we have denominated the *vena contracta*, more exactly than can be done by direct measurement; that is, having carefully measured the quantity of discharge in any given time, and having at the same time computed the theoretic discharge due to the same interval; the ratio of these two quantities will also be the ratio between the contracted and actual orifice. M. Hachette, in all these determinations, followed strictly the methods pointed out by D. Bernoulli, and neglected none of those precautions that were necessary to diminish the errors of the experiments. He measured the time by means of an excellent seconds watch, and the orifices he employed were made and measured by a very correct artist; and particular care and means were employed to assure the perfect stability of level in the fluid, that is to say, to assure its remaining constantly the same during each experiment; but his observations were made upon rather a small scale, both with respect to the size of the vessel and the column of issuing fluid; on the contrary, the times were very considerable, lasting in some instances for more than an hour.

It appears from a table given at the conclusion of this author's memoir, that the smallest contraction observed during these experiments, and which answered to the smallest aperture, was ·78 of the orifice; the diameter of this latter being only ·039 of an inch; but for diameters more than ·39 of an inch, the contraction became nearly constant, viz. between ·60 and ·63; which agree very nearly with Bossut's deductions. It appeared to vary a little with the altitude of the fluid, but was independent of the direction of the jet.

Considerable difference will, doubtless, be noticed between these contractions, and still more if they be compared with some preceding experiments by Borda, Bernoulli, &c. Much of this deviation must certainly be attributed to the different sizes of the orifices employed, and to the pressures under which the discharges were made; but the author points out another circumstance, which had indeed been perceived by Bernoulli, and which necessarily has a considerable influence in producing those discrepances. According to this author, the waste of the fluid, every thing else being the same, is the smallest when the surface in contact with the fluid is convex. It increases when the surface becomes plane, and is greatest when it is concave. Accordingly, M. Hachette found the waste varied about one-twentieth part when the copper disc containing the orifice, which was *plano-convex*, was simply turned, the one side or the other, towards the fluid; and in some instances the difference was much greater.

Form of the fluid vein.

93. We have seen that the form of the orifice, when the sides are thin, has no sensible influence on the expenditure, unless it contains re-entering angles, but it has a marked effect upon the exterior surface of the fluid vein. As the contraction diminishes with the dimensions of the orifice, it was natural to suppose that when the fluid vein escaped between the two sides of a salient angle, the contraction ought to increase in proportion to the distance from the summit of the angle; so that a section made at a little distance from the plane of the orifice, and parallel to that plane, will be terminated, not by two straight lines, but by two curve arches, convex to each other, and this is what actually happens. Hence, when the contour of the orifice is a regular polygon, each side of that polygon becomes the base, not of a plane, but of a surface, which viewed externally is convex from the orifice to the contracted section. The concavity of the surface, after having reached its maximum between these two sections, diminishes as we approach the contracted section, and even changes beyond it, in consequence of the velocity acquired, into a very evident convexity, so as

to show a salient edge where there was before a hollow. The hollow, and the salient edge that succeeds it, are produced on the middle of the side which we examine, and are situated in a perpendicular plane on the same side. When the contour of the orifice presents a re-entering angle, an edge hollow at first, and convex afterwards, passes by the summit of this angle.

Additional tubes.

94. We have already seen, in the experiments of M. Bossut, the increase of discharge that may be obtained by employing a cylindric pipe instead of the simple orifice; and it has also been shown, in our abstract of the deductions of Venturi, that by a particular construction of the additional tube the discharge may be increased in the ratio of 10 to 24; but even this, it appears from M. Hachette's Memoir, is not the maximum flow that it is possible to obtain; a discharge, exceeding that stated by Venturi, having been actually produced by a construction due to M. Clement.

We have explained the principles on which M. Bossut endeavours to account for this remarkable phenomenon; but this is not considered by M. Hachette to be satisfactory. According to this author, the sole, or at least, the principal cause of it, is the adhesion of the fluid to the sides of the tube; that is to say, the force which produces the capillary and other similar phenomena. The following experiments were made to confirm this hypothesis:

Experiment 1. The fluid in motion was mercury, and the pipe was iron. When the mercury was pure, having then no affinity for the iron, the former flowed in the same manner as through a simple aperture; but when the mercury was covered with a pellicle formed of an alloy of tin and other metals, this alloy covered the inside of the pipe, and then the mercury flowed in a full stream.

Exp. 2. Here the fluid was water, and the pipe was coated within with wax. In this case also the discharge took place the same as in the simple orifice. But it is always possible to cause the water to moisten the wax; which being done, the water was found to fill the tube, owing to the wax being replaced by the first coat of water that covered it.

Hence the reason why a disc of glass at last adheres to water, with the same force whether it be covered or not with a coat of wax, for as soon as the wax is wetted it is merely the action of the water on the water which determines the phenomenon, as Laplace has explained it in his Theory of Capillary Attraction.

Discharge in a vacuum.

95. Another no less important fact which M. Hachette has determined is, that in a vacuum, or in air rarefied to a certain degree, the phenomenon of the increased discharge of pipes ceases to take place. To show this, the author having caused water to run in a full stream through a tube under a receiver of an air-pump, and having rarefied the air in the receiver, the fluid vein was observed to detach itself from the sides of the pipe, when the internal pressure was reduced to 23 centimetres of mercury, the external pressure being 76 centimetres. By thus diminishing the internal pressure, the effect of the external pressure is increased, which increase is transmitted to the pipe by means of the fluid contained in the vessel, and to which is to be added the pressure of the fluid itself. In this manner such a degree of rarety and additional velocity may be obtained, as to detach the fluid vein from the sides of the pipe, in the same manner as a sufficient force detaches a disc from the surface of a fluid to which it adheres.

The same experiment was repeated in the open air. The fluid vein was not detached from the sides of the pipe, except under a column of water whose vertical height was 22·8 metres, so that the difference between the superior and inferior pressure was (22·8 — 10.33 = 12·47 metres of water), or 91 centimetres of mercury.

Thus the flowing out of the water in a vacuum, or in rarefied air, agrees perfectly with the hypothesis of M. Hachette, and does not prove, as might be supposed, that the phenomena of pipes are owing to the pressure of the air in which that fluid flows, an opinion which would obviously be inconsistent with the two experiments (art. 94); for in these the action of the air was the same, and yet the phenomena were different, according to the nature of the fluid and the matter of which the pipe was composed.

When the fluid vein has been detached by rarefying the air, as stated above, if we allow the air to enter again into the receiver, the water does not in consequence again begin to flow in a full stream; that is, the contraction of the vein which took place in the rarefied air, continues to subsist, although the pressure of the atmosphere be restored. This, the author thinks, leads to the conclusion, that the adhesion of the fluid to the sides of the pipe takes place only at the commencement of the motion, before the fluid has acquired a sensible velocity in a direction which separates it from the side; an opinion which M. Hachette verifies by the following experiment. The water flowed in a full stream through a pipe outside of the receiver of an air-pump, when a small hole being made in the pipe very near the orifice, and the external air thereby entering the pipe, interposed itself between the water and the sides of the pipe, the contraction of the vein takes place in the inside of the tube, and the water ceases to flow in a full stream. This being the case, the opening made was exactly shut, but the adhesion of the water and the pipe was not again produced, the flowing of the water continuing the same as if the pipe had been removed. This experiment succeeded equally, whatever was the direction of the jet, but in order to the success of it, it is very necessary to be careful not to agitate the apparatus; for a very small lateral motion of the fluid vein causes it to adhere again to the moist sides of the pipe. It was probably in consequence of Venturi having neglected this precaution, that this author, in his work on the Lateral Communication of Motion in Fluids, arrives at a different result.

When water, or any other fluid, flows through a pipe in a full stream, the action exercised on the interior particles of the vein by those which are in contact with the inside of the pipe, produces the double effect of dilating the vein and diminishing its velocity.

When the length of the pipe is not sufficient to render the latter of these effects sensible, the dilation of the vein produces a considerable increase in the expenditure. Thus, when a circular orifice of 9½ millimetres in diameter, has given under a pressure of 142 millimetres a contraction of ·37, that is, when the vein is contracted to ·63 of the area of the aperture, it is sufficient to add to this orifice a pipe of equal diameter, and 6 millimetres in length to obtain under a pressure of 30 millimetres a contraction of only ·07.

96. When the length of the pipe is considerable in respect of its diameter, the velocity of the fluid particles is sensibly retarded by the action of those which are in contact with the inside of the pipe; and the consequence is a diminution of expenditure, which destroys a part, and sometimes even surpasses the whole augmentation produced by the dilation of the vein. For example, if in the last case, we increase considerably the length of the pipe, the expenditure will become much less.

If we fit a pipe to a given orifice, so that a portion of the pipe penetrates through the aperture into the inside of the vessel in which the fluid is retained, and if beside, the pipe be very thin towards the extremity at which the liquor is introduced, the effect will be the same as when the orifice is made in a surface convex towards the inside of the vessel; that is to say, the expenditure will be diminished. Borda has remarked, with reference to this subject, that when large pipes with thin sides, entirely plunged into the fluid, are employed, the expenditure is diminished by at least one half.

97. When we employ a capillary tube with a thin end, the cause just mentioned, joined to the diminution of the velocity arising from the length of the tube (which is always very great in comparison with its diameter), ought to produce a very considerable diminution of expenditure. This conjecture M. Hachette verified by means of a capillary tube, whose length was 49·3 millimetres, and its diameter 1·19 millimetres. This tube terminated in a cone towards its extremity, occasioned under a pressure of 24 centimetres a diminution of ·6 in the expenditure. And if we increase indefinitely the length of a capillary tube, we at last reach a limit, beyond which the liquid flows out only drop by drop; but this limit varies with the height of the fluid above the orifice.

As a proof that the contraction of the vein diminishes with the height of the fluid, or with the pressure resulting from, it is shown, that while an orifice of 27 millimetres diameter gave, under a pressure of 15 centimetres, a contraction of about ·40, that is, a contracted section of ·60, the same orifice, under a pressure of 16 millimetres, gave only a contraction of ·31. Consequently, since the fluid vein has a tendency to contract in proportion as the pressure increases, it was natural to imagine, that when a pipe was employed, the fluid by pressures always increasing, ought to tend more and more to detach itself from the inside of the pipe, and at last to separate itself altogether; and this was found to be actually the case. The pressure necessary to produce the separation, diminishes, as was to be expected, as the length of the pipe increases. It is less for a conical pipe than for a cylindrical one, and decreases at the same time with the angle at the vertex of the cone.

M. Hachette found, with reference to this subject, that in a pipe of 6 millimetres length, and $9\frac{1}{2}$ in diameter, it was still superior to 30 millimetres. This, therefore, destroys in part, the opinion maintained by Professor Vince, Phil. Trans. for 1795, that the flow cannot take place, with the tube full, in pipes shorter than 6 millimetres.

7 98. When the height of the fluid above the orifice becomes very small, the fluid vein at last acquires a particular form, very different from that which it had before, and which seems to be independent of the form of the orifice; for M. Hachette observed them alike with orifices and pipes of all figures and sizes. These the author denominates *secondary veins.*

Hydraulics.

If we make the height of the fluid decrease indefinitely after having obtained secondary veins, we at last reach a limit beyond which the flow ceases to be uninterrupted. The author has particularly examined the law of this last phenomenon, in the case when cylindrical capillary tubes are employed as pipes. Six experiments made upon similar tubes of different lengths, and of the same diameter, appear to prove that the limit in question is proportional to the lengths of the tubes.

Nature of different fluids.

99. The experiments above referred to were made with water; but most of the phenomena remain the same when mercury is substituted in place of the water. Thus, for example, the contraction relative to an orifice of one millimetre in diameter, with thin sides, and that which is given under a pressure of 24 centimetres, by a capillary tube of 49·3 millimetres in length, and 1·19 millimetres in diameter, will be, for mercury as for water, the first ·31 and the other ·60.

Alcohol, whose molecules adhere less to each other than those of water, flows out more readily; and for the same reason the pressure necessary to detach a fluid vein from the inside of a pipe is less for alcohol than for water.

When oil is employed, its viscosity increases very considerably the duration of the flow of the fluid, through small orifices. Through an orifice of one millimeter in diameter, the time of the flow of the latter in comparison to the time when water was used, was as three to one.

The nature of the fluid, therefore, is one of the principal causes on which depends the continuity or discontinuity of the jet through capillary tubes.

When water was employed, the thread, or fine stream, remained continuous at all pressures, for a tube with a diameter equal to, or greater than, a millimeter. But when oil was used, the flow through a similar tube, whose length did not exceed five centimetres, was only drop by drop, under a pressure of a column of oil of more than a metre in height.

Effect of the medium.

100. With respect to the medium in which the experiments are made, it may influence the flow in two ways: first, by modifying the pressures on the orifice of the liquid under consideration; and, secondly, by opposing a certain resistance to the emission of the fluid or to its motion. In order that the first two of these effects may become sensible, it is necessary that the vertical pressure, exercised from the top to the bottom on the upper surface of the fluid, and the pressure in a contrary direction on the exterior surface of the orifice or pipe should be very different from each other. This happens when we leave the upper part of the vessel containing the liquor exposed to the open air, and place the orifice or the pipe through which the fluid flows, under the receiver of an air-pump, in which the air may be rarefied at pleasure. By means of this artifice, and by diminishing progressively the elastic force of the air under the receiver, we may observe the same phenomena as those produced in the open air, by the gradual augmentation of the height of the fluid.

We may thus even have the advantage of determining the result of a very considerable pressure at little expence; and it was by this method that M. Hachette was able to ascertain the diminution of the expenditure under a pressure equivalent to 10 metres of water for capillary tubes terminated in cones towards the orifices; a diminution that was found the same as for pipes with thin sides and of a large diameter entirely plunged into the fluid.

If, instead of increasing the pressure, we wished to diminish it, it would obviously be sufficient to leave the given orifice or pipe exposed to the air, and to put the other surface of the liquid in contact with air rarefied under the receiver of an air-pump.

Resistance opposed to the issue.

It remains for us to speak of the resistance opposed to the issue, and to the motion of the fluid vein by the surrounding medium. Some philosophers have thought that we ought to ascribe to that resistance the changes of form which the vein experiences under variable pressures; but this conjecture is destroyed by the experiment of M. Hachette, as he observed no difference in the form of the fluid veins produced by the flow of water and mercury through a triangular orifice in the air and in a vacuum.

The flow of the liquid through small cylindrical tubes, seems entirely to depend upon the resistance and density of the surrounding medium. Dr. Matthew Young had remarked, that in this case, if we place the apparatus under the receiver of an air-pump, the flow continually decreases with the density of the air, and that in the open air the vein runs in a full stream, filling the pipe; while in a vacuum it detaches itself from the sides of the pipe. (Irish Transactions, vol. vii.) But he does not appear to have suspected the difference which exists in this respect between tubes of a great and of a small diameter.

M. Hachette states, that a tube 6·6 millimetres in diameter, only gave two different products for all densities of the air, according as the fluid vein filled, or did not fill the pipe; but, when he employed a tube whose diameter was reduced to 3 millimetres, he obtained, the same as Dr. Young, an expenditure varying with the density of the air. Dr. Young concluded from his experiments, that this expenditure reached its maximum when the elastic force of the air is equivalent to the weight of the liquid contained in the pipe; and that in this case, the liquid fills the pipe, but this conclusion appears very doubtful. All that we can be sure of is, that for tubes of a very small diameter, when we diminish the elastic force of the air beyond a certain limit, the expenditure continually decreases; M. Hachette supposes, with much probability, that in this case the vein fills only a part of the pipe; and he ascribes this effect to the compression due to the air which endeavours to enter into the pipe, to replace that which the motion of the fluid has necessarily carried off. When the diameter of the tube augments, a double current of air may be established, and the effect of which we are speaking, ceases to take place.

When the fluid impinges against fixed obstacles.

We shall conclude this account of M. Hachette's experiments by the following, on the impediment to the motion of the fluid, as arising from the impinging of the issuing vein on fixed planes, at different distances from the orifice. Daniel Bernoulli thought that these obstacles did not change the quantity of fluid that issued out, and he mentions in the fourth section of his Hydrodynamics, an experiment in support of his opinion; the influence of such an object, however, is very obvious in the following experiments:

M. Hachette caused a fluid vein flowing out of a great vessel by a circular horizontal orifice, 20 millimetres in diameter, to fall into a vessel placed at a considerable distance from the aperture. The level of the fluid in the vessel was depressed in 10′ 20″, about six decimetres. The plane face of an obstacle was then presented at different distances from the orifice, on which the jet fell perpendicularly.

These distances, expressed in millimetres, being

128^{m} 80^{m} 50^{m} 24^{m} 4^{m}.

The corresponding times of sinking six decimetres,

10′ 21″ 10′ 25″ 10′ 26″ 11′ 13″ 15′ 24″;

which shows that, at the distance of 128 millimetres, or 5·039 inches, the obstacle produces no sensible effect; but at 4 millimetres, or ·157 of an inch, the time is increased rather more than one half.

§ IV. *On the exhaustions of vessels.*

101. In all the preceding experiments the water was kept constantly at the same height, but it is also important to know the relation which has place between the theoretic and actual discharge, or between the computed time and the actual time, in which a given vessel will empty itself through a given aperture, or tube, when the supply from above is cut off. Bossut made some experiments with a reference to this comparison, but as, towards the latter part of the evacuation, the water necessarily runs very slow and irregular, it was thought best not to attend to the actual discharge of all the water, but merely to note the time employed in sinking the level of the fluid in the reservoir a certain quantity.

The vessel used in these experiments was a prism three feet square at the bottom; the apertures were circular, and made in the horizontal base of the reservoir; the first height of the water above it was 11 feet 8⅝ inches, and the time was observed in which the surface descended 4 feet and 9 feet successively.

The following are the results of these experiments:

1. The water issuing through a circular orifice 1 inch in diameter, in a thin horizontal plate, the surface was depressed 4 feet in 7 minutes 25½ seconds.

2. The aperture being increased to 2 inches, the surface was depressed 4 feet in 1 minute 52 seconds.

3. The 1 inch orifice being again replaced, the water sunk 9 feet in 20 minutes 24½ seconds.

4. The water issuing again through the 2 inch aperture, the surface was depressed 9 feet in 5 minutes 6 seconds.

If we compute the time requisite to depress the water 4 and 9 feet, respectively, with the given apertures, on the principle explained (art. 70), substituting, agreeably to the preceding deductions, ·62 a, instead of the actual area a, we shall find the times to be, 1st experiment = 443·04 seconds; 2d = 110·76 seconds; 3d = 1221·2 seconds; and 4th = 305·25; which agree very nearly with the experimental results, as will appear by reducing the former time to seconds, and placing the computed and actual times under one point of view, as in the following table:

Primitive height of the fluid 11 feet 8⅝ inches; area of the base of the vessel 9 square feet.	Diameter of actual orifice.	Diameter of contracted orifice.	Depression of the water	Time deduced from experiment.	Time deduced from computation
		Inches.		Seconds.	Seconds.
	1 inch	·088333	4 feet	445·5	443·04
	1 inch	·083333	9 feet	1224·5	1221·20
	2 inches	·166666	4 feet	112	110·76
	2 inches	·166666	9 feet	306	305·25

It appears from these experiments, and computed results, the agreement between the two is as near as it is possible to expect in cases of this kind, and sufficiently so for all practical purposes.

We may remark once again on concluding this short abstract of Mr. Bossut's experiments on the exhaustion of vessels, that the results have been given in the old French foot and inch, and that to make them correspond with English experiments, or to apply them to any computation in which the dimensions are given in English measures, we must reduce the altitude of the reservoir, so that the nominal velocity of the descent of a body passing through it by the force of gravity, may be the same in both cases.

§ V. *On the motion of water through long pipes.*

102. The investigations and tabulated experiments given in the preceding sections, must only be considered as applicable to the discharge of water through simple orifices and short pipes; for when we have to conduct this or any other fluid from one place to another, to any considerable distance, as from a reservoir to any given place, the actual discharge is very much diminished, in consequence of the friction and other obstacles, the due estimation of which it is almost impossible to introduce into our theoretical investigations. In short, it is only to experiment that we can look with any confidence for the requisite data, in order to decide in any practical case the proper dimensions to be observed, according to the height of the head of water, and the distance to which the fluid is to be conveyed.

The most complete series of experiments we possess of this kind are those performed by M. Bossut, the detail of which is given at considerable length in his Hydrodynamics, vol. ii. The means employed by this author in carrying on his experiments will be understood from the following description. EFGD, fig. 7, shows a vertical section of the principal reservoir, which contained about 30 cubic feet of water, and which communicated by means of a close tube with a secondary reservoir HKLM, containing about 6 cubic feet, and which was preserved constantly at the same height by regulating accordingly the supply of water from the principal source FEDG. To the secondary reservoir was fixed a tin pipe O, about 8 or 9 inches in diameter, the other end of which entered into a box of the same metal, shut on all sides, but having two apertures, A and B, either of which might be opened at pleasure. To these, tin pipes were fixed, respectively, 1½ and 2 inches in diameter; and these pipes might be lengthened at pleasure, and were actually, in the following experiments, extended to different lengths, from 30 feet to 180 feet. Fig. 8 shows a horizontal section of the same reservoir and its appendages, the conduit pipes mentioned above being represented by *a t*, and *b u*.

EXPERIMENTS.—*Series* I.

Detail of experiments.

103. *Constant height of the water in the secondary reservoir, above the axis of the tube, 1 foot, the diameter being 1½ inches.*

1. The pipe being 30 feet long, the discharge in 45 seconds was 2084 cubic inches of water.

2. The pipe being 60 feet in length, the discharge in 45 seconds was 1468 cubic inches.

3. The pipe being 90 feet long, the discharge in 45 seconds was 1190 cubic inches.

4. The pipe being lengthened to 120 feet, the discharge in 50 seconds was 1126 cubic inches.

5. The pipe being farther lengthened to 150 feet, in 50 seconds, there was discharged 982 cubic inches of water.

6. With 180 feet of pipe, the discharge in 50 seconds was only 877 cubic inches.

EXPERIMENTS.—*Series* II.

104. *The constant height of the water above the axis of the conduit pipe, 2 feet; the diameter of that pipe being still the same, viz. 1½ inches.*

7. The length of the pipe being 30 feet, there was received in 50 seconds, 3388 cubic inches of water.

8. With the pipe lengthened to 60 feet, the discharge in 50 seconds was 2407 cubic inches.

9. The length of pipe 90 feet, the discharge in 50 seconds was 1960 cubic inches.

10. The length of pipe 120 feet, discharge in 60 seconds, or 1 minute, 2011 cubic inches.

11. The length of pipe 150 feet, discharge in 1 minute 1762 cubic inches.

12. The length of pipe 180 feet, discharge in 1 minute 1538 cubic inches.

EXPERIMENTS.—*Series* III.

105. *The constant height of water 1 foot above the axis of the tube, the diameter of the latter being 2 inches.*

13. Length of pipe 30 feet, time 70 seconds, discharge 8960 cubic inches.

14. Length of pipe 60 feet, time 70 seconds, discharge 6492 cubic inches.

15. Length of pipe 90 feet, time 70 seconds, discharge 5290 cubic inches.

16. Length of pipe 120 feet, time 75 seconds, discharge 4930 cubic inches.

17. Length of pipe 150 feet, time 75 seconds, discharge 4358 cubic inches.

18. Length of pipe 180 feet, time 75 seconds, discharge 3899 cubic inches.

EXPERIMENTS.—*Series* IV.

106. *The constant height of the water above the axis of the pipe, 2 feet; diameter of the latter 2 inches.*

19. Length of pipe 30 feet, time 60 seconds, discharge 11219 cubic inches.

20. Length of pipe 60 feet, time 60 seconds, discharge 8190 cubic inches.

21. Length of pipe 90 feet, time 60 seconds, discharge 6812 cubic inches.

22. Length of pipe 120 feet, time 65 seconds, discharge 6375 cubic inches.

23. Length of pipe 150 feet, time 65 seconds, discharge 5668 cubic inches.

24. Length of pipe 180 feet, time 65 seconds, discharge 5103 cubic inches.

TABLE X. *Of the above results reduced to the discharges due to 1 minute.*

N° of Exp.	Length of the conduit pipe.	Height of the head of water.	Discharge in 1 minute through the pipe $1\frac{1}{3}$ inch in diameter.	Discharge in 1 minute through the pipe 2 inches in diameter.
1...13	30 feet	1 foot	2778	7680
2...14	60 feet	1 foot	1957	5564
3...15	90 feet	1 foot	1587	4534
4...16	120 feet	1 foot	1351	3944
5...17	150 feet	1 foot	1178	3486
6...18	180 feet	1 foot	1052	3119
7...19	30 feet	2 feet	4066	11219
8...20	60 feet	2 feet	2888	8190
9...21	90 feet	2 feet	2352	6812
10...22	120 feet	2 feet	2011	5885
11...23	150 feet	2 feet	1762	5232
12...24	180 feet	2 feet	1583	4710

Comparison of results.

107. If we compute, by means of our tabular results (art. 75), the discharges due to 1 minute through short tubes of $1\frac{1}{3}$ inches, and 2 inches diameter, the heads of water above them being 1 foot, and 2 feet, we shall find, that an

Altitude 1 foot, diameter $1\frac{1}{3}$ inches, gives 6330 cubic inches.
Altitude 2 feet, same tube, . . . 8939 ditto.
Altitude 1 foot, diameter 2 inches, . 14243 ditto.
Altitude 2 feet, same tube, . . . 20112 ditto.

Which show that the expence of the fluid diminishes very considerably as the length of the pipe increases.

In order to form a general idea of the ratio of this diminution of discharge, let us represent in all cases the quantity yielded with a short tube by 100; then we shall have the proportional discharges at 30 feet, 60 feet, 90 feet, &c. exhibited in the fourth terms of the following analogies.

The tube of $1\frac{1}{3}$ inches diameter, the altitude of the water being 1 foot:

6330 : 2778 :: 100 : 43·89.
6330 : 1957 :: 100 : 30·91.
6330 : 1587 :: 100 : 25·07.
6330 : 1351 :: 100 : 21·34.
6330 : 1178 :: 100 : 18·61.
6330 : 1052 :: 100 : 16·62.

The same tube, under 2 feet altitude, will give

8939 : 4066 :: 100 : 45·48.
8939 : 2888 :: 100 : 32·31.
8939 : 2352 :: 100 : 26·31.
8939 : 2011 :: 100 : 22·50.
8939 : 1762 :: 100 : 19·71.
8939 : 1583 :: 100 : 17·70.

The tube of 2 inches diameter, under a head of water of 1 foot, will give

14243 : 7680 :: 100 : 53·92.
14243 : 5564 :: 100 : 39·06.
14243 : 4534 :: 100 : 31·83.
14243 : 3944 :: 100 : 27·69.
14243 : 3486 :: 100 : 24·48.
14243 : 3119 :: 100 : 21·90.

The same tube, under a head of water of 2 feet:

20112 : 11219 :: 100 : 55·78.
20112 : 8090 :: 100 : 40·72.
20112 : 6812 :: 100 : 33·87.
20112 : 5885 :: 100 : 29·26.
20112 : 5232 :: 100 : 26·81.
20112 : 4710 :: 100 : 23·41.

108. We may perceive, by means of the above numbers, the gradual diminution of discharge as the length of the pipe increases; let us then examine the question under a physical point of view, and endeavour to determine the cause of this diminution, and the law which it observes with respect to the diameter and length of the conduit pipe.

In the first place it is evident that there will always be asperities in the interior of the pipes, however perfect we may suppose them polished, which will retard the natural velocity of those particles of the fluid coming in contact with them, and even independent of this, the natural adhesion between the water and the metal, will retain, or tend to retain, the particles contiguous to it in a state of rest, which state is only destroyed by the velocity of the other particles. The effect, therefore, of this must necessarily be to diminish the expence of water, and that so much the more, according to the length of the pipe through which the fluid passes.

If we consider the fluid as constituted of an indefinite number of thin threads, it will be only those in contact with the sides of the tube (which we may denominate *the lateral threads)* that will be, by the above supposition, retarded by the causes we have suggested, but they soon communicate, in consequence of the natural adhesion of the particles of water to each other, a like retardation to those in contact with them, and hence by successive steps every thread is ultimately effected from the surface to the centre of the moving column; still, however, as the retardation is less from one thread to another, the centre threads will be less retarded than those in contact with them, and so on to the edges where the velocity is the least. The diminution of the expence will not, therefore, be proportional to the length of the tubes, for the longer these are, the less difference there will be between the velocities of two contiguous threads, and therefore the less relative diminution; which is conformable to the results of the preceding experiments, where it will be seen, that for every increase of 30 feet in the pipe, the difference between that discharge and the one preceding it, is continually diminished.

109. M. Bossut, in the work above referred to, enters into an ingenious investigation in order to determine the relation which has place between the length of the tube and the corresponding discharge. He assumes an indefinite line to represent the length of his tube, and at the same time the axis or absciss of a curve, upon which, at distances corresponding to his several experimental lengths of 30, 60, 90, &c. feet, he supposes ordinates to be erected to denote the several discharges due to these lengths, and then, by means of these six given ordinates, he interpolates, according to a principle given by M. Charles, under the article "In-

terpolation EncyclopédieMethodique," other ordinates; and ultimately conceives a curve to pass through their several summits, and it is the equation of this curve which expresses the relation sought.

M. Bossut, by these means, finds the curve in question, within most practical limits, to be such that the ordinates are, very nearly, reciprocally as the square roots of the abscisses; whence it follows,

That the discharges due to equal intervals of time through horizontal tubes of the same diameter, and under the same height of water, but of different lengths, not differing greatly from each other, will be very nearly in the inverse ratio of the square roots of these lengths.

Having therefore ascertained, from actual experiment, the discharge at any distance from the reservoir, that is, for any length of tube, we may readily compute the discharge that will be obtained in any time, and for any length of pipe, by means of the preceding theorem. It may, moreover, be observed, that it appears by a comparison of the preceding experiments with the tabular results (art. 75), that the discharge at the distance of 30 feet is about half of what would take place through a short cylindrical tube; and this discharge, we have seen, is to the theoretic discharge, calculated on a supposition of there being no contraction, in the ratio of 13 to 16. We have, therefore, all the data requisite for computing the expence of water, from a pipe of any length, of any proposed diameter, and due to any given height of the level in the reservoir.

110. Let us suppose, for example, that the water in a reservoir is 4 feet above the insertion of a conduit pipe, the latter 120 feet in length and 3 inches in diameter, to find its discharge per minute.

First, our table (art. 75) gives, for the discharge through a short tube 1 inch in diameter, and with the given head of water, 7070 cubic inches, therefore a tube 3 inches in diameter will yield $7070 \times 9 = 63630$ cubic inches in the same time; consequently, from what is stated above $\frac{63630}{2} = 31815$ cubic inches will be the expence through a tube 30 feet in length; whence $\sqrt{120} : \sqrt{30} :: 31815 : 15907\frac{1}{2}$ cubic inches the discharge in one minute through a pipe 120 feet in length, as required.

111. The above determinations, however, it must be acknowledged, do not possess all the accuracy that is desirable, although they may be sufficiently exact for most practical cases.

Of the two tubes employed in the preceding experiments, that of $1\frac{1}{2}$ inches diameter gave less water in proportion than that of two inches under the same height of reservoir, and the same length of tube, a result which was to be expected when we reflect that the surface of the fluid subjected to friction is greater in proportion as the diameter of the tube diminishes.

Again, the greater the height of water we have in the reservoir, the less in proportion is the diminution of discharge; but it is very difficult to trace the law of this diminution. We shall not attempt it in this place, deferring our remarks on this subject till we have examined at greater length the experimental results exhibited in the following sections.

§ VI. *On the discharges through inclined tubes.*

112. When the tube is vertical, or inclined at a certain angle, gravity will have a tendency to accelerate the velocity of the fluid. The combination therefore of this force, with that due to the pressure of the column of water, and the negative force arising from the friction above considered, will serve to modify in any manner at pleasure the quantity of discharge under any proposed conditions.

Vertical tubes.

Fig. 9.

Let MQRN, fig. 9, denote a long cylindrical vertical tube, adapted to the bottom of the reservoir ABCD, kept constantly full to the height MH. If now, in the first instance, we make abstraction of the friction along the internal surface of the tube; it is evident that the velocity of the fluid at the moment it is about to pass the section MN, being that due to the height HM; each particle, if at that instant it became an isolated body, and only submitted to the action of gravity, would, when it had arrived at any point Q of the vertical MQ, have acquired a velocity due to the height HQ. But the several particles having a natural adhesion to each other, they form, in consequence of this force, a column which fills entirely the tube, without leaving in it any vacant space whatever, and they therefore necessarily all move with the same velocity along the whole space MQ. Now, as the velocity produced by the pressure of the water in the reservoir above the section MN, is always the same, and that on the other hand each particle receives so much the greater effect from gravity, as it is farther below the same section MN; it is evident, that the lower particles must accelerate the velocity of the upper ones in such a manner, that the velocity of every particle in the column becomes uniformly the same.

The lower particles therefore of the fluid have a tendency from the adhesion, or viscosity of the fluid, to accelerate the velocity of those above them; and on the same principle, it is obvious that the upper particles ought to retard the natural velocity of those below them; the velocity therefore with which the fluid issues, is greater than that which is due to the simple pressure at the section where the vertical pipe is inserted into the reservoir, and less than the natural velocity of the lower particles; that is, than that due to the whole altitude of the pipe, *plus* the pressure at the section MN. The conclusion, therefore, which some authors have drawn, viz. that the velocity of discharge is that due to the whole altitude of the column, is incorrect, except when the descending tube is very short, in which case gravity acts freely, and impresses its whole energy on every particle of the descending fluid.

The same effects have place in long tubes inclined at different angles, viz. the water will be discharged from these with a greater velocity, than that due to the height of the fluid above the section of their insertion in the reservoir, and with a less velocity than that due to the height of the head of water above their lower extremity.

Experiments connected with this subject.

It is very difficult to make experiments upon the discharges of long vertical pipes; and on this account M. Bossut confined his experimental researches to a sloping inclined tube, forming an angle with the horizon of $6\frac{1}{2}°$; the following are the results for three different lengths of this tube.

EXPERIMENTS.—*Series* V.

113. *Constant height of the water in the reservoir above the centre of the insertion of the tube* 10 *inches.*

1. Length of the tube 177 feet; time 45 seconds, discharge 4346 cubic inches, or 5795 cubic inches in one minute.

2. Length of the tube 118 feet; time 45 seconds; discharge 4351 cubic inches, or 5801 cubic inches in one minute.

3. Length of the tube 59 feet; time 45 seconds; discharge 4356 cubic inches, or 5808 cubic inches per minute.

If the above tube had been short, viz. about 2 inches in length, so as to have acted merely as an additional tube, as detailed in art. 75, its discharge in one minute would have been 5779 cubic inches; we see, therefore, the discharges, instead of following the ratio of the square roots of the height of the fluid above their lower extremity, become in this case actually less as the length of the pipe is increased, although the depth of the fluid also necessarily increases in the same proportion. We derive, however, from these experiments this important conclusion:

That when a pipe, similar to the above, in polish and diameter, is inclined from the horizon at an angle of about $6\frac{1}{2}^\circ$, it will, whatever may be its length, give a constant discharge, equal to that which belongs to a short additional tube placed at the same distance below the head of water in the reservoir; the additional friction being, at the above inclination, about counterbalanced by the additional action of gravity upon the descending fluid. And we may hence infer that whatever may be the diameter of any tube, some inclination may be found, such, that whatever be the length of the tube, the discharge will be always the same as that which would take place through a short additional tube; and it appears also highly probable that there is only one inclination for any given diameter that possesses this peculiar property. We are aware that some authors have arrived at a different result, but we conceive the experiments above referred to are sufficient to justify the above conclusion.

§ VII. *On the motion of water in curved tubes.*

Curved tubes.

114. It may be seen, by a comparison of the preceding experimental results with what may be considered our theoretical deductions, or with what would have been our deductions, independent of practical data, that even in the simple case of tubes perfectly uniform, and having a uniform position, either horizontal or inclined, we should have been led into the greatest errors with respect to their computed rate of discharge.

If, therefore, in this comparatively simple instance, theory alone is insufficient to meet the general problem, it must naturally be expected to be still more defective when we suppose tubes to receive all kinds of curvature. In such cases as these, nothing but the results of a well conducted course of experiments are capable of furnishing the requisite data.

In connection with this inquiry, M. Bossut begun by having constructed for this purpose a tube of lead 50 feet in length, and 1 inch interior diameter, the bore very perfect, and the thickness of metal $\frac{1}{12}$ of an inch: to the end of this pipe was soldered a short tube of 2 inches interior diameter, which communicated with the lesser of the two reservoirs described in art. 102, and represented in fig. 7 and 8. To this short tube a stop-cock was attached, by means of which the running of the water might be stopped at pleasure. The opening in this was a circular, and $1\frac{1}{2}$ inch in diameter: there were, moreover, several small holes in the upper side of the lead pipe, intended to facilitate the escape of any particles of air which the water might bring with it.

EXPERIMENTS.

Boss peri

115. *The diameter of the pipe, as stated above,* 1 *inch, and the length* 50 *feet.*

1. The tube being placed horizontally, and the head of water above its axis 4 inches, the rate of discharge per minute was 576 cubic inches.

2. The head of water being increased to 1 foot above the axis of the tube, the discharge in 1 minute was 1050 cubic inches.

3. The pipe being now put into a serpentine form, as denoted by the line OQZ, &c. (fig. 10), but the plane of the curve still retaining a horizontal position, the discharge in 1 minute, with a head of water of 4 inches, was 540 cubic inches.

Fig.

4. Every thing remaining as in the last experiment, except the head of water being increased to 1 foot, the rate of discharge per minute was 1030 cubic inches.

5. The plane of the curve being now rendered vertical, but its two extremities being still placed in the same horizontal line, and no point of the curve rising above the same, the discharge per minute, with a head of water of 4 inches, was 520 cubic inches.

Note. In this experiment no discharge could be obtained till the small holes were opened that were made in the upper parts of the curve; the air accumulated in them, preventing the flow of the water: it was, therefore, only after these had been opened for some little time that the above uniform discharge was obtained.

6. Every thing remaining the same, except the head of water being increased to a foot, the discharge per minute was 1028.

Coup expe ment

116. Before we attempt to make any remarks, or to draw any conclusions from the results given by the above experiments, we think it will be better to enlarge the series by the addition of others reported by M. Couplet, in the "Memoires de l'Academie" for the year 1732, the latter being in fact the actual results of observations on the discharges of certain pipes belonging to fountains, &c. in the neighbourhood of Paris.

In the first instance, we may remark that the principal reservoir was 2 feet square at the bottom, and 2 feet 8 inches deep, from which proceeded a vertical tube of lead 6 inches in diameter, and in length about 6 feet; to this was soldered another tube of lead 4 inches in diameter and 17 feet 4 inches in length; at the extremity of this was again soldered an iron pipe of the same diameter, and nearly at right angles to it, which latter formed the remainder of the conduit, except that at its other extremity a leaden pipe of the same diameter, and 6 feet 3 inches long, ascended and discharged the water: the entire length of the pipe, measuring it along all its bendings, was 296 toises 5 feet 4 inches.

1. Understanding, now, the height of water to denote the difference of level between the upper surface of the water in the reservoir and the discharging aperture, it was found, that with a height of 9 inches, the discharge

was $\frac{1}{28{\cdot}5}$ of what would have taken place through an additional short tube of the same diameter, or of the discharge independent of friction, the head of water being the same. Under a head of 21 inches the discharge was $\frac{1}{26{\cdot}53}$; and, under 31 inches, it was $\frac{1}{25{\cdot}79}$.

2. With the same reservoir, and with a tube principally of iron, the whole length of which was 285 toises, and diameter 6 inches, the ratio of the effective discharge to that, through a short additional tube, was found, under a head of 3 inches, to be $\frac{1}{12{\cdot}35}$; and, under a head of 5¼ inches, $\frac{1}{11{\cdot}37}$.

3. Here, about 50 toises of the conduit pipe next the reservoir was stone, and the other part lead, its diameter throughout was 5 inches, and its entire length 1170 toises. This pipe was tried under five different heads of water, viz. $5\frac{7}{12}$ inches, 11⅓ inches, 1 ft. 4¾ in. 1 ft. $9\frac{1}{12}$ in. and 2 ft. 1 in. and the corresponding ratios were $\frac{1}{23{\cdot}1}$, $\frac{1}{20{\cdot}98}$, $\frac{1}{19{\cdot}49}$; $\frac{1}{18{\cdot}78}$, $\frac{1}{18{\cdot}46}$.

4. In the following experiments, the diameters and lengths of the pipes were very considerable, the lengths being about 600 toises, and the diameters 18 inches and 12 inches. To the same reservoir there were adapted two tubes of the former dimensions, and three of the latter. They were all similarly situated, and gave the same results; that is to say, each of the two 18-inch pipes gave under a head of 12 feet $1\frac{1}{12}$ inch, $\frac{1}{6{\cdot}05}$ of the discharge that would have been made through a short additional tube, or the discharge independent of friction; and each of the three 12-inch tubes $\frac{1}{10{\cdot}08}$, under the same head of water.

Couplet's experiments.

5. Another tube of iron 18 inches in diameter, and 790 toises in length, gave under a head of water of 4 feet 7½ inch, the fraction $\frac{1}{10{\cdot}11}$.

6. And lastly, a pipe 12 inches in diameter, and 2340 toises in length, yielded under a head of 20 feet 3 inches, $\frac{1}{19{\cdot}34}$ of the discharge due to a short additional tube of the same diameter.

117. In order to the more ready comparison of the results of the several experiments related above, it will be very convenient to bring them all under one point of view, and to reduce the several results to the same form, which may be done by converting the cubic inches, in which M. Bossut has expressed his discharges, into the proportional part that each bears to that through a short additional tube. This reduction is shown in the last column of the following table; the other results require no explanation.

Couplet and Bossut.

TABLE XI. *Of the mean results of the experiments of M.M. Bossut and Couplet, on the discharge of water through long tubes, horizontal, inclined, and bent.*

N° of Experiment.	Length of tube.	Diameter of tube.	Position, &c. of the tube.	Height of the head of water.	Ratio of the effective discharge to the discharge independent of friction.
1	50 feet.	1 inch.	Lead, placed horizontally and straight.	0 ft. 4 in.	$\frac{1}{3{\cdot}55}$
2				1 0	$\frac{1}{3{\cdot}18}$
3	50 feet.	1 inch.	Lead, various sinuosities, with the plane of the curve horizontal.	0 4	$\frac{1}{3{\cdot}78}$
4				1 0	$\frac{1}{3{\cdot}43}$
5	50 feet.	1 inch.	The same pipe, but the plane of the curve vertical.	0 4	$\frac{1}{3{\cdot}93}$
6				1 0	$\frac{1}{3{\cdot}44}$
7	180 feet.	1½ inch.	Tin, rectilineal and horizontal.	1 0	$\frac{1}{6{\cdot}01}$
8				2 0	$\frac{1}{5{\cdot}64}$
9	180 feet.	2 inch.	Tin, rectilineal and horizontal.	1 0	$\frac{1}{4{\cdot}57}$
10				2 0	$\frac{1}{4{\cdot}27}$
11	177 feet.	1⅓ inch.	Tin, rectilineal, but inclined to the horizon, at an angle of 6½°.	20 11	$\frac{1}{5}$
12	118 feet.	1⅓ inch.	Tin, inclined as above.	13 4⅔	$\frac{1}{4}$
13	59 feet.	1⅓ inch.	Tin, inclined as above.	6 8⅓	$\frac{1}{2{\cdot}82}$

Couplet and Bossut.

TABLE XI.—*continued.*

Nº of Experiment.	Length of tube.	Diameter of tube.	Position, &c. of the tube.	Height of the head of water.	Ratio of the effective discharge to the discharge independent of friction.
14	297 toises.	4 inches.	Iron, with various bendings, both horizontal and vertical.	0 9	$\frac{1}{28{\cdot}5}$
15				1 9	$\frac{1}{26{\cdot}53}$
16				2 7	$\frac{1}{25{\cdot}79}$
17	285 toises.	6 inches.	Iron, with various bendings, both horizontal and vertical.	3 0	$\frac{1}{12{\cdot}35}$
18				5 3	$\frac{1}{11{\cdot}37}$
19	1170 toises.	5 inches.	Partly stone, and partly metal, with various bendings, both horizontal and vertical.	0 $5\frac{1}{12}$	$\frac{1}{23{\cdot}1}$
20				0 11⅓	$\frac{1}{20{\cdot}98}$
21				1 4¾	$\frac{1}{19{\cdot}49}$
22				1 $9\frac{1}{12}$	$\frac{1}{18{\cdot}78}$
23				2 1	$\frac{1}{18{\cdot}46}$
24	600 toises.	12 inches.	Iron, with various bendings, both horizontal and vertical.	12 1¼	$\frac{1}{10{\cdot}08}$
25	600 toises.	18 inches.	Ditto, ditto.	12 1¼	$\frac{1}{6{\cdot}05}$
26	790 toises.	18 inches.	Ditto, ditto.	4 7½	$\frac{1}{10{\cdot}11}$
27	2340 toises.	12 inches.	Ditto, ditto.	20 3	$\frac{1}{19{\cdot}34}$

The preceding table furnishes us with much useful practical information, but we shall defer any comparison of the results which it offers till we have given the same author's experiments on the motion of water in open canals.

118. To the above we shall add, in concluding this section, the following table of M. Bossut, relative to the discharges through pipes with different adjutages.

Experiments with different adjutages

TABLE XII.—*Of discharges through pipes fitted with different adjutages.*

Head of water.		Length of pipe.	Diameter of pipe.	Dimensions of the orifices.	Ratio of the real to the theoretic discharge.	Ratio of the height due to the velocity to the head of water.	Cubic inches discharged per minute.
Ft.	In.		Inch.	Twelfths of Inches.			
24	7	161	1	7½	0·045	0·002	242
23	9	192	1	5½	0·075	0.006	230
19	3	193	1	6¼	0·068	0·005	222
19	9	188	1	6½	0·061	0·004	237
19	10	146	1	2½ by 7	0·089	0·008	168
29	1	187	1¼	7½ by 5½	0·505	0·001	588
8	0	1069	1½	Two adjutages, each ½ inch.	0·435	0·189	1686
24	7	278	1¼	3½	0·396	0·157	458
32	7	314	1¼	Two adjutages, each $\frac{5}{12}$ inches.	0·227	0·052	1232
30	5	446	1½	2 by 6½	0·037	0·001	636
26	3	506	1½	4	0·447	0·200	696
27	0	668	1½	5½	0·301	0·091	900
30	0	812	1½	11	0·048	0·002	600
10	5	194	1	5	0·377	0·139	576
10	11	462	1	5½	0·332	0·109	576
10	0	420	1¼	7	0·162	0·028	483

§ VIII. *On the motion of water in open canals.*

peri- nts on pen canals.

119. The motion of water in open canals is no less a subject of interest than that of its motion in close pipes, and accordingly M. Bossut, in the very general view which he took of practical hydraulics, made this an important part of his inquiry, and with much labour carried on a course of experiments which is far from being the least useful part of his undertaking. The reservoir was kept constantly at the same height during the same experiment, and the canal was supplied by a sluice at the bottom of the reservoir. The former was 5 inches in breadth, and by means of a sluice-gate or slider, it could be opened ½ an inch, an inch, two inches, &c. at pleasure; its whole length was 105 feet. In order to measure the velocity of the water in the canal, the author employed small wheels, fixed at certain distances from the sluice; sometimes at five equal distances of 21 feet each, and at others at three equal distances of 35 feet each. These wheels, as soon as the fluid struck them, began to turn, indicating very accurately the arrival of the fluid in those points; and thus, with the assistance of a seconds and a half-seconds pendulum, the time from the beginning of the motion to the water reaching these several points, was estimated with a considerable degree of precision.

The results of these experiments are shown in the following table: the first column of which gives the number of the experiment; the second the height of water in the reservoir, the position of the canal, viz. whether horizontal or inclined; and in the latter case the difference between the horizontal levels of its extremities is indicated by the words *slope of the canal.* In the same column is also shown the breadth, and vertical depth of the sluice; and the third column exhibits the time in seconds that the water was in passing over the corresponding number of feet in the fourth, after first opening the orifice.

Note.—The signs + or — show that the time a little exceeded, or fell short, of the number of seconds to which these signs are subjoined.

TABLE XIII. *Of experiments on the motion of water in open horizontal canals.*

Nº of Experiment.	Dimensions, slope, &c.	Ft. in.	Time in seconds.	Nº of feet run.	Nº of Experiment.	Dimensions, slope, &c.	Ft. in.	Time in seconds.	Nº of feet run.
I.	Head of water	11 8	2	21	II.	Head of water	7 8	3 —	21
	Horizontal, or slope of canal	0 0	5 —	42		Slope of canal	0 0	7	42
	Breadth . .	0 5	10 —	63		Breadth . .	0 5	13 —	63
	Depth . . .	0 0½	16 —	84		Depth . . .	0 0½	20 —	84
			23 +	105				28 +	105
III.	Head of water	3 8	3 +	21	IV.	Head of water	11 8	2	21
	Slope of canal	0 0	9	42		Slope of canal	0 0	4	42
	Breadth . .	0 5	17 +	63		Breadth . .	0 5	7	63
	Depth . . .	0 0½	29 +	84		Depth . . .	0 1	11	84
			38 +	105				16¼	105
V.	Head of water	7 8	2 +	21	VI.	Head of water	3 8	3 —	21
	Slope of canal	0 0	5	42		Slope of canal	0 0	6 +	42
	Breadth . .	0 5	9	63		Breadth . .	0 5	11 +	63
	Depth . . .	0 1	14	84		Depth . . .	0 1	18 +	84
			20	105				26	105

120. By a comparison of the above results, it appears that the time employed by the water in running successively through each 21 feet, forms very nearly an arithmetical progression, so that the series may be continued *ad libitum*, and the time corresponding to any other distance, not included in the series, interpolated.

Since the vein of issuing fluid is reduced by contraction in the ratio of 8 : 5, the velocity in the canal, independent of friction, ought to be to the velocity due to the altitude of the fluid in the same ratio. If, therefore, H represent the height of the fluid, we shall have

$$h = \frac{5^2}{8^2} \text{H} = \frac{25}{64} \text{H}$$

for the height due to the actual velocity; this velocity, therefore, by the laws of falling bodies, will be

$$v = 2 \sqrt{h\,g},$$

g denoting the descent due to gravity in one second; consequently $\text{S} = 2\,t\,\sqrt{h\,g}$ will denote the space which ought to be passed over, independent of friction and resistance in t seconds; whence

$$t = \frac{\text{S}}{2\sqrt{h\,g}} = \frac{5\,\text{S}}{8\sqrt{\text{H}\,g}}$$

will be time requisite, under these circumstances, for the fluid to pass over the space S. Substituting in this formula the French value of g, viz. $g = 15$, we shall find the times of passing over the 105, to be

6·35″, 7·83″, 11·33″, 6·35″, 7·18″, 11·33′;

while the experimental times are

23″, 28″, 38″, 16″, 20″, 26″;

Results compared.

from which it appears that the velocity of the water is very much retarded by the resistances it meets with, but that the retardation is less as the height of the orifice is increased.

It may not be amiss to observe here, that notwithstanding the very considerable retardation which the velocity of the fluid experienced in its motion along the canal, yet M. Bossut found that the supply of the discharge from the canal, was constantly the same whatever its length might be; in which respect, therefore, the two cases of open canals and close pipes are essentially different.

Table XIII. *continued.—Experiments on the motion of water in open inclined canals*

Nº of Experiment.	Dimensions, slope, &c.	Time in seconds.	Nº of feet run.	Nº of Experiment.	Dimensions, slope, &c.	Time in seconds.	Nº of feet run.
	Ft. in.				Ft. in.		
	Head of water 11 8	4	35		Head of water 7 8	4 +	35
VII.	Slope of canal 0 3	11 +	70	VIII.	Slope of canal 0 3	14 +	70
	Height of sluice 0 0½	22	105		Height of sluice 0 0½	26	105
	Head of water 3 8	6 +	35		Head of water 11 8	3½	35
IX.	Slope of canal 0 3	18 +	70	X.	Slope of canal 0 6	11½	70
	Height of sluice 0 0½	34 +	105		Height of sluice 0 0½	21	105
	Head of water 7 8	4 +	35		Head of water 3 8	6	35
XI.	Slope of canal 0 6	14	70	XII.	Slope of canal 0 6	18 —	70
	Height of sluice 0 0½	25 +	105		Height of sluice 0 0½	31 +	105
	Head of water 11 8	3	35		Head of water 7 8	4 —	37
XIII.	Slope of canal 0 6	8	70	XIV.	Slope of canal 0 6	9 +	70
	Height of sluice 0 1	15	105		Height of sluice 0 1	19 —	105
	Head of water 3 8	5 —	35		Head of water 11 8	3 —	37
XV.	Slope of canal 0 6	13 —	70	XVI.	Slope of canal 1 0	7½	70
	Height of sluice 0 1	23 —	105		Height of sluice 0 1	14	105
	Head of water 7 8	4 —	35		Head of water 3 8	5 —	37
XVII.	Slope of canal 1 0	9	70	XVIII.	Slope of canal 1 0	12	70
	Height of sluice 0 1	16	105		Height of sluice 0 1	21	105
	Head of water 11 8	2 +	35		Head of water 7 8	4 —	35
XIX.	Slope of canal 2 0	7	70	XX.	Slope of canal 2 0	9 —	70
	Height of sluice 0 1	13	105		Height of sluice 0 1	15 —	105
	Head of water 3 8	4½	35		Head of water 11 8	2 +	35
XXI.	Slope of canal 2 0	10½	70	XXII.	Slope of canal 4 0	6½	70
	Height of sluice 0 1	17½	105		Height of sluice 0 1	12	105
	Head of water 7 8	3 +	35		Head of water 3 8	4 +	35
XXIII.	Slope of canal 4 0	8	70	XXIV.	Slope of canal 4 0	9 +	70
	Height of sluice 0 1	13	105		Height of sluice 0 1	15 +	105
	Head of water 11 8	2 +	35		Head of water 7 8	3 +	35
XXV.	Slope of canal 6 0	6	70	XXVI.	Slope of canal 6 0	7 +	70
	Height of sluice 0 1	10	105		Height of sluice 0 1	12	105
	Height of water 3 8	4	35		Head of water 11 8	2 +	35
XXVII.	Slope of canal 6 0	9 —	70	XXVIII.	Slope of canal 9 0	6 —	70
	Height of sluice 0 1	14 —	105		Height of sluice 0 1	9	105
	Head of water 7 8	3 +	35		Head of water 3 8	4 —	35
XXIX.	Slope of canal 9 0	6½	70	XXX.	Slope of canal 9 0	8 —	70
	Height of sluice 0 1	10	105		Height of sluice 0 1	12 —	105
	Head of water 3 8	3½ +	35		Head of water 11 8	4½	35
XXXI.	Slope of canal 9 0	7½	70	XXXII.	Slope of canal 9 0	9½	70
	Height of sluice 0 1	11½	105		Height of sluice 0 0½	15	105
		1	21			1½ +	21
	Head of water 11 8	3½	42		Head of water 7 8	4 +	42
XXXIII.	Slope of canal 11 0	6	63	XXXIV.	Slope of canal 11 0	6½ +	63
	Height of sluice 0 0½	8½	84		Height of sluice 0 0½	9 +	84
		10½ +	105			11½ +	105
		2 +	21			1	21
	Head of water 3 8	5	42		Head of water 11 8	2½	42
XXXV.	Slope of canal 11 0	8	63	XXXVI.	Slope of canal 11 0	4½	63
	Height of sluice 0 0½	11	84		Height of sluice 0 1	6½	84
		14	105			8½	105
		1½ +	21			1½	21
	Head of water 7 8	3½	42		Head of water 3 8	4 +	42
XXXVII.	Slope of canal 11 0	5½	63	XXXVIII	Slope of canal 11 0	6½	63
	Height of sluice 0 1	7½	84		Height of sluice 0 1	9 —	84
		9½	105			11	105
		1	21			1½ —	21
	Head of water 11 8	2½	42		Head of water 7 8	3	42
XXXIX.	Slope of canal 11 0	4 +	63	XL.	Slope of canal 11 0	5 —	63
	Height of sluice 0 1½	6	84		Height of sluice 0 1½	6½ +	84
		7½ +	105			8½	105
		1½ +	21			1½	21
	Head of water 3 8	3½	42		Head of water 3 8	3½	42
XLI.	Slope of canal 11 0	5½ +	63	XLII.	Slope of canal 11 0	5½ —	63
	Height of sluice 0 1½	7½	84		Height of sluice 0 1½	7½	84
		10	105			10	105

Hydraulics.

121. From the nature of the preceding experiments, it is obvious, that only the velocity of the first portion of water that issued from the reservoir was observed. In order to compare this velocity with that of the current, after it was completely established, M. Bossut made the experiments detailed in the following table, in which, the time in seconds, of passing over certain spaces, are shown for both cases. The former, viz. the time, in the first case was shown by means of the little wheels above mentioned; and in the second by the aid of small pieces of cork, which followed exactly the current. The first portion of the canal was always run over in less time than any of the other divisions, and the velocity did not become sensibly uniform till the declivity was about the 10th part of the length of the canal.

Table XIII. *continued.—Experiments made in order to compare the velocity of the first portion of water with that of the established current.*

N° of Experiment.	Dimensions, slope, &c.	First water.		Established course.	
		Time in seconds.	Number of feet run.	Time in seconds.	Number of feet run.
XLIII.	Ft. In. Head of water . . 4 0 Slope $\frac{1}{10}$ of length Height of sluice . . 0 1	10	100	8	100
		20 +	200	17	200
		31 −	300	26	300
		42 −	400	35	400
		52½	500	43 +	500
		62 +	600	52	600
XLIV.	Head of water . . . 4 0 Slope $\frac{1}{16}$ of length Height of sluice . . 0 2	8	100	7	100
		17	200	14½	200
		26	300	22	300
		35 −	400	29 +	400
		43 +	500	37 −	500
		52 −	600	44 +	600
XLV.	Head of water . . . 2 0 Slope $\frac{1}{16}$ of length Height of sluice . . 0 1	11	100	10	100
		23	200	20	200
		35	300	30	300
		46 +	400	40	400
		58	500	49	500
		69	600	58	600
XLVI.	Head of water . . . 2 0 Slope $\frac{1}{10}$ of length Height of sluice . . 0 1	9	100	8 −	100
		19	200	16	200
		29	300	24	300
		39	400	32	400
		49	500	40	500
		58	600	48	600
XLVII.	Head of water . . . 1 0 Slope $\frac{1}{10}$ of length Height of sluice . . 0 1	12 +	100	12	100
		25½	200	23 +	200
		39	300	33	300
XLVIII.	Head of water . . . 1 0 Slope $\frac{1}{10}$ of length Height of sluice . . 0 2	11 −	100	9	100
		22	200	18 −	200
		32½	300	27	300
XLIX.	Head of water . . . 0 4 Slope $\frac{1}{10}$ of length Height of sluice . . 0 1	15	100	13	100
		31	200	26½	200
		47	300	39½	300
L.	Head of water . . . 0 4 Slope $\frac{1}{10}$ of length Height of sluice . . 0 2	13½	100	11½	100
		26¼	200	23	200
		39½	300	33½	300

By a comparison of these results with each other, it will appear that the velocity of the first portion of the fluid is always less than that of the established current, and that the one has to the other a ratio that is nearly constant, the canal being the same. This in fact must necessarily be the case; for the asperities and other inequalities of the canal being the same, the water issuing always from the same sluice, will experience the same obstacles, and ought to establish its course regularly and permanently, according to the same law, whatever may be the head of water and the slope of the canal. But when the canals are not the same, we cannot expect the same relation will obtain, as the primitive velocities in two different canals may be very unequal.

When the slope of a canal is very small, neither the primitive velocity nor the established velocity will be uniform at all distances; and different canals will require different slopes to produce an uniform current. In that employed by M. Bossut, he found that neither the primitive nor permanent velocity could be rendered uniform, till the slope amounted to about $\frac{1}{10}$th part of the length of the canal. And even with this slope, the first division of it was passed over in less time than any of the others.

§ IX. *Application of the preceding experimental data to the determination of general formulæ.*

Investigation of general formulæ.

122. In this investigation we shall follow the steps of Du Buat, and assume as a general principle, that whether in pipes, or in open canals, such a slope may be found that the water will flow uniformly in them; and that when this uniformity is obtained, the accelerating force of gravity is exactly equal to the resistances which the fluid experiences from the friction against the sides of the pipe or canal, and the natural viscidity of the fluid itself.

Let s denote that part of the length of any pipe whose declivity is one inch; that is, let $\frac{1}{s}$ denote the ratio of the perpendicular of a right angled triangle to the hypothenuse; this ratio being such, that the velocity of the fluid is uniform. Call the velocity V; and conceive the head of water which produces this velocity to be divided into two parts; the one that which would be necessary, independent of friction, to produce the given velocity, and the other will then be that part of the head that is employed in overcoming the resistances only.

We have seen (art. 83) that in short additional tubes $V = 6{\cdot}504 \surd H$ *feet;*
whence, supposing H to be expressed in inches, we have $V = 6{\cdot}504 \surd 12 \times \surd H$,
or $H = \frac{V^2}{495{\cdot}6}$.

This, therefore, is the height of the head of water due to the velocity V, and the remainder after this deduction must be considered as a declivity to be distributed over the whole length of the pipe; calling, therefore, the whole length of the pipe l, and the height of the head of water H + D, that is, supposing D to denote the remainder after deducting H, we shall have

$$\frac{D}{l} = \frac{1}{s}.$$

If we now consider the resistances which bring the velocity of the stream to uniformity, we shall find that they ought to be proportional to the square of the velocity; for while the impulses on all the asperities of the pipe are as the velocity, the number of striking particles will be also in the same proportion. We may therefore express the resistances by the symbol $\frac{V^2}{m}$, m being some number to be determined by experiment.

If $2g$ be taken to denote the accelerating force of gravity on any particle, $\frac{2g}{s}$ will be the accelerating power, by which it would urge it down the pipe, whose slope is $\frac{1}{s}$; whence, by the principle of uniform motion, or the equality of the accelerating force and the resistance, we shall have

$$\frac{V^2}{m} = \frac{2g}{s}, \text{ or } V \surd s = \surd 2gm.$$

Inv tion ral f

Therefore, the product of the uniform velocity into the square root of the reciprocal of the square root of the slope is a constant quantity $= \surd 2gm$, for any given pipe, and the primary formula for all the uniform velocities of one and the same pipe is

$$V = \frac{\surd 2gm}{\surd s}.$$

By comparing this formula with the results of experiments, Du Buat found, that even with respect to a pipe or canal, which was uniform throughout, this was not true. He found, in the first place, that in the same canal, the product $V \surd s$ increased faster than $\surd s$; that is, that the velocity increased more rapidly than the square root of the reciprocal of the slope; in fact, that the resistances did not increase in the ratio of the square of the velocity, a remark which we have made in two or three places, in detailing the experiments of M. Bossut.

123. Du Buat, therefore, next endeavoured to determine some constant function of s, which, by addition or subtraction, should render the equation more consistent with actual practice. The formula which he assumes is $V = \frac{\surd 2gm}{X}$,
X denoting any function of s; and after numerous trials and reflections, he found a value of X, which corresponded with a great variety of slopes and velocities, from motions almost imperceptible to the greatest velocities which could be produced by gravity alone in a vertical pipe, and when he compared them together, he found a very evident relation between the resistances and the magnitudes of the section. That is, in two canals, which had the same slope and the same head of water, the velocity was greatest in the canal which had the greatest section relative to its perimeter; this also might have been expected, *à priori*, and we have seen it obviously marked in the experiments previously detailed.

It is therefore evident, that the resistance of each particle is in the direct proportion of the whole resistance, and in the inverse proportion of the number of particles which receives equal shares of it; consequently it is as the perimeter directly, and as the section inversely. Therefore, in the expression $\frac{V^2}{m}$ above assumed, the quantity m can only be constant in the same pipe or canal: in different pipes it will vary with the relation of the section to the perimeter, because we have seen that the resistances diminish in proportion as this relation increases.

In cylindric pipes the section is the area of a circle, and the perimeter of the section is the circumference of the circle, and the quotient arising from dividing the one by the other is equal to one-half of the radius. In rectangular pipes or canals, and even in irregular channels, there is still some line which, multiplied by the perimeter of the section, will give an area equal to the area of the section, as in the circle the product of half the radius by the circumference is equal to the area of the circle; this line Du Buat denominates the mean radius, and it is called by Dr. Robison the *Hydraulic mean depth.*

Denoting this mean depth by d, since the resistances increase as the ratio of the perimeter to the area of the section increases, the quantity m ought to be proportional to d, and consequently $\sqrt{d}$ ought to be proportional to $\sqrt{2\,g\,m}$, and therefore in every case the expression $\frac{\sqrt{2gm}}{\sqrt{d}}$ should be a constant quantity.

124. Du Buat, however, found by examining this formula with the result of actual experiment, that $\sqrt{2mg}$ was neither proportional to $\sqrt{d}$, nor to any power or root of d, but that it increased less and less in proportion as the square root of d increased. In very great beds of channels, indeed, the above proportion nearly obtained, but it would not answer in cases of less magnitude. This led the author to consider, that by diminishing d by a small quantity, such as would make no sensible alterations in canals of great extent, but which would, notwithstanding, become sensible in others of less dimensions, he might at least approximate to the actual value, and, after many comparisons, he found, that if in every case $\sqrt{d}$ be diminished by 0·1 of an inch, the calculated discharges would agree very exactly with the results of experiment. Instead, therefore, of $\sqrt{d}$, he makes use of $\sqrt{d} - 0{\cdot}1$, and finds this quantity always proportional to $\sqrt{2\,m\,g}$; that is, he finds $\frac{\sqrt{2mg}}{\sqrt{d} - 0{\cdot}1}$ = *a constant quantity, or very nearly so;* its utmost limits being between 297 and 287 in all sections, from that of a very small pipe to that of a small canal. In the larger canals and rivers it diminishes still more, but is in no case less than 256.

If we assume the former number, viz. 297, we shall have for the resulting equation

$$\frac{\sqrt{2mg}}{\sqrt{d} - 0{\cdot}1} = 297,$$

$$m = \frac{297^2}{2g}(\sqrt{d} - 0{\cdot}1)^2;$$

or

$$m = \frac{88209}{362}(\sqrt{d} - 0{\cdot}1)^2;$$

the French value of g, viz. $15\frac{1}{12}$ feet, or 181 inches, being here retained in order to follow the author the better in his subsequent comparisons.

Or, if we make $\frac{88209}{362} = 243{\cdot}7 = n$, we may write the above value of m,

$$m = n(\sqrt{d} - 0{\cdot}1)^2;$$

the resistance, therefore, which was expressed by $\frac{V^2}{m}$, will now become

$$R = \frac{V^2}{n(\sqrt{d} - 0{\cdot}1)^2},$$

where n is an invariable abstract number, given by the nature of the resistance which water sustains from its bed, and which indicates its intensity.

Lastly, since $m = n(\sqrt{d} - 0{\cdot}1)^2$, we have $\sqrt{2gm} = \sqrt{2mg}(\sqrt{d} - 0{\cdot}1)$;

and the expression for the velocity V which water acquires, and maintains along any channel whatever, now becomes

Hydraulics.

Investigation of general formulæ.

$$V = \frac{\sqrt{2ng}(\sqrt{d} - 0{\cdot}1)}{X};$$

in which X is a variable quantity, but some constant function of the slope $\frac{1}{s}$.

We shall not follow M. Du Buat in his determination of this function; we shall merely state that upon the supposition, 1st, that X must be sensibly proportional to $\sqrt{s}$, when s is very great; 2d, that it must in all cases be less than $\sqrt{s}$; 3d, that it must deviate from the proportion of $\sqrt{s}$ so much the more as s is smaller; 4th, that it must not vanish when the velocity in infinite; and, finally, that it ought to agree with any series of well conducted experiments, for every variety of channels and slopes, M. Du Buat found these conditions would be fulfilled by taking

$$X = \sqrt{s} - \text{hyp. log.}\ \sqrt{(s + 1{\cdot}6)};$$

so that, substituting this in our preceding formula, we shall have

$$V = \frac{297(\sqrt{d} - 0{\cdot}1)}{\sqrt{s} - \text{h. l.}\ \sqrt{(s + 1{\cdot}6)}};$$

or

$$V = \frac{297(\sqrt{d} - 0{\cdot}1)}{\sqrt{s} - \frac{1}{2}\text{h. l.}\ (s + 1{\cdot}6)}.$$

125. This point being established, our author next supposes that there is a constant portion of the accelerating force employed in overcoming the viscidity of the fluid and producing the mutual separation of its filaments.

We may express this part of the accelerating force by $\frac{1}{S}$ part of that slope which constitutes the whole of it. If this were not employed in overcoming the resistance it would produce a velocity, which (on account of the resistance) is not produced. So that in reasoning upon this, as we did upon the real velocity, its value will be

$$\frac{\sqrt{2ng}(\sqrt{d} - 0{\cdot}1)}{\sqrt{S} - \text{h. l.}\ \sqrt{S}},$$

a quantity which must always be subtracted from the velocity already determined.

Hence we shall have

$$V = \frac{\sqrt{2ng}(\sqrt{d} - 0{\cdot}1)}{\sqrt{s} - \text{h.l.}\ \sqrt{(s + 1{\cdot}6)}} - \frac{\sqrt{2ng}(\sqrt{d} - 0{\cdot}1)}{\sqrt{S} - \text{h.l.}\ \sqrt{S}};$$

or

$$V = (\sqrt{d} - 0{\cdot}1) \times \left\{ \frac{\sqrt{2ng}}{\sqrt{s} - \text{h.l.}\ \sqrt{(s + 16)}} - \frac{\sqrt{2ng}}{\sqrt{S} - \text{h.l.}\ \sqrt{S}} \right\}$$

In this formula, the latter term being composed only of constant quantities, viz. S, n, and g, it may be expressed by a single letter or number; the value of which, as determined from numerous comparisons and experiments, was found by Du Buat to be = 0·3 of an inch, very nearly; and by substituting this value, our latter equation become

$$V = (\sqrt{d} - 0{\cdot}1) \times \left\{ \frac{\sqrt{2ng}}{\sqrt{s} - \text{h. l.}\ \sqrt{(s + 1{\cdot}6)}} - {\cdot}3 \right\};$$

or

$$V = \frac{\sqrt{2ng}(\sqrt{d} - 0{\cdot}1)}{\sqrt{s} - \text{h. l.}\ \sqrt{(s + 1{\cdot}6)}} - {\cdot}3(\sqrt{d} - 0{\cdot}1).$$

If we replace g by its numerical value in French and English inches, the above becomes

Final expression for the velocity.

$$V = \frac{297\,(\surd d - 0{\cdot}1)}{\surd s - \text{h. l. } \surd(s + 1{\cdot}6)} - {\cdot}3\,(\surd d - 0{\cdot}1) \text{ French;}$$

$$V = \frac{307\,(\surd d - 0{\cdot}1)}{\surd s - \text{h. l. } \surd(s + 1{\cdot}6)} - {\cdot}3\,(\surd d - 0{\cdot}1) \text{ English.}$$

It may be convenient to recall here to the recollection of the reader the quantities which the letters in these formulæ are intended to represent, viz.

V is the mean velocity in inches per second of any current of water running uniformly, in a pipe or open canal, whose section, figure, and slope, are constant, but its length indefinite.

d is the mean radius, or hydraulic mean depth; that is, the quotient arising from dividing the section of the pipe or canal, in square inches, by its perimeter, expressed in linear inches.

s is the slope of the pipe, or of the surface of the current, being the denominator of the fraction expressing the ratio between the sine and radius of the angle of the pipe's inclination from the horizontal level; and it is found by dividing the whole length of the pipe by the height of the fluid which remains, after deducting that which would be sufficient to produce (independent of resistance) the uniform velocity; or by dividing the whole length of the pipe by the difference in the horizontal levels of its two extremities.

g denotes the space through which a heavy body descends in the first second by the force of gravity in inches; it is therefore = 182 French inches, or 193 English.

n is an abstract number, determined by experiment to be = 243·7.

h. l. represents the hyperbolic logarithm of any number or quantity to which it is prefixed, and may be found when a table of these numbers is not at hand, by multiplying the common logarithm of any number by 2·3026.*

126. The following table contains the result of the actual experiments, from which the above formulæ were deduced, and the comparison of the real velocities with the velocities computed by the formula appertaining to it.

It consists of two distinct sets, the first on pipes, and the second on rivers and canals. In the first set, column 1, contains the number of the experiment; the second the length of the tube, the third the height of the reservoir, the fourth the values of s deduced from the second and third columns; the fifth shows the observed velocities, and the sixth the velocities as calculated from the formula.

In the second set, column 2 gives the area of the section of the channel; the third the perimeter of the section in contact with the water; the fourth $\surd d$, or the square root of the mean radius; the 5th the denominator s of the slope; the sixth the observed mean velocities; and the seventh the mean velocities by the formula.

In the last ten experiments on large canals and a natural slope, the sixth column shows the observed velocities at the surface.

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* The most extensive table of hyperbolic logarithms yet published, is that given in Barlow's Mathematical Tables, containing h. l of all numbers from 1 to 10,000.

Comparison of theoretical and experimental velocities.

TABLE XIV. *Containing a comparison of the velocities calculated by Du Buat's formula, with the velocities observe in the experiments of Couplet, Bossut, and Du Buat, on pipes, canals, and rivers.*

EXPERIMENTS ON PIPES.

Experiments of Du Buat. Pipe ⅔ds of a line in diameter placed vertically.

$\surd d = 0{\cdot}117851.$

N° of experiment.	Length of pipe in inches.	Height of reservoir in inches.	Values of s in inches.	Observed velocities in inches.	Calculated from the formula, in inches.
1	12	16·166	0·75636	11·704	12·006
2	12	13·125	0·9307	9·753	10·576
Pipe 1½ line in diameter, placed vertically. $\surd d = 0{\cdot}176776.$					
3	34·166	42·166	0·9062	45·468	46·210
4	34·166	38·333	0·9951	43·156	43·721
5	34·166	36·666	1·0396	42·385	42·612
6	34·166	35·333	1·07805	41·614	41·714
Same pipe horizontal.					
7	34·166	14·583	2·5838	26·202	25·523
8	34·166	9·292	4·0367	21·064	19·882
9	34·166	5·292	7·03597	14·642	14·447
10	34·166	2·083	17·6378	7·320	2·351

Du Buat's experiments on close pipes.

TABLE XIV.—*continued.*

N° of experiment.	Length of pipe in inches.	Height of reservoir in inches.	Values of *s* in inches.	Observed velocities in inches.	Calculated from the formula in inches.
Pipe 2 lines in diameter, placed vertically. $\surd\, d = 0\cdot204124.$					
11	36·25	51·250	0·854509	67·373	64·945
12	36·25	45·250	0·963382	59·605	60·428
13	36·25	41·916	1·038080	57·220	57·838
14	36·25	38·750	1·120473	54·186	55·321
Same pipe with a slope of $\frac{1}{1\cdot3024}$.					
15	36·25	33·500	1·291741	51·151	50·983
Same pipe horizontal.					
16	36·25	15·292	2·79005	33·378	33·167
17	36·25	8·875	4·76076	25·430	24·553
18	36·25	5·292	7·89587	19·940	18·313
19	36·25	2·042	20·01636	10·620	10·492
Pipe 2·9 lines in diameter, placed vertically. $\surd\, d = 0\cdot245798.$					
20	36·25	53·250	0·952348	85·769	85·201
21	36·25	50·250	1·006424	82·471	82·461
22	36·25	48·333	1·044400	81·646	80·698
23	36·25	48·333	1·044400	79·948	
24	36·25	47·916	1·052952	81·027	80·318
25	36·25	44·750	1·124052	76·079	77·318
26	36·25	41·250	1·215688	73·811	73·904
Same pipe with a slope $\frac{1}{1\cdot3024}$.					
27	36·25	37·500	1·332332	70·822	70·138
Same pipe horizontal.					
28	36·25	20·166	2·43034	51·956	50·140
29	36·25	9·083	5·26858	33·577	32·442
30	36·25	7·361	6·45035	28·658	28·801
31	36·25	5·000	9·35730	23·401	23·195
32	36·25	4·916	9·50972	22·989	22·974
33	36·25	4·833	9·66522	21·679	22·754
34	36·25	3·708	12·4624	19·587	19·550
35	36·25	2·713	16·3135	16·631	16·324
36	36·25	2·083	21·6639	14·295	14·003
37	36·25	1·625	27·5102	12·680	12·115
38	36·25	0·833	52·3427	7·577	8·215

Comparison of theoretical and experimental velocities.

TABLE XIV.—*continued.*

Pipe 1 inch in diameter, horizontal, $\surd d = 0·5$.

N° of experiment.	Length of pipe in inches.	Height of reservoir in inches.	Values of *s* in inches.	Observed velocities in inches.	Calculated from the formula in inches.
39	117	36·000	5·65026	84·945	85·524
40	117	26·666	7·48002	71·301	72·617
41*	138·5	20·950	10·32149	58·808	60·034
42	117	18·000	10·78798	58·310	58·472
43*	138·5	6·000	33·19623	29·341	29·663
44*	737	23·700	33·66578	28·669	29·412
45	737	14·600	54·26340	21·856	22·056
46	737	13·700	57·77718	20·970	21·240
47	737	12·320	64·15725	19·991	19·950
48	737	8·96	87·86790	16·625	16·543
49*	737	8·96		16·284	
50*	737	7·780	101·0309	15·112	15·232
51*	737	5·930	132·1617	13·315	13·005
52*	737	4·20	186·0037	10·671	10·656
53*	737	4·20		10·441	
54*	138·5	0·700	257·8663	8·689	8·824
55*	737	0·500	1540·76	3·623	3·218
56	737	0·150	5113·42	1·589	1·647

Note.—In the experiments marked with an asterisk, the pipe discharged itself into water. In all the other experiments it discharged itself into air.

Experiments of BOSSUT.
Horizontal pipe 1 inch in diameter, $\surd d = 0·5$.

57	600	12	54·5966	22·282	21·975
58	600	4	161·3120	12·223	11·576

Pipe 1⅓ inch in diameter, horizontal, $\surd d = 0·57735$.

59	360	24	19·0781	48·534	49·515
60	720	24	33·6166	34·473	35·130
61	360	12	37·0828	33·160	33·106
62	1080	24	48·35416	28·075	28·211
63	1440	24	64·1806	24·004	24·023
64	720	12	66·3020	23·360	23·345
65	1800	24	78·05318	21·032	21·182
66	2160	24	92·9474	18·896	19·096
67	1080	12	95·87567	18·943	18·749
68	1440	12	125·6007	16·128	15·991
69	1800	12	155·4015	14·066	14·119
70	2160	12	185·2487	12·560	12·750

Pipe 2·01 inches in diameter, horizontal, $\surd d = 0·7088723$.

71	360	24	21·47087	58·903	58·803
72	720	24	35·80824	43·000	43·136
73	360	12	41·27586	40·322	39·587
74	1080	24	50·41193	35·765	35·096
75	1440	24	65·1448	30·896	30·096
76	720	12	70·14263	29·215	29·796
77	1800	24	79·84866	27·470	26·639
78	2160	24	94 7·9006	27·731	24·079
79	1080	12	99·4979	23·806	23·400
80	1440	12	129·0727	20·707	20·076
81	1800	12	158·75116	18·304	17·788
82	2160	12	188·5179	16·377	16·097

Couplet's experiments on pipes.

TABLE XIV.—*continued.*

Experiments of COUPLET, *at Versailles.*

Pipe 5 inches in diameter, with various bendings,

$\sqrt{d} = 1{\cdot}118034.$

N° of experiment.	Length of pipe in inches.	Height of reservoir in inches.	Values of s in inches.	Observed velocities in inches.	Calculated from the formula in inches.
83	84·240	25·000	3378·26	5·323	5·287
84	84·240	24·000	3518·98	5·213	5·168
85	84·240	21·083	4005·66	4·806	4·887
86	84·240	16·750	5041·61	4·127	4·225
87	84·240	11·333	7450 42	3·154	3·388
88	84·240	5·583	15119·96	2·010	2·254

Pipe 18 inches in diameter, with various bendings,

$\sqrt{d} = 2{\cdot}12132.$

89	43·200	145·083	304·9734	39·159	40·510

Experiments on canals.

EXPERIMENTS WITH A WOODEN CANAL.

Trapezium canal.

N° of experiment.	Area of the section of the canal in inches.	Perimeter of canal in contact with the water in inches.	Values of $\sqrt{d}$ in inches.	Values of s.	Mean observed velocity.	Calculated mean velocity.
90	18·84	13·06	1·20107	212	27·51	27·19
91	50·60	29·50	1·30960	212	28·92	29·88
92	83·43	26·00	1·79130	412	27·14	28·55
93	27·20	15·31	1·33290	427	18·28	20·39
94	39·36	18·13	1·47342	427	20·30	22·71
95	50·44	20·37	1·57359	427	22·37	24·37
96	56·43	21·50	1·62007	427	23·54	25·14
97	58·74	28·25	1·86955	432	28·29	29·06
98	100·74	28·53	1·87910	432	28·52	29·23
99	119·58	31·06	1·96219	432	30·16	30·60
100	126·20	31·91	1·98868	432	31·58	31·03
101	130·71	32·47	1·00637	432	31·89	32·32
102	135·32	33·03	1·02407	432	32·32	31·61
103	20·83	13·62	1·23667	1728	8·94	8·58
104	34·37	17·00	1·42188	1728	9·71	9·98
105	36·77	17·56	1·44708	1728	11·45	10·17
106	42·01	18·69	1·49924	1728	12·34	10·53

Rectangular canal.

107	34·50	21·25	1·27418	458	20·24	18·66
108	86·25	27·25	1·77908	458	28·29	26·69
109	34·50	21·25	1·27418	929	13·56	11·53
110	35·22	21·33	1·28499	1412	9·10	10·01
111	51·75	23·25	1·49191	1412	12·10	11·76
112	76·19	26·08	1·70921	1412	14·17	13·59
113	105·78	29·17	1·90427	1412	15·55	15·24
114	69·00	25·25	1·65308	9288	4·59	4·46
115	155·25	35·25	2·09868	9288	5·70	5·86

Comparison of theoretical and experimental velocities.

TABLE XIV.—*continued.*

EXPERIMENTS ON RIVERS AND CANALS.

Experiments on the canal of Jard.

N° of experiment.	Area of the section of canal.	Perimeter of canal in contact with the water.	Values of $\sqrt{d}$.	Values of s.	Observed velocities.	Computed velocities.
116	16252	402	6·3583	8919	17·42	18·77
117	11905	366	5·7032	11520	12·17	14·52
118	10475	360	5·3942	15360	15·74	11·61
119	7858	340	4·8074	21827	9·61	8·38
120	7376	337	4·6784	27648	7·79	7·07
121	6125	324	4·3475	27648	7·27	6·55

Experiments on the river Hayne.

N° of experiment.	Area of the section of the river.	Border in contact with the water.	Values of $\sqrt{d}$.	Values of s.	Observed velocity at the surface.	Calculated mean velocity.
122	31498	569	7·43974	6048	35·11	27·62
123	38838	601	8·03879	6413	31·77	28·76
124	30905	568	7·37632	32951	13·61	10·08
125	39639	604	8·10108	35723	15·96	10·53

127. The striking coincidence between most of the preceding calculated and observed velocities, is a strong indication of the accuracy of the equations by which the former are computed; at the same time it must be acknowledged, that it does not possess that simplicity of construction, nor facility of application so desirable in analytical formulæ.

In the first place we may observe, that in the equation

$$V = \frac{\sqrt{2ng}\,(\sqrt{d} - 0{\cdot}1)}{\sqrt{s} - \text{h. log. } \sqrt{(s + 1{\cdot}6)}} - {\cdot}3\,(\sqrt{d} - 0{\cdot}1),$$

s is itself a function of V, and of H′, the height of the head of water.

For it is obvious from the nature of our investigation that $\frac{1}{s}$, the slope of the pipe, depends upon the part of the head of water which remains after that portion has been deducted which is equivalent to the production of the given velocity: this height we have found (art. 122) to be $H = \frac{V^2}{495{\cdot}6}$,

or $$H = \frac{V^2}{505},$$

according to the determination of Du Buat.

Wherefore, what we have denoted by D, article (122)

is $$D = H' - \frac{V^2}{505},$$

H′ denoting the whole head of water.

Hence, since $\frac{D}{l} = \frac{1}{s}$, or $s = \frac{l}{D}$,

we have $$s = \frac{505\,l}{505\,H' - V^2};$$

consequently, when only the dimensions of the pipe and the head of water are given, we must, in the general formula, substitute $\frac{505\,l}{505\,H' - V^2}$ instead of s; whence this formula becomes

$$V = \frac{\sqrt{2ng}\,(\sqrt{d} - 0{\cdot}1)}{\sqrt{\frac{505\,l}{506\,H' - V^2}} - \text{h. log. } \sqrt{\left(\frac{505\,l}{505\,H' - V^2} + 1{\cdot}6\right)}}$$
$$- {\cdot}3\,(\sqrt{d} - 0{\cdot}1);$$

where it is evident that we can only determine V by a very tiresome and laborious approximation.

While, therefore, we cordially join with Dr. Robison and others in our admiration of the skill and ingenuity by which these results were obtained, we must qualify our approbation with respect to the utility of them. Even in the comparison of any given experiment with the computed result, for which the general formula is more peculiarly adapted, it has been thought necessary to compute extensive tables of the value of the expression $\sqrt{s}$ — h. log. $\sqrt{(s + 1{\cdot}6)}$ for different values of s; but in the natural case, viz. where the dimensions of the pipe and the head of water only are given to determine the discharge, these tables are of little or no use, and the problem itself becomes one of great difficulty.

There are, however, some cases in which we may avail ourselves of the original formula, without much danger of error, viz. where the pipe is of very considerable length, and the velocity of discharge in consequence very small: in these cases the subtraction of the head of water appertaining to the actual discharge will almost disappear, in comparison with the whole head, and we may therefore divide the whole length of the pipe by the entire difference of level between the surface of the reservoir and the point of discharge, for our value of s, as in the following examples:

PRACTICAL EXAMPLES.

128. Exam. 1. Let it be required to determine the velocity and quantity of discharge through a pipe 14930 feet in length, its diameter being $4\frac{1}{2}$ inches, and the slope of the pipe to the point of delivery 51 feet.

In this example $d = \frac{4\frac{1}{2}}{4} = 1\cdot125$, and $\frac{1}{s} = \frac{51}{14930}$, or $s = 292\cdot745$;
$\sqrt{s} = 17\cdot11$, $s + 1\cdot6 = 294\cdot345$, and $\frac{1}{2}$ h. log. $294\cdot345 = 2\cdot643063$;
whence the denominator of our formula

$$V = \frac{307\,(\sqrt{d} - 0\cdot1)}{\sqrt{s} - \text{h. log.} \sqrt{(s + 1.6)}} - 0\cdot3\,(\sqrt{d} - 0\cdot1),$$

becomes $17\cdot11 - 2\cdot643063 = 14\cdot466937$.

Again, $\sqrt{d} = \sqrt{1\cdot125} = 1\cdot06066$, and $\sqrt{d} - 0\cdot1 = \cdot96066$;
consequently our numerator is

$$307 \times \cdot96066 = 294\cdot92262,$$

and $\cdot3 \times (\sqrt{d} - 0\cdot1) = \cdot288196$;
therefore

$$\frac{307\,(\sqrt{d} - 0\cdot1)}{\sqrt{s} - \text{h.log.} \sqrt{(s + 16)}} = \frac{294\cdot92262}{14\cdot466937} = 20\cdot673;$$

Wherefore from 20·67300
Subtracting $3\,(\sqrt{d} - 0\cdot1) =$ ·28819

There remains V = 20·38481

The area of the orifice is $(4\frac{1}{2})^2 \times \cdot7854 = 15\cdot9043$;

therefore $\frac{20\cdot3848 \times 15\cdot9043 \times 60}{1728} = 11\cdot257$ cubic feet, the discharge per minute.

The numbers given in the above example are those actually appertaining to one of the pipes which supply the city of Edinburgh with water, and the actual supply of this pipe at its maximum is 11·333 cubic feet per minute; the error, therefore, between the actual and computed discharge is very inconsiderable.

It is however to be observed, that in consequence of our having determined the value of s, by dividing the whole length by the whole height, instead of dividing the former by the latter, reduced for the height due to the velocity of discharge, s is a little less than it ought to be, which defect entering into the denominator, gives the velocity rather greater than it ought to be. This correction being made, therefore, our approximation would not be quite so close as we have found it.

Exam. 2. As another case, suppose the length of a pipe to be 21350 feet, its diameter 7 inches, and the head of water 222 feet, to find its delivery per minute.

Here $d = \frac{7}{4}$, and $\sqrt{d} - 0\cdot1 = 1\cdot2228$, and

$$s = \frac{21350}{222} = 96\cdot17, \text{ and } s + 1\cdot6 = 97\cdot77;$$

$\sqrt{s} = 9\cdot8066$, and $\frac{1}{2}$ h. log. $97\cdot77 = 2\cdot29130$;
whence $\sqrt{s} -$ h. log. $\sqrt{(s + 1\cdot6)} = 7\cdot5153$;
also $307\,(\sqrt{d} - 0\cdot1) = 307 \times 1\cdot2228 = 375\cdot3996$.

$$\text{Hence } \frac{307\,(\sqrt{d} - 0\cdot1)}{\sqrt{s} - \text{h. log.} \sqrt{(s + 1\cdot6)}} = \frac{375\cdot3996}{7\cdot5153} = 49\cdot964.$$

From 49·964
Take $\cdot3\,(\sqrt{d} - 0\cdot1) =$ ·367

There remains V = 49·597 the velocity;

Hydraulics.
Examples.

whence $\frac{49\cdot597 \times 7^2 \times \cdot7854 \times 60}{1728} = 64\frac{7}{12}$, the number of cubic feet which ought to be discharged per minute.

It appears that this computed discharge so far exceeds the actual discharge of the pipe of which the above are the dimensions, that we must look to some permanent defect in the pipe; the reduction of the head for the actual velocity would decrease the computed quantity, but by no means sufficiently to make it approximate to the observed effect, which is very little more than one half of that given by the computation.

Exam. 3. Mr. Watt found, from very careful measurements of a canal, that it was 7 feet wide at the bottom, and 18 feet at the surface, and 4 feet deep; and that it had a declivity of 4 inches in a mile. Its velocity at the surface was 17 inches per second, 14 inches in the middle, and 10 at the bottom; the mean velocity being 13·3. Let it be required to compare this with the formula.

From these dimensions we find the perimeter of the section in contact with the water = 20·6 feet; and the area of the section = 50 square feet;

whence $\frac{50}{20\cdot6} = 2\cdot4272$ feet $= 29\cdot126$ inches;
therefore $d = 29\cdot126$, and $\sqrt{d} - 0\cdot1 = 5\cdot297$.

Again, $\frac{5280 \times 12}{4} = 15840$ inches $= s$.

Substituting these values in our formula, viz.

$$V = \frac{307\,(\sqrt{d} - 0\cdot1)}{\sqrt{s} - \text{h.log.} \sqrt{(s + 1\cdot6)}} - \cdot3\,(\sqrt{d} - 0\cdot1),$$

it becomes

$$V = \frac{307 \times 5\cdot297}{125\cdot8 - 9\cdot671} - \cdot3\,(5\cdot297) = 12\cdot41;$$

which does not differ essentially from that determined from observation.

Formula of different authors.

129. We shall here conclude by observing, that Dr. Robison proposes to render the preceding formula of Du Buat a little more simple by writing $2\frac{1}{4}$ com. log. $\sqrt{(s + 1\cdot6)}$ instead of h. log. $\sqrt{(s + 1\cdot6)}$; we thus have

$$V = \frac{307\,(\sqrt{d} - 0\cdot1)}{\sqrt{s} - \text{log.} \sqrt{(s + 1\cdot6)}} - \cdot3\,(\sqrt{d} - 0\cdot1).$$

Dr. Young also proposes the formula

$$V = 153\,(\sqrt{d} - 0\cdot2) \sqrt{\left(\frac{h}{l + 45\,d}\right)} + 1\cdot6\left(\frac{h}{l + 45\,d}\right)^{\frac{7}{8}} - \cdot001,$$

where l is the length of the pipe, h the height of the whole head of water, and d the diameter of the pipe.

In this formula, our s is equal to $\frac{l + 45\,d}{h}$, and it may be applied to rivers, by taking $\frac{1}{s}$ to denote the sine of their inclination.

M. Langsdorf, in his "Theorie der Hydraulik," published in 1794, recommends the substitution of the formula $V = \sqrt{478\,h}$, instead of that of Du Buat, which reduced to Engligh measure, gives $h = \frac{V^2}{509}$; and when the pipe is bent, the effect of the bending, according to this author, may be found by adding into one sum, the squares of the sines, then denoting this sum by Σ, $s = \dfrac{l}{h - \left(\dfrac{V^2}{509} + \dfrac{V\,\Sigma^2}{3000}\right)}$;

from which we deduce

$$V = \surd\left(\frac{509\,d\,h}{d + \frac{1}{43}\,l + 0{\cdot}16\,d\,\Sigma}\right)$$

in English inches.

§ X. *On the quantity of water discharged over weirs, and through openings in reservoirs, when those openings reach to the surface.*

Discharges over weirs.

130. Since we assume that the velocity with which water discharges itself from any head, is always as the square root of the height of each particle from the surface, it necessarily follows, that theoretically, the quantity discharged through a rectangular notch, which reaches to the surface, is two-thirds of what would issue through an equal orifice placed at the whole depth below the level of the fluid; for the velocity being every where as the square root of the height, we may represent the several velocities by the ordinates of a parabola, which have the same relation with their abscisses, and consequently the sum of all the velocities will have the same relation with the sum, supposing them all equal to the greatest, as the area of a parabola has to its circumscribing rectangle.

This, however, is a mere theoretical result, and is like most others of the same kind relating to the practice of Hydraulics, of very little use without the aid of experiments; we propose, therefore, to add a few of the best experiments on this subject.

The following experiments of Messrs. Brindley and Smeaton were made on a great scale; that is, the flow took place from very extensive surfaces, and are therefore of so much the greater value. Column 1 contains the width of the notches, the second their depth, the third the quantity of water discharged, and the fourth the time.

Experiments by Brindley and Smeaton.

TABLE XV. *Of Brindley's and Smeaton's experiments on the discharge of water through notches reaching to the surface.*

Width of the notch in inches.	Depth in inches.	Cubic feet discharged.	Time of discharge.
6	1	20	7min. 16sec.
6	$1\frac{3}{8}$	20	4 55
6	$1\frac{5}{16}$	20	2 19
6	$3\frac{1}{8}$	20	1 33
6	$6\frac{1}{2}$	20	0 30
6	5	20	0 46
6	$1\frac{1}{4}$	20	5 26
6	$1\frac{5}{8}$	20	3 55
6	$5\frac{5}{8}$	20	0 42

Fig. 11.

In a weir or jetty the water is commonly projected over a board called the waste-board, as AB (fig. 11); we must here, therefore, measure the depth of the orifice by the depth of the upper edge of the plank AB, below the level surface of the water in the river or reservoir. Du Buat made four accurate experiments on this subject, the result of which is given in the following table, reduced to English inches.

TABLE XVI. *Containing the experiments of Du Buat on the quantity of water discharged over weirs; reduced to English inches.*

Common breadth of the orifice = $18\frac{1}{3}$ *inches.*

Depth of the orifice in English feet.	Observed discharge in cubic feet.	Calculated discharge from the formula $C = 11{\cdot}4172\,l\,H^{\frac{3}{2}}$
1·778	506	524
3·199	1222	1218
4·665	2153	2155
6·753	3750	3771

Where C denotes the content or quantity of water discharged, l the length of the waste-board, and H its depth.

131. Dr. Robison examined a numerous set of experiments, in which the discharge considerably exceeded the quantity given by Du Buat's formula, which led him to that above given. To the same author we also owe the following table.

Depth of the upper edge of the waste-board below the surface, in English inches.	Computed number of cubic feet discharged per minute, for every inch of the waste-board.
1	0·403
2	1·140
3	2·095
4	3·225
5	4·507
6	5·925
7	7·466
8	9·122
9	10·884
10	12·748
11	14·707
12	16·758
13	18·895
14	21·117
15	23·419
16	25·800
17	28·258
18	30·786

132. The above column is computed on a supposition that the water from which the discharge is made is in a state of stagnation, and if, therefore, it reach the opening with any velocity, we must multiply the opening by the velocity of the stream, and the above discharge will be increased by a quantity equal to that product.

In order to calculate the discharge of rectangular orifices reaching to the surface, Eytelwein represents the velocity, which varies as the square root of the height, by the ordinates of a parabola, as we have stated above; then taking the case of a lake, the orifice made into it being 3 feet wide, and 2 feet deep, and the contraction being introduced, reducing the area to 5·1, he finds

$$\tfrac{2}{3}\sqrt{H} \times 5\cdot1 = \tfrac{2}{3}\sqrt{2} \times 5\cdot1 = 4\cdot8 \text{ feet}$$

the mean velocity; and consequently

$$4\cdot8 \times 6 = 28\cdot8 \text{ cubic feet,}$$

the discharge due to one second; and generally denoting the co-efficient for the contraction by C, we shall have

$$Q = \tfrac{2}{3} C \times H \times W \times \sqrt{H}$$

for the cubic feet discharged per second from any opening whose height is H and width W.

And the same formula becomes, when Q and W are given,

$$H^{\frac{3}{2}} = \frac{3\,Q}{2\,C \times W} \text{ or } H = \left(\frac{3\,Q}{2\,C\,W}\right)^{\frac{2}{3}},$$

and $$W = \frac{3\,Q}{2\,C \times H^{\frac{3}{2}}}.$$

133. The following table contains the co-efficients of this author for different cases connected with this inquiry. The whole velocity due to the height is 8·04, and the diminished velocity arising from contraction is shown as below.

Eytelwein's co-efficients.

Table XVII. *Containing Eytelwein's co-efficients for different orifices.*

N°	Nature of the orifices employed.	Ratio between the theoretical and real discharge.	Co-efficients for finding the velocities in English feet.
1	For the whole velocity due to the height	1 to 1·00	8·04
2	For wide openings whose bottoms are on a level with that of the reservoir	1 to 0·961	7·70
3	For sluices with walls in a line with the orifice	1 to 0·961	7·70
4	For bridges with pointed piers	1 to 0·961	7·70
5	For narrow openings whose bottoms are on a level with that of the reservoir	1 to 0·861	6·90
6	For smaller openings in a sluice with side walls	1 to 0·861	6·90
7	For abrupt projections and square piers of bridges	1 to 0·861	6·90
8	For openings in sluices without side walls	1 to 0·635	5·1

§ XI. *On the pressure which water exercises against the sides of tubes in which it is in motion.*

134. We have already, in (art. 23) Hydrostatics, offered some remarks upon the pressure which a stagnant fluid exercises against the sides and bottom of the vessels in which it is contained, and on the thickness of metal, or other material, necessary to resist it; in the present section we propose to consider the pressure of water in motion, particularly as it relates to cylindrical tubes.

The thickness of the pipes through which water is conveyed from one point to another is generally determined in practice by experiment, or rather by imitating, as nearly as circumstances will allow, some other work of a similar kind. In fact, as the pressure of the fluid is generally very far from the actual forces that such pipes have to support, and there being necessarily many other considerations to be attended to besides the actual fluid pressure, such tubes commonly far exceed in thickness what would be required to resist the former force; still, however, it may be interesting in some cases, to know the amount of this force, and some economy may perhaps, in certain instances, be practiced by a just valuation of it; we propose, therefore, to bestow a few remarks on this subject in the present section; in which, however, we shall confine ourselves to the most usual case, viz. of cylindrical pipes.

Let us conceive a horizontal cylindrical tube adapted to a given reservoir, and that this reservoir is kept constantly at the same height; also that the water moves freely in the tube, without experiencing any resistance. Then it is evident that if we except the pressure which takes place from the weight of the horizontal column, the tube will experience no effort, for the velocity of the water having a free horizontal direction, there can result from it no force which is exercised against the sides of the tube; but in order to put this conclusion to the test of experiment, M. Bossut proceeded according to the following principles.

Experiments.

In a great reservoir the author caused to be placed a horizontal tube, 3 feet in length, and $\frac{1}{4}$ of an inch in diameter; and towards its middle was made a small lateral hole, intended to form a jet d'eau, the pipe being so adjusted that the hole might be turned upwards, downwards, or sideways, by merely turning the tube on its axis.

The water was now retained in the reservoir to the height of 4 feet above the tube, and the extreme end of the latter shut, when immediately the jet d'eau was formed; and when the hole was upwards, the jet rose very nearly to the level of the surface of the water in the reservoir. This effect being well observed, with the hole turned in different directions, the end of the tube was opened, the water began to flow, and the jet ceased to act almost entirely; it was only observed, that when the hole was below, the water escaped from it by drops. The cessation of the jet under the latter circumstance clearly demonstrated the cessation of the pressure of the water against the sides of the tube.

135. This being established, let us consider a tube fixed in the same manner as we have supposed above, but instead of being entirely open so as to give a free motion to the fluid, conceive a part only of the extreme end to be opened, and that the water can therefore only escape through this small aperture. Then, as in this case, the force which causes the water to pass from the reservoir into the tube is constantly the same, but the water has not the means of escaping with the velocity which this force requires, a part of that force

will be exercised against the sides of the tube, and cause a pressure which it is our object to determine.

Fig. 12.

Investigation.

136. Let us conceive such a tube and reservoir to be represented by fig. 12, and suppose the column of water EN, to be divided into an infinite number of vertical lamina GF, *fg*, equal to each other. Then, as we neglect the friction, it is evident that all the points of the same lamina will have the same velocity; and moreover, that this velocity is common to all the laminæ of which we suppose the column to be composed. It is also evident, that if *q r* represent the section of the contracted vein, in escaping from the orifice *p n*, the velocity of the column of water in the tube, will be to the velocity of the contracted vein, as the area of the latter to the area of the section of the tube, or of the column in it.

Calling, therefore, *h* the constant height BH of the reservoir, D the diameter of the tube, and *d* that of the contracted vein; the velocity of the latter is as the square root of *h*, and may be denoted by $m\sqrt{h}$; consequently the velocity in the tube will be

$$\frac{m\,d^2\sqrt{h}}{D^2}.$$

This being established, in the same manner as the velocity $m\sqrt{h}$ is produced by the pressure of a column *h*, the velocity $\frac{m\,d^2\sqrt{h}}{D^2}$, will be that due to the pressure of a column whose height is $\frac{d^4\,h}{D^4}$.

Since, then, every lamina which is in contact with the end of the tube PN, has a tendency to move with the velocity $m\sqrt{h}$, while it actually only moves with the velocity $\frac{m\,d^2\sqrt{h}}{D^2}$, it must evidently exercise upon P*p* or N*n* which supports it, a force equal to the difference between the pressures which produce the velocities $m\sqrt{h}$ and $\frac{m\,d^2\sqrt{h}}{D^2}$.

Now this pressure, from the nature of fluids, is distributed equally in every direction throughout the entire mass or column EN, and consequently against every point in the sides of the tube. The pressure, therefore, exercised upon every point of the tube, will be

$$h - \frac{d^4\,h}{D^4}, \text{ or } \frac{h\,(D^4 - d^4)}{D^4}.$$

This result obviously agrees with our former deduction; for supposing the water to escape in a full column, then we shall have $d = D$, and the expression for the pressure vanishes, or becomes zero, as we found in the preceding article.

137. Supposing that we have made in the sides of the tube a little opening, we may readily find the quantity of water which it ought to yield in a given time. For the expences through the same opening, and in the same interval, are proportional to the square roots of the heights of the reservoir, or which is the same thing, to the square roots of the pressures. Whence, calling Q the discharge, which is made during a certain time, with the proposed opening under the pressure *h*, *q* that which is made during the same time under the pressure $h - \frac{d^4\,h}{D^4}$, we shall have

$$Q : q :: \sqrt{h} : \sqrt{\left\{h - \frac{d^4\,h}{D^4}\right\}};$$

whence we deduce

$$q = Q \times \frac{\sqrt{(D^4 - d^4)}}{D^2},$$

where, since Q is determinable from the principles previously explained, the value of *q* also becomes known.

Incl tubes

This theory will equally obtain for inclined tubes, provided that in the last case the opening *p n*, through which the water escapes, be very small with regard to PN. If this condition have not place, the velocity with which the fluid issues through *p n*, will not be that due to the whole height of the water in the reservoir; and we must therefore, in such cases, determine *h* on different principles.

138. By means of this theory, we may determine at least approximatively, the thickness which we ought to give to conduit pipes furnished with different adjutages, in order that they may effectually resist the pressure of the water passing through them. The pressure exerted against each point of the circumference of the section of our tube EN, being expressed by $h - \frac{d^4\,h}{D^4}$, it is evident, that in order to sustain that pressure, the tube ought to have the same thickness as if the water were at rest under the altitude $h - \frac{d^4\,h}{D^4}$; and we have seen (art. 23) that this problem is always resolvable. After all, however, as we have before observed, the pressure of the water is only one of many circumstances that require to be considered in determining the thickness of water pipes; our investigations founded on the above principles will therefore only point out to us a limit beyond which we must not diminish them; but in most cases common prudence will dictate a greater thickness of metal than would result from such computations.

Thic of le leade pipe

139. The following table contains the thickness of leaden and iron pipes, according to the general practice of French engineers.

Leaden Pipes.		Iron Pipes.	
Diameter in inches.	Thickness in lines.	Diameter in inches.	Thickness in lines.
1	2½	1	1
1½	3	2	3
2	4	4	4
3	5	6	5
4½	6	8	6
6	7	10	7
7	8	12	8

In this table the results are given merely with reference to the diameters of the tubes, and are independent of their lengths, or the height of the head of water they may have to support; it is obvious, therefore, that they can only be considered as results sanctioned by practice, which are independent of any particular computation founded on the fluid pressure.

140. In order to submit the formula found above,

viz. $q = Q \times \frac{\sqrt{(D^4 - d^4)}}{D^2}$ to experiment, M. Bossut having stopped the end of one of his horizontal tubes, made a small hole in the side of the same tube, about 3½ lines in diameter, and measured the discharge due to a certain time; this quantity being denoted by Q, he computed, by means of the above formula, the quantity q which ought to be discharged from the same lateral hole when the end of the tube was opened: he also measured the actual discharge in the latter case, and the following table shows the results of these practical and computed discharges.

Height of the reservoir in feet.	Length of the tube in feet.	Computed discharge for one minute, in cubic inches.	Observed discharge in cubic inches.
1	30	176	171
1	60	186	186
1	90	190	190
1	120	191	191
1	150	192	193
1	180	193	194
2	30	244	240
2	60	259	256
2	90	264	261
2	120	267	264
2	150	268	265
2	180	269	266

The agreement here between the computed and practical discharges is as accurate as can be expected in researches of this kind, where experiments are subject to so many inequalities; the greatest error is that in experiment (1), which amounts to $\frac{5}{176}$ or $\frac{1}{35}$th part of the computed discharge.

141. By means of the same formula, viz.

$$q = Q \times \frac{\sqrt{(D^4 - d^4)}}{D^2},$$

we may determine the discharge due to any tube, by means of the discharges made through a small orifice, of given dimensions in its side, for the values of q and Q being determined, we have

$$Q : q :: 1 : \sqrt{\left(1 - \frac{d^4}{D^4}\right)},$$

$$Q^2 : q^2 :: 1 : \left(1 - \frac{d^4}{D^4}\right),$$

$$\frac{d^4}{D^4} = \frac{Q^2 - q^2}{Q^2}, \text{ or } \frac{d^2}{D^2} = \frac{\sqrt{(Q^2 - q^2)}}{Q}.$$

Now the ratio of $1 : \frac{d^2}{D^2}$ denotes that of the velocity of the fluid in the pipe, abstracting from friction, to that of its actual velocity; we have, therefore, only to compute what ought to be the discharge in the former case, and then that quantity, multiplied by the fraction $\frac{d^2}{D^2}$, will give the actual discharge.

ple.

142. In order to illustrate what we have stated above by example, let us suppose a pipe 2 inches in diameter to be placed horizontally, the head of water 3 feet, and that a circular opening in its side of 6 lines diameter yielded in one minute 1000 cubic inches of water; to determine the actual discharge of the pipe itself in the same time.

Here, if we compute the discharge due to an *orifice* of ½ an inch in diameter, and with a head of 3 feet, by means of table III. (art. 75), we shall find it 1178 = Q, and we have given $q = 1000$; whence

$$\frac{d^2}{D^2} = \frac{\sqrt{(Q^2 - q^2)}}{Q} = \cdot 5289.$$

Again, the discharge due to an *additional tube* of 2 inches diameter, and with a head of water of 3 feet, is, by table VII. = 24504 cubic inches per minute: wherefore 24504 × ·5289 = 12952, the actual discharge per minute in cubic inches.

§ XII. *On the theory and construction of jets d'eau.*

Jets d'eau.

143. The construction of jets d'eau and fountains, it must be acknowledged, is rather a subject of curiosity than of utility; still, however, as it is an inquiry intimately connected with the general science of Hydraulics, we think it ought not to be passed over unnoticed in this treatise.

When water is projected from an orifice or tube, in any direction, either vertical or oblique, it follows the same laws as any other projected body, and even approaches nearer to the circumstances required by the parabolic theory, as given (art. 44), DYNAMICS, than a solid body; consequently, the ranges it will take under different velocities and directions, the height to which it will ascend, &c. may be computed on the principles explained in the article above referred to. We know that, theoretically, the velocity with which a fluid issues from an orifice, ought to be equal to that which a heavy body would acquire in falling through the altitude of the fluid above the point of projection; but we have also seen, in the preceding sections, that there are many circumstances necessary to be considered, which, individually and collectively, serve to modify the above theoretical deduction. It is also equally evident, that the resistance of the air will have an effect in retarding the motion of the issuing stream, and, consequently, that it would be in vain to attempt a theoretical computation, unsupported by experimental data; and various experiments have accordingly been made relative to this inquiry, by Desaguliers, Mariotte, Bossut, &c. We shall, however, confine our remarks principally to those of the latter.

Bossut's experiments. Figs. 13 and 14.

144. The reservoirs employed by Bossut are represented in fig. 13 and fig. 14. To each of these was adapted, horizontally, a tin tube OE, shut or stopped at the extremity E, but opened at the other end into the principal vessel ABCD. The tubes were each 6 feet long; that represented at fig. 13 being 3⅔ inches in diameter, and that of fig. 14 only ¼ of an inch.

In the points F in both figures was fixed an adjutage 2 lines in diameter; in G another, 4 lines in diameter; and in H a third, of 8 lines. Moreover, in fig. 13, in the point K, there was fixed a conical tube KM, of which the height was 5 inches 10 lines; the diameter at bottom being 9 lines, and at top 4 lines. In I, the tube IN was cylindrical, its height being 5 inches 10 lines, and its diameter 4 lines.

The following are the results of these experiments.

The water was retained constantly in the reservoir exactly 11 feet above the upper side OF of the tube OE.

1. The orifice F being opened, the jet ascended in a fine column to the height of 10 feet 10 lines; but when slightly inclined, it rose to 10 feet 4 inches 6 lines.

2. The jet through the orifice G, same figure, rose vertically 10 feet 5 inches 10 lines; and when slightly inclined, it ascended to 10 feet 7 inches 6 lines.

3. The vertical jet through H rose 10 feet 6 inches 6 lines, and when a little inclined, it ascended to 10 feet 8 inches.

4. The vertical jet through the conical tube KM, ascended only 9 feet 6 inches 4 lines; but when slightly inclined, it reached 9 feet 8 inches 6 lines; this height being still measured from the upper side of the tube.

5. The vertical jet through the cylindrical tube IN, rose only 7 feet 1 inch 6 lines; and when inclined, it reached no more than 7 feet 3 inches 6 lines.

The three following experiments were made on the apparatus represented in fig. 14.

The height of water in the reservoir the same as before.

6. The vertical jet through F rose to 9 feet 11 inches.

7. The vertical jet through G rose to 9 feet 7 inches.

8. The vertical jet through H rose to 7 feet 10 inches.

In two other experiments M. Bossut opened an orifice, ½ an inch in diameter, in the vertical side of his reservoir, and found, in the first place, when the head of water above it was 9 feet, that the range on a horizontal plane, 4 feet 3 inches and 7 lines below the orifice, was 12 feet 3 inches 3 lines.

And when the head of water was reduced to 4 feet the range on the same plane was 8 feet 2 inches 8 lines.

We shall be able to form a comparison between these several results with greater facility, by reducing them into the form of a table, as follows:

TABLE XVIII. *Showing the heights to which jets rise through adjutages of different dimensions, the altitude of the reservoir being* 11 *feet, reckoning from the upper surface of the horizontal tube.*

Diameter of the tube.	Nature of the orifice.	Reference to figs. 13 and 14.	Diameter of the orifice of the adjutage.	Altitude of the jet when vertical.	The same, when inclined.	Remarks.
Inches. lin.			Inches. lin.	Ft. in. lin.	Ft. in. lin.	
3 8	Simple orifice.	F, fig. 13.	.... 0 2	10 0 10	10 4 6	Vertical very fine.
3 8	Ditto.	G, ditto.	.... 0 4	10 5 10	10 7 6	Vertical jet fine, not much enlarged at top.
3 8	Ditto.	H, ditto.	.... 0 8	10 6 6	10 8 0	The vertical jet much enlarged at top; the inclined one less so, and more beautiful.
3 8	Conical tube.	M, ditto.	Length 5 10 Lower diam. 0 9 Upper, do. 0 4	9 6 4	9 8 6	Vertical jet fine.
3 8	Cylindric tube.	N, ditto.	Length 5 10 Diameter 0 4	7 1 6	7 3 6	The vertical jet fine.
0 9½	Simple orifice.	F, fig. 14.	.... 0 2	9 11 0	......	Very fine jet.
0 9½	Ditto.	G, ditto.	.... 0 4	9 7 10	......	The jet much enlarged at top.
0 9½	Ditto.	H, ditto.	.... 0 8	7 10 0	......	Column much broken.

Deductions.

145. By a comparison of the results of these experiments with each other, and with those of Mariotte, M. Bossut finds that *the differences between the height of the reservoir and that of the jet, in different experiments, are to each other very nearly as the square roots of the heights of the jets.* Whence the height of the reservoir and the height of the jet being given in one experiment, we may find the height of reservoir requisite to produce any proposed height of jet in another. Thus, for example, the height of the reservoir in any case being denoted by a and the height of the jet by b, to find the height d of another reservoir, which shall give a jet whose height is c: we shall have

$$\sqrt{b} : \sqrt{c} : a - b : \frac{(a-b)\sqrt{c}}{\sqrt{b}},$$

the difference between the jet and the reservoir; therefore

$$d = c + \frac{(a-b)\sqrt{c}}{\sqrt{b}} = \frac{c\sqrt{b} + a\sqrt{c} - b\sqrt{c}}{\sqrt{b}}$$

will be the height required.

Again, let a, b, and d be given to determine c.

To distinguish c as an unknown quantity, let us denote it by x, and we shall have

$$\sqrt{b} : \sqrt{x} :: a - b : d - x,$$

or $$b : x :: (a-b)^2 : (d-x)^2;$$

this gives

$$b(d-x)^2 = (a-b)^2 x;$$

or $$b(d-x)^2 + (a-b)^2(d-x) = d(a-b)^2;$$

whence

$$(d-x)^2 + \frac{(a-b)^2}{b}(d-x) = \frac{d(a-b)^2}{b},$$

or

$$d - x = \frac{-(a-b)^2}{2b} \pm \sqrt{\left\{\frac{(a-b)^4}{4} \right.} \quad d$$

or

$$x = d + \frac{(a-b)^2}{2b} \mp \sqrt{\left\{\frac{(a-b)^4}{4b^2} + \frac{d(a-b)^2}{b}\right\}}.$$

146. With respect to the height to which a jet will ascend, it appears, by a comparison of the preceding tabulated results with each other, and with those of Mariotte, that when there is a proper supply, the larger jet will rise the highest, the height of the head of water being the same; and it is obvious, that when the orifices are the same, the height of the jet will depend upon the height of the reservoir. There are, therefore, certain relations between the height of the reservoir, the diameter of the tube, and that of the adjutage, in order that the effect, under given circumstances, may be the greatest possible.

M. Bossut assumes, from the results of various experiments, that if such relations have been determined in any one case, and the velocity of the water in the pipe ascertained, any other jet will have the greatest effect; that is to say, will rise to the greatest height, when the velocity of the water in its pipe is equal to that previously found; that is, the velocity of the water in the pipe is always a constant quantity in all tubes when the effect is the greatest possible.

If, therefore, D be taken to denote the diameter of any tube, and d that of its adjutage, when under the head of water h, the effect is the greatest; and v be taken to denote the velocity in the pipe, we shall have

$$v = \frac{d^2}{D^2} 2\sqrt{hg}.$$

For the velocity at the adjutage $= 2\sqrt{hg}$, and the velocity in the tube is to that at the orifice as the area of section of the latter to that of the former, therefore

$$D^2 : d^2 :: 2\sqrt{gh} : v = \frac{d^2}{D^2} 2\sqrt{gh}.$$

In the same manner, if D′, d', h' and v', be assumed to denote the corresponding quantities in any other tube, we shall have

$$v' = \frac{d'^2}{D'^2} 2\sqrt{h'g};$$

whence, according to the assumed principle, if we suppose $v = v'$, we have

$$\frac{d^2}{D^2}\sqrt{h} = \frac{d'^2}{D'^2}\sqrt{h'};$$

or $$D^2 : D'^2 :: d^2\sqrt{h} : d'^2\sqrt{h'}.$$

That is, the squares of the diameters of the tubes ought to be to each other in the ratio compounded of that of the squares of the diameters of the adjutage and the square roots of the heights of the water in the reservoirs.

Consequently, knowing the dimensions that will give the greatest height of the jet in any one case, the same may be determined in any other proposed case by a simple proportion.

In giving this rule, however, the reader will of course understand that it is only meant to be implied that the diameter of the tube must not be less than that given for the [illegible]rtion, it may obviously be greater

on of effect; but the enlargement

ht of the jet.

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ng deductions.

Hydraulics.

The height of water in the reservoir was 3 feet 2 inches 11 lines, the diameter of the tube 1 inch, and the diameters of the adjutages with the corresponding height of the jet were as below, viz.

Maximum effect of jets d'eau.

Diameters of the adjutages.	Height of the jet.		
	Feet.	Inches.	Lines.
1	3	1	6
2	3	1	8
3	3	2	0
4	3	1	7
5	3	1	5
6	3	0	4
7	2	10	6

It is obvious here, that the greatest height is when the diameter of the adjutage is 3 lines; but Bossut supposes that a greater height might have been obtained by increasing the adjutage to 3¼ lines, and hence computes a sort of standard, due to a reservoir of 16 feet, and an adjutage whose diameter is 6 lines; and by this means finds for the required diameter

$$D' = \frac{Dd'^4}{d}\sqrt{\frac{h'}{h}} = 28\tfrac{1}{2} \text{ lines}.$$

By assuming this result, which agrees very nearly with the determination of Mariotte, he computes the following table, which is intended to facilitate the practical determination of the proper dimensions in the construction of fountains and similar works:

Heights of the jets in feet.	Heights of the reservoir in feet.		Discharge per minute in cubic inches.	Diameters of the tubes in lines.
Feet.	Feet.	Inches.		
5	5	1	1536	21
10	10	4	2160	26
15	15	9	2688	28
20	21	4	3120	31
25	27	1	3504	33
30	33	0	3888	34
35	39	1	4224	36
40	45	4	4560	37
45	51	9	4748	38
50	58	4	5184	39
55	65	1	5472	40
60	72	0	5760	41
65	79	1	6100	42
70	86	4	6188	43
75	93	9	6528	44
80	101	4	6816	45
85	109	1	7156	46
90	117	0	7294	47
95	125	1	7584	48
100	133	4	7824	49

§ XIII. *On the percussion of water, with reference to the motion of water wheels.*

Definitions.

148. Water wheels are distinguished into four kinds, viz. *undershot wheels*, *overshot wheels*, *breast wheels*,

and *horizontal wheels*, and each of these several species are again variously constructed for the purpose of avoiding certain imperfections, or for the sake of adjusting them to the particular circumstances under which they have to act. It does not, however, belong to the present article to enter at length into a description and an examination of the best practical construction; this will fall in its proper place in our general alphabetical arrangement; our only business in this place is to take a general view of these machines as first movers, and to determine, theoretically and experimentally, the laws of their motion.

Overshot wheel.
Fig. 15.

149. An *overshot wheel* of the common kind, is represented in (fig. 15), where ABCD is the rim of the wheel, having a number of buckets, *a*, *b*, *c*, *d*, arranged round its circumference. When the wheel is in a state of rest upon its axis O, and the water is introduced into the bucket *c*, from the horizontal mill course or canal EF, the weight of the water in the bucket acting at the end of a lever equal to *m* O, puts the wheel in motion in the direction *c d*. When the subsequent bucket *b* comes into the position *c*, it is also filled with water, and so on with the rest. When the bucket *c* reaches the situation of *d*, its mechanical effect to turn the wheel is increased, being now equal to the weight of water, acting at the end of the lever *n* O, equal to the distance of its centre of gravity *d*, from a vertical line passing through the axis O; so that the mechanical effect of the water in the bucket increases all the way to B, and of course diminishes while the buckets are moving from B to C. The buckets, however, between B and C have not the same power upon the wheel as those between A and B; for the water begins to fall out of the buckets before they arrive at B, which are almost completely empty when they reach the point H. The constructions of the buckets, therefore, as shown in the figure, is very improper, as it not only allows the water to escape before it has reached the point B, where its mechanical effect ought to be a maximum; but also to escape completely, long before the buckets have reached the lowest point C of the wheel. The power, therefore, of an overshot wheel must depend in a great measure on the form given to the buckets, which should be so constructed that they may retain the greatest possible quantity of water when they arrive at the point of greatest mechanical effect B; they ought, moreover, to retain their water the longest possible, after passing this point. These considerations have given rise to a variety of constructions by different artists, which we intend to describe at length when we have to consider the general article WATER WHEELS; but at present, a simple definition and an explanation of the principles of operation is all that is requisite, and what is stated above is sufficient for that purpose.

Undershot wheel.
Fig. 16.

150. An *undershot wheel*, is a wheel with a number of float boards, or plane surfaces arranged around its circumference, for the purpose of receiving the impulse of the water, which is conveyed to the under part of the wheel from an inclined canal. A wheel of this kind, of the common construction, is shown in (fig. 16), where AB is the wheel with 24 float boards. Here *c d* is a float board receiving the impulse of the water which moves with great velocity, in consequence of having fallen from a considerable height down the inclined mill course MN. The construction in this case is much more simple than in the overshot wheel, and there is therefore less room for the display of ingenuity in the engineer; but even in this we shall see, in the article referred to, that different constructions have been proposed in order to accommodate the wheel to some particular circumstances under which it may be proposed to act.

Hor
whe
Fig

151. *Horizontal water wheels.* These differ in no respect from the common undershot wheels, except in the circumstance of their being placed horizontally instead of vertically. The mill course is constructed nearly in the same manner for both. The principal object of this form of the water wheel, is to save machinery, by placing the mill stone directly on the vertical shaft of the wheel. The water wheel must therefore move with a very great velocity, in order to enable the mill stones to perform their work. The water is turned into a horizontal direction before it strikes the float boards, which may be either vertical or inclined to the radius, as in undershot wheels.

Horizontal water wheels are frequently constructed in such a manner that the float boards have a great inclination to the radius. In this case the water is not turned into a horizontal direction, but is made to strike the float boards perpendicularly, as in (fig. 17), where AB is the wheel, MN the mill course, discharging its contents perpendicularly upon the float board C, which ought to have a surface more than twice the area of the section of the stream.

Bre
whe
Fig.

152. A *breast wheel*, is a wheel in which the water is delivered at an intermediate point between the upper and under part of a wheel with float boards. It is generally delivered at a point below the level of the axis, as in (fig. 18); but sometimes at a point higher than the axis. The wheel represented in the figure is one of Mr. Smeaton's construction; AB is a portion of the wheel, MN the canal which conveys the water to the wheel, MOP the curvilinear mill course accurately fitted to the extremities of the float boards, and *c d* the shuttle moved by the pinion *a*, for the purpose of regulating the admission of water upon the wheel.

Theory of water wheels.

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153. The varying force which communicates motion to water wheels, and the resistances occasioned by friction, tenacity, and various other causes, render the application of the theory to practice in these cases extremely difficult. For this reason, perhaps, it happens that the art of constructing machines to be moved by the force of water has been almost wholly practical; the best improvements having generally been deduced from constant observation and reiterated experiments.

Since the theory of Mechanics is confessedly inadequate to the complete investigation of every circumstance in the motion of hydraulic machines, it has been supposed that the same laws of motion would not extend to all branches of Mechanics, but that different principles were to be accommodated to the motion and mutual action of different kinds of bodies. If this were truly the case, the science of Mechanics would fall far short of that superior excellence and extent it is generally allowed to possess. But it is highly probable that there is no kind of motion, whether of the most simple or complicated nature, but what may be referred to the principles established in our treatise of

Mechanics; and if we are not enabled to investigate the effects from the data in every case which can arise, the deficiency must not be imputed to a defect in the first principles of Mechanics, but to a want of method of applying our mathematical knowledge to the particular circumstances under discussion; to which we may also add a want of the whole of the requisite data, and our inability to reduce to computation the various causes of resistance and obstruction to the moving forces. For these reasons we shall not dwell long upon the theory of the motion of water wheels, but shall merely exhibit a few propositions; the consideration of which may enable the student to appreciate more exactly the value and importance of the experimental researches we shall subsequently detail.

Proposition I.

154. If a stream of water impinge on the float-boards of an undershot wheel, and escape from it the very instant after it has made its impact, the quantity of water which actually impinges against the wheel will be to the whole quantity which passes by it in a given time, as the difference between the velocities of the water and wheel to the absolute velocity of the water.

Let WH (fig. 19) represent the wheel, BA the stream of water, and let the float-board F first receive the impact from the water at F, and quit it at C; also let DF be to FC as the absolute velocity of the water to the velocity of the float-board.

Then when F arrives at C, the particle at D will have passed at F, and, taking DE = FC, all the water in the space DE will pass by the wheel without impinging against it; for it cannot impinge on the float at F, because the float emerges from the water at C; neither can it impinge on the subsequent float, because it has already passed it. Therefore the whole quantity of water which passes by the wheel in a given time, is to that which actually impinges against it as DF to EF; that is, as the velocity of the water to the difference between the velocities of the water and float, as stated in the proposition. Hence again it follows, that the force of the impinging water is as the square of the difference of the velocities of the water and wheel. For the force is as the relative velocity into the quantity of impinging matter, and the latter is manifestly as the relative velocity; therefore the force will be as the square of the relative velocity.

Problem I.

155. Let W be a weight, fastened to a line which is wound round the horizontal axis of an undershot wheel; A the altitude of the column of water equal to the force of impact of the water on the wheel when the wheel is quiescent; V the velocity with which the water impinges on the float-boards; v the velocity of the circumference of the wheel; R the radius of the wheel, and r the radius of the axle, to determine the velocity v, all the other quantities being given.

The relative velocity with which the water strikes the wheel is $V - v$; whence, because the force of the stroke is as the square of the relative velocity, we have

$$V^2 : (V - v)^2 :: A : F = \frac{A(V - v)^2}{V^2},$$

the force of the water to turn the wheel, when its velocity is v; whence

$$F \times V^2 = A(V - v)^2,$$

or

$$v = V\left(1 - \sqrt{\frac{F}{A}}\right).$$

But the acceleration of the wheel ceases when the force of the water to turn it is equal to the force or the weight which opposes it; that is, when

$$F . R = W r, \text{ or when } F = \frac{W . r}{R}.$$

Hence, substituting this value for F, in the preceding value of v, there results

$$v = V\left(1 - \sqrt{\frac{W . r}{A . R}}\right),$$

for the velocity of the wheel when its acceleration ceases.

Maximum effects.

156. When the weight W is supposed variable, and it is required to determine it, so that its momentum may be a maximum, we must find

$$v W = WV\left(1 - \sqrt{\frac{W r}{AR}}\right) = \text{a maximum};$$

or

$$W - W^{\frac{3}{2}}\sqrt{\frac{r}{AR}} = \text{a maximum};$$

which gives

$$W = \frac{4\,AR}{9\,r}.$$

And in a similar manner we shall find that the greatest momentum generated in the ascending weight will be $= \frac{4}{27} AV$; also that

1. When the momentum of the ascending weight W is a maximum, that weight will be $\frac{4}{9}$ of the weight which would, if suspended from the axle, balance the force of the stream.
2. And when the momentum of the ascending weight is a maximum, the velocity will be $\frac{1}{3}$ of the velocity of the stream.
3. When the uniform velocity of the ascending weight W is a maximum, the radius of the wheel will be $= \frac{9\,W r}{4\,A}$.

For the radius of the wheel being R, and that of the axle r, the uniform velocity of the ascending weight will be found by multiplying together

$$V\left(1 - \sqrt{\frac{W r}{AR}}\right) \times \frac{r}{R};$$

that is, the velocity of W will be equal to

$$\frac{r}{R} V\left(1 - \sqrt{\frac{W r}{AR}}\right);$$

which, supposing R variable, and making the fluxion $= o$ gives

$$R = \frac{9\,W r}{4\,A}.$$

Proposition II.

Effect as depending upon the quantity of water.

157. If the velocity of the stream be given, the effect will be as the quantity of water expended.

For it appears from the deductions of the last problem, the greatest effect is as $\frac{4}{27} AV$; that is, since $\frac{4}{27}$ is constant, the effect is in a ratio compounded of the force of impact and the velocity of the stream; but the force of water is as the quantity of water expended and velocity conjointly; consequently the effect is as the quantity of water expended and the square of the velocity; or, if the velocity be given, as the quantity of water expended.

Whence we may draw the following conclusions:

1. When the expence of water is the same, the effect will be as the square of the velocity.

2. The expence of water being the same, the effect will be, theoretically, as the height of the head of water.

3. The aperture being the same, the effect will be as the cube of the velocity. For the effect is as Q (the quantity of water) into the square of the velocity; but the quantity is obviously as the velocity, therefore the effect is, *cæteris paribus*, as the cube of the velocity.

Remark.

158. If all the water which passes by an undershot wheel actually impinged against it, as supposed by some authors, the force of the stream would be simply in the direct ratio of the relative velocity. For the number of particles which, according to this hypothesis, strike the wheel in a given time would be given, whatever might be the velocity of the wheel. The hypothesis itself is, however, absolutely erroneous, and it would, therefore, be useless to draw from it its depending conclusions. It is true, that even on the preceding supposition, the conclusions drawn from theory do not agree with those experimentally determined; but the approximation is nevertheless much nearer than if we had adopted the latter principle. We have found that the maximum effect is produced when the wheel moves with one-third of the velocity of the stream; but in actual practice, the velocity of the wheel, when the machine is in the greatest perfection, will be between one-third and one-half of the velocity of the stream, as appears both by the experiments of Smeaton and Bossut. The discrepance between these results, it is clear, arises from this, that the water does not all escape the instant after it has made the impact, but is confined in the channel for some time; so that the succeeding water, which would otherwise pass by the wheel, inefficaciously drives the confined water against the float-boards, and therefore acts in the same manner as if it actually impinged against the wheel. Experiments show, that when the most work is done in a given time, the velocity of the wheel is nearer the half than a third of the velocity of the stream in a confined canal; but it is nearer one-third than a half in an open river, where the water has more facilities to escape.

More accurate investigation.

The discrepances between the results of experiments and the conclusions drawn from the preceding theory, doubtlessly arise, as before suggested, from our not being able to introduce into our computations the several circumstances attending the different soliciting and impeding causes of the motion, one of which, and the most important of them, we have already mentioned above. It may be observed also, that other conditions ought to enter into our investigation, according to the strict principles of Dynamics, which have been entirely omitted, in consequence of our not being able to estimate them correctly. Suppose, for instance, the moving force generally to consist of the impact and the weight of such portion of the water as may tend to move the wheel by its gravity; this entire moving force, whether determined by theory or experiment, may be denoted by A. After the moving force which determines the motion of the wheel has been found, the resistance opposed to this force should also be computed; for on the relation between the moving force and the resistance will depend the acceleration of the machine. The resistance will arise, first, from the inertia, which may be thus estimated; let the distance of the centre of gyration of the wheel from that of the motion be ρ, the weight of the wheel P, and the other quantities according to the notation above adopted: then will the inertia of the wheel which resists the communication of motion to the circumference be expressed by $\frac{P\rho^2}{R^2}$ (§ XIII. DYNAMICS); and in nearly a similar manner the inertia of any parts of the system may be obtained, knowing the weight, figure, and position, by the proper rules for determining the centre of gyration of different bodies.

In the second place, if the machine be one that raises water, or other weights, the weight raised, allowing for its mechanical effect on the point, the acceleration of which is sought, must be subducted from the moving force before found, and this will be a constant quantity.

Thirdly, there are likewise other kinds of resistances homogeneal to weight, as for instance friction, tenacity, &c. which vary in some ratio of the velocity of the machine; and in order to proceed with the investigation, the exact quantity of weight to which the friction is equal, when the wheel moves with a given velocity, must be known, as well as the variation of the resistances in respect of the velocities, circumstances that must be experimentally determined. If then the force equivalent to the friction, &c. be subtracted from the moving force, the remainder may be considered as effective moving power; and this, divided by the inertia of the mass moved, will give the force which accelerates the circumference.

Thus, if when the velocity of the wheel's circumference is V, the friction be equal to the weight Q applied at the circumference, and varies as the nth power of the velocity, we shall have the force which accelerates the circumference, as follows:

$$\frac{A}{W+P} \times \left(\frac{V-v}{V}\right)^2 \times \frac{R^2}{\rho^2} - \frac{WRr}{W+P\rho^2} - \frac{R^2 Q v^n}{V^n (W+P)};$$

whence, supposing x to be the space which has been described by the circumference when the velocity is v, and $g = 16\frac{1}{12}$, the principles of acceleration will give us this equation,

$$\frac{v\dot{v}}{2g} = \dot{x} \left\{ \frac{A}{W+P} \times \left(\frac{V-v}{V}\right)^2 \times \frac{R^2}{\rho^2} - \frac{WRr}{W+P\rho^2} - \frac{R^2 Q v^n}{V^n (W+P)\rho^2} \right\};$$

and from this, if v be determined in functions of x, and constant quantities, the velocity communicated to the wheel may be ascertained.

It is only by taking all the circumstances into the account that an exact coincidence can be produced between theory and experiment. The difficulty, however, of introducing these considerations precisely according to their individual effect is so great, that we can look for little assistance from a calculation founded on such principles; it will be much more likely to answer our purpose to determine all these circumstances, or what are equivalent to them, from actual experiments, and by means of the data thence obtained, to proceed with our theoretical researches.

Problem II.

159. To compute the effective weight of water in the buckets of any proposed overshot wheel.

An overshot wheel unloaded, is considered as perfectly in equilibrio upon its axis; but when it is loaded the equilibrium is destroyed; because the weight of the water lies all upon one moiety of the circumference, which causes it to preponderate as we have stated (art. 149).

Let ABC (fig. 20) represent an overshot wheel, of which the buckets a, b, c, B, d, e, are loaded with water, and let there be supposed an equal quantity of fluid W in each bucket. Now the centre of gravity of the bucket a, being supposed directly over the axle C of the wheel, will have no tendency to give it a rotatory motion, but will merely cause it to press more firmly upon its pivots, as it would if a weight a' equal to that of the water in the bucket a, hung directly below the axle. But the second bucket, whose centre of gravity is b, acts in the same manner upon the wheel, as would an equal weight b' hanging freely at the point f; consequently, its effort will be proportional to the product

$$w \times \mathrm{C}f = w \sin a \mathrm{C} b$$

(w denoting the weight of the water); and the same will hold with respect to the bucket e.

The water in the buckets c and d, act in the same manner as would the equal weights c' d' hanging freely at the point g; their joint effort will therefore be proportional to

$$2\,w \times c\,\mathrm{I} = w \times 2 \sin a \mathrm{C} c;$$

and the water in the bucket B, acting at the extremity of the radius, will have its effect proportional to

$$w \sin 90° = w\,\mathrm{R},$$

(R being the radius of the wheel).

1. Hence, first, if there be 12 buckets about the wheel, as in the figure, the arcs $a\,b$, $b\,c$, c B, &c. will each contain 30°; and we shall have

$$(0 + 2 \sin 30° + 2 \sin 60° + 1)\,w =$$
$$(0 + 1 + \sqrt{3} + 1)\,w = 3{\cdot}7320508\,w,$$

for the effective weight of water on the wheel, while its real weight is 6 w; so that the real weight of the water is to its effective weight as

6 : 3·732, &c.; or as 1 : ·6220, &c.

2. If the number of buckets were 24, and all supposed equally full, then pursuing a similar course, we should find

$$(2 \sin 15° + 2 \sin 30° + 2 \sin 45° + 2 \sin 60° + \text{\&c.}\ 2 \sin 90°)\,w = 7{\cdot}585754\,w$$

for the effective weight of water in the buckets: now the real weight in this case is 12 w; whence the ratio of the real to the effective weight will be as

12 : 7·585754, or as 1 : ·6322.

3. If we suppose the number of buckets to be so increased that the weight may be considered as equally diffused over the semi-circumference a BN of the wheel; then will the sum of all the efforts to turn the wheel be equal to the sum of all the distances, as c I into the corresponding weights at c; that is by the nature of the centre of gravity as the semicircle a BN × C k, k being the centre of gravity of the semicircular arc.

Now, by (art. 122) Statics, C k = ·63662 CB; consequently the actual weight will be to the effectual weight as the arc a BN to ·63662 × arc a BN, or as 1 : ·63662; this being the ratio to which the others approximate.

Proposition III.

Maximum effect of an overshot wheel.

160. In an overshot wheel the machine will be in its greatest perfection, when the diameter of the wheel is two-thirds of the height of the water above the lowest point of the wheel.

If the wheel ABN revolve with a velocity, such as a body would acquire in falling through the altitude x above the upper part a of the wheel, the water will fall into the buckets without an impulse (because then $\mathrm{V} - v = o$), and produce its effect by its weight only.

Let $h = x + a\,\mathrm{N}$, the height of the supply of water above the lowest point N of the wheel; then, as we have already observed, the sum of all the effective forces of the water, in all the buckets of the semicircle a BN, will be equal to the semicircle a BN × C k, k being the centre of gravity of the semicircular arc; that is, it will be equal to the quantity of water

$$a\,\mathrm{BN} \times \frac{\mathrm{C}\,k}{\mathrm{CB}},$$

acting at the point B.

$$\text{But} \qquad \mathrm{C}\,k = \frac{\mathrm{BC}.\,a\,\mathrm{N}}{\mathrm{arc}.\,a\,\mathrm{BN}} = \frac{\mathrm{BC}}{a\,c\,\mathrm{B}};$$

consequently the force of all the water to turn the wheel, is equal to the quantity of water

$$a\,\mathrm{BN} \times \frac{\mathrm{BC}}{a\,c\,\mathrm{B}},\ \text{acting at B} = \text{quantity of water 2 BC},$$

or a N acting at B.

Hence, if BC = R, the whole force of water will vary as R × a N, or as R × $(h - x)$.

But the velocity of the wheel is $2\sqrt{g\,x}$, and consequently the force of water to produce the rotatory motion of the wheel is $\mathrm{R}\{2\sqrt{g\,x} + (h - x)\}$; this being made a maximum, will require

$$h^2 x - 2\,h\,x^2 + x^3 \text{ a maximum};$$

whence we obtain $x = \frac{1}{3}\,h$;

consequently the height of the supply of water above the upper part of the wheel, should be one-third of the whole height, and the diameter of the wheel two-thirds of the height, that the machine may produce its maximum effect.

The velocity of an overshot wheel, when the water produces its effect by its weight only, and the machine is in its state of greatest perfection, is to the velocity of an undershot wheel as $\sqrt{3}$ to 1; on the supposition that all the water escapes from the undershot wheel the moment after it makes its impact.

For h being the height of the water, the velocity of the overshot wheel will be equal to $\sqrt{\frac{4}{3}\,g\,h}$; and the velocity of the undershot wheel equal to $\sqrt{\frac{4}{9}\,g\,h}$; consequently the former is to the latter as $\sqrt{\frac{1}{3}} : \sqrt{\frac{1}{9}}$ or $\sqrt{3} : 1$.

Proposition IV.

Overshot and undershot compared.

161. The efficacy of an overshot wheel is to that of an undershot wheel, the weight of the water, aperture and diameter being given, as 13 to 5 nearly.

If A denote the altitude of the column of water, the weight of which is equal to the force of impact on the undershot wheel, when quiescent, since the velocity of the wheel is equal to one-third of that of the stream (art. 156-2), we shall have the analogy

$$3^2 : (3-1)^2 :: A : \tfrac{4}{9} A,$$

the weight which is equivalent to the force of the stream, when the machine is in its greatest perfection: whence, if V be the velocity acquired in falling down the altitude A; the momentum of the wheel will be equal to $\tfrac{4}{9} A \times \tfrac{1}{3} V = \tfrac{4}{27} AV$, when in its best state.

In the overshot wheel the weight of the water acting at the circumference is equivalent to $\tfrac{2}{3}$ A, and the velocity $= \surd \tfrac{2}{3} g A$, when the machine is in its best state. But $\surd \tfrac{2}{3} g A = V \surd \tfrac{1}{3}$; consequently the momentum of the overshot wheel $= \tfrac{2}{3} A \times V \surd \tfrac{1}{3} = \tfrac{2}{3} AV \surd \tfrac{1}{3}$, and is to the undershot wheel as $\tfrac{2}{3} \surd \tfrac{1}{3}$ to $\tfrac{4}{27}$, or as $\tfrac{3}{2} \surd 3$ to 1, or as 2·5987 to 1, or as 13 to 5 nearly.

Bossut's experiments on the motion of water wheels.

Description of the apparatus.

162. The experiments of which we here propose to give the detail, were made with different views, viz. to find the effect produced by a greater or less number of float-boards; the maximum effect, as depending upon the ratio of the velocities of the stream and of the wheel; the arc of immersion; the direction of the float-boards, &c.

Fig. 21.

The first series of experiments was made on the wheel AGFH (fig. 21), which had at first 48 float-boards or paddles, and these were successively reduced to 24 and 12. All the paddles were about 5 inches broad, and between 4 and 5 inches long, viz. in the direction of the radii of the wheel. The axis of the wheel was horizontal and cylindrical, for receiving the parallel envelopes of the cord COS, which, passing over the pulley O, thus raised up the weight P.

The exterior diameter of the wheel BK, was 3 feet 1 inch 10 lines; the diameter of the axis, independent of the cord, 2 inches; and the diameter of its axle 2½ lines; the diameter of the pulley 3 inches 8 lines, and of its axle 2⅔ lines; the diameter of the cord 2 lines.

The wheel was placed about 50 feet from the reservoir, and the velocity of the stream determined in the same manner as in (article 119). The wheel had just room to turn freely in the canal; and the number of revolutions were counted from the time that the velocity of the ascending weight became uniform, which was generally after about four or five revolutions of the wheel.

163. *Results of the first series of experiments.*

EXPERIMENTS 1—12.

(Wheel as in fig. 21.)

N°	Velocity, &c.	Number of float boards.	Weight raised in lbs.	Time of motion in seconds.	Number of revolutions.
1	Velocity of the stream $\frac{300}{33} = \frac{100}{11}$ feet per second.	48	12	60	33¼
2		48	16	60	28½
3		24	12	60	29
4		24	16	60	25½
5		12	12	60	25½
6		12	16	60	19¼
7	Velocity of the stream 10 feet per second.	48	12	48	34
8		48	16	48	31¼
9		24	12	48	30⅓
10		24	16	48	28½
11		12	12	48	25
12		12	16	48	23

It appears from these experiments that the weight raised being the same, the wheel will turn with a greater velocity when the number of paddles are 48, than when they are 24, and greater with 24 than when they are reduced to 12. Whence we may therefore conclude, that in all similar cases it will be advantageous to give at least 48 paddles, provided the circumference of the wheel be such that its strength be not diminished by the insertion of such a number.

At the distance of 50 feet from the reservoir, where the wheel was placed, the water rose above the bottom of the canal about 13 or 14 lines, and the greatest depth to which the float-boards were plunged was about 13 lines, which gives for the arc of immersion about 25°., or rather 24° 54′. In great wheels of about 20 feet diameter, which are turned by a rapid current, the arc immersed seldom exceeds 25 or 30 degrees, and these wheels have seldom more than 40 float-boards; but it appears from the above results, that their number might be advantageously increased.

164. In the second course of experiments the wheel employed is represented in figs. 22 and 23; the floats being secured by an iron pin; between the two iron rims, of which one appears in the vertical section in each figure, and so contrived that they might be set at any angle, as represented in fig. 23. The breadth of the iron rims B*b* = 9 lines, the thickness 1 line; the floats were of plate iron, about ½ a line thick. The diameter BF of the wheel = 3 feet, the breadth of the floats 5 inches, and the height BA = 6 inches; the bare diameter of the cylindrical axis 2½ inches, and that of its spindle 3 lines; the diameter of the pulley 3 inches 8 lines, and that of its spindle or axis 2⅔ lines; the diameter of the cord 2 lines; and the weight of the whole machine was 44 pounds. The breadth of the stream or canal was 12 or 13 feet, and its depth 7 or 8 inches; and the wheel was erected on a sort of bridge, that no impediment might be made to the motion and action of the water.

165. *Results of the second series of experiments.*

EXPERIMENTS 13—16.

(With the wheel, fig. 22.)

N°	Velocity, &c.	Number of float boards.	Weight raised in lbs.	Time of motion in seconds.	Number of revolutions.
13	Velocity 68½ feet per second	48	24	60	27$\frac{17}{18}$
14		24	24	60	27$\frac{7}{48}$
15	The floats immersed 4 inches.	24	40	40	15$\frac{28}{48}$
16		12	40	40	13$\frac{14}{48}$

Here it will be observed that the wheel raised the same weight, with a velocity sensibly greater, with 24 floats than with 12, but its motion was very little accelerated by using 48, beyond what it was with 24. The arc plunged in the water was 77°, 53′. With the above depth of immersion, therefore, we ought to give at least 24 floats; but with a greater depth of immersion they may be reduced in number. In practice it is usual to give, M. Bossut says, only 8 or 10, and sometimes less, to the wheels of mills placed on rivers: which number is, he asserts, too little: they would act better with 12 or 18.

We have determined, theoretically, the velocity which a wheel ought to have in comparison with that of the stream, in order to produce the greatest possible effect; this determination is put to the test of experiment in the two following series.

In the first table the narrow canal was employed; viz. that made use of in the first experiments of this author; and in the second, the broad canal employed in the last series.

The wheel in both cases was that shown in fig. 22; but in the first there were 48 floats, and in the second only 24.

166. *Results of the third series of experiments.*

EXPERIMENTS 17—28.

(With the wheel, fig. 22.)

N°	Velocity, &c.	Weight raised in lbs.	Duration of motion in seconds.	Number of revolutions.
17	Velocity of the stream, $\frac{300}{27}$ feet per second. Number of floats, 48.	$30\frac{1}{2}$	40	$22\frac{12}{48}$
18		31	40	$22\frac{4}{48}$
19		$31\frac{1}{2}$	40	$21\frac{42}{48}$
20		32	40	$21\frac{32}{48}$
21		$32\frac{1}{2}$	40	$21\frac{20}{28}$
22		33	40	$21\frac{8}{48}$
23		$33\frac{1}{2}$	40	$20\frac{14}{48}$
24		34	40	$20\frac{32}{48}$
25		$34\frac{1}{2}$	40	$20\frac{21}{48}$
26		35	40	$19\frac{44}{48}$
27		$35\frac{1}{2}$	40	$19\frac{15}{48}$
28		36	40	$18\frac{28}{48}$

Since the time employed in these experiments is the same for all, the greatest effect will be estimated by the greatest product of the weight into the number of revolutions; this condition gives the maximum effect when $20\frac{21}{48}$ revolutions are made in 40 seconds with a load of $34\frac{1}{2}$ lbs., as in experiment 25.

In order to compare this with the velocity of the stream, we shall find that, since the water runs 300 in 27 seconds, it will pass over 5334 inches in 40 seconds. Also the wheel having 3 feet or 36 inches diameter, its circumference is

$$36 \times \cdot 7854 = 113 \cdot 0976 \text{ inches};$$

whence $113 \cdot 0976 \times 20\frac{21}{48} = 2311$ inches passed over by any point in the circumference of the wheel; thus we shall have experimentally the greatest effect when the velocity of the stream is to that of the wheel as 5334 to 2311.

But if we take the centre of impression of the water on the floats, the diameter will be reduced to 34 inches, and the ratio of the two velocities will be as 5334 to 2183 nearly; which differs not very essentially from that of 5 : 2.

But it may be asked, does this ratio remain the same if we have regard to the resistances which are omitted in the preceding valuation. This question reduces itself to this: We have two consecutive and similar quantities, $M \cdot v$ and $N \cdot v'$, which express each the product of a load; by its velocity we suppose that $M \cdot v$ is a maximum, and consequently $M v > N v'$. Now, in order to introduce the resistances, the weights M and N must be conceived to be each augmented by a certain quantity. Suppose, then, that M becomes $M + m$, and that N becomes $N + n$, it is required to determine whether the same velocity which renders $M v$ a maximum will give also $(M + m) v$ a maximum, or whether $(M + m) v > (N + n) v'$. It is evident that this will not be generally the case, as it will depend upon the ratio of m and n to each other. But, supposing (what is in fact very nearly the case) that the forces $m v$ and $n v'$ of the resistances are as the forces $M v$ and $N v'$, we shall have

$$m v : n v' :: M v : N v';$$

whence we draw

$$M v + m v : N v' + n v' :: M v : N v';$$

consequently,

$$M v + m v > N v' + n v'.$$

This result, as we have stated above, is certainly not generally or geometrically true; but, at the same time, it is of that kind that little error will arise in practice by adopting it.

Hence, then, we may conclude, that when a wheel has 48 float-boards, and is turning in a course of little depth, the velocity of its circumference ought to be about $\frac{2}{5}$ths of that of the stream, in order to produce its maximum effect.

167. *Results of the fourth series of experiments.*

Experiments continued.

EXPERIMENTS 29—45.

(Wheel as in fig. 22.)

N°.	Velocity, &c.	Weight raised in lbs.	Duration of motion in seconds.	Number of revolutions.
29	Velocity of the stream, $68\frac{1}{2}$ feet per second. Number of floats, 24. Depth of immersion 4 inches.	30	40	$17\frac{22}{48}$
30		35	40	$16\frac{25}{48}$
31		40	40	$15\frac{28}{48}$
32		45	40	$14\frac{12}{48}$
33		50	40	$13\frac{34}{48}$
34		55	40	$12\frac{37}{48}$
35		56	40	$12\frac{28}{48}$
36		57	40	$12\frac{10}{48}$
37		58	40	$12\frac{10}{48}$
38		59	40	$12\frac{1}{48}$
39		60	40	$11\frac{40}{48}$
40		61	40	$11\frac{30}{48}$
41		62	40	$11\frac{12}{48}$
42		63	40	$11\frac{7}{48}$
43		64	40	$10\frac{41}{48}$
44		65	40	$10\frac{25}{48}$
45		66	40	$10\frac{5}{48}$

Here, by multiplying each respective weight by the corresponding number of revolutions, we shall find that the greatest product will be that in experiment 39, viz. $60 \times 11\frac{40}{48} = 710$; and comparing the velocity of the wheel in this experiment with that of the stream, we shall find the ratio to be again very nearly that of 2 to 5, as before.

168. The next object M. Bossut undertakes to examine, relates to the inclination of the floats when they take any position, such as that shown in fig. 23,

instead of being directed towards the centre as in the two preceding cases.

We have already explained that by means of the mechanism in part shown in the figure, the angle of their direction might be changed at pleasure.

The experiments reported in the first of the three following tables were made on the narrow canal, and the others on that employed in the experiments immediately preceding.

The word *direct*, indicates that the floats were directed to the centres, as in figs. 21, 22, and the abbreviations; inclin. 8^d, inclin. 12^d, &c. signify that the floats made with the radius such or such an angle.

Experiments on oblique floats.

169. *Results of the fifth series of experiments.*

EXPERIMENTS 46—51.

(Wheel as in fig. 23.)

N°	Velocity, &c.	Number of float boards, 48.	Weight raised in lbs.	Duration of motion.	Number of luti
46	The velocity of the stream, $\frac{300}{27}$ feet per second.	Direct.	34	40	$20\frac{26}{48}$
47		Inclin. 8^d	34	40	$19\frac{20}{48}$
48		Do. 8^d	38	40	$17\frac{6}{48}$
49		Do. 12^d	34	40	$19\frac{10}{48}$
50		Do. 12^d	38	40	$17\frac{22}{48}$
51		Do. 16^d	34	40	$20\frac{24}{48}$

It appears from the above results, that when the floats are directed towards the centre in the case of a straight and narrow canal, the action of the wheel is more advantageous than when they are inclined at an angle of 8°; that the last are less advantages than when the inclination is 12°; and that the last inclination is less favourable than when it is increased to 16 degrees; in fact there is very little difference in the action between this last inclination and that in which the floats are direct.

When the floats are direct, the percussion acts with the greatest effect; and when they are oblique, the water, besides striking the floats, rises up of them, and remains hanging upon them as it were for a certain time, thus counterbalancing by its weight in some measure what was lost by the oblique percussion; and this compensation will be so much the more complete according to the angle of inclination, and according to the depth and velocity of the stream. It is difficult, however, to say what the inclination ought to be, as depending upon the latter data.

170. *Results of the sixth series of experiments.*

EXPERIMENTS 52—58.

(Wheel as in fig. 23.)

N°	Velocity, &c.	Number of float boards, 48.	Weight raised in lbs.	Duration of motion.	Number of revolutions.
52	Velocity of the stream, $68\frac{1}{2}$ feet per second. Depth of immersion, 3 inches.	Direct.	20	60	$17\frac{5}{48}$
53		Do.	32	60	$10\frac{24}{48}$
54		Inclin. 10°	20	60	$16\frac{12}{48}$
55		Do. 10°	32	60	$11\frac{3}{48}$
56		Do. 20°	20	60	17
57		Do. 20°	32	60	$11\frac{40}{48}$
58		Do. 30°	20	60	$11\frac{7}{48}$

It ought to be remarked here, that several irregularities in the stream prevented the same degree of accuracy in these experiments as in the following.

EXPERIMENTS 59—62.

(Wheel as in fig. 23.)

N°	Velocity, &c.	Number of float boards, 48.	Weight raised in lbs.	Duration of motion.	Number of revolutions.
59	Velocity, $68\frac{1}{2}$ ft. per second. Depth of immersion, 4 inches.	Direct.	40	40	$13\frac{17}{48}$
60		Inclin. 15°	40	40	$14\frac{21}{48}$
61		Do. 30°	40	40	$14\frac{20}{48}$
62		Do. 37°	40	40	$14\frac{15}{48}$

It appears, that in circumstances analogous to those of the last table, the most advantageous inclination lies between 15° and 30°. There is here, as in all similar cases, a degree of obliquity which we must not surpass, because by this means we lose more by the percussion than is gained by the weight of the suspended water.

We conclude our account of M. Bossut's experiments on this subject by the following series on the motion of overshot water wheels.

Expe ments overs wheel

The wheel employed in the following experiments is shown by AEDB, fig. 24; the canal which brought the water to the wheel was horizontal; it was 5 inches in breadth, the height of the water was constantly the same, and the supply 1194 cubic inches per minute.

The diameter of the wheel was 3 feet; that of its axis or cylinder 2 inches 7 lines, and of the axis on which it turned $2\frac{1}{2}$ lines. The pulley O the same as in the preceding experiments; the number of buckets was 48, their depth 3 inches, and breadth 5 inches.

It may be proper to observe here, as in the preceding detail, that the time was counted from the period when the motion of the wheel became uniform, which was commonly after 5 or 6 revolutions.

171. *Result of the seventh series of experiments.*

EXPERIMENTS 1—8.

(Wheel as in fig. 24.)

Weight raised in lbs.	Time of motion in seconds.	Number of revolutions.
11	60	$11\frac{44}{48}$
12	60	$11\frac{14}{48}$
13	60	$10\frac{24}{48}$
14	60	$9\frac{40}{48}$
15	60	$9\frac{10}{48}$
16	60	$8\frac{11}{48}$
17	60	$8\frac{3}{48}$
18	60	$7\frac{18}{48}$

In putting 19 lbs. for the weight, the wheel turned very slowly; when it was 20 lbs. the wheel stopped, although a slight motion was commenced by the hand, in order to make it take the water.

When there was no load, the wheel made $40\frac{1}{4}$ turns in a minute.

If we multiply each load raised by the corresponding number of turns of the wheel, that which had 17 lbs. for its load was the greatest. In which case the velocity of the wheel is very nearly that required by the theory.

Since when the effect is the greatest, the wheel makes $8\frac{5}{16}$ revolutions in a minute, while it would make in the same time $40\frac{1}{4}$ without any weight, it follows that the velocity requisite for the greatest effect, is to the velocity which the wheel will take when it has no load to raise, as $8\frac{5}{16}$ to $40\frac{1}{4}$, or as 1 to 5 nearly.

Smeaton's experiments on water wheels.

172. A valuable course of experiments and observations relative to the action of differently constructed water wheels, was undertaken and completed by the celebrated English engineer, Smeaton, of which an ample detail was published in the 51st volume of the Philosophical Transactions, many years after the course had been completed, and after the greater part of the deductions obtained from them had been verified by actual constructions on a large scale. They are thus rendered doubly valuable, because it frequently happens that conclusions drawn from experiments carried on, as such experiments as these must necessarily be, on a small scale, are not found to obtain when they are submitted to actual practice in the constructions of mills and other machinery. In the series we are about to describe, this source of error is avoided by the best of all possible criterions, the actual success of the various constructions in many pieces of machinery, erected under the direction of this celebrated engineer.

Of undershot wheels.

173. The machine for conducting these experiments is represesented in fig. 25, where ABCD is the lower cistern for receiving the water, after it has quitted the wheel, and for supplying the upper cistern DE; the water being raised by a pump to any required height, that height is shown by a small rod FG, divided into inches and parts; with a float at the bottom to move the rod up and down, as the surface of the water rises and falls. HI is a rod by which the sluice is drawn up, and stopped at any height required by means of a pin or peg K, which fits several holes, placed in the manner of a diagonal scale on the face of the rod HI. GL is the upper part of the pump rod, for drawing the water out of the lower cistern, in order to raise and keep up its surface at the desired height in the cistern DE, to supply the water expended by the aperture of the sluice. MM is the arch and handle for working the pump, which is limited in its stroke by N, a piece for stopping the handle from raising the piston too high; being in like manner prevented from going too low by meeting the bottom of the barrel. O is a cylinder on which a cord winds, and which being conducted over the pulley P and Q, raises R, the scale into which the weights are put for trying the power of the water. S, T, are the two standards which support the wheel, and are made to slide up and down, to adjust the wheel as near as possible to the floor of the conduit; W, the beam which supports the scale and pulleys. This, in the figure, is represented as but little higher than the machine, which, however, is merely for bringing the figure into a narrow compass; in the machine itself it was placed 15 or 16 feet higher than the wheel.

Fig. 26 is a section of the same machine, where the same parts are marked with the same letters as in the preceding figure. Besides which, XX is the pump barrel, being 5 inches diameter, and 11 inches long. Y is the piston, and Z the fixed valve. GV is a cylinder of wood, fixed on the pump rod, and reaches above the surface of the water; this being of such thickness that its section is half the area of the pump barrel, and will cause the surface of the water to rise in the head, as much while the piston is descending as while it is rising, and will therefore keep the guage rod, FG, more uniformly to its height; *a a* shows one of the two wires which serve as directors to the float, that the guage rod, FG, may be kept perpendicular; *b* is the aperture of the sluice; *c* A a kant-board for throwing the water more directly down the opening *c d* into the lower cistern; and *c e* is a sloping board for bringing back the water that is thrown up by the floats of the wheel. Without entering more minutely into the particulars of the construction of this machine, the reader will readily comprehend the nature of its operation, and we shall therefore now proceed to an account of the experiments themselves.

The velocity of the water.

174. Mr. Smeaton's first object was to determine the velocity of the water, and with this view he proceeded as follows: First, let the wheel be put in motion by the water, but without any weights in the scale, and suppose the number of turns in a minute to be 60; now it is evident, were the wheel free from friction and resistance, that 60 times the circumference of the wheel would be the space through which the water would have moved in a minute, with the velocity with which it struck the wheel; but the wheel being encumbered with friction and resistance, and still making 60 revolutions per minute, it is obvious that the velocity of the water must be greater than 60 circumferences before it met the wheel. Let now the cord be wound round the cylinder, but contrary to the usual way, and put a weight in the scale; this counter weight will assist the wheel in turning the same way as it would have been turned by the water; put therefore as much weight into the scale as, without any water, will cause it to turn somewhat faster, as, for example, 63 times per minute; let it now be tried again with the water assisted by the weight, and if the wheel still, from the action of the water, be farther accelerated, it will be obvious that the water has a greater velocity than 63 revolutions per minute; by proceeding thus, the motion of the wheel may at length be found so that the water being let on will not accelerate it, and then we may conclude that the velocity of the water and of the circumference of the wheel are equal to each other. It was in this manner Mr. Smeaton estimated the velocity of the impinging fluid.

Maximum of effect.

175. In finding the greatest product or maximum effect, the author proceeded as follows: Having found by trials what weight gave the greatest product, by multiplying each weight by the number of revolutions in a given time, he found what weight in the scale, when the cord was on the contrary side of the cylinder, caused the wheel to make the same number of revolutions the same way without water; then it is evident that this weight is nearly equal to all the friction and resistance taken together; and, consequently, that

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the weight added to the counter weight will be equal to the weight that could have been raised, if the machine had been devoid of friction and resistance, and which, multiplied by the height to which it was raised, will give the greatest effect of that power.

It would be useless to enter into a minute description of the pump and other parts of the apparatus; it will be sufficient to observe, that their construction was such that the quantity of water discharged for any given time, or for any given number of strokes of the pump, might be very accurately determined.

Specimens of a set of experiments.

Sluice drawn to the first hole.	Inches.
Water above the floor of the sluice	30
Strokes of the pump per minute	39½
The head raised by 12 strokes	21
The wheel raised the empty scale and made it turn 60 times per minute	60
With a counter weight of 1lb. 8 oz. it made	85
Ditto, tried with water	86

N°	Weight.		Revol. per minute.	Product.
	lb.	oz.		
1	4	0	45	180
2	5	0	42	210
3	6	0	36¼	217½
4	7	0	33¾	236¼
5	8	0	30	240
6	9	0	26½	238½
7	10	0	22	220
8	11	0	16½	181½
9	12	0	Ceased working.	

Counter weight for 30 revolutions, without water, 2 oz. in the scale.

The area of the head of water 105·8 inches.

Weight of empty scale and pulley 10 oz.

Circumference of the cylinder 9 inches.

Ditto of the water wheel 75 inches.

Maximum effect computed.

176. Here, the circumference of the wheel being 75, we have $\frac{75 \times 86}{60} = 107{\cdot}5$ inches, or 8·96 feet the velocity per second of the water, which is that due to a head of 15 inches; and this we shall call the virtual head.

The area of the head being 105·8 inches, we shall have from the known specific gravity of water, 6·126 ounces for the weight of one inch deep of the head, and this multiplied by the whole depth 21 inches, gives 80·43 lbs. for the weight of water due to 12 strokes of the pump; therefore, 12 : 39·5 :: 80·43 : 264·7 lbs. the weight of water expended per minute.

Mr. Smeaton now estimates, as this weight of water has descended 15 inches in a minute, that the product of 264·7 × 15 = 3970 will express the power of the water to produce its mechanical effect, which was as follows:

The velocity of the wheel at the maximum, as appears by the above reductions, was 30 revolutions per minute, which, multiplied by 9 inches, the circumference of the cylinder, makes 270 inches; but as the weight was hung by a pulley and double line, the weight was only raised half this; viz. 135 inches.

	lb.	oz.
The weight in the scale at the maximum	8	0
Ditto of scale and pulley	0	10
Counter-weight and pulley	0	12
Sum of resistance	9	6 = 9·375

Now, 9·375 × 135 = 1266 is the maximum effect produced; therefore the ratio of the power to the effect at the maximum, is as 3970 : 1266, or as 10 : 3·8.

But Mr. Smeaton observes, though this is the greatest single effect producible from the power mentioned, by the impulse of the water on an undershot wheel, yet as the whole power of the water is not exhausted by it, this will not be the true ratio between the power and the water, and the sum of all the effects producible by it; for as the water must necessarily leave the wheel, with a velocity equal to that of the circumference of the wheel, it is plain that some part of the power of the water must remain after quitting the wheel. The velocity of the wheel at 30 revolutions per minute, is 3·123 feet per second, which answers to a head of 1·82 inches, which, multiplied by the expence per minute, viz. 264· lb. gives 481 for the power remaining in the water; and this deducted from the original power, becomes 3970 — 481 = 3489, for the power employed in producing the effect 1266; consequently, the power spent, to the effect produced, is as 3489 : 1266, or as 11 to 4 nearly. The velocity of the water striking the wheel has been determined to be equal to 86 circumferences per minute, and the velocity of the wheel at the maximum to be 30; the velocity therefore of the water to that of the wheel, is as 86 to 30, or as 20 to 7. The load at the maximum has been shown to be equal to 9 lb. 6 oz. and that the wheel ceased moving with 12 lb. in the scale; to which, adding the weight of the scale, 10 oz. the proportion will be nearly as 3 to 4 between the load at the maximum and that with which the wheel is stopped.

It is somewhat remarkable, Mr. Smeaton remarks, that through the velocity of the wheel in relation to the water, turns out greater than ⅓d of the velocity of the water, yet the impulse of the water, in the case of a maximum, is more than double what is assigned by theory; that is, instead of $\frac{4}{9}$ths of the column, it is nearly equal to the whole column. It must be remembered, therefore, that in the present case, the wheel was not placed in an open river, where the natural current, after it has communicated its impulse to the float, has room on all sides to escape, as the theory supposes; but in a conduit, or race, to which the float being adapted, the water cannot otherwise escape than by moving along with the wheel.

It is observable, that the wheel moving in this manner, as soon as the water meets the float, it receives a sudden check, and rises up against the latter like a wave against a fixed object; so that when the sheet of water is not even ¼ of an inch thick before it meets the float, yet this sheet will act on the whole surface of a float, whose height is 3 inches, and consequently were the float no higher that the thickness of the sheet of water, as the theory also supposes, a great part of the force would have been lost, by the water's dashing over.

177. The following table contains the results of 27 sets of experiments, conducted and reduced after the manner above explained.

Table of experiments.

TABLE *of Smeaton's experiments on undershot wheels.*

Number.	Height of water in the cistern.	Turns of the wheel unloaded.	Virtual head thence deduced.	Turns at the maximum.	Load at the equilibrium.	Load at the maximum.	Water expended per minute.	Power.	Effect.	Ratio of power to effect.	Ratio of the velocity of the water and wheel.	Ratio of the load at the equilibrium, to the load at the maximum	Experiments.
			inches.	inches.	lbs. oz.	lbs. oz.							
1	33	88	15·38	30	13 10	10 9	275	4357	1411	10 : 3·24	10 : 3·4	10 : 7·75	At the 1st hole.
2	30	86	15·0	30	12 10	9 6	264·7	3970	1266	10 : 3·2	10 : 3·5	10 : 7·4	
3	27	82	13·7	28	11 2	8 6	243	3329	1044	10 : 3·15	10 : 3·4	10 : 7·5	
4	24	78	12·3	27·7	9 10	7 5	235	2890	901·4	10 : 3·12	10 : 3·55	10 : 7·53	
5	21	75	11·4	25·9	8 10	6 5	214	2439	735·1	10 : 3·02	10 : 3·45	10 : 7·32	
6	18	70	9·95	23·5	6 10	5 5	199	1970	561·8	10 : 2·82	10 : 3·36	10 : 8·02	
7	15	65	8·54	23·4	5 2	4 4	178·5	1524	442·5	10 : 2·9	10 : 3·6	10 : 8·3	
8	12	60	7·29	22	3 10	3 5	161	1173	328	10 : 2·8	10 : 3·77	10 : 9·1	
9	9	52	5·47	19	2 12	2 8	134	733	213·7	10 : 2·9	10 : 3·65	10 : 9·1	
10	6	42	3·55	16	1 12	1 10	114	404·7	117	10 : 2·82	10 : 3·8	10 : 9·3	
11	24	84	14·2	30·75	13 10	10 14	342	4890	1505	10 : 3·075	10 : 3·66	10 : 7·9	At the 2d hole.
12	21	81	13·5	29	11 10	9 6	297	4009	1223	10 : 3·01	10 : 3·62	10 : 8·05	
13	18	72	10·5	26	9 10	8 7	285	2993	975	10 : 3·25	10 : 3·6	10 : 8·75	
14	15	69	9·6	25	7 10	6 14	277	2659	774	10 : 2·92	10 : 3·62	10 : 9	
15	12	63	8·0	25	5 10	4 14	234	1872	549	10 : 2·94	10 : 3·97	10 : 8·7	
16	9	56	6·37	23	4 0	3 13	201	1280	390	10 : 3·05	10 : 4·1	10 : 9·5	
17	6	46	4·25	21	2 8	2 4	167·5	712	212	10 : 2·98	10 : 4·55	10 : 9	
18	15	72	10·5	29	11 10	9 6	357	3748	1210	10 : 3·23	10 : 4·02	10 : 8·05	At the 3d hole.
19	12	66	8·75	26·75	8 10	7 6	330	2887	878	10 : 3·05	10 : 4·05	10 : 8·1	
20	9	58	6·8	24·5	5 8	5 0	255	1737	541	10 : 3·01	10 : 4·22	10 : 9·1	
21	6	48	4·7	23·5	3 2	3 0	288	1064	317	10 : 2·99	10 : 4·9	10 : 9·6	
22	12	68	9·3	27	9 2	8 6	359	3338	1006	10 : 3·02	10 : 3·97	10 : 9·17	At the 4th.
23	9	58	6·8	26·25	6 2	5 13	332	2257	686	10 : 3·04	10 : 4·52	10 : 9·5	
24	6	48	4·7	24·5	3 12	3 8	262	1231	385	10 : 3·13	10 : 5·1	10 : 9·3	
25	9	60	7·29	27·3	6 12	6 6	355	2588	783	10 : 3·03	10 : 4·55	10 : 9·45	5th.
26	6	50	5·03	24·6	4 6	4 1	307	1544	450	10 : 2·92	10 : 4·9	10 : 9·3	
27	6	50	5·03	26	4 15	4 9	360	1811	534	10 : 2·95	10 : 5·2	10 : 9·25	6th.
1	2	3	4	5	6	7	8	9	10	11	12	13	14

From the above experiments Mr. Smeaton deduces the following practical maxims and observations:—

1. That the virtual or effective head being the same, the effect will be nearly as the quantity of water expended. This appears by a comparison of the 4th, 8th, and 10th columns of the preceding table.

2. That the expence of water being the same, the effect will be nearly as the height of the virtual or effective head; as appears from a comparison of the same columns as above.

3. That the quantity of water expended being the same, the effect is nearly as the square of its velocity. This follows from a comparison of columns 3, 8, and 10.

4. The aperture being the same, the effect is nearly as the cube of the velocity of the water, as appears from columns 3, 8, and 10.

5. On comparing columns 2 and 4, it is evident that the virtual head bears no certain proportion to the head of water; but that when the aperture is greater, or the velocity of water that issues from it is less, they approach nearer to a coincidence: and, consequently, in the large openings of mills and sluices, where great quantities of water are discharged from moderate heads, the head of water, and the virtual head, determined from the velocity, will nearly agree with each other.

6. On comparing the several proportions between the power and effect in column 11, the most general is that of 10 to 3; the extremes 10 to 3·2, and 10 to 2·8; but as it is observable, that where the quantity of water, or its velocity, is a maximum, that is, where the power is the greatest, the 2d ratio is greatest also; we may therefore properly allow the proportion subsisting in large works to be as 3 to 1.

7. The proportions of velocities between the water and wheel, in column 12, are contained within the limits of 3 to 1, and 2 to 1; but as the greater velocities approach the limit of 3 to 1, and the quantity of approach to that of 2 to 1, the best general proportion will be that of 5 to 2.

After the experiments above referred to were performed, the wheel which had in the first instance 24 floats, was reduced to 12, which caused a diminution in the effect, on account of a greater quantity of water escaping between the floats and the floor; but a circular float being employed, of such a length that one float entered the curve before the preceding one quitted it, the effect came so near to the former, as not to give hopes of increasing it by employing 24 in this particular wheel.

On overshot wheels.

Overshot wheels.

178. In the former part of this article we have considered the impulse of a confined stream acting on undershot wheels. Let us now follow Mr. Smeaton in his examination of the power and application of water when acting by its gravity on overshot wheels.

The alterations in the machinery already described, in order to accommodate it to the present case, was chiefly as follow:

In fig. 26, the sluice I *b* being shut down, the rod HI was unscrewed and taken off. The undershot water wheel was taken off its axis, and instead of it the overshot wheel of the same diameter was put in its place. The wheel was 2 inches in the shroud or depth of the bucket, and the number of buckets was 36. The standards S and T, fig. 25, were raised half an inch, so that the bottom of the wheel might be clear of stagnant water. A trunk for bringing the water on the wheel was fixed according to the dotted lines *f g* (fig. 26). The aperture was adjusted by a shuttle *k i*, which also closed up the upper end of the trunk when the water was to be stopped.

179. *Specimen of a set of experiments.*

Head 6 inches; 14½ strokes of the pump per minute; 12 ditto = 80 lbs.; weight of scale 10½ oz.; counterweight for 20 turns, besides the scale, 3 oz.

N°	Weight in the scale.	Number of revolutions.	Products.	Observations.
1	0 lb.	60	...	Threw most part of the water out of the wheel.
2	1	56	...	
3	2	52	...	
4	3	49	147	Received the water more quickly.
5	4	47	188	
6	5	45	225	
7	6	42½	255	
8	7	41	287	
9	8	38½	308	
10	9	36½	328½	
11	10	35½	355	
12	11	32¾	360¼	
13	12	31¼	375	
14	13	28½	370½	
15	14	27½	385	
16	15	26	390	
17	16	24½	392	
18	17	22¾	386¾	
19	18	21¾	391½	
20	19	20¾	394¼	Maximum.
21	20	19¾	395	
22	21	18½	388	
23	22	18	396	Worked irregularly.
24	23	..	...	Overset by its load.

In these experiments the head being 6 inches, and the height of the wheel 24 inches, the whole descent was 30 inches: the expence of water was 14½ strokes of the pump, which is equivalent to 96⅔ lbs. and 96⅔ × 30 = 2900, the estimated power. Taking the 20th experiment for the maximum, we shall have 20¾ revolutions per minute, which raised the weight 93·37 inches in the same time. The weight of the scale and its load, with the counter-weight, makes 20½ lbs.; consequently, according to Mr. Smeaton's principle,

$$20\tfrac{1}{2} \times 93{\cdot}37 = 1914, \text{ the effect.}$$

The ratio, therefore, of the power and effect will be as 2900 : 1914, or nearly as 3 to 2. But if we compute the power from the height of the wheel only, we shall have 96⅔ lbs. × 24 = 2320 for the power; and this will be to the effect as

2320 : 1914, or as 5 to 4 *nearly*.

180. The following table contains the results of 16 sets of experiments, carried on and reduced after the manner above explained.

TABLE *of Smeaton's experiments on overshot wheels.*

Table of experiments.

Nº	Whole descents.	Water expended per minute.	Revolutions per minute at the maximum.	Weight raised at the maximum.	Power of the whole descent.	Power of the wheel.	Effect.	Ratio of the whole power and effect.	Ratio of the power, wheel, and effect.	Mean ratio.
1	27	30	19	6½ lbs.	810	720	556	10 : 6·9	10 : 7·7	10 : 8·1
2	27	56⅔	16¼	14½	1530	1360	1060	10 : 6·9	10 : 7·8	
3	27	56⅔	20¾	12½	1530	1360	1167	10 : 7·6	10 : 8·4	
4	27	63⅓	20½	13½	1710	1524	1245	10 : 7·3	10 : 8·2	
5	27	76⅔	21½	15½	2070	1840	1500	10 : 7·3	10 : 8·2	
6	28½	73⅓	18¾	17½	2090	1764	1476	10 : 7	10 : 8·4	10 : 8·2
7	28½	96⅔	20¼	20½	2755	2320	1868	10 : 6·8	10 : 8·1	
8	30	90	20	19½	2700	2160	1755	10 : 6·5	10 : 8·1	10 : 8·2
9	30	96⅔	20¾	20½	2900	2320	1914	10 : 6·6	10 : 8·2	
10	30	113⅓	21	23½	3400	2720	2221	10 : 6·5	10 : 8·2	
11	33	56⅔	20¼	13½	1870	1360	1230	10 : 6·6	10 : 9·0	10 : 8·5
12	33	106⅔	22¼	21½	3520	2560	2153	10 : 6·1	10 : 8·4	
13	33	146⅔	23	27½	4840	3520	2846	10 : 5·9	10 : 8·1	
14	35	65	19¾	16½	2275	1560	1466	10 : 6·5	10 : 9·4	10 : 8·5
15	35	120	21½	25½	4200	2880	2467	10 : 5·9	10 : 8·6	
16	35	163½	25	26½	5728	3924	2981	10 : 5·2	10 : 7·6	
1	2	3	4	5	6	7	8	9	10	11

181. *Of the ratio between the power and effect.*

1. The effective power of the water must be reckoned on the whole descent, because it must be raised that height, in order to be in a condition of producing the same effect a second time.

The ratios between the powers so estimated, and the effects at the maximum deduced from the several sets of experiments, are exhibited in one view in col. 9, and it appears that those ratios vary, from that of 10 to 7·6 to that of 10 to 52, that is nearly from 4 : 3 to 4 : 2. In those experiments where the heads of water and quantities expended are least, the proportion is nearly as 4 to 3; but where the heads and quantities are greatest, it approaches nearer to that of 4 to 2; and by a medium of the whole, the ratio is nearly that of 3 to 2.

2. We have seen before, in the observations on the effects of undershot wheels, that the general ratio of the power to the effect, when greatest, was 3 to 1; the effect, therefore, of overshot wheels, under the same circumstances of quantity and fall, is at a medium double to that of an undershot wheel.

3. The powers of water computed from the height of the wheel only, compared with the effects, as in column 10, appear to observe a more constant ratio; for if we take the medium of each class which is set down in column 11, we shall find the extremes to vary only between the ratio 10 : 8·1 to 10 : 8·5, and as the second term of the ratio gradually increases, from 8·1 to 8·5 by an increase of head from 3 inches to 11 inches, the excess of 8·5 above 8·1 is to be imputed to the superior impulse of the head of water at 11 inches above that of 3 inches; so that if we reduce 8·1 to 8 on account of the impulse of the 3 inch head, we shall have the ratio of the power computed on the height of the wheel only, to the effect at a maximum, as 10 : 8, or as 5 : 4, nearly. And from the equality of the ratio between the power and effect, subsisting where the constructions are similar, we must infer, that the effects, as well as the powers, are as the quantities of water and perpendicular heights multiplied respectively together.

Of the height of the wheel as depending on the descent of the water.

On the height of the wheel.

182. We have already seen, from the preceding observations, that the effect of the same quantity of water, descending through the same perpendicular space, is double when acting by its gravity on an overshot wheel, to what the same produces when acting by its impulse on an undershot wheel. It also appears that by increasing the head from 3 inches to 11 inches, that is, the whole descent from 27 inches to 35, or in the ratio of 7 to 9 nearly, the effect is advanced only from 8·1 to 8·4, that is, 7 : 7·26, and, consequently, the increase of effect is not ¼th of the increase of perpendicular height. Hence it follows that the higher the wheel is in proportion to the whole descent, the greater will be the effect, because it depends less upon the impulse of the head, and more on the gravity of the water in the buckets; and if we consider the obliquity with which the water issuing from the head strikes the buckets, we shall not be at a loss to account for the little advantage that arises from its impulse, and shall immediately see how little consequence this impulse is to the effect of an overshot wheel. However, says Mr. Smeaton, as every thing has its limits, so has this, for thus much at least is desirable, "that the water should have a somewhat greater velocity than the circumference of the wheel in coming on it;" for otherwise the wheel will not only be retarded by the buckets

striking the water, but thence dashing a part of it over, so much of the power will be lost. The velocity which the circumference of the wheel ought to have, being known by the following deductions, the head requisite to give the water its proper velocity is easily computed from the proper rules of Hydrodynamics, and will be found much less than what is generally practised.

On the velocity of the wheel to produce the greatest effect.

Velocity of the wheel.

183. If a body be let fall freely from the surface of the head to the bottom of the descent, it will employ a certain time in falling, and in this case the whole action of gravity is spent in giving the body a certain velocity; but if this body in falling be made to act on some other body, so as to produce a mechanical effect, the falling body will be retarded, because a part of the action of gravity is then spent in producing the effect, and the remainder only in giving motion to the falling body; and therefore the slower the body descends, the greater will be the portion of the action of gravity applicable to the producing of the mechanical effect, and in consequence the greater that effect will be.

If a stream of water fall into the bucket of an overshot wheel, it is there retained till the wheel by moving round discharges it, and therefore the slower the wheel moves the more water each bucket will receive, so that what is lost in speed is gained by the pressure of a greater quantity of water acting in the bucket at one time; and if considered only in this light, the mechanical power of an overshot wheel to produce effects will be equal, whether it move quick or slow; but if we attend to what has been just observed with regard to the falling body, it will appear that so much of the action of gravity, as is employed in giving the wheel and water in it a great velocity, must be subtracted from its pressure on the buckets; so that through the product made by multiplying the number of cubic inches of water acting in the wheel at one time by its velocity, will be the same in all cases; yet as each cubic inch, when the velocity is greater, does not press so much on the bucket as when it is less, the power of the water to produce effects will be greater in the less velocity; and hence we are led to this general rule. *That, cæteris paribus, the less the velocity of the wheel the greater will be its effects.*

A confirmation of this principle, together with the limits it is subject to in practice, may be deduced from the foregoing set of experiments.

2. Again, from the same experiments it appears, that when the wheel made about 20 turns in a minute, the effect was nearly at the greatest; when it made 30 turns, the effect was diminished about $\frac{1}{20}$th part; but when it made 40, it was diminished about $\frac{1}{4}$th; when it made less than 18½, its motion was irregular; and when it was loaded so as not to admit of its making 18 turns, the wheel was overpowered by its load.

3. It is an advantage in practice, that the velocity of the wheel should not be diminished farther than what will procure some solid advantage in point of power; because, *cæteris paribus*, as the motion is slower, the buckets must be made larger, and the wheel being more loaded with water, the stress upon every part of the work will be increased in proportion. *The best velocity for practice is, therefore, when such a wheel as is here used makes 30 revolutions per minute; that is, when the velocity of the circumference is a little more than 3 feet per second.*

4. Experience confirms, that this velocity of 3 feet per second is applicable to the highest overshot wheels as well as the lowest; and, all other parts of the work being properly adapted, will produce very nearly the greatest effect possible: it is also, however, certain, from experience, that high wheels may deviate farther from this rule than low wheels can admit to do; for a wheel of 24 feet high may move at the rate of 6 feet per second without losing any considerable part of its power, while, on the other hand, Mr. Smeaton observes, that he has seen a wheel of 33 feet high that has moved very steadily and well with a velocity but little exceeding 2 feet per second.

Of the load of an overshot wheel when the effect is a maximum.

184. The maximum load for an overshot wheel is that which reduces the circumference of the wheel to its proper velocity, and this will be known by dividing the effect it ought to produce in a given time by the space intended to be described by the circumference of the wheel in the same time: the quotient will be the resistance overcome at the circumference of the wheel, and is equal to the load required, the friction and resistance of the machinery included.

Of the maximum velocity of an overshot wheel.

185. The greatest velocity that the circumference of an overshot wheel is capable of, depends jointly upon the diameter or height of the wheel, and the velocity of falling bodies; for it is evident that the velocity of the circumference can never be greater than to describe a semi-circumference, while a body let fall from the top of the wheel will descend through its diameter; nor indeed quite so great, as a body descending through the same perpendicular space cannot perform the same in so small a time, when passing through a semicircle, as would be done in a perpendicular line. Thus, if a wheel be 16 feet 1 inch in diameter, a body would fall through that diameter in one second: this wheel, therefore, never can arrive at a velocity equal to the making one turn in two seconds; but, in reality, an overshot wheel can never come near this velocity; for when it acquires a certain speed the greatest part of the water is prevented from entering the buckets; and the rest, at a certain point of its descent, is thrown out by the centrifugal force. This appears to have been the case in the three first experiments of the foregoing specimen; but as the velocity, when this begins to happen, depends on the forms of the buckets, as well as on other circumstances, the utmost velocity of overshot wheels is not to be determined generally: and indeed it is less necessary in practice, as it is in this circumstance incapable of producing any mechanical effect, for reasons already given.

On the greatest load that can be overcome by overshot wheels.

186. The greatest load an overshot wheel will overcome, considered abstractedly, is unlimited or indefinite: for as the buckets may be of any given capacity, the more the wheel is loaded the slower it turns; but the slower it turns the more will the buckets be filled with water; and, consequently, though the diameter of the wheel and quantity of water expended are both limited, yet no resistance can be assigned which it is not able to overcome; but in practice we actually find a limit which cannot be passed, for the buckets must necessarily be of some given capacity, and, consequently, such a

resistance will stop the wheel as is equal to the effort of all the buckets in one semi-circumference filled with water.

The structure of the buckets being given, the quantity of this effort may be assigned; but it is not of much consequence to the practice, as in this case also the wheel loses its power; for though there is the exertion of gravity on a given quantity of water, yet being prevented by a counter balance from moving, it is capable of producing no mechanical effect.

But, in reality, an overshot wheel generally ceases to be useful before it is loaded to that pitch; for when it meets with such a resistance as to diminish its velocity to a certain degree, its motion becomes irregular; yet this never happens till the velocity of the circumference is less than two feet per second, where the resistance is equable, as appears not only from the preceding experiments, but also from others on large wheels, of which we have cited one instance.

187. Having thus examined the different effects of the power of water, when acting by its impulse, and by its weight, under the titles of undershot and overshot wheels, we might proceed to examine the effect when the impulse and weight are combined, as in the several kinds of breast wheels, &c. but what has been already stated, being carefully attended to, the application of the same principles to these mixed cases, will be easy, and reduce what is to be said on this head, to a very narrow compass; for all kinds of wheels, where the water cannot descend through a given space, unless the wheel move with it, are to be considered of the same nature as an overshot wheel, according to the perpendicular height that the water descends from; and all those that receive the impulse or shock of the water, whether in a horizontal, perpendicular, or oblique direction, are to be considered as undershots. And, therefore, a wheel which the water strikes at a certain point below the surface of the head, and after that descends in the arc of a circle, pressing by its gravity on the wheel, *the effect of such wheel will be equal to the effect of an undershot wheel, whose head is equal to the difference of level between the surface of the water in the reservoir, and the point where it strikes the wheel; added to that of an overshot wheel, whose height is equal to the difference of level between the point where it strikes the wheel, and the level of the tail-water.*

It is here supposed, that the wheel receives the shock of water at right angles to its radii; and that the velocity of its circumference is properly adapted to receive the utmost advantage of both these powers, otherwise a reduction must be made on that account. (See Phil. Tran. Abr. vol. xi.)

§ XIV. *On the resistance of fluids to the rectilinear motion of bodies.*

188. In our historical chapter we have referred to the theories of Newton, D. Bernoulli, Euler, &c. relative to the resistance and percussion of fluids. We have also stated, that although these display great talent and ingenuity, they involve, with the exception of the first, intricacies of calculation little suited to practical researches; and the first, though much more simple, is founded upon an hypothesis so obviously erroneous as to lead to results not to be depended upon in all cases. In fact, the common theory of the resistance of fluids will apply with no better success to actual experiment, than the parabolic theory of projectiles to the ranges of balls and shells from pieces of ordnance. But on the same principles as the latter is commonly retained in books of Mechanics, for the sake of the curious geometrical properties which it involves, and as being the foundation on which a correct theory must be established, so most writers on Hydrodynamics generally appropriate a few pages to discussing the laws of the resistance of fluids, founded upon what is called the common hypothesis, viz. *that the fluid consists of an infinite number of parallel threads in the direction of its motion, each of which communicates its impact on the body without impeding the action of the others.* This, as we have said, is far from being rigorously true, and, in many cases, the results derived from it but ill accord with the determination of actual experiments; yet since, as an abstract theory, it furnishes several curious theorems, and as it may even be useful in examining deductions drawn from the experiments we shall subsequently detail, we shall make no apology for bestowing a few columns to these investigations.

PROPOSITIONS.

Propositions.

189. If a fluid, of which all the particles move with the same velocity, impinge perpendicularly on two planes at rest, the percussive forces on these two planes will be as their areas. For since all the particles of the fluid, by the hypothesis, move with the same velocity, the impulsion will be as the number of particles that impinge in a given time, and this will obviously be as the planes.

190. If two fluids of the same kind move with different velocities, and strike perpendicularly two given planes at rest, the forces will be as the product of the planes by the squares of the velocities of the fluids. For, supposing the planes to be the same, the impulsions in a given time will be as the product of the number of impinging particles, by the velocity of each, but the number of particles which strike in a given time will be as the velocity; therefore, when the planes are the same, the forces are as the squares of the velocities; and we have seen that when the velocities are the same, the impulsions are as the planes; therefore, universally, the forces will be as the product of the planes and squares of the velocities.

REMARKS.

Remarks.

191. In the above propositions we have supposed the planes to be at rest, and the fluid in motion; but we should obviously have come to the same conclusion if we had assumed the fluid to be stagnant, and the planes to have been in motion with the velocities we have given to the fluids. Hence, then, we may conclude, that whether a fluid in motion strike a plane that is at rest, or a plane be moved with any velocity in a stagnant fluid, the percussion in the former instance, or the resistance in the latter, will vary as the plane into the square of the velocity. The imperfection of our first hypothesis, however, is very obvious in the above deductions; for it supposes that after a particle strikes the body, or the body a particle, the action of that particle entirely ceases; whereas, each particle, after it has come in contact with the plane,

diverges and acts upon the particles behind it, and thus obviously causes some difference between theory and experiment. Another cause of error is, that we suppose the particles of the fluid to fall in behind the body as fast as the body moves forward; which is true as to sense, while the velocity of the body is inconsiderable; but when the motion of the body is very rapid, a kind of vacuum is left behind the body, and it has, therefore, to overcome, besides the natural resistance of the particles, the actual gravity of those parts which are left without a counter-pressure behind the body.

PROPOSITIONS.

Propositions.

192. If two fluids impinge perpendicularly on two planes P and P′ with the velocities V and v, while the planes themselves are moving parallel to the direction of the fluids with the velocities V′ and v', the resistances will be as the planes into the squares of the difference, or the squares of the sum of the velocities, according as the motions of the planes and bodies are in the same or in opposite directions; that is, we shall have, denoting the forces by F and F′,

$$F : F' :: P(V \pm V')^2 : P'(v \pm v')^2.$$

For we have seen that it is the same whether we suppose the planes at rest and the fluids in motion, or the fluids at rest and the planes in motion; it is, therefore, only the relative velocities of the two which is important to be considered; now the relative velocities between the planes and the fluids will obviously be in this case the difference or sum of the velocities, according as the two motions conspire with, or are opposed to, each other. The resistances, therefore, will obviously vary as the square of the difference, or sum of the velocities, the planes being the same; and if these also be different, then the resistances will vary as the square of the difference, or of the sum, of the velocities into the planes, as is obvious.

Inclined plane.

193. If a fluid impinge on two equal planes, the one of which is perpendicular to the direction of the fluid, and the other oblique to it, the impulsions on the former plane will be to that on the latter perpendicular to its surface, as the square of radius to the square of the sine of the angle of the plane's inclination from the line of direction of the fluid. Or, if two equal planes be in motion in a stagnant fluid, with equal velocities, one in a direction perpendicular to its own surface, and the other inclined to it; in any given angle the resistances experienced by the two planes perpendicularly to their surfaces, will be respectively as the square of radius, to the square of the sine of the angle of inclination.

For the number of fluid particles which come in contact with the inclined plane will be to the number of those which meet the direct plane as radius to the sine of the plane's inclination; moreover, the effect of the impulsion in the former case, in a direction perpendicular to the surface of the plane, will, from the resolution of forces, be reduced in the ratio of radius to the sine of the angle of direction or inclination; therefore, the resistance, as arising from these two causes, will obviously be reduced in the ratio of radius squared to sine squared; that is, the resistance to the direct plane to that of the inclined plane, will be as 1 to sine squared, assuming radius equal to unity.

We have here supposed the planes and velocities to be equal to each other; if these be different, then, since we have seen that, *cæteris paribus*, the resistances are as the planes into the square of the velocities, the ratio of the resistances of two planes moving with different velocities, and with different inclinations, will be as the planes into the squares of the velocity, and squares of the sines of the planes' inclinations.

That is, if P and P′ denote the areas of two planes, v and v' their velocities, and a and a' the angles which they form with the lines of their directions; then will the resistances perpendicular to their surfaces be to each other as $P v^2 \sin^2 a$ to $P' v'^2 \sin^2 a'$; or if the first angle $a = 90°$; then, because $\sin 90° = 1$, we have $P v^2$ to $P v'^2 \sin^2 a'$ for the ratio of the two resistances. It will be observed, that we have spoken above only of the resistance perpendicular to the plane; if we were to estimate the resistance in the direction of the motion, we should find it reduced in the ratio of the cube of the radius to the cube of the sine of the angle of inclination. Let, for example, AB (fig. 27) represent any plane oblique to the direction of the fluid, of which fluid QD denotes one of the filaments. Now, in the first place, the number of filaments that meet AB in this inclined position, will be to the number that would meet an equal direct plane, as BC : AB. Secondly, if the force QD be resolved into the two, ED and QE, the former (in consequence of the supposed facility of escape after impact) will be the only one that produces any effect. But ED : QD :: BC : AB; whence the perpendicular force ED, to the force on a direct plane equal to AB, will be as $BC^2 : AB^2$. Lastly, the force ED being resolved into the two EF, FD, the latter only will be effective in the direction QD; and here again, FD : ED :: BC : AB : consequently, the force on all the particles of the planes will, from a combination of all these causes, be reduced in the ratio of radius cubed to sine cubed of the angle of the plane's inclination; or as $1 : s^3$; denoting radius by unity, and the sine of the angle of inclination by s. Knowing, therefore, the value of the resistance in any one case, that due to any other may be obtained by means of the above analogies. The method employed by M. Bossut for estimating the measure of the resistance, or of the percussion of a fluid moving with a given velocity, will be explained in a subsequent article; at present we shall merely examine the relative values in different cases, results which are wholly independent of any assumed unit of measure, or mode of estimation.

PROBLEMS.

194. The isosceles triangle ABC (fig. 27-*a*), at rest, being exposed to the action of a fluid, of which the direction is perpendicular to its base AB, it is required to determine the ratio of the impulsion which the triangle will receive parallel to its altitude CD, to the direct and perpendicular impulsion which it receives on its base AB.

Let F denote the force of direct impulsion against AD or DB; and f the perpendicular impulsion against AC or CB; then, by the preceding article,

$$F : f :: \text{rad.}^2\, AD : \sin^2 ACD : AC;$$

$$\text{or} \quad F : f :: AC.^2\, AD : AD.^2\, AC,$$

$$\text{or} \quad F : f :: AC : AD;$$

$$\text{that is,} \quad f = \frac{F \times AD}{AC}.$$

known, and, consequently, also the resultant of these two forces.

The same method will apply to any solid, by decomposing the force which results perpendicular against one of its elements into three forces, each respectively parallel to three rectangular axes, or to any three axes A, B, C, given in position.

As the calculation which this general method requires is long and intricate, without being very difficult, and more especially as, after all, it must be considered rather as a subject of curiosity than of real utility, we shall limit our illustration of it to two or three of the most simple cases.

Semicircle. Fig. 28.

Let F′QF (fig. 28) denote any curve with two branches equal to each other QF′, QF, and symmetrical with respect to the axis QC, which is struck by the fluid in a direction parallel to this axis.

Draw to the axis QC the two ordinates PF, $p f$, indefinitely near to each other, and the right line F r parallel to QC: let PF $= y$, $f r = \dot{y}$, F $f = \dot{s}$; the velocity of the fluid $=$ V; and let F denote the perpendicular impulsion of a fluid moving with a velocity U against a given plane surface A at rest. Then the perpendicular impulsion against $f r$ will be expressed by

$$\frac{\mathrm{F}\cdot\mathrm{V}^2\dot{y}}{\mathrm{A}\cdot\mathrm{U}^2} = n\,\mathrm{V}^2\dot{y},$$

(by making the constant co-efficient $\frac{\mathrm{F}}{\mathrm{A}\cdot\mathrm{U}^2} = n$).

This co-efficient is called the *co-efficient of the percussion.* Now, the impulsion which results perpendicularly against F f, being resolved into two forces, the one directed according to FP, and the other in the direction F r, and the first of these forces being destroyed, by an equal and opposite force, which proceeds from the point F, it follows that the impulsion against F f, in the direction F r, is

$$n\,\mathrm{V}^2\dot{y} \times \frac{\dot{y}^2}{\dot{s}^2} = \frac{n\,\mathrm{V}^2\,\dot{y}^3}{\dot{s}^2};$$

it is then only requisite to eliminate $\dot{s}$ from this formula, by means of the equation of the curve, in order to reduce it to the proper form for integration.

Suppose, for example, the FQF′ to be a semicircle, of which the radius CQ $= a$, then we shall have

$$\dot{s}^2 = \frac{a^2\,\dot{y}}{a^2 - y^2};$$

and the general formula $\frac{n\,\mathrm{V}^2\,\dot{y}}{\dot{s}^2}$, becomes

$$\frac{n\,\mathrm{V}^2\,\dot{y}\,(a^2 - y^2)}{a^2};$$

of which the fluent is

$$n\,\mathrm{V}^2 y - \frac{n\,\mathrm{V}^2\,y^3}{3\,a^2},$$

which is the value of the impulsion against the indeterminate arc QF, in the direction QC; making $y = a$, the above becomes

$$n\,\mathrm{V}^2\,a - \tfrac{1}{3}\,n\,\mathrm{V}^2\,a = \tfrac{2}{3}\,n\,\mathrm{V}^2\,a,$$

for the sum of all the impulsions on the quadrant QF; and as the perpendicular impulsion against the radius a will be obviously $n\,\mathrm{V}^2 a$; it appears that the impulsion against the quadrant QF, is two-thirds of the direct impulsion against the radius, and, consequently, the impulsion against the semi circumference FF′ will be two-thirds of that against the diameter; as we have already determined (art. 195).

As a second example, let QF denote a parabolà of which the perimeter is p. Then we shall find

$$\dot{s}^2 = \dot{x}^2 + \dot{y}^2,$$

or,

$$\dot{s}^2 = \frac{\dot{y}^2\,(p^2 + 4\,y^2)}{p^2};$$

and whence we have

$$\frac{n\,\mathrm{V}^2\,\dot{y}^3}{\dot{s}^2} = \frac{n\,\mathrm{V}^2\;p^2\,\dot{y}}{p^2 + 4\,y^2}.$$

The fluent may be put under the form

$$n\,\mathrm{V}^2\!\int\!\frac{\frac{1}{4}\,p^2\,\dot{y}}{\frac{1}{4}\,p^2 + y^2} = n\,\mathrm{V}^2 \times \text{arc to tan } y \text{ and radius } \tfrac{1}{2}\,p.$$

In order to determine the impulsion for any solid of revolution, it will only be necessary to multiply the impulsion found above, in the general formula, for the element of the curve by the circumference generated by that element, supposing the curve to revolve about its axis, that is, by a circumference whose radius is y. If we put 3·14159, &c. $= \pi$, this circumference will be $\pi\,y$, and hence the general expression for the impulsion on a solid of revolution becomes

$$\frac{\pi\,n\,\mathrm{V}^2\,y\,\dot{y}^3}{\dot{s}^2};$$

substituting in this equation for $\dot{s}^2$, as given by the equation of the curve, and it will be in a state ready for integration.

We found the expression for the quadrant to be

$$\int\!\frac{n\,\mathrm{V}^2\,\dot{y}\,(a^2 - y^2)}{a^2};$$

wherefore, for the solid formed by the revolution of a quadrant, about one of its radii, that is, for the hemisphere, it will be

$$\int\!\frac{\pi\,n\,\mathrm{V}\,\dot{y}\,y\,(a^2 - y^2)}{a^2};$$

which being determined, by the usual rules, gives

$$\tfrac{1}{2}\,\pi\,n\,y^2\,\mathrm{V}^2 - \frac{\pi\,n\,\mathrm{V}^2\,y^4}{4\,a^2}.$$

This formula, when y becomes equal to a, reduces to

$$\tfrac{1}{2}\,\pi\,n\,y^2\,\mathrm{V}^2 - \tfrac{1}{4}\,\pi\,n\,\mathrm{V}^2\,a^2 = \tfrac{1}{2}\,\pi\,n\,\mathrm{V}^2\,a^2,$$

for the impulsion against the convex surface of the hemisphere.

Now, the impulsion against the base of the same (its area being $\pi\,a^2$) will be $\pi\,n\,\mathrm{V}^2\,a^2$; therefore, the impulsion against the convex surface is to the direct impulsion against the base of the hemisphere, as 1 to 2, as we have already found it in (art. 195).

If the arc were that of a parabola, whose parameter is p, the formula

$$\frac{\pi\,n\,\mathrm{V}^2\,y\,\dot{y}^3}{\dot{s}^2}?$$

would become

$$\frac{\pi\,n\,\mathrm{V}^2\,p^2\,y\,\dot{y}}{p^2 + 4\,y^2};$$

and, therefore,

$$\pi\,n\,\mathrm{V}^2\!\int\!\frac{p^2\,y\,\dot{y}}{p^2 + 4\,y^2} = \frac{\pi\,n\,\mathrm{V}^2\,p^2}{8} \times \text{h. log.}\left\{\frac{p^2 + 4\,y^2}{p^2}\right\}$$

will be the sum of all the impulsions sought.

It will be seen, in the subsequent section, that the solution of the problem, given in (art. 194), agrees pretty well with experiment, when the angle has any considerable magnitude, viz. for angles between 60° and 90°. But for angles of incidence, that are much below 60°, the theory and experiment no longer give the same results; the percussion diminishing less experimentally, than it ought to do according to theory.

The other two problems (arts. 195 and 196) are merely intended to show the manner of applying the theory to curve surfaces; they do not give results which correspond with those found from experiment; but the deviation here from actual practice, is contrary to that in the former case. Experiment, for example, shows, according to M. Bossut's results, that the impulsion against the semi-circumference is only a little more than half that of the perpendicular impulsion against the diameter; while, according to theory, it ought to be two-thirds. It follows, therefore, that all the solutions which have been given to the problem of the solid of least resistance, can only be considered as ingenious analytical speculations, and must not be introduced into any practical constructions, without the greatest caution.

197. Let it now be required to investigate the circumstances attending the descent of a heavy spherical body in a fluid, viz. to determine the impulsion which its motion produces on the bottom of the vessel containing the fluid.

It is obvious, in the first place, that the descending body will, every instant of its descent, strike the fluid with the velocity acquired at that point of its path, and that this impulsion, which, from the nature of fluids, is transmitted in all directions, will thus operate against the bottom of the vessel. It is proposed to determine its amount at any proposed instant.

Let the radius of the body $= a$,
Its mass or weight $= \text{P}$,
The weight of the fluid displaced by it $= \text{P}'$,
The space described vertically $= s$,
The velocity at the end of this space $= u$,
The co-efficient of percussion $= n$,
The semi-circumference of a circle to radius 1 $= \pi$.

These substitutions being made, it is obvious that the body is urged downwards every instant by the excess of its own weight over that of an equal bulk of the fluid and that of the resistance.

Now, the expression for the resistance (art. 196) is $\frac{\pi n a^2 u^2}{2}$, and, consequently, the accelerating force will be

$$\text{P} - \text{P}' - \frac{\pi n a^2 u^2}{2};$$

therefore, from the usual formula for accelerating forces,

$$\text{P} u \dot{u} = \left(\text{P} - \text{P}' - \frac{\pi n a^2 u^2}{2}\right) \dot{s};$$

or,
$$\dot{s} = -\frac{\text{P}}{\pi n a^2} \times -\left\{\frac{2 \pi n a^2 u \dot{u}}{2\text{P} - 2\text{P}' - \pi n a^2 u^2}\right\};$$

of which the fluent is

$$s = \text{C} - \frac{\text{P}}{\pi n a^2} \times \text{h. log. } (2\text{P} - 2\text{P}' - \pi n a^2 u^2),$$

C being a constant arbitrary or correction.

We shall arrive at the value of C, by observing that when $s = o$, $u = o$; which gives

$$s = \frac{\text{P}}{\pi n a^2} \times \text{h. log. } \left(\frac{2\text{P} - 2\text{P}'}{2\text{P} - 2\text{P}' - \pi n a^2 u^2}\right);$$

therefore, if E be taken to denote the number of which the hyperbolic logarithm is 1, and we multiply s by h. log. E, we shall have

$$s \,\text{h. log. E} = \frac{\text{P}}{\pi n a^2} \times \text{h. log. } \left(\frac{2\text{P} - 2\text{P}'}{2\text{P} - 2\text{P}' - \pi n a^2 u^2}\right);$$

or, passing to the numbers,

Hydraulics.

$$u^2 = 2\left(\frac{\text{P} - \text{P}'}{\pi n a^2}\right) \times \left\{1 - \text{E}^{\frac{-\pi n a^2 s}{\text{P}}}\right\};$$

therefore, the expression for the resistance, which, when the velocity is u, we have found to be

$$\frac{\pi n a^2 u^2}{2},$$

becomes, in the present case,

$$(\text{P} - \text{P}') \times \left(1 - \text{E}^{\frac{-\pi n a^2 s}{\text{P}}}\right).$$

As the quantity E is greater than unity, the expression $\text{E}^{\frac{-\pi n a^2}{\text{P}}.s}$ becomes less and less as s increases, and is zero, when s is infinite; therefore, in this supposititious case only, the impulsion on the bottom of the vessel ever becomes equal to $\text{P} - \text{P}'$.

This result is employed to illustrate the sinking of the barometer in rainy weather; for the distance from which the drops of rain fall is never very great, and, therefore, their action on the surface of the earth is always less than their absolute weight; whereas, while they are held in solution they press with their actual weight; the mercury in the barometer, therefore, ought to fall during rain, which is agreeable to the result that Leibnitz obtained from his hypothesis.

§ XV. *Experiments on the percussion of fluids, and the resistances which they oppose to the motion of bodies.*

Bossut's *Experiments on the percussion of fluids.*

Bossut's experiments

198. We have stated, in the preceding section, the inadequacy of pure theory to meet the general question of the percussion and resistance of fluids; much must necessarily depend upon accurate experimental data, and considerable pains and labour have accordingly been bestowed upon this subject by different authors; of which we propose to give some abstracts in the present section, commencing with the experiments of M. Bossut.

The questions which this philosopher proposed to himself were as follows:

1. What is the proper measure of the perpendicular percussion of a fluid?

2. Are the perpendicular percussions that are made with the same velocities, proportional to the section of the impinging vein, or to the surfaces on which indefinite fluids impinge?

3. Are the perpendicular percussions made on the same surface with different velocities, proportional to the squares of the velocities?

4. Are percussions made on different oblique planes (all other circumstances being the same) as the squares of the sines of the angles of incidences?

Apparatus.
Figs. 29 and 30.

199. The apparatus with which M. Bossut performed the following experiments, is represented in (fig. 29), or (fig. 30). The beam AB was 3 feet 6 inches in length: at one of its extremities B, was attached a scale, and at the other a circular plate of copper, 2½ inches in diameter; this was well polished; its plane prolonged, passed through the axis of motion, and its centre corresponded very exactly with the centre of the additional tube PQ pq, through which the water passed out of the vessel VXYZ; by means of the

two screws shown in the stem of the balance, the beam AB and its attached plate might be adjusted to any height, or to any distance from the orifice of the tube $p\,q$; and as the beam turned freely on its axis o, it might be inclined at any angle at pleasure.

In the first series of experiments, the constant height of the water TA, was 4 feet above the centre of the plate A, and the latter was 1 inch distant from the end of the tube $p\ q$, in order that the water might issue freely, which it did in a full stream.

Experiments.

Experiments. Figs. 29 and 30.

1. The diameter of the tube $p\ q$ being 10 lines, the weight S (fig. 29) necessary to balance the effort of the water was 12608 grains.

2. The diameter of the tube $p\ q$ being 10 lines, and the angle of inclination TAB (fig. 30) being 60°, the weight S = 12248 grains.

3. The diameter of the tube $p\ q$ being reduced to 6 lines, the weight S (fig. 29) was 4484 grains.

4. The diameter being still 6 lines, and the beam inclined at 60°, as in (fig. 30), the weight S was 4315 grains.

The constant height TA being now reduced to 2 feet, the following results were obtained:

5. Diameter of the tube 10 lines (fig. 29), S = 6306 grains.

6. Diameter 10 lines, angle TAB = 60° (fig. 30), S = 6125 grains.

7. Diameter 6 lines (fig. 29), S = 2243 grains.

8. Diameter 6 lines, angle TAB = 60° (fig. 30), S = 2138 grains.

Comparison of the results.

200. By comparing experiments 1 and 3 with each other, as also 5 and 7, it appears, that while the velocities are equal, the perpendicular percussions are nearly proportional to the surfaces struck by the fluid.

201. Comparing together experiments 1 and 5, as also 3 and 7, we see that the perpendicular percussions against the same surface, are very nearly to each other as the height of the fluid above the centre of percussion, or which is the same, very nearly as the squares of the velocities of the fluids; which is conformable to the theoretical hypotheses.

202. When the water strikes the plate obliquely, as in experiments 2, 4, 6, 8, there results from the percussion a force perpendicular to the plate, which force, according to theory, may be represented by

$$F \times \frac{B \sin^2 TAB}{A \cdot R^2},$$

R denoting the radius; and F the force which the same plate receives when it is perpendicular to the action of the fluid; and B the oblique surface covered by the fluid. This force being resolved into its perpendicular direction, has for the lever by which it acts, the arm of the balance CA, while the weight S has only the effective distance CL. Since, then,

$$B : A :: \text{rad} : \sin TAB,$$

and

$$AC : CL :: \text{rad} : \sin TAB;$$

and that

$$F \times \frac{B \sin^2 TAB}{A \cdot \text{Rad.}^2} \times CA = S \times CL,$$

it follows that F = S.

Hence, according to theory, we ought always to have the same weight in the scale, whether the balance be horizontal or inclined at any angle whatever; which is a conclusion not borne out by these experiments; the weight being less when the angle TAB is 60°, than when it is a right angle; and the former, M. Bossut says, diminishes more and more, as the angle becomes less and less. Our theoretical formulæ, therefore, in which we introduce the sine of the angle of incidence, requires certain modifications, which can only be obtained experimentally.

Bossut's experiments on the resistance of indefinite fluids.

203. In the year 1775, Bossut, in company with D'Alembert and Condorcet, undertook a series of experiments on the resistances opposed to the motion of vessels in small canals, and in waters of indefinite extent, with respect to the dimensions of the bodies on which the experiments were made. Several small vessels were constructed of different forms and dimensions; and on the side of the reservoir, which was 100 feet in length, 53 feet in breadth, and 6½ feet deep, was erected a mast 76 feet in height, at the top of which was fixed a brass pulley, perfectly moveable, and another similar one at bottom. Each of these pulleys were 5 inches in diameter. Under the lower pulley passed a line, which was thence carried over the upper pulley; the upper end having a weight attached to it, while the lower end being brought along the surface of the water, was fixed successively to the different vessels on which the experiments were to be made. The weight being then left to descend by its gravity, drew the boat along the reservoir. The resistance was estimated by the time the respective bodies were in passing over 50 feet in the centre part of the reservoir; the first part of the motion was made from a more distant point; but as the velocity did not become uniform in the first moments, only 50 feet space was employed in estimating the effect. The velocity had always become uniform before it arrived at the point whence the time of the motion began to be counted; and it terminated when the body had passed over 50 feet.

204. Five different formed bodies were employed, which may be denoted by A, B, C, D, E; and of these forms there were different dimensions; these may be distinguished for each letter, by N° 1, 2, 3, &c. By these means we may state the form and dimensions of the bodies, as follows:

A. Rectangular parallelopipedon, with one end in the form of an isosceles wedge, of which the breadth was 1 foot, the length, independent of the end, 4 feet; length of the triangular end, or altitude of the triangle, 2 feet.

A. { N° 1, rectangular end foremost.
    { N° 2, triangular prow foremost.

B. Rectangular parallelopipedon, with one end of a trapezoidal form, instead of triangular, as in vessel A. The breadth of this vessel 2 feet; length, independent of the trapezoid, 4 feet; length of trapezoidal end, 2 feet, and smallest end 1 foot.

In this vessel the trapezoid always formed the stern, and the other end had isosceles triangular prows attached to it, of the following dimensions.

B. {
3 The prow, rectangular end.
4 Ditto, triangular, altitude 6 inches.
5 Ditto.......... ditto 1 foot.
6 Ditto.......... ditto 1½ foot.
7 Ditto.......... ditto 2 feet.
8 Ditto.......... ditto 2½ feet.

Experiments on indefinite fluids.

C. Rectangular parallelopipedon, of which the length was 6 feet 1 inch, and the breadth 1 foot 7⅔ inches.

C. 9 The prow, the rectangular end.
10 Triangular prow, height of triangle 9 inches 9½ lines.
11 Ditto.......... ditto 19 inches 8 lines.
12 Cylindrical prow, radius 9 inches 9½ lines.

D. The vertical section of this vessel was of a mixtilinear form, sides curved; of which the greatest breadth was 1 foot 7 inches 8 lines; and length six feet. This vessel was tried only under one form, and is marked D, N° 13.

E. N° 14, was made after a model of a ship of the line, the greatest breadth of its vertical section and length, the same as D, N° 13.

The depth of the floatation is specified in the following table.

205. When the vessel is at rest, the fluid all round is of the same height, and forms what is called the line of floatation; but when it is in motion, the fluid rises before the prow, and along the sides of the vessel, and falls behind; this wave, if we may so denominate it, is greater about the midships than at the extremities. These two cases are denoted in the table by the words *central* and *lateral*, and their height is expressed in lines; when, as is sometimes the case, the water at the angles of the prow is lower than the line of floatation, it is indicated by the sign — *minus*.

TABLE I. *Of experiments by Bossut.*

Form.	Depth of immersion.	Central elevation of the fluid.	Lateral elevation of the fluid.	Moving weights expressed in marcs.	Time employed in passing over 50 feet in half seconds.
	Inches.	Lines.	Lines.		
A, N° 1.	12	21	15	12	43·70
	12	26	19	16	38·37
	12	34	26	20	34·75
	12	....	....	24	32·16
A, N° 2.	12	29	0	12	30·80
	12	36	....	16	27·25
	12	38	— 4	20	24·50
	12	39	— 6	24	23·40
B, N° 3.	12	18	15	16	50·11
	12	25	21	24	44·54
	12	36	26½	32	38·80
	12	37	30	40	34·80
	12	44	37	48	32·33
B, N° 4.	12	16	7	16	50·00
	12	21	11	20	43·50
	12	27	17	24	38·50
	12	30	19	28	37·00
	12	34	22	32	34·92
B, N° 5.	12	22	....	16	42·68
	12	24	12	20	38·18
	12	28	11	24	35·56
	12	33	9	28	32·95
	12	....	....	32	31·60
B, N° 6.	12	22	....	16	39·00
	12	24	11	20	35·75
	12	27	....	24	32·62
	12	33	....	28	30·37
	12	38	....	32	28·87
B, N° 7.	12	24	6	16	37·70
	12	29	0	20	33·45
	12	35	....	24	30·70
	12	40	....	28	28·50
	12	45	....	32	27·60
B, N° 8.	12	29	—	16	36·62
	12	33	—	20	33·40
	12	38	—	24	30·81
	12	45	—	28	28·62
	12	52	—	32	27·00
C, N° 9.	7⅚	24½	....	16	41·75
	7⅚	32½	....	20	37·80
	7⅚	38	....	24	34·75
	7⅚	46	....	30	32·50
	7⅚	50	....	36	29·90
C, N° 9.	12½	15	....	16	52·00
	12½	18	....	20	46·05
	12½	20½	....	24	42·07
	12½	26	....	30	37·25
	12½	32½	....	35	35·18
C, N° 9.	15⅚	15	....	20	50·75
	15⅚	18	....	24	46·50
	15⅚	24	....	32	41·00
	15⅚	33	....	40	36·50
	15⅚	39	....	48	33·69
C, N° 10.	7⅚	23	....	12	38·18
	7⅚	28	10	14	35·37
	7⅚	30	12	16	33·90
	7⅚	35	....	18	31·95
	7⅚	39	....	20	30·45
C, N° 10.	12½	16	2	12	47·90
	12½	24	5	16	42·06
	12½	30	8	20	37·68
	12½	36	10	24	34·90
	12½	39½	....	28	32·37
C, N° 10.	15⅚	11	....	12	54·60
	15⅚	15½	0	16	47·20
	15⅚	18½	....	20	42·00
	15⅚	20	....	22	40·25
	15⅚	23	....	24	38·40
C, N° 11.	7⅚	32	....	14	32·12
	7⅚	36	....	16	30·41
	7⅚	40	....	18	29·20
	7⅚	44	....	20	28·10
	7⅚	50	....	24	26·25

Experiments on indefinite fluids.

TABLE I.—*continued.*

Form.	Depth of immersion.	Central elevation of the fluid.	Lateral elevation of the fluid.	Moving weights expressed in marcs.	Time employed in passing over 50 feet in half seconds.
	Inches.	Lines.	Lines.		
C, N° 11.	$12\frac{1}{2}$	18	− 2	12	42·85
	$12\frac{1}{2}$	24	....	16	37·15
	$12\frac{1}{2}$	35	....	20	30·00
	$12\frac{1}{2}$	40	....	24	31·10
	$12\frac{1}{2}$	45	....	28	29·30
C, N° 11.	$15\frac{5}{6}$	....	....	12	46·87
	$15\frac{5}{6}$	....	....	14	44·00
	$15\frac{5}{6}$	18	− 5	16	41·50
	$15\frac{5}{6}$	24	....	18	39·12
	$15\frac{5}{6}$	30	....	20	36·94
C, N° 12.	$7\frac{5}{6}$	26	....	12	36·00
	$7\frac{5}{6}$	33	....	16	32·00
	$7\frac{5}{6}$	40	....	20	29·60
	$7\frac{5}{6}$	46	....	24	27·40
C, N° 12.	$12\frac{1}{2}$	20	....	12	44·00
	$12\frac{1}{2}$	26	....	16	38·50
	$12\frac{1}{2}$	32	....	20	35·40
	$12\frac{1}{2}$	38	....	24	32·69
C, N° 12.	$15\frac{5}{6}$	15	....	12	49·31
	$15\frac{5}{6}$	20	....	16	42·62
D, N° 13.	$12\frac{2}{3}$	18	....	12	54·65
	$12\frac{2}{3}$	22	....	16	47·94
	$12\frac{2}{3}$	$26\frac{1}{2}$	....	20	43·14
	$12\frac{2}{3}$	30	....	24	38·92
	$12\frac{2}{3}$	36	....	30	34·82
	$12\frac{2}{3}$	42	....	36	32·72
D, N° 13.	$15\frac{11}{12}$	11	....	10	65·56
	$15\frac{11}{12}$	13	....	12	60·70
	$15\frac{11}{12}$	18	....	16	53·05
	$15\frac{11}{12}$	22	....	20	47·65
	$15\frac{11}{12}$	27	....	24	43·06
	$15\frac{11}{12}$	33	....	30	39·25
E, N° 14.	$12\frac{2}{3}$	30	....	8	31·18
	$12\frac{2}{3}$	38	....	10	27·90
	$12\frac{2}{3}$	48	....	12	26·50
E, N° 14.	$15\frac{11}{12}$	33	....	8	36·69
	$15\frac{11}{12}$	$36\frac{1}{2}$	....	10	32·90
	$15\frac{11}{12}$	40	....	12	30·75

Deductions from the experiments.

206. From a comparison of these several experiments with each other, and with a reference to those reported in the beginning of this section, M. Bossut concludes,

1. That the resistances of the same body, of whatever figure, which divides a fluid with different velocities, are very nearly proportional to the squares of those velocities.

2. That the direct and perpendicular resistances of plane surfaces, are with the same velocities nearly proportional to the areas of the surfaces.

3. That the resistances which arise from oblique impulses are not in the ratio of the squares of the sines of the angles of incidence; but that when these angles are between 60° and 90°, the common theory may be employed as an approximation, by observing that it always gives resistances a little less than experiment; and much less as the angle diminishes.

4. That the absolute measure of the direct and perpendicular resistance of a plane surface in an indefinite fluid, is the weight of a column of the fluid, which has for its base the area of the surface, and for its height the altitude due to the velocity. The resistance is much greater, and nearly double in a mill course, which conveys the water to the float boards of an undershot wheel.

5. That the tenacity of the water is a force which may be regarded as indefinitely small, in relation to that which a body experiences in striking the water, particularly when the velocity is considerable.

Experiments on the resistance of bodies in narrow canals.

207. It having been known, or at least generally asserted, by persons in the habit of navigating vessels on canals, that the labour was much greater when the water was shallow than when it was deep, and as in the preceding course of experiments the extent of the surface of the fluid might be considered as indefinite with respect to the dimensions of the body, M. Bossut undertook a second series of experiments on close and open canals; which being constructed of different breadths and depths, were placed in the same reservoirs as that above described, and were conducted in exactly the same manner. The bodies also were the same, and are therefore denoted by the same numbers in the following, as in the preceding table; but another vessel was now added. It was a rectangular parallelopipedon, its breadth $19\frac{2}{3}$ inches, length 2 feet 1 inch; this will be denoted by N° 15, and N° 16, was the same in all respects, but in its length, which was 4 feet.

TABLE II. *Of experiments on narrow canals.*

1st canal, opened at both ends, breadth 2 feet $4\frac{1}{2}$ inches, depth $15\frac{1}{6}$ inches.

Form.	Depth of immersion.	Central elevation of the fluid.	Lateral elevation of the fluid.	Moving weights expressed in marcs.	Time employed in passing over 50 feet in half seconds.
	Inches.	Lines.	Lines.		
N° 1.	12	22	12	16	50·86
	12	16	21	20	46·95
	12	30	24	24	42·30
	12	35	30	28	39·80
	12	40	34	32	38·65
N° 3.	12	18	....	32	68·83
	12	24	18	40	62·75
	12	30	24	48	58·33
	12	34	26	56	54·80
N° 15.	$12\frac{1}{2}$	18	12	24	69·86
	$12\frac{1}{2}$	24	18	32	61·80
	$12\frac{1}{2}$	30	21	40	55·65
	$12\frac{1}{2}$	36	24	48	52·50

TABLE II.—*continued.*

Form.	Depth of immersion.	Central elevation of the fluid.	Lateral elevation of the fluid.	Moving weights expressed in marcs.	Time employed in passing over 50 feet in half seconds.
	Inches.	Lines.	Lines.		
N° 16.	12½	18	12	24	68·79
	12½	22	16	32	61·40
	12½	26	20	40	54·29
	12½	30	24	48	49·85
N° 13.	12⅔	15	9	24	61·30
	12⅔	18	12	32	53·45
	12⅔	24	18	40	47·45
	12⅔	....	....	48	44·31

2d canal, breadth 3 feet 4 inches, depth 16⅙ inches.

N° 1.	12	24	18	16	45·10
	12	30	24	20	41·00
	12	36	24	24	38·50
	12	48	40	32	34·81
N° 3.	12	24	20	32	60·10
	12	30	24	40	54·75
	12	36	28	48	51·00
N° 16.	12½	24	18	24	63·37
	12½	29	21	32	54·60
	12½	34	24	40	49·62
	12½	38	30	48	46·25
N° 13.	12⅔	24	20	24	53·00
	12⅔	30	24	32	46·00
	12⅔	36	30	40	42·00

3d canal, breadth 6 feet 3 inches, depth 15⅙ inches.

N° 1.	12	24	20	16	40·00
	12	28	23	20	35·00
	12	32	26	24	33·00
N° 3.	12	30	24	32	47·83
	12	38	30	40	44·00
	12	....	....	48	40·55
N° 16.	12½	26	22	24	51·07
	12½	34	....	32	46·10
	12½	39	32	40	43·00
	12½	48	40	48	40·00
N° 13.	12⅔	24	20	24	45·60
	12⅔	34	28	32	40·10
	12⅔	44	34	40	36·10

4th canal, breadth indefinite, depth 15⅓ inches.

N° 3.	12	36	30	32	41·66
	12	42	34	40	37·56
	12	48	....	48	35·37
N° 16.	12½	36	30	32	41·50
	12½	42	36	40	37·00
	12½	48	42	48	35·25

TABLE II.—*continued.*

Experiments on canals of indefinite breadth.

5th canal, breadth indefinite, depth 2 feet 3¼ inches.

Form.	Depth of immersion.	Central elevation of the fluid.	Lateral elevation of the fluid.	Moving weights expressed in marcs.	Time employed in passing over 50 feet in half seconds.
	Inches.	Lines.	Lines.		
N° 3.	12	30	24	32	40·65
	12	39	33	40	35·60
	12	46	38	48	32·55
N° 16.	12½	30	18	32	39·00
	12½	36	24	40	35·15
	12½	42	32	48	31·83

6th canal, breadth 28½ inches, depth 15½ inches.

N° 2.	12	14	....	16	42·80
	12	18	....	20	39·37
	12	24	....	24	37·05

7th canal, breadth 28½ inches, depth 15⅓ inches.

N° 5.	12	18	12	32	64·83
	12	24	18	40	59·60
	12	30	24	48	54·50

Canals shut at each end.

8th canal, shut at both ends, breadth 28½ inches depth 15½ inches.

N° 1.	12	24	18	16	49·00
	12	30	24	20	45·12
	12	36	30	24	41·94
N, 3.	12	18	12	32	69·80
	12	24	18	40	64·20
	12	30	24	48	60·60

Deductions.

208. From the results deduced by a comparison of these experiments with those given (art. 207), we may conclude, that in narrow canals, the resistances are proportional to the squares of the velocities, following the same law as in a fluid of indefinite extent; and that the resistances in narrow canals, and canals which have little depth, is greater than in fluids of indefinite extent.

The cause of this is obvious; for when the velocity of the body is considerable, the fluid which that body pushes before it has not the means of expanding itself on every side, but forms a current more or less rapid as the velocity of the body is more or less great. If the body entirely filled the canal, it would push all the water before it like the piston of a pump; but as in the narrowest canals there is always some room for the water to run both below the boat and at its sides, a part of the fluid escapes, while another part is driven back; and thus a variety of contrary currents are formed, by which the resistance is increased. This augmentation of resistance is produced, not only by heaping up the fluid on its anterior part, but also by the want of hydrostatical support behind.

These results will furnish an important lesson to the engineer, by pointing out to him the advantage of making all canals of navigation as wide and deep as is consistent with proper economy.

Vessels with angular prows.

209. We shall conclude this account of M. Bossut's labours with the following table, containing the results of sixty-nine experiments, made principally with a view of determining the comparative resistances of angular prows of different degrees of inclination, an object of considerable importance in naval architecture.

The first five were made on a vessel, in the form of a rectangular parallelopipedon, the length of which was 4 feet, breadth 2 feet, depth in the fluid 2 feet, and part above the same 7 inches.

The other 64 were made with 14 different vessels, in the form of a prism, the prows being isosceles wedges, having their vertices forward.

These vertical angles were varied from 12° to 24°, 36°, &c. to 180°; the length of each vessel, independent of the prow, was 4 feet, breadth 2 feet, depth of immersion 2 feet, part above the fluid 7 inches, all in French feet and measures.

N° of the experiments.	Angles of the prow.	Weight of the mover.	Time of describing 96 feet uniformly.	N° of the experiments.	Angles of the prow.	Weight of the mover.	Time of describing 96 feet uniformly.
		lbs.				lbs.	
1	180°	61·8	78·08	36	96°	60·5	63·00
2		112·5	57·51	37		110·5	46·45
3		162·5	47·44	38		162·5	38·05
4		212·5	41·49	39		212·5	32·66
5		262·6	37·32	40		262·5	29·27
6	168°	62·5	77·50	41	84°	60·5	60·55
7		112·5	56·95	42		110·5	44·56
8		162·5	47·22	43		162·5	35·78
9		212·5	41·26	44		212·5	31·25
10		262·5	37·12	45		262·5	27·51
11	156°	62·5	75·09	46	72°	62·5	57·50
12		112·5	56·15	47		112·5	42·75
13		162·5	46·44	48		162·5	34·85
14		212·5	41·03	49		212·5	29·65
15		262·5	36·52	50		262·5	25·86
16	144°	62·5	73·38	51	60°	60·5	55·45
17		112·5	54·75	52		112·5	40·04
18		162·5	45·35	53		162·5	33·05
19		212·5	39·58	54		212·5	28·25
20		262·5	37·57	55		262·5	24·77
21	132°	60·5	72·08	56	48°	62·5	52·51
22		111·5	53·25	57		112·5	38·05
23		162·5	43·75	58		162·5	31·61
24		212·5	38·26	59		212·5	27·56
25		262·5	34·30	60		262·5	24·30
26	120°	62·0	63·32	61	36°	62·5	51·15
27		112·5	50·84	62		112·5	36·96
28		162·5	41·84	63		162·5	30·53
29		212·5	36·62	64	24°	61·5	49·48
30		262·5	32·77	65		112·5	35 76
31	108°	61·5	65·85	66		162·5	30·23
32		112·5	48·75	67	12°	62·5	49·38
33		162·5	39·50	68		112·5	35·33
34		262·5	31·46	69		162·5	30·01
35		610·5	31·05	70		——	——

Deductions.

210. The direct resistance on the end of the vessel, without an isosceles prow, may be represented by any arbitrary number (as for example 10,000), and by this means the comparative resistances with differently formed prows numerically expressed.

If the angle of the prow, or that formed at the fore part of the vessel or summit of the triangular section, be x, the resistance is found to be 10,000 $\cos^2 x$. To correct this expression by means of the foregoing table, we may observe, that when the angle x undergoes any variation of 12°, each of the angles at the base of the isosceles prow will vary 6°; and denoting this latter variation by q, M. Bossut deduces from the table the addative correction $3{\cdot}153 \left(\frac{x}{q}\right)^{3\cdot25}$, x and q being expressed in decimal parts of the radius $= 1$. Thus the whole resistance against the prow, estimated in the opposite direction to that of the motion of the boat, is equal to

$$10{,}000 \cos^2 x + 3{\cdot}153 \left(\frac{x}{q}\right)^{3\cdot25},$$

which formula will be sufficiently exact when the extreme angle of the prow is greater than 12°; but when the angle is less than 12°, the formula errs in excess.

Du Buat's experiments.

211. We have had frequent occasion to mention the interesting experiments and deductions of this author, on the discharge and motion of fluids, and those which he has made on their resistance, are no less valuable; we shall, therefore, endeavour to lay before our readers some detail of them in the present section. The first object that M. Du Buat had in view was to determine the resistance experienced by an immoveable surface, when struck by an isolated vein of fluid, whose area was either greater than or equal to the area of the surface.

From a series of experiments directed wholly to this determination, but rather on a small scale, the author found that the height due to the impulse is the same as that due to the velocity. Bossut found it to be proportional to the velocity; but the height of the column appeared to be that due to double the velocity. Du Buat accounts for this difference, by stating that the vertical vein of fluid in Bossut's experiments, of which we have already given the detail, enlarged itself in striking the surface upon the balance; and that the fluid filaments taking a horizontal direction after they had made their impact on the plate, the resistance as determined by the experiments was not merely the impulse of the vein whose diameter was that of the orifice, but also the additional pressure of a ring of the fluid of a certain extent around the circular base of the vein in contact with the resisting surface.

Having established this point, the next object of the author was to determine the amount of the resistance when an immoveable surface is placed in a fluid of indefinite extent. The apparatus employed for this purpose was a tin box, presenting a surface of one square foot. Its thickness was about one-third of an inch, and it was shut on all sides except a small opening in its posterior surface, into which was soldered the horizontal branch of a tin tube, bent at right angles, 1⅓ inch in diameter, which received a float to indicate the height to which the water rose within the tube. By means of a bar of iron, about 10 feet long, which could be attached to the back of the box, it could be fixed at any depth, within this limit, below the surface of the current. Five holes, each about 1 line in diameter, were perforated in the front side of the tin box. The first was in the middle; the second equi-distant, in a horizontal line from the middle and the edge; and the third in the same horizontal line, but only ⅝ of an inch distant from the edge; the fourth was quite close to the edge; and the fifth in the lower angle of the square. When the box was fixed, and the current of water allowed to enter one or more of these holes, the fluid rose in the vertical branch of the tin tube to a height which indicated that due to the pressure of the fluid. In this manner Du Buat made several experiments which gave very singular results. The pressure was not only found to diminish from the centre of the surface to the edge or margin, but it became nothing at a certain distance from the centre, and afterwards negative to the margin; that is, when the water entered through the central hole, it rose to a certain height in the tin tube at right angles to the stream. This height diminished when the water entered at a hole nearer to the margin; the height became nothing at a certain distance; and still nearer the margin, the fluid was actually depressed in the tin tube below the surface of the current.

Hydraulics.

In order to explain this remarkable anomaly, Du Buat observes, that the real pressure against any part of the surface is only the difference of the pressure against the part considered separately from the rest of the surface, and the height due to the variable velocity with which the water moves along the surface which is struck; that is, the water, in escaping along the surface, always diminishes the real pressure of the current; but towards the edges it becomes more powerful than the real current, and therefore that point is less pressed than if the fluid were immoveable, and it will therefore sink in the tin tube.

The next object was to ascertain the mean pressure on the whole surface; in order to which Du Buat shut up the holes already mentioned, and perforated the same surface with 625 holes, disposed symmetrically in a square, whose side was 25. By exposing this surface to the current, it appeared that the height of the water in the tube was 25½ lines, when the mean velocity was 3 feet per second, which is that due to a height of 21½ lines. Hence, in these experiments, the height due to the real resistance is equal to 1·186 times the height due to the velocity. In like manner, as the foremost part of a body, or that immediately opposed to the action of a fluid in motion, is subjected to a greater pressure than when the fluid is stagnant, so will the after part of the body be relieved from a certain portion of the natural pressure from the fluid not falling instantaneously in behind; this is called the *non-pressure*; and by a similar contrivance, Du Buat endeavoured to ascertain its measure, and found, first, that the diminution of pressure increased considerably by lengthening the body; secondly, that it increased a little from the circumference of the body to the centre; and, lastly, that the diminution of pressure is proportional to the area of the surface that is submerged.

Comparison between the motion of the body and that of the fluid.

212. In the theory which we have advanced, relative to the resistance and percussion of fluids, we have stated that the effects are the same, whether we suppose the fluid to be at rest, and the body in motion; or whether the latter be at rest, and the former in motion: but we have at the same time hinted, that this is not true in practice. Du Buat undertook a course of experiments in order to examine this question in a clear and satisfactory manner; and from these he concluded, first, that the phenomena are by no means the same, when the body is at rest, as when it is in motion; secondly, that in the latter case, the pressure does not diminish so sensibly from the centre to the circumference, and instead of a negative pressure towards the sides, that the pressure is then so great as to have for its measure a third of the height due to the velocity, which shows that the water runs along the interior surface either with less velocity, or with greater uniformity. Thirdly, that the pressure diminishes in a less ratio than the square of the velocity, when this is less than three or four feet per second. Fourthly, that the mean pressures are measured by the exact height due to the velocities, instead of being equal to 1·186 times that height, as in the former case. Lastly, that the diminutions of pressure decrease a little from the centre to the circumference, in the same order as the pressures themselves.

Bodies oscillating in a fluid.

213. When globes, or any other bodies, oscillate in a fluid, they drag with them a certain quantity of the fluid; therefore, in order to compare the results of experiments made after this manner, the quantity of

such fluid ought to be determined, which it is obvious can only be done by means of observations, directed entirely to this inquiry. M. Du Buat, with this view, employed globes of wood, lead, and glass. These he caused to vibrate in a vessel 51 inches in length, 17 inches wide, and 14 inches deep; they were entirely immersed about 3 inches below the surface, and the wire by which they were suspended, was as delicate as their weight would permit. From these experiments the general conclusion was, that a globe oscillating in water, drags with it, both before and behind, a portion of the fluid whose volume is ·585 times that of the globe.

Other experiments of a similar kind were made with various plane surfaces of tin; as, for instance, cylinders oscillating in the plane of their axis; triangular and quadrangular prisms also oscillating in the planes of their axis; cubes oscillating directly; cubes oscillating by the common section of two of their bounding planes, by a solid angle, &c. &c. We cannot, however, attempt to follow the author in his detail of these varied experiments.

Vessels in narrow canals.

214. We have given at some length the experiments of Bossut relative to the resistance opposed to vessels in narrow canals. Du Buat made various comparisons between these and similar experiments, and from the several results he deduces the following formula:

$$R = \frac{K}{\frac{C}{b} + 2}, \text{ or } R = \frac{8{\cdot}46}{\frac{C}{b} + 2},$$

in which C is the area of the section of the canal, b the area of the section of the vessel, and R the resistance; the resistance in an indefinite fluid being 1, and K a constant quantity = 8·46.

In order to compare this formula with experiment, the author selected the results arising from five kinds of prismatic boats, several feet in length, and terminated both fore and aft by a plane surface.

The several dimensions of these were as follow:

N° 1 had its immersed part a rectangle of 1 foot base, and 1 foot height.

N° 2 had 2 feet upon the base, and 1 foot of immersion in height.

N° 4 and 5 had 19⅔ inches base, and 1 foot 5½ lines immersed.

N° 6 had its greatest section like that of the midship section of a ship of the line, and the area of the immersed part was 190 inches.

TABLE *of the results of Du Buat's comparisons on the resistance of boats on narrow canals.*

Canal 28½ inches wide, and 15⅙ inches deep.

Number of the vessels.	Ratios of the sections or values of $\frac{C}{b}$.	Observed resistances, or the values of R.	Values of R, as deduced from the formula.
1	3·00	1·66	1·69
2	1·50	2·50	2·41
4 and 5	1·76	2·25	2·25
6	2·275	1·94	1·97

Canal 40 inches wide, and 15⅙ inches deep.

1	4·212	1·33	1·36
2	2·106	2·11	2·05
4 and 5	2·476	1·90	1·89
6	3·192	1·92	1·62

M. Du Buat concludes, from these experiments, that a canal cannot be considered as of indefinite extent, unless its width be at least 4½ times that of the vessel, or, what is the same thing, that when this is the ratio between the width of the vessel and the canal, the vessel experiences the same resistance as if it moved on the open sea.

In order to confirm this result, the following experiments were made; in which $\frac{C}{b}$ and R in the last column are calculated by reducing the canal to 4½ times the breadth of the vessel.

Canal 75 inches wide, and 15½ inches deep.

Number of the vessels.	Ratio of the sections or values of $\frac{C}{b}$.	Observed resistances, or the values of R.	Values of R, as deduced from the formula.
1	5·81	1·053	1·08
2	4·036	1·384	1·40

Canal of indefinite width, and 15⅓ inches deep.

2	5·75	1·125	1·09
5	5·53	1·143	1·12

Canal of indefinite width, and 27¼ inches deep.

2	4·46	1·1	1·00

215. With regard to the oblique resistance, it does not appear to be the same in narrow canals as in a fluid of indefinite extent; and an angular prow, added to a prismatic vessel, produces a less diminution of the resistance, as the canal becomes more narrow. M. Du Buat expresses the resistance of an angular prow in a narrow canal by the following formula:

$$r = \frac{R - \left(R - \frac{R}{q}\right)\left(\frac{C}{b} - 1\right)}{5{\cdot}46},$$

where R is the resistance of a plane prow in a narrow canal, r the resistance of an angular prow on the same base, q the ratio between the resistances of these two prows, in an indefinite fluid; and $\frac{C}{b}$ the ratio of the sections of the canal and vessel. When the boat N° 2 had an angular prow of 45°, and moved in a canal 28½ inches wide and 15½ deep, the resistance was 4·42, whereas the formula gives 4·444.

When the same boat had an angular prow of 14° 3′ the resistance was 3·2, and the formula gives 3·25. When the canal was shut at both ends the resistance of the boat N° 1 was very nearly the same as when it was open; but with N° 2 the resistance was considerably augmented. The effects in this case become very complicated, particularly in short canals; but in actual practice, when the locks of canals are at two miles distance, the part of the canal between them may be considered as of indefinite length, and the resistance computed accordingly.

Experiments by Col. Beaufoy on the oscillation of differently formed bodies in a fluid.

216. We have in the above account of Du Buat's experiments, alluded to some which this author made on

the oscillation of bodies in water. The following by Col. Beaufoy, though not made with precisely the same views, are nearly of the same kind, and give rise to several curious results: we extract them from No. 58 of Thomson's "Annals of Philosophy." These were made by means of a pendulum rod, with bodies of different forms attached to its lowest extremity, and hanging in water. The body being thus suspended, it is obvious, that when the pendulum is drawn aside to a certain point, and then let go, the less resistance the fluid offers to the attached body, the greater will be the ascending arc described by the pendulum; and *vice versâ;* consequently, the greater or less resistance will be measured by the arc of vibration, at least some idea may hence be formed of the comparative resistance of the different bodies; but the slowness and inequality of the motion, and the passage of the figures through the water, not being rectilinear, are certainly so many causes of error, or of the inadequacy of the results for furnishing correct practical data.

The following experiments were made with a pendulum 5 feet 5·85 inches long; the lower extremity being immersed 12·7 inches.

The solids were two inches in diameter, and as much in length, with the exception of the double cone, which was four inches long, and when lengthened by a cylinder, it measured six inches (the same remark is applicable to the elliptical spindle.) The sphere, when cut in halves, and separated by a cylinder, measured four inches.

TABLE I.	Ratio of resistances.
Resistance of a cube, the angle being opposed to the fluid	1000
Resistance of the same, with the side opposed to the fluid	877
Resistance of a cylinder	1000
Resistance of a sphere	574
Resistance of a sphere cut in halves, and lengthened by a cylinder	238
Resistance of the base of a cone	1000
Resistance of the vertex, the angle being 53°08′	467
Resistance of the base of a wedge	1000
Resistance of its vertex, or edge, the angle being 53°08′	512
Resistance of a double cone	1000
Resistance of the same, lengthened by a cylinder	380
Resistance of an elliptical spindle	1000
Resistance of the same, lengthened by a cylinder	735

The following experiments were made on solids, the greatest diameter of each of which was 2 inches, and length 7 inches.

TABLE II.	
1. Elliptic spindle	1000
2. Circular spindle	847
3. Double circular spindle, the greatest breadth being $\frac{1}{4}$th from the foremost end	648
4. Ditto, ditto, greatest breadth being $\frac{5}{12}$th from the foremost end	603
5. Ditto, ditto, greatest breadth being $\frac{1}{3}$d from the foremost end	587
6. Ditto, ditto, greatest breadth being $\frac{2}{5}$ths from the foremost end	540

The numbers expressing the ratio of the resistances in this and the preceding table must only be understood to apply each to the same solid, and the manner in which they were obtained was as follows: Taking, for example, the elliptic spindle, No. 1, table II,—the pendulum rod, without any attached body, being brought up to a certain point, the measure of the ascending arc was ascertained; the spindle then being attached, the new ascending arc was found in the same manner. Call the difference of the two ascending arcs in seconds d, and the difference between the first of these, and the ascending arcs in experiments 2, 3, &c. d' d'' d''', &c. then assuming the first difference 1000, we have

$$d : d' :: 1000 : \frac{d' \times 1000}{d},$$

$$d : d'' :: 1000 : \frac{d'' \times 1000}{d}, \text{ \&c. \&c.}$$

which fourth terms, converted into numbers, are those expressed in the table.

We give these experiments, not for the sake of the deductions that have been drawn from them, but because they seem to indicate one of the most simple methods of carrying on an extensive series, from which some valuable information might be obtained.

Professor Vince's experiments on the resistance of fluids.

Vince's experiments.

217. The experiments of Mr. Vince, on the resistance of fluids, were published in 1798, in the Philosophical Transactions. They were made with bodies moving at a considerable distance under water, and from the well-established character of their author we may conclude that they were made with all the accuracy of which such experiments are susceptible. They differ in many respects in their results from those which have been detailed in the preceding articles of this treatise by Bossut, Buat, &c.; and we therefore think it important to enter upon them at some length, in order to give our readers an opportunity of estimating the discrepancy, and drawing in some measure their own conclusions from a comparison of the different results.

The series of experiments of Mr. Vince may be considered under two distinct heads, viz. as they relate to the action of water at rest on a body moving in it; and to the action of a fluid in motion on a body at rest. The following table exhibits the results of the first set of experiments, in which the 1st column shows the angle at which the plane struck the fluid; the 2d, the resistance as determined by experiment in the direction of the motion in troy ounces; the 3d, the theoretic resistance obtained by assuming the perpendicular resistance to be the same as by experiment; and the 4th the power of the sine of the angle to which the resistance is proportional.

TABLE I. *Showing the resistance of a plane surface moving in a fluid with a velocity of ·66 of a foot per second, and inclined at different angles to the line of motion.*

Inclination of the plane to the line of motion.	Resistances by experiment.	Resistances by theory.	Power of the sine of inclination to which the resistance is proportional.
10°	0·0112	0·0012	1·73
20°	0·0364	0·0093	1·73
30°	0·0769	0·0290	1·54
40°	0·1174	0·0016	1·54
50°	0·1552	0·1045	1·51
60°	0·1902	0·1476	1·38
70°	0·2125	0·1926	1·42
80°	0·2237	0·2217	2·41
90°	0·2321	0·2321	——

The 4th column was computed as follows: Let s denote the natural sine of the angle to radius 1; r the resistance at that angle, and suppose r to vary as s^m; then

$$1^m : s^m :: 0{\cdot}2321 : r;$$

whence $$s^m = \frac{r}{0{\cdot}2321},$$

and $$m \log. s = \log. r - \log. 0{\cdot}2321;$$

consequently, $$m = \frac{\log. r - \log. 0{\cdot}2321}{\log. s};$$

and, therefore, by substituting for r and s their several corresponding values, the respective values of m, as given in the preceding table, may be determined.

Comparison of theory and experiment.

218. We have seen that, according to theory, the resistances ought to vary as the cube or third power of the sine of the angle of inclination, whereas it appears from the above results, as well as from those of M. Bossut, that the resistance decreases, from an angle of 90°, in a less ratio than that, but not as any constant power, nor as any function of the sine and cosine of the angle at present discovered. Hence, the actual resistance is always greater than that which is deduced from theory, assuming the perpendicular resistance to be the same; which doubtless arises from our supposing the effect of each particle to cease with its first impact, and as a consequence of this, that we neglect entirely in our theoretic investigation all that part of the resolved force which is parallel to the plane; also the resistance of the particles which escape from the plane into the surrounding fluid, may probably tend to increase the actual resistance above that which is given by the theory. But as this latter circumstance affects the resistance at all angles, and as we do not know the quantity of it, it is impossible to say how it may affect the resistances at different angles.

The area of the planes employed in these experiments was 3·73 inches, and according to theory the perpendicular resistance equal to a column of the fluid having 3·73 inches for its base, and its altitude equal to the space through which a heavy body must fall to acquire the velocity of ·66 of a foot. Now that space is ·08124 inches, consequently, the weight of the column = 0·1598 oz. troy; but the actual resistance was found to be 0·2321 oz. Hence, the actual resistance of the planes is to the resistance given by the theory as 0·2321 : 0·1598, or as 3 to 2 nearly.

Disagreement of these and previous deductions.

Our readers will perceive that this result is very different from that deduced from the experiments carried on by D'Alembert, Condorcet, and Bossut; and as we can have no doubt of the accuracy of the latter experiments, while at the same time we have every reason to rest with confidence on those of Mr. Vince, we must attribute the disagreement to the different circumstances under which the experiments were performed. Now one of the most striking differences appears to be, that those of Professor Vince were made at some depth below the surface of the fluid, while the others were made on the surface itself; and it is not difficult to imagine, that the fluid may escape much more freely from the surface on which it impinges in the latter case than in the former, and, consequently, that experiment and theory ought to approach nearer to a co-incidence in this case than when the surface is wholly immersed in the fluid.

219. Mr. Vince's next experiments were made on two hemispheres, or semi-globes, which were made to revolve with their flat sides foremost. The diameter of each was 1·1 inch, the distance of the centre of resistance from the axis was 6·22 inches, and they moved with a velocity of ·542 feet per second. The resistance, as found by experiment, was ·08339 oz. which, by the common theory, ought to be ·05496. oz. Wherefore, the resistance by experiment is to the resistance from theory as ·08339 to ·05496, that is again nearly as 3 to 2. But when the convex surface moves forward with the same velocity, the resistance was ·034 oz. Consequently, the resistance on the spherical side of a hemisphere is to the resistance on its base as ·034 to ·08339, which does not very well agree with the ratio determined by the theory, that is to say of 1 to 2. Moreover, according to the theory, there is no difference between the resistance of a semi-globe and that of a perfect sphere: the figure of the hinder part of the body not entering into the investigation, whereas in actual experiment this figure has a very considerable influence. Mr. Vince, therefore, took two cylinders of the same diameter as the two semi-globes, and of the same weight; and by giving them the same velocity, he found the resistance to be ·07998 oz.; therefore, the resistance on the flat side of a hemisphere is to the resistance of a hemisphere of the same diameter, and moving with the same velocity, as ·08339 to ·07998. This difference can obviously arise only from the action of the fluid, on the back part of the semi-globe being less, when moving with its flat side foremost, than on the back of the cylinder; in consequence of which the former suffers the greatest resistance.

The resistance of cylinders determined directly from experiments on planes, agrees very well with the foregoing deductions. The resistance, *cæteris paribus*, varies as the square of the velocity very nearly, and may be taken so for all practical purposes, according to the results of various experiments performed by our author on different bodies both in air and water. Hence, for different planes, the resistance varies as the areas into the squares of the velocity. Now, the resistance of planes whose area is 3·73 inches, moving with a velocity of ·66 feet per second, was found to be 0·2321 oz. The area of the end of the two cylinders was $$2 \times (1{\cdot}1)^2 \times {\cdot}7854 = 1{\cdot}9 \text{ inch};$$
and their velocity was ·542 feet per second; to find, therefore, the resistance of the cylinders from that of the planes, we have

$$0{\cdot}66^2 \times 3{\cdot}73 : 0{\cdot}542^2 \times 1{\cdot}9 : 0{\cdot}2321 : 0{\cdot}07973$$

for the resistance on the cylinders, which differs but little from 0·07998, as determined from actual experiment.

To get the resistance on a perfect globe, we must consider that when the back part is spherical, the resistance is greater than when it is flat in the ratio of ·08339 to ·07998; hence, the resistance on a globe is to the resistance on a semi-globe in the same ratio; but the resistance on the semi-globe was ·034 oz.; hence,

$${\cdot}07998 : {\cdot}08339 :: {\cdot}034 : {\cdot}0354,$$

the resistance of a globe; consequently, the resistance of a globe is to the resistance of a cylinder of the same diameter, and moving with the same velocity in water, as $${\cdot}0354 : {\cdot}07998, \text{ or as } 1 : 2{\cdot}23.$$

Let us now compare the actual resistance of a globe with the resistance assumed by the theory. In the first place, the absolute quantity of resistance has been found to be greater than that which we employ in theory, in the ratio ·2321 to ·1598; but by theory the resistance of the globe is to the resistance of the cylinder as 1 to 2, or as 1·115 to 2·23: hence, by theory, we make the resistance of the globe too great in the ratio of 1·115 to 1; and it is too small from the preceding consideration in the ratio of 0·1598 to 0·2321; therefore, the actual resistance of the globe is to the resistance in theory, as

0·2321 : 0·1598 × 1·115 :: 0·2321 : 0·1782,

which is nearly in the ratio of 4 to 3.

Thus far we have considered the resistance of a body moving in a fluid at rest; we shall now examine the case in which the body is at rest and the fluid in motion.

220. The following table contains the results of the experiments, made by the same author, with reference to the above determination. In which the first column shows the angle between the direction of the fluid, and that of the plane on which it impinged; the second contains the results as deduced from experiment, and the third the resistances, as computed by the theory.

TABLE II. *Showing the resistance of a plane surface struck by a fluid in motion, and inclined to the same at various angles.*

Inclination of the plane to the line of fluid's motion.	Resistances from experiment.			Resistances by theory.		
	oz.	dwts.	grs.	oz.	dwts.	grs.
90°	1	17	12	1	17	12
80°	1	17	0	1	16	22
70°	1	15	12	1	15	6
60°	1	12	12	1	12	11
50°	1	18	10	1	18	17
40°	1	4	10	1	4	2
30°	0	18	18	0	18	18
20°	0	12	12	0	12	12
10°	0	6	4	0	6	12

The third column is computed by assuming, that the resistance is as the sine of the angle of inclination, and the agreement between the results thus obtained and those determined experimentally, is as accurate as can possibly be expected, the small deviation in some cases being no more than may very fairly be attributed to slight inaccuracies in conducting the experiments, or in the observations.

Having obtained these results, Mr. Vince next proceeds to consider what the whole perpendicular resistance by experiment is, when compared with that by the theory. In the latter, he observes, the resistance is equal to a column of the fluid whose base is ·045 inch, and altitude 45·1 inches; the weight of such a column is 1 oz. 1 dwt. 10 gr. Hence, the resistance by theory is to the resistance by experiment as 1 oz. 1 dwt. 10 gr. to 1 oz. 17 dwt. 12 gr.; or as 514 : 900. In the next place, in order to compare this resistance with the resistance of a plane moving in a fluid, it is to be observed, that it is shown above that the resistance of the fluid in motion acting on the plane at rest, is to the resistance by theory, as 900 to 514; and we have before seen that the resistance by theory, is to the actual resistance of a plane body moving in a fluid, as 1598 to 2321: hence, the resistance of a fluid in motion on a plane at rest, is to the same plane moving with the same velocity in a fluid at rest as 900 × 1598 to 514 × 2321, or as 6 to 5 nearly, which agrees with the deductions of Du Buat.

Now, we know that the actual effect on the plane must be the same in both cases; and the difference can only, we conceive, arise from the action of the fluid behind the body in the latter case.

We must here bring to a conclusion our detail of these experiments. Mr. Vince pushes his examination on to other inquiries, but for these we must refer the reader to the volume of the Phil. Trans. above quoted.

The society for the advancement of Naval Architecture made a series of valuable experiments on this subject, with a surface of 40 square feet, moving with different velocities, from which the following are abstracted, viz.

Velocities.	Resistance when moving in its own direction.
1 mile per hour	0·563 lbs.
2	1 992
4	6·642
6	12·839
8	19·856.

From these, and other experiments, made by this society, it appeared that the direct resistance varied as $v^{2\cdot106}$.

Dr. Young's formulæ.

221. Dr. T. Young has bestowed considerable attention to the subject of the resistance of fluids, and has given, in the Journal of the Royal Institution, a valuable table, containing a comparison of different formulæ, with the experiments of Eytelwein, Bossut, and those of the society for the encouragement of Naval Architecture. In these formulæ *a* denotes the angle of inclination, and R the risistance.

Formula (A) deduced by Dr. Young, is

$$R = \cos^2 a + \tfrac{1}{10} \tan a.$$

Formula (B) as deduced from theory, is

$$R = \tfrac{2}{10} + \tfrac{1}{100} \tan. a + 288 \cos^2 a \div 360° + a°.$$

Formula (C) as deduced from experiment, is

$$R = \cos^2 a + \cdot 0000004217\, a^{3\cdot10},$$

in which the last term is a little less than one millionth of the cube of the angle of incidence expressed in degrees.

Eytelwein's formula is

$$R = \cos^2 a + \cdot 4 \text{ vers. } a.$$

TABLE, *containing Dr. Young's comparison of different formulæ and experiments.*

Angles of inclinations or values of *a*	Cos ^2a.	Tan *a*.	Formula (A).	Formula (B).	Formula (C).	Eytelwein's formula.	Bossut's experiments.	Experiments of the Society for Naval Architecture.
0	1·0000	·000	1.0000	1·0000	1·0000	1·0000	1·0000	
6	·9890	·105	·9995	·9824	·9891	·9910	·9893	
12	·9568	·212	·9780	·9492	·9580	·9656	·9578	
18	·9045	·325	·9370	·9022	·9086	·9241	·9084	
24	·8346	·445	·8791	·8438	·8449	·8690	·8446	
30	·7500	·577	·8077	·7769	·7710	·8036	·7710	
36	·6544	·726	·7270	·7049	·6919	·7308	·6925	
42	·5523	·900	·6423	·6317	·6135	·6551	·6148	
48	·4478	1·111	·5589	·5606	·5414	·5802	·5433	
54	·3455	1·376	·4831	·4985	·4816	·5103	·4800	
60	·2500	1·732	·4232	·4407	·4403	·4500	·4404	·347
66	·1654	2·346	·4000	·3924	·4231	·4026	·4240	
72	·0955	3·078	·4033	·3869	·4344	·3719	·4142	·269
78	·0432	4·705	·5137	·4166	·4816	·3600	·4063	·222
84	·0109	9·514	·9623	·5875	·5658	·3693	·3999	

The details given in the preceding pages, appear to be those most to be depended upon for accuracy, and for utility, of all that has been written on this extensive subject; and our object has necessarily been to present to the reader a selection of experiments which shall best accord with the limits and plan of this work.

§ XVI. *On the oscillatory motion of water in a siphon.*

Oscillations of a fluid.

222. It has been shown (art. 80), DYNAMICS, that if a body, or pendulum, suspended by a line, be made to describe small circular arcs, by oscillating about a fixed point O, or in any arc of a cycloid, all these oscillations will be *isochronous*, that is, they will each be performed in the same time. It has also been shown, that the times of vibration of pendulums of different lengths, are as the square roots of those lengths, and we shall see the motion of water which balances itself in a siphon, is of the same kind.

In a rectangular siphon.

Fig. 31.

223. Suppose it were required to determine the motion of oscillation of a fluid in a siphon, as KLMN (fig. 31), the interior being throughout of an uniform diameter, and the whole composed of one horizontal and two ascending vertical branches. Let us conceive at first, that the fluid, in a state of rest, occupies the space ALND; then the two surfaces AB, CD, are on the same level; and let us suppose now, that from some cause, such for example as the action of a piston, the liquid is forced to descend to GH, in the branch MN, and, consequently, to rise in the branch KL to EF. This being supposed, let us imagine the piston to be removed, and the water left to the action of its own gravity. It is clear that the water will rise and fall alternately, forming oscillations similar to those of the pendulum. Let P (fig. 32) be a pendulum, of which the length OP is the half of the length $x\ y\ z$ of the column of fluid, and which describes at the lowest point I, arcs PI, equal to the spaces EA. Now, the force which causes the water to oscillate, is the excess of the weight of the fluid contained in one of the branches of the siphon, over that of the weight of the water contained in the other branch. Thus, when the fluid ascends to EF in the branch KL, and, consequently, decends to GH in the branch NM, this force is the weight of the column ESTF; or the double of the weight of the column EABF. It is, therefore, to the weight of all the water, as EA : $x\ y\ z$, or as AE : OP. Whence it follows; 1st. that the length $x\ y\ z$, being a constant quantity, the force which causes the water to oscillate, is always proportional to the space it has to describe, and that, consequently, the oscillations of the water are isochronous (art. 28), DYNAMICS. 2dly. That these oscillations are of the same duration as those of the pendulum P; for the force which causes the pendulum to describe the small arc PI, is to the gravity of the pendulum, as PI to PO, or as AE is to OP. The water and the pendulum are then solicited by the same force, and ought, consequently, to make their oscillation in the same time.

Fig. 32.

Hence, since the oscillations of water follow the same laws as those of the pendulum, if we augment or diminish the column of water, the time of these oscillations will also be augmented, and the times will be in the proportion of the reciprocals of the square roots of the lengths.

Siph any

224. Let us now examine the circumstances of the motion of water in a siphon of any form.

Fig.

Let KLMN (fig. 33) be a siphon of any form, containing a fluid which having been elevated towards K by some exterior cause, is arrived at the indeterminate position EFGH, where it is left freely to the action of its own weight or gravity; and let it be required to determine the nature of the oscillations with which it will in consequence be affected. In order to this we ought to know the directions that the particles of the fluid take in their motion. We may suppose here, either that these laminæ are horizontal, or that they are perpendicular to the curve K g h M, regarded as the axis of the siphon. The calculation, on the first hypothesis, is similar to that by which we estimate, theoretically, the equation between the height of the water in a vessel and the velocity of its discharge through a given aperture, because we may consider for an instant EFGH as a vessel or tube, and EF as the upper surface of the fluid, and GH the lower. The second hypothesis is that adopted by many of the most celebrated writers; the investigation in this case is as follows:

Let us imagine that the fluid EFGH is parted into an infinite number of laminæ, of which LO o l is one; these being all equal to each other, and perpendicular at every point to the curve $g f h$: gravity, which acts vertically, on each of these laminæ, may be resolved into two forces; the one perpendicular to the curve, which may be neglected, and the other in the direction of the curve; this being the only one that is effective.

Let the force of gravity .... $= 2\,g$,
the surface EF .... $=$ P,
the surface GH .... $=$ Q,
the section LO .... $= y$,
the arc $g f$ of the curve $g f h$ .... $= x$,
the radius .... $= 1$,
the angle $m f n$ which the curve makes at f with the vertical .... $= p$,
the velocity of the surface EF .... $= u$,
the height due to this velocity .... $= r$,
the velocity of the section LO ..... $= v$.

These substitutions being made, it is clear that if the particles did not impede each other's motion, the velocity v at the end of the time t (since the part of gravity which acts in the direction $f n$ is $2\,g \cos p$), ought to be

$$v + 2\,g \cos p\, \dot{t}.$$

But as, in consequence of the action of the particles, it becomes $v + \dot{v}$, it follows that the different laminæ urged with the velocity

$$2\,g \cos p\, \dot{t} - \dot{v},$$

would, by the principle of D'Alembert, produce an equilibrium; we shall therefore have

$$\int \dot{x}\,(2\,g \cos p\, \dot{t} - \dot{v}) = 0;$$

whence, putting instead of $\dot{t}$ its equal $\frac{\dot{x}}{v}$, and for v its equal $\frac{\text{P}\,u}{y}$, we have

$$\frac{2\,g\,y\,\text{M}}{\text{P}\,u} \int \cos p\, \dot{x} - \text{P}\,\dot{u} \int \frac{\dot{x}}{y} + \text{P}\,u\,y\,\dot{x} \int \frac{\dot{y}}{y^3} = 0.$$

These fluents ought to answer to the entire curve $g f h$. Let then

$$\int \cos p\, \dot{x} = \text{F};$$

$$\int \frac{\dot{x}}{y} = \text{N};$$

and let us also assume

$$\int \frac{\dot{y}}{y^3} = \frac{1}{2\,\text{P}^2} - \frac{1}{2\,\text{Q}^2};$$

and the preceding equation will then become

$$\text{F}\,.\,\text{Q}^2 y\,\dot{x} - \text{P}^2\,\text{Q}^2\,\text{N}\,\dot{r} - r\,y\,\dot{x}\,(\text{Q}^2 - \text{P}^2) = 0, \quad (1)$$

a formula which expresses the motion of the fluid due to any time t.

Now if, at the first instant of the motion, the fluid occupy the space VZRN, and that we call z the space $k g$ passed over by the surface EF during the time t; it is obvious, since the nature of the curve KGHM is given, and the two spaces VZRN, EFGH, occupied successively by the fluid are equal to each other, that the quantities F, N, P, Q, $y\,\dot{x}$ may be expressed in functions of z and constant quantities.

Hydraulics.

Therefore, the velocity of the surface EF, or the altitude r, due to this velocity, will be also a function of z; and in like manner t will be a function of z, from the equation $\dot{t} = \frac{\dot{z}}{u}$.

Let us apply these results to some particular examples.

Semicircular siphon.

225. *Suppose the siphon to be cylindrical, and bent into the curve of a semicircle; the diameter* KM *being horizontal.*

We shall have in this case $\text{P} = \text{Q} = y$, and each of these quantities will be constant. Let there be drawn to the diameter KM, the ordinate $f q$, and call the radius CK equal to unity; the semi-circumference $\text{K}\,g\,h\,\text{M} = m$; the given arc $g f h$, to which the water corresponds in all positions of the siphon $= n$; $f q = s$; and the indeterminate arc $\text{K}\,g f = w$.

Then we shall have

$$\cos p = \frac{\dot{s}}{\dot{w}} = \frac{\text{flux.} \sin w}{\dot{w}} = \cos w,$$

$$\textstyle\int \dot{x} \cos p = \int \dot{x} \cos w = \int \dot{w} \cos w = \sin w + \text{C},$$

a fluent which ought to be taken between the limits $w = z$, and $w = z + n$;

Now $\quad \text{F} = \sin(z + n) - \sin z;$

also $\quad \text{N} = \frac{n}{p};\ y\,\dot{x} = \text{P}\,\dot{z};$

whence equation (1), in this case, becomes

$$\dot{z}\,\{\sin(z + n) - \sin z\} - n\,\dot{r} = o.$$

Wherefore, supposing the fluid to depart from the point K (and consequently $r = o$, when $z = o$), we shall have

$$n\,r = \cos n - 1 + \cos z - \cos(z + n);$$

which gives r, or the height due to the velocity of the surface EF.

If we make $z = m - n$, or if we suppose the anterior surface GH to come to M, we shall again find $r = o$. Whence it follows that the fluid will then have lost all its velocity, and will re-descend; and thus continue to oscillate between the two extremes K and M of the siphon.

The time of the vibrations will be found, by means of the equation,

$$\dot{t} = \frac{\dot{z}}{u} = \frac{\dot{z}}{2\sqrt{g\,r}} =$$

$$\frac{\sqrt{n}}{2\sqrt{g}} \times \frac{\dot{z}}{\sqrt{\{\cos n - 1 + \cos z - \cos(z + n)\}}};$$

the fluents of which must be taken in such a manner, that t may vanish when $z = o$, and that t may receive its complete value when $z = m - n$.

Rectilinear siphon.

226. Let now the siphon be composed of three rectilinear tubes of equal diameters, of which the intermediate one is horizontal and the two others inclined.

Fig. 34.

Draw through the points f and m (fig. 34), the verticals $f i$ and $m s$, and call f the cosine of the given angle K$f i$; m the cosine of the given angle M $m s$; P the constant and perpendicular section of each tube; l the given length $g f m h$ of the space occupied by the fluid; b the given length Kf of the first inclined tube; c the given length of the horizontal tube; and z the space passed over by the fluid during any time t.

Now, as the law of continuity is not observed from one tube to the other, we may apply to each of them the reasonings advanced in the general solution; and we shall thus find, that for the tube EFOL, the quantity

$$\int \dot{x} \cos p = f(b - \dot{z});$$

and that for the tube GHNR,

Rectilinear siphon.

$$\int \dot{x} \cos p = m h \times m = m(z + l - b - c);$$

and, finally, for the horizontal tube,

$$\int \dot{x} \cos p = 0.$$

Subtracting the second expression from the first (because the sign of m is contrary to that of f), the remainder

$$f(b - z) - m(z + l - b - c)$$

will be the value of F for the entire siphon; moreover, the quantity

$$\int \frac{\dot{x}}{y} \text{ or N will be here } \frac{l}{\text{P}};$$

consequently (observing that $y = \text{P}$, $\dot{z} = \dot{x}$, and $\text{Q} = \text{P}$), the general equation (1) becomes

$$\dot{z}\{f(b - z) - m(z + l - b - c)\} - l\dot{r} = 0;$$

whence we obtain

$$r = \frac{f + m}{2l} \times \left\{\frac{2(fb + mb + mc - ml)z}{f + m} - z^2\right\};$$

wherefore, because

$$\dot{t} = \frac{\dot{z}}{u} = \frac{\dot{z}}{2\sqrt{gr}},$$

we shall have $t =$

$$\frac{\sqrt{l}}{\sqrt{(2g(f+m))}} \times \int \frac{\dot{z}}{\sqrt{\left\{\frac{2z(fb+mb+mc-ml)}{f+m} - z^2\right\}}},$$

a fluent which depends generally upon the quadrature of the circle.

For the sake of abridgment, let us represent the coefficient of z by 2 A; and call B the quadrant of the circle, whose radius is A; also T the time employed in describing the space A.

Then we shall have

$$t = \frac{\sqrt{l}}{\text{A}\sqrt{\{2g(f + m)\}}} \times \int \frac{\text{A}\dot{z}}{\sqrt{(2\text{A}z - z^2)}},$$

$$\text{and} \quad \text{T} = \frac{\sqrt{l}}{\sqrt{\{2g(f + m)\}}} \times \frac{\text{B}}{\text{A}}.$$

Now $\frac{\text{B}}{\text{A}}$ is a constant quantity, whatever may be the value of A; therefore, T is a constant quantity; and, consequently, the entire oscillations of the fluid are isochronous, whatever may be their amplitudes.

Let L represent the length of a simple pendulum which describes small arcs of a circle; C the initial distance of this pendulum from the vertical; D the quadrant of a circle, whose radius is L; z the circular arc which the pendulum describes in the time t with the velocity v; and T the time which it employs to arrive at the vertical.

Then we shall have

$$v\dot{v} = \frac{2g(\text{C} - z)\dot{z}}{\text{L}};$$

and, consequently, by taking the fluents

$$v^2 = \frac{2g(2\text{C}z - z^2)}{\text{L}};$$

therefore,

$$t = \int \frac{\dot{z}}{v} = \frac{\sqrt{\text{L}}}{\sqrt{2g}} \times \int \frac{\dot{z}}{\sqrt{(2\text{C}z - z^2)}},$$

$$\text{and} \quad \text{T} = \frac{\sqrt{\text{L}}}{2g} \times \frac{\text{D}}{\text{C}}.$$

Equating this value of T with that previously determined for the oscillations of the fluid, and observing that $\frac{\text{B}}{\text{A}} = \frac{\text{D}}{\text{C}}$, we shall have $\text{L} = \frac{l}{f + m}$; this expresses, therefore, the length of a pendulum which makes its oscillations in the same time as the fluid.

This is the result given by John Bernoulli, without demonstration, in vol. iii. p. 125 of his works.

When the two ascending tubes are vertical, then $m = 1$, and $f = 1$, and the expression

$$\text{L} = \frac{l}{f + m} = \frac{l}{2} = \tfrac{1}{2}l;$$

that is to say, the length of the pendulum which makes its oscillations in the same time as the fluid, is then equal to half the length of the column of fluid; which agrees with our preceding deduction.

227. Newton, in his Principia, considers, in connection with the oscillatory motion of water in a siphon, the undulatory motion of an indefinite fluid mass.

Thus, A, C, E, &c. (fig. 35) being the tops or highest points of fluid waves, and B, D, F, &c. the intermediate cavities that separate them, he supposes that the succession of waves is made in such a manner, that the highest parts A, C, E, &c. become afterwards the lowest, and that the force which causes them to ascend and descend is always the weight of the elevated water; consequently, admitting this hypothesis, the oscillations of the waves may be immediately referred to those of water in a siphon. If, therefore, we suppose a pendulum, of which the length is equal to half the distance between any two consecutive highest and lowest points, then the highest points will become the lowest in the same time as the pendulum will make one oscillation; and, again, in the same time, the lowest points will have become the highest. The pendulum will therefore make two vibrations during the time of one complete undulation; that is to say, during the time that each wave passes from one highest point to the next in succession; the distance of which two points expresses, therefore, the length of each complete undulation. And as the pendulum, whose length is four times the preceding, will have the time of vibrations doubled, our author concludes, that the time of one undulation in an indefinite fluid is equal to the time in which a pendulum would perform one vibration; the length of the pendulum being equal to the distance of any two consecutive waves, or to the length of one undulation. Therefore, the distance between the tops of two consecutive waves being given, the time of their motion may be determined on the same principles as the times of vibration of a pendulum of a given length.

Newton, it must be observed, gives this only as an approximation, and it is obviously nothing more; for it supposes that the particles of the fluid ascend and descend in right lines, as in the rectangular siphon, instead of the curvilinear motion which they actually take.

The motion of waves, however, may be determined analytically, by introducing into our investigations the general laws of the motions of fluids; but the reduction of the resulting equations is necessarily very intricate and laborious. We may refer our readers to a paper on this subject, by La Place, in the Memoirs of the Academy of Sciences for 1776; to another by Lagrange, in the Memoirs of Berlin for 1781; and to the last chapter in the "Traité de Mecanique," by Poisson; to which latter work we have been indebted for the sketch of this general method given in the following section.

§ XVII. *General equations for the equilibrium and motions of incompressible fluids.*

228. In order to determine the circumstances attending the motion of fluids on the general principles alluded to in the conclusion of the last article, it will be necessary, in the first place, to take a general view of the conditions of their equilibrium, and to reduce our preceding determinations connected with this inquiry into the form of general equations; a proceeding which we purposely avoided in the early part of this treatise, for the sake of simplifying our deductions, and thereby rendering them more intelligible to such of our readers who may not be familiar with fluxional operations and the calculus of variations.

Let x, y, z denote the rectangular co-ordinates of any point of the fluid mass ABCD, parallel to the three axes Ox, Oy, Oz (fig. 36), and conceive, for the sake of perspicuity, the plane Oxy to be horizontal, and, consequently, Oz vertical. Let us also represent by X, Y, and Z, the sum of the components (respectively parallel to the axes of x, y, and z) of the accelerating forces which act on the given point; and conceive the fluid mass to be divided into an indefinite number of infinitely small elements, by planes parallel to the co-ordinates and indefinitely near to one another. These elements will thus become rectangular parallelopipeds, which will have their faces parallel to the axes and equal to the fluxions or differentials of the co-ordinates. The two horizontal faces of the element which correspond to the co-ordinates x, y, and z, and which is that shown in the figure, will be equal to $\dot{x}\dot{y}$; its vertical height will be equal to $\dot{z}$; and its volume will, therefore, obviously be expressed by $\dot{x}\dot{y}\dot{z}$. The density of the same, whether constant or variable, with reference to the whole mass, may be considered as constant for this element; and if we denote it by d, and the element of the mass by $\dot{m}$, we shall have

$$\dot{m} = d \,.\, \dot{x}\dot{y}\dot{z}.$$

If the fluid be homogeneous, d will be a constant quantity; and it will be a function of x, y, z, if the density be variable. In the same manner we may regard the forces X, Y, and Z, as constant, throughout the entire extent of the element $\dot{m}$; and, consequently, the products $\mathrm{X}\dot{m}$, $\mathrm{Y}\dot{m}$, $\mathrm{Z}\dot{m}$, will express the motive forces of the particle $\dot{m}$ parallel to the axes Ox, Oy, Oz.

This being established, we may observe, that the parallelopiped $\dot{x}\dot{y}\dot{z}$ is pressed from without, inwards, on its six faces, by the contiguous particles of the fluid, and that the pressure which it experiences ought to produce an equilibrium with the three forces

$$\mathrm{X}\dot{m},\ \mathrm{Y}\dot{m},\ \text{and}\ \mathrm{Z}\dot{m}.$$

Let us denote by p, the vertical pressure on an unit of surface, on the upper face of the element $\dot{x}\dot{y}\dot{z}$; that is, on the face $\dot{x}\dot{y}$, in the direction $c\,b$, which vertical pressure will vary with the position of the element we are considering. The quantity p, which is supposed to denote this value for every element, is therefore variable, and a certain function of x, y, z; consequently, when x and y remain constant, and z becomes $z + \dot{z}$, p will become $p + p\dot{z}$, or $p + \frac{\dot{p}}{\dot{z}}\dot{z}$, and will obviously express the pressure on an unit of surface, as referred to the lower horizontal face of the element $\dot{x}\dot{y}\dot{z}$; whence it follows, that the vertical and opposite pressures sustained by the element on its upper and lower surfaces, in the directions $c\,b$ and $c'\,b'$, will be equal to

$$p\,\dot{x}\dot{y},\ \text{and}\ \left(p + \frac{\dot{p}}{\dot{z}}\,\dot{z}\right)\dot{y}.$$

The first of these obviously tends to sink the element, and the second to elevate it; and, consequently, it follows, since all the particles are supposed in equilibrio, that the excess of the second of these pressions over the first, must be equal to the vertical force $\mathrm{Z}\,\dot{m}$; this consideration leads us to the equation

$$\frac{\dot{p}}{\dot{z}}\,\dot{x}\dot{y}\dot{z} = \mathrm{Z}\,\dot{m};$$

and exactly in the same manner we obtain

$$\frac{\dot{q}}{\dot{x}}\,\dot{x}\dot{y}\dot{z} = \mathrm{X}\,\dot{m},$$

$$\frac{\dot{r}}{\dot{y}}\,\dot{x}\dot{y}\dot{z} = \mathrm{Y}\,\dot{m},$$

where q and r represent the pressures on an unity of surface, when referred to the faces parallel to the planes of $x\,z$, and $y\,z$, and nearest to those planes.

Substituting, in these equations, the value of $\dot{m}$, and suppressing afterwards the common factor $\dot{x}\,\dot{y}\,\dot{z}$, they become

$$\frac{\dot{p}}{\dot{z}} = d\,\mathrm{Z},\ \frac{\dot{q}}{\dot{y}} = d\,\mathrm{Y},\ \frac{\dot{r}}{\dot{x}} = d\,\mathrm{X}.$$

Now, if the elements into which we have supposed the mass to be divided, were solids instead of being fluids, there would be no necessary relation between the pressures which each of the faces of our parallelopiped would have to sustain, provided only that those which were opposite were equal to each other; but as the element partakes of the fundamental property of fluids, and being, by the hypothesis, in equilibrio, each of the forces due to an unit of surface, and which have been denoted by p, q, and r, are equal to each other, or at least the difference between them must be indefinitely small. In fact, the pressure which the contiguous particles exert on any one of the faces of the element, is transmitted to the other faces, through the medium of the fluid particle of which the element is composed; whence it follows, that if we represent by $p\,\dot{x}\,\dot{y}$, the pressure which has place on one of the horizontal faces, the pressure transmitted to the vertical faces ought, in the same manner, to be represented by $p\,\dot{x}\,\dot{z}$, $p\,\dot{y}\,\dot{z}$; and, therefore, in order to have the entire pressure $q\,\dot{x}\,\dot{z}$ exerted on the face parallel to the plane of $x\,z$, and nearest to that plane, we must add to the term $p\,\dot{x}\,\dot{z}$, the pressure due to the motive forces $\mathrm{X}\,\dot{m}$, $\mathrm{Y}\,\dot{m}$,

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$Z\,\dot m$, of the element. Now, although we may not know the actual value of these forces, it is, nevertheless, evident, that it is an infinitesimal of the third order, the same as the element $\dot m$; the quantity $q\,\dot x\,\dot z$ is therefore equal to $p\,\dot x\,\dot z$ *plus*, an infinitesimal of the third order; whence, dividing by $\dot x\,\dot z$, it will appear, that q differs from p by an infinitely small quantity, and the same conclusion obviously applies to the quantities p and r. We may, therefore, suppose p, q, and r, to be equal to each other, and modifying accordingly our last three equations, they become

$$\frac{\dot p}{\dot z} = d\,Z, \frac{\dot p}{\dot y} = d\,Y, \frac{\dot p}{\dot x} = d\,X. \qquad (1)$$

These are the general equations of the equilibrium of fluids. Let us now proceed to establish, on similar principles, the general equations of their motion.

Motion of fluids.

229. We shall here, as before, denote by x, y, z the co-ordinates of any point of the fluid mass, also still represent by X, Y, Z the components of the forces which act on this point, parallel to the axes x, y, z. Then the quantities X, Y, and Z are simply functions of x, y, and z, while the forces of which they are the components, change not their intensities during the motion, and are directed towards fixed centres; but when these forces are directed towards moveable centres, and when they proceed from the mutual attraction of the fluid particles, the values of X, Y, and Z, will contain the time in their expressions; so that denoting at any instant by t, the time from the origin of the motion, the values of X, Y and Z, will be general functions of x, y, z and t.

Let us resolve into the directions of the axes $O\,x$, $O\,y$, $O\,z$, the velocity of the point which answers to the co-ordinates x, y, z, and let v, u, w, be the respective components parallel to these axes; then u, v, w will be unknown functions of x, y, z, t: they will depend upon x, y, z, because, that for the same instant, or with the same value of t, the velocity varies from one particle to another, both in magnitude and direction; and they will depend on the time t, because that in the same place, or with the same values of x, y and z, the velocity varies with the time. If we wish to compare together the velocity of one and the same fluid particle in two consecutive instants, we must suppose that the variable t becomes $t + \dot t$, and at the same time that the co-ordinates x, y, z, become respectively

$$x + u\,\dot t,\; y + v\,\dot t,\; z + w\,\dot t;$$

for in consequence of these velocities u, v, w, the same particle which answers to the co-ordinates x, y, z, at the end of the time t, will answer to the co-ordinates

$$x + u\,\dot t,\; y + v\,\dot t,\; z + w\,\dot t,$$

at the end of the time $t + \dot t$.

It follows, therefore, that in order to have the variation of the quantities u, v, w, relatively to the same particle, we must take the fluxion with reference to t, and with reference to x, y, z, taking $u\,\dot t$, $v\,\dot t$, $w\,\dot t$, for the increase of these last variables. We shall thus have

$$\dot u = \frac{\dot u}{\dot t}\,\dot t + \frac{\dot u}{\dot x}\,u\,\dot t + \frac{\dot u}{\dot y}\,v\,\dot t + \frac{\dot u}{\dot z}\,w\,\dot t,$$

$$\dot v = \frac{\dot v}{\dot t}\,\dot t + \frac{\dot v}{\dot x}\,u\,\dot t + \frac{\dot v}{\dot y}\,v\,\dot t + \frac{\dot v}{\dot z}\,w\,\dot t,$$

$$\dot w = \frac{\dot w}{\dot t}\,\dot t + \frac{\dot w}{\dot x}\,u\,\dot t + \frac{\dot w}{\dot y}\,v\,\dot t + \frac{\dot w}{\dot z}\,w\,\dot t.$$

Eq of mo ves

230. Let us now, as in the investigation relative to the equilibrium, conceive the fluid mass to be divided into an indefinite number of infinitely small rectangular parallelopides, of which the sides or faces are respectively parallel to the planes of the axes $O\,x$, $O\,y$, $O\,z$.

The volume of the element which answers to the co-ordinates x, y, z, will have, as in the former case, for its expression $\dot x\,\dot y\,\dot z$, and we may regard the density of this element as constant through the entire volume of it. Representing this density as before by d, the mass of the element will be $d\,\dot x\,\dot y\,\dot z$, or $\dot m = d\,\dot x\,\dot y\,\dot z$.

Represent also by p, the pressure which the fluid exercises on an unit of surface on the different faces of the elementary rectangular parallelopiped, and which is the same on all sides, according to the fundamental property of fluids.

Now, the quantities d and p, as well as the velocities u, v, w, are unknown functions of the variables x, y, z, t, and the former five quantities, viz. u, v, w, d, and p, are the unknowns required to be determined. When we have found the means of expressing them in terms of x, y, z, and t, the state of the fluid mass will be determined for any instant, for we shall then know the velocity, direction, the density of the fluid, and the pressure which it exercises in any point taken at pleasure, either on the surface, or in the interior of the mass. It remains, therefore, for us to establish the equations from which these five quantities may be determined.

Three of these five equations are given immediately, by the principle of D'Alembert; in fact, the velocities lost during the time $\dot t$, by the particle submitted to the action of the three forces X, Y, Z, are

$$X\,\dot t - \dot u,\; Y\,\dot t - \dot v,\; Z\,\dot t - \dot w;$$

for $\dot u, \dot v, \dot w$, express the actual increases of velocities which take place during this instant; and $X\,\dot t$, $Y\,\dot t$, $Z\,\dot t$, are those which would obtain in the same time from the action of the forces X, Y, Z, if the particles were entirely free.

Dividing now by $\dot t$, in order to have the accelerating forces capable of producing these velocities, and denoting them by X′, Y′, Z′, we shall have

$$X - \frac{\dot u}{\dot t} = X',$$

$$Y - \frac{\dot v}{\dot t} = Y',$$

$$Z - \frac{\dot w}{\dot t} = Z'.$$

Or, substituting for $\dot u, \dot v, \dot w$, their values as above determined, these equations become

$$X - \frac{\dot u}{\dot t} - \frac{\dot u}{\dot x}\,u - \frac{\dot u}{\dot y}\,v - \frac{\dot u}{\dot z}\,w = X',$$

$$Y - \frac{\dot v}{\dot t} - \frac{\dot v}{\dot x}\,u - \frac{\dot v}{\dot y}\,v - \frac{\dot v}{\dot z}\,w = Y',$$

$$Z - \frac{\dot w}{\dot t} - \frac{\dot w}{\dot x}\,u - \frac{\dot w}{\dot y}\,v - \frac{\dot w}{\dot z}\,w = Z'.$$

Equations investigated.

Now, according to the principle above cited, there will be an equilibrium in the fluid mass if every particle of it be impressed with forces capable of producing the velocity lost or gained at each instant. Therefore, the general equations appertaining to the equilibrium of fluids, as determined (art. 229), ought to be established by taking X′, Y′, Z′, for the accelerating forces, parallel to the co-ordinates, which act on all the points of an element, and which are in the equations above referred to, represented by X, Y, and Z; consequently, we shall have

$$\frac{\dot p}{\dot x} = d\,X', \quad \frac{\dot p}{\dot y} = d\,Y', \quad \frac{\dot p}{\dot z} = d\,Z'.$$

Or, re-establishing the values of X′, Y′, and Z′, and dividing by d,

$$\left.\begin{aligned}\frac{1}{d}\frac{\dot p}{\dot x} &= X - \frac{\dot u}{\dot t} - \frac{\dot u}{\dot x}u - \frac{\dot u}{\dot y}v - \frac{\dot u}{\dot z}w,\\ \frac{1}{d}\frac{\dot p}{\dot y} &= Y - \frac{\dot v}{\dot t} - \frac{\dot v}{\dot x}u - \frac{\dot v}{\dot y}v - \frac{\dot v}{\dot z}w,\\ \frac{1}{d}\frac{\dot p}{\dot z} &= Z - \frac{\dot w}{\dot t} - \frac{\dot w}{\dot x}u - \frac{\dot w}{\dot y}v - \frac{\dot w}{\dot z}w.\end{aligned}\right\} \quad (a)$$

Each of the elements into which we have supposed the fluid mass to be divided, will change its form during the instant $\dot t$; and it would also change its volume if it were compressible; but as its mass ought always to remain the same, it follows, that if we investigate what its volume and density ought to become at the end of the time $t + \dot t$, their product ought to be the same as at the end of the time t; equating therefore to zero, the variation of this product, there will result a new equation of the motion.

In order to form this equation, let us consider the rectangular parallelopiped, of which the volume is expressed by $\dot x\,\dot y\,\dot z$ at the end of the time t, and ascertain the form that this portion of the fluid will take at the end of the time $t + \dot t$.

6-a. Let m (fig. 36-a) be the summit or angle of this parallelopiped, which answers to the co-ordinates x, y, z; and let also $m\,n$, $m\,l$, $m\,k$, be the three sides adjacent to this summit, and respectively parallel to the three axes O x, O y, O z, then

$$m\,n = \dot z, \quad m\,l = \dot y, \quad m\,k = \dot x.$$

Let us also conceive that e, f, g, h, are the other four angles, and that during the instant $\dot t$, the eight points, m, n, l, k, e, f, g, h, are transferred to $m', n', l', k', e', f', g', h'$, then will the polyëdron, of which these last points are the angles, be also a parallelopiped.

In order to demonstrate this, let us determine and compare with each other the lengths and directions of the twelve sides $m'\,n'$, $m'\,l'$, &c.

The co-ordinates x, y, z, of the point m, will become, after the instant $\dot t$,

$$x + u\,\dot t, \quad y + v\,\dot t, \quad z + w$$

which quantities are, therefore, the co-ordinates of the point m': we may deduce those of any other summit, by merely replacing x, y, and z, by the primitive co-ordinates of that summit. Thus, the primitive co-ordinates of the point n are x, y, and $z + \dot z$; and, substituting these for x, y, and z, we shall have for the co-ordinates of n',

$$x + u\dot t + \frac{\dot u}{\dot z}\dot z\,\dot t,$$

$$y + v\dot t + \frac{\dot v}{\dot z}\dot z\,\dot t,$$

$$z + \dot z + w\dot t + \frac{\dot w}{\dot z}\dot z\,\dot t.$$

Knowing thus the co-ordinates of the two points m' and n', it will be easy to determine the length of the side $m'n'$, from the known formula

$$m'n' = \sqrt{\left\{\left(\frac{\dot u}{\dot z}\right)^2\dot z^2\dot t^2 + \left(\frac{\dot v}{\dot z}\right)^2\dot z^2\dot t^2 + \left(\dot z + \frac{\dot w}{\dot z}\dot z\,\dot t\right)^2\right\}}$$

Taking the square root, and neglecting the infinitesimals of the third order, we shall have

$$m'\,n' = \dot z + \frac{\dot w}{\dot z}\dot z\,\dot t.$$

The co-ordinates of e' are deduced from those of m', and the co-ordinates of f' from those of n', by writing $x + \dot x$, and $y + \dot y$, instead of x and y; consequently, the length of the side $e'f'$ is deduced in the same manner as that of the side $m'\,n'$. This gives

$$e'f' = \dot z + \frac{\dot w}{\dot z}\dot z\,\dot t + \frac{\ddot w}{\dot z\,\dot x}\dot x\,\dot z\,\dot t + \frac{\ddot w}{\dot z\,\dot y}\dot y\,\dot z\,\dot t;$$

whence, neglecting the last terms, which are of the third order, the value of $e'f'$ will be the same as that of $m'n'$. In the same manner, it appears that the sides $k'h'$, and $l'g'$, are equal to the same side $m'\,n'$; so that, in fact, we have

$$m'\,n' = e'f' = k'\,h' = l'\,g'.$$

If we change z into y, and w into v, in the values of $m'\,n'$, it will evidently become that of $m'\,l'$, viz.

$$m'\,l' = \dot y + \frac{\dot v}{\dot y}\dot y\,\dot t.$$

Changing, in the same manner, z into x, and w into u, we shall have the value of $m'\,k'$, which will be

$$m'\,k' = \dot x + \frac{\dot u}{\dot x}\dot x\,\dot t.$$

We shall thus have

$$m'\,l' = k'\,e' = h'f' = n'\,g',$$
$$m'\,k' = n'\,h' = g'f' = l'\,e'.$$

We see, therefore, that the sides, which were equal to each other in the primitive parallelopiped, are still equal to each other after it has changed its form; and the parallelism of its sides is a consequence of their equality. The element, therefore, which we are considering, preserves, at the end of the instant $\dot t$, the form of a parallelopiped; but it is no longer rectangular, as at the commencement of that instant.

We shall have the volume of this element by multiplying one of its faces; for example, the face $m'\,k'\,e'\,l'$ by the perpendicular $n'\,q'$, let fall from the summit n' upon this face.

The area of the parallelogram $m'\,k'\,e'\,l'$ is equal to the product of its two sides $m'\,k'$, and $m'\,l'$, multiplied by the sine of the angle $l'\,m'\,k'$, which they include; the perpendicular $n'\,q'$ is equal to the side $m'\,n'$, multiplied by the sine of the angle $n'\,m'\,q'$; therefore, the volume of our second parallelopiped will be equal to

$$m'\,n'.\ m'\,l'.\ m'\,k'.\ \sin l'\,m'\,k'.\ \sin n'\,m'\,q'.$$

Now, the angles $l'\,m'\,k'$ and $n'\,m'\,q'$, being right angles in the primitive parallelopiped, they can in the second only differ from a right angle by a quantity indefinitely small, and the sine of such an angle differ from unity only by quantity, also indefinitely small, of the second order. If, therefore, we neglect the terms of the fifth order, we must consider each of those sines as equal to unity; whence the preceding product reduces to $m'n'.\,m'l'.\,m'k'$.

Equations established.

Writing now for $m'n'$, $m'l'$, $m'k'$, the values above given, performing the multiplication, and neglecting all the infinitely small quantities of the fifth order, we shall, for this volume, have

$$\dot{x}\dot{y}\dot{z}\left(1+\frac{\dot{u}}{\dot{x}}\dot{t}+\frac{\dot{v}}{\dot{y}}\dot{t}+\frac{\dot{w}}{\dot{z}}\dot{t}\right).$$

Such is, therefore, at the end of the time $t+\dot{t}$, the volume of the element which at the end of the time t was $\dot{x}\dot{y}\dot{z}$. The density d being in the case where that quantity is variable, a function of x, y, z and t, it follows, that when t becomes $t+\dot{t}$, and that in the same time x, y and z are changed into $x+u\dot{t}$, $y+v\dot{t}$, and $z+w\dot{t}$, it becomes

$$d+\frac{\dot{d}}{\dot{t}}\dot{t}+\frac{\dot{d}}{\dot{x}}u\dot{t}+\frac{\dot{d}}{\dot{y}}v\dot{t}+\frac{\dot{d}}{\dot{z}}w\dot{t}.$$

We now multiply this density by the preceding expression for the volume, and the product will give the mass of the element at the end of the time $t+\dot{t}$; from which subtracting the mass at the end of the time t, viz. $d\dot{x}\dot{y}\dot{z}$, we shall have the variation of this mass, which variation ought to be equal to zero. Neglecting, therefore, the terms containing $\dot{t}^2$, and suppressing the factors $\dot{x}\dot{y}\dot{z}$ and $\dot{t}$, which are common to every term, we find

$$\frac{\dot{d}}{\dot{t}}+\frac{\dot{d}}{\dot{x}}u+\frac{\dot{d}}{\dot{y}}v+\frac{\dot{d}}{\dot{z}}w+d\left(\frac{\dot{u}}{\dot{x}}+\frac{\dot{v}}{\dot{y}}+\frac{w}{\dot{z}}\right)=0;$$

or, which is the same,

$$\frac{\dot{d}}{\dot{t}}+\frac{\dot{d}u}{\dot{x}}+\frac{\dot{d}v}{\dot{y}}+\frac{\dot{d}w}{\dot{z}}=0. \qquad (b)$$

This equation, and the three equations (a), are four of the five equations we had proposed to establish, and these appertain equally to every species of fluids, whether heterogeneous or homogeneous, compressible or incompressible; but they will now require certain modifications to render them applicable to these several cases.

1. If the fluid be heterogeneous and incompressible, then equation (b) may be resolved into two others; for, in this case, not only the mass of each element ought to remain constantly the same, but also the density and volume are invariable: equalling, therefore, separately to zero, the variation of the density and the volume, we shall have the two equations

$$\left.\begin{array}{l}\dfrac{\dot{d}}{t}+\dfrac{\dot{d}}{\dot{x}}u+\dfrac{\dot{d}}{\dot{y}}v+\dfrac{\dot{d}}{\dot{z}}w=0\\[2ex]\dfrac{\dot{u}}{\dot{x}}+\dfrac{\dot{v}}{\dot{y}}+\dfrac{\dot{w}}{\dot{z}}\qquad\qquad\;=0\end{array}\right\}\qquad(c)$$

From which two equations, and the three equations a, the five unknown qualities p, d, u, v, w, may be determined.

2. When the fluid is homogeneous and incompressible, d is given, and the first equation c disappears, and there remains the second equation (c), and the three marked a, for determining the four unknowns, p, u, v, w.

In like manner we might show that the same equations may be rendered applicable to compressible and elastic fluids, but we shall reserve this for our treatise on Pneumatics, where such fluids will form the subject of our investigations.

It appears, from the above deductions, that whatever may be required with respect to the motion of incompressible fluids, whether homogeneous or heterogeneous, we shall always be able to establish as many equations as there are unknown quantities; but unfortunately these equations, appertaining to the partial differences between four variables, viz. the co-ordinates x, y, z, and the time t, their general integration cannot be obtained by any means at present known; and when even we arrive at their integration, after simplifying them by certain hypotheses, it still remains to determine, from the state of fluidity at the origin of the motion, the arbitrary functions which their integrals contain: this is also a subject of considerable difficulty.

This general view of the equilibrium and motion of fluids may be interesting to some of our analytical readers, although we are fearful it will be of little use to them in any investigation connected with practical determinations. We have shown above, the insuperable difficulties which attend the solution of the equations involving the unknown quantities, and unless this solution were possible, the equations themselves are of little value; yet, as some of our readers might have considered the treatise incomplete without some such general view, we have chosen the above as being the most perspicuous and scientific we remember to have seen. M. Poisson pursues his inquiry to a greater length, but the little advantage that can be derived from it has induced us to stop at the point where the establishment of the equations is completed. The reader, therefore, who is desirous of extending his inquiries on this head to a greater length, is referred, for the requisite information, to his work already cited.

§ XVIII. *Various hydrodynamic experiments and machines.*

231. All the theoretical investigations and deductions of the preceding treatise, have been made to depend upon one general principle; namely, "that fluids press equally in all directions;" this fundamental property being admitted, all our deductions have been demonstrated to be necessary consequences of this one general law; it is, therefore, not absolutely requisite to prove experimentally the truth of these results, yet, as it may be satisfactory to see the theory and practice go hand in hand in cases of so much importance, and as many of our investigations depend upon a calculus, with which every one is not acquainted, we propose to bestow a few pages upon an experimental illustration of the most interesting cases, and on an explanation of the principles and constructions of certain hydrostatic and hydraulic engines, where their operations are independent of pneumatic pressure, or but slightly connected with it. Such as have an equal relation both to the theory of Hydrodynamics and Pneumatics, as the various kinds of pumps, &c. will be described in our treatise on the latter science.

I. *Fluids press equally in all directions.*

1. If a soft, or frangible substance, be exposed to a greater pressure in one direction than another, it will either change its form, or be broken in pieces; but if the force with which a body is pressed be equally applied to every part, it will preserve its form if it be soft, and will not be broken if it be frangible. Hence it follows, that if any body be exposed to a pressure sufficiently powerful to change its shape, or to crush it in pieces, and if it preserve its form and remain whole under this pressure, we may conclude that the pressure is equal in every direction.

The following experiment has been devised to prove this fundamental property in fluids.

Let a piece of soft wax, of an irregular figure, and an egg, be placed in a bladder, or in a bottle of India rubber, filled with water; then if the bladder be laid in a strong box, and a cover be put upon it, so as to be entirely supported by the bladder, and any large weight be laid on the cover, although it shall amount to three or four hundred pounds, yet will not the wax be found to have changed its shape, nor will the egg be broken, notwithstanding they must each have sustained a pressure equal to that enormous weight.

II. *The pressure on the bottoms of vessels of different forms depends wholly on the altitude of the fluid, and the area of the bases, whatever may be their forms or dimensions.*

1. This is demonstrated by means of the apparatus shown in (figs. 12, 13, 14, and 15), and described in art. (12), Hydrostatics; for, notwithstanding the inequality of dimensions and the dissimilarity of figure, the bottom in each of the three cases gives way immediately the fluid has attained the same altitude. And a slight change in the arrangement of the valve would show that the lateral pressure is equal to the vertical at the same depth; which is also one of our theoretical deductions.

Mr. Ferguson, in his Lectures, shows the above effect experimentally in a different manner, as follows:

2. Let AB (fig. 37) be a box which contains any small quantity of water, as, for example, about a pound, and let *a b c d e* be a glass tube, fixed to the end C of the beam of a balance, and the other end to a moveable bottom, by which the water in the box is supported, the bottom and wire being equal in weight to an empty scale suspended at the other extremity of the balance.

Now, if a pound weight be put into the empty scale, it will cause the bottom to rise a little, and the water will appear at the lower end of the tube *a*. The water will therefore press with a force of one pound upon the bottom. If another pound be put into the scale, the water will ascend to *b*, twice as high as the point *a* above the bottom of the vessel. If a third, fourth, fifth, &c. pound be put into the scale, the water will rise each time to *c*, *d*, *e*, &c.; the distance *a b*, *b c*, *c d*, &c. being equal to one another. This result will be obtained, however small the bore of the glass tube may be, at least within certain limits; and since, when the water is at *b*, *c*, *d*, *e*, the pressures upon the bottom are successively twice, three times, four times, and five times as great as when the water was contained within the box; it follows that the pressures upon the bottom of the vessel depend wholly on the height of the water in the glass tube, and not upon the quantity which it contains. If a long narrow tube, therefore, be fixed on the top of a cask, and if both the cask and the tube be filled with water, then, although the tube may be so small as not to hold a pound of the fluid, the pressure caused by the water in it will be sufficient, in many cases, to burst the vessel in pieces, for the pressure is the same as if the cask were continued up in its full size to the top of the tube, and filled with water to that height. It follows, therefore, from this principle, that any quantity of water, however small, may be made to produce a force of any given magnitude, however great. This has been already illustrated by means of the hydrostatic bellows (art. 13), Hydrostatics.

III. *Experiments on the pressure of fluids at different depths below their surfaces.*

Pressure at different depths.

1. The pressure of a fluid at different depths, as it arises from the gravity of its particles, may be very simply and satisfactorily exhibited as follows: attach a bag made of leather, or a bottle of India rubber, filled with mercury, to the extremity of a glass tube, so that the mercury may just enter the tube when the bag is held in the air. Then, by immersing the bag to different depths, the mercury will be forced higher and higher into the tube; and the elevation of it will always be found proportional to the depth of immersion.

Pressure at great depths.

2. As the pressure of a column of water of about 33 feet, or 34 feet, on a square inch of surface, is equal to about 15 lbs. it follows that at the depth of as many fathoms the pressure will be increased in a six-fold proportion, and so on for greater depths; the enormous pressure that may thus be produced is shown by a well-known experiment, viz. of letting down an empty bottle, well corked, to a great depth in the sea; in which case the pressure of the water forces in the cork, and the bottle, when brought up, is always filled with water. Mr. Campbell, on his voyage home from the Cape of Good Hope, tried this experiment. He drove, very tight, into an empty bottle, a cork, which was so large that half of it remained above the neck. A cord was then tied round the cork, and fastened to the neck of the bottle, and a coating of pitch was put over the whole. When it was let down to about the depth of fifty fathoms, it appeared from the additional weight, that it was instantaneously filled, and, upon drawing it up, it was found that the cork had been forced in, and, of course, the bottle filled with water. The experiment was again repeated, the bottle was corked as before, with the additional precaution of making a sail-needle pass through it, so as to rest on each side on the neck, the whole being pitched over as before; the bottle was then let down to about the same depth, and on bringing it up, it was found to be filled with water, although the cork still remained in its first position, and that no part of the pitch seemed to have been disturbed. Mr. Campbell is inclined to account for this phenomenon, by supposing the water to have been forced through the pores of the glass: that it must either have entered in this way, or through the pitch and cork, is obvious; and it is, perhaps, difficult to say which, for the Florentine experiment, spoken of in the commencement of this treatise, shows that water may be forced through the pores of gold; and there seems, therefore, to be no reason why it should not be equally liable to pervade

Hydrodynamics.

those of glass.—Campbell's "Travels in the South of Africa."

IV. *Description of the sea gage.*

Sea gage.

1. The sea gage, an instrument for measuring the depth of the sea, when it is too great to be effected with a fathom-line, is in some measure connected with this subject. The description of this instrument, the invention of Desaguliers and Dr. Hales, is as follows:

Fig. 38.

AB (fig. 38) is the gage bottle, into which is cemented the gage tube E*f* in the brass cap at G. The upper end of the tube E is hermetically sealed, and the lower end *f* is immersed in mercury, marked G, on which swims a small quantity of treacle. On the top of the bottle is screwed a tube of brass GH, pierced with several holes to admit the water into the bottle AB. K is a body hanging by its shank L, in a socket N, with a notch on the side at *m*, in which is fixed the catch *l* of the spring *s*, and passing through the hole L in the shank of the weight K, prevents its falling off when once hung on. On the top, in the upper part of the brass tube at H, is fixed a large empty ball, or full-blown bladder I, which must not be so large but that the weight K may be able to sink the whole under water.

The instrument thus constructed is used in the following manner. The weight K being hung on, the gage is let fall into deep water, and sinks to the bottom; the socket N is somewhat longer than the shank L, and therefore, after the weight K comes to the bottom, the gage will continue to descend till the lower part of the socket strikes against the weight; which gives liberty for the catch to fly off the hole, whereby the weight K is disengaged. This being done, the ball or bladder I, instantly buoys up the gage to the top of the water. While the gage is below, the water, having free access to the treacle and mercury in the bottle, will, by its pressure, force it up into the tube; and E*f*, the height to which it has been forced by the greatest pressure, viz. that at the bottom, will be shown by the mark in the tube which the treacle leaves behind it, and which is its only use. This shows into what space the whole air in the tube E*f* is compressed, and, consequently, the height or depth of water; which is reciprocally proportional to the space occupied by the air, as shown (art. 17), PNEUMATICS.

2. If the gage tube E*f* were of glass, a scale might be drawn on it by the point of a diamond, showing by inspection what height the water stands above the bottom; but the length of 10 inches is not sufficient for fathoming great depths, because, when all the air in such a length of tube is compressed into half an inch, the depth of the water will be not more than 634 feet, which is not $\frac{1}{8}$th of a mile. If, to remedy this, we make use of a tube 50 inches long, which, for strength, may be a musket barrel, and supposing the air compressed into an hundredth of an inch, then we shall have 1 : 99 :: 400 : 39600 inches, or 3300 feet. Even this is but little more than half a mile. But it is reasonable to suppose the cavities of the sea bear some proportion to the mountainous parts of the land; some of which are more than four miles above the earth's surface; therefore, to explore such great depths, Dr. Hales contrived a new form for his sea gage, or rather for the gage tube in it, which is as follows. BD*c*F (fig. 38-*a*) is a hollow metallic globe, communicating at the top with a long tube AB, whose capacity is one ninth of that of the globe. On the lower part at D, it has also a short tube DE, to stand in the mercury and treacle. The air contained in the compound gage tube is compressed by the water as before, but the degree of compression, or the height to which the treacle has been forced, cannot here be seen through the tube; therefore, to answer that end, a slender rod of metal or wood, with a knob on the top of the tube AB, will receive the mark of the treacle, and show it when taken out.

If the tube AB be 50 inches long, and of such a bore that every inch in length shall be a cubic inch of air, and the contents of the globe and tube together equal to 500 cubic inches; then when the air is compressed within one hundredth part of the whole, it is evident that the treacle will not approach nearer than 5 inches to the top of the tube, which will answer to a depth of 3300 feet of water, as above. Twice this depth will compress the air into half that space, viz. $2\frac{1}{2}$ inches, corresponding to a depth of 6600 feet, which is a mile and a quarter; and halving the space again, that is, reducing it to $1\frac{1}{4}$ inch, will give a depth of $2\frac{1}{2}$ miles, which is probably very nearly the greatest depth of the sea.

V. *Principle of the hydrostatic press.*

1. It has been shown (art. 4), HYDROSTATICS, if two pistons be applied to two openings in a vessel containing an incompressible fluid, that the powers acting on those pistons will be in equilibrio when they are respectively proportional to the areas of the pistons, so that, if (as the simplest case) we suppose a vessel of water to have two circular openings at top, one of half an inch diameter, and the other ten inches; since the areas of these are as 400 to 1, a force of 1 lb. applied at the smaller orifice will balance a force of 400 lbs. applied at the greater; or, a pressure of 1 lb. at the less orifice will produce a pressure of 400 lbs. at the greater.

The vessel being closed on all sides, however, it is immaterial in what part the piston acts, and, therefore, in what direction the force of the machine is applied. Let ABCD (fig. 39) represent a section of a strong iron cylinder closed on all sides, and having a moveable piston EF fitting water tight, to which is fixed the strong iron piston rod HG, which passes through the end AB, but so contrived as to be also water tight. Let *a b c d* be a small tube opening into the part of the vessel AEFD, and having a valve at *a b* opening inwards. Now, suppose the space AEFB to be filled with water, as also a part of the small tube *a b c d*; then, according to the principle above explained, the pressure of a single pound applied at *h*, to the piston rod shown in the figure, will produce a pressure upon EF, which will be to the former as the area of the cylinder ABCD to the area of the tube *a b c d*; if, therefore, these have the proportions above supposed, the one pound at *h* will act with a force of 400 lbs. at H. All things being the same, an equal force will be exerted at H′; but the former will be a pressing force, and the latter a drawing one. The most general application of the machine is with the pressing force, but it is also frequently made to act in the opposite case, as in drawing up piles, driven down for the purpose of constructing locks to canals, and even in drawing up trees by their roots; it acts also in this manner at the iron cable manufactory of Messrs. Brunton and Co. for the purpose of trying the strength of their cables and bars of iron, and in various other cases.

It may be proper to observe, that we have been speaking merely of the principle on which the machine acts, and that no adequate idea of its immense power can be formed from what is above stated, nor even of its operation; on these latter points, however, we shall endeavour to be a little more explicit in what follows. In the first place, it is necessary to remark, that instead of the force being applied at the smaller orifice by the hand, as indicated in the figure, it is produced by injecting water into that pipe by means of a forcing pump; or there may be two, three, or more such pumps, as the nature of the operation, or power proposed to be accumulated, may appear to require.

2. Suppose now, the diameter of the cylinder to be 12 inches, and the diameter of the piston of the small pump or injector, only one quarter of an inch, then the proportion between the two surfaces or ends of the said pistons, will be as 1 to 2304; and, supposing, as above, the intermediate space between them to be filled with water, a force applied to the small piston will be multiplied 2304 times upon the larger one. If, therefore, as may be easily accomplished, the small piston or injector be forced down, when in the act of forcing, with a pressure equal to 20 cwt. the piston of the great cylinder will act with a force equal to 20 × 2304 = 46080 cwt. = 2304 tons, which enormous power may be employed either in drawing or pressing, and applied in either case, in the most convenient manner imaginable, in much less time than the same could be produced by any other known means whatever. The effect produced by other machines, which commonly depend upon various combinations, is in a great measure counteracted by an accumulated complication of parts, which renders them incapable of being usefully extended beyond a certain degree; but in machines acting upon, or constructed on the principles above described, every difficulty of this kind is obviated, and their power subject to no finite restraint; for the power of any machine of this kind may be increased, *ad libitum*, either by extending the proportion between the diameter of the injector, and the great cylinder, or by applying a greater power acting on the lever of the small pump. It is only necessary that the cylinder be made sufficiently strong to resist the pressure proposed to be applied to it. Besides the immense power gained by this kind of machine, it has also many conveniences peculiarly its own. It may, for example, be made the vehicle of communication between one machine and another, let their distances and their local situations be what they may; thus, suppose two small tubes, or cylinders, in the inside of each of which is a piston, made water and air tight, a tube may be conveyed under ground, or otherwise, from the bottom of one cylinder to the other, to form a communication between them, notwithstanding their distance be ever so great. This tube being filled with water, or other fluid, until it touches the bottoms of the two pistons, if we depress the piston of one cylinder, the piston of the other will be raised; and, in the same manner, the forcing pump of the machine in question, may act at any distance from the point where the force is to be applied. Another advantage attending this kind of machine, is, that it may be made of great strength and power within a small compass, and that it is easily transferred to any place where its operation may be desired. Having given the above description of the principle on which the operation of this machine depends, we must refer to the article Press in our alphabetical arrangement, for its construction and application to specific purposes. It may be proper to observe, that although the merit of this invention appears to be unquestionably due to Mr. Bramah, yet the principle of its operation has been known for more than two centuries, and a machine to be constructed upon it was first proposed by Pascal, in his "Traites de l'Equilibre des Liqueurs, et de la Pesanture de la Mass de l'Air," in 1664; where he speaks of it as a new sort of machine, or mechanical power, equal in value to the lever or the screw.

Hydraulic engines.

It is not our intention, under this head, to treat of hydraulic machines generally; but of those only which are employed for the purpose of raising water; those which are put in motion by water, in order to produce a mechanical effect, applicable to other purposes, as flour mills, saw mills, flax mills, &c. will be treated of in their proper places in our alphabetical arrangement.

Of Hydraulic engines, whose action is independent of pneumatic pressure.

VI. *Archimedes' screw, or water snail.*

Archimedes' screw.

Figs. 40 and 41.

1. This is a machine for raising water, which consists either of a pipe wound spirally round a cylinder (fig. 40), or of one or more spiral excavations formed by means of spiral projections, from an internal cylinder, as in (fig. 41), covered by an external coating, so as to be water tight; this screw is one of the most ancient and, at the same time, ingenious machines we know, being truly worthy of the name it bears. We have seen, however, in our historical chapter, that there is some doubt whether its name be justly applied. Though simple in its general manner of operation, its theory is attended with some difficulties, which could only be conquered by the modern analysis. It was first stated correctly, as far as we have been able to ascertain, by M. Pitot, in the "Memoires de l'Academie Royale des Sciences," and afterwards more elaborately by Euler, in "Novi Comment. Petro," tom. v. Later attempts by Langsdorf, in his "Handbuch der Maschinenlehre," and some other authors, are not to be relied on. That the nature of this curious machine may be the better understood, we shall here state generally, its manner of operation.

If we conceive a flexible tube to be rolled regularly about a cylinder from one end to the other, this tube or canal will be a screw or spiral, in which we suppose the intervals of the spires or threads to be equal. The cylinder being placed with its axis in a vertical position, if we put in at the upper end of the spiral tube a small ball of heavy matter, which may move freely, it is certain that it will follow all the turnings of the screw from the top to the bottom of the cylinder, descending always, as it would have done had it fallen in a right line, along the axis of the cylinder, except that it will occupy more time in running through the spiral. If the cylinder be placed with its axis horizontally, and we again put the ball into one opening of the canal, it will descend, following the direction

Archimedes screw.

of the first demi-spire; but when it arrives at the lowest point of this portion of the tube, it will stop. It must be remarked, that though its gravity has no other tendency than to make its descent in the demi-spire, the oblique position of the tube, with respect to the horizon, is the cause that the ball, by always descending, is always advancing from the extremity of the cylinder whence it commenced its motions to the other extremity. It is impossible that the ball can ever advance more towards the farther, or, as we shall call it, the second extremity of the cylinder, if the cylinder, placed horizontally, remain always immoveable; but if, when the ball is arrived at the bottom of the first demi-spire, we cause the cylinder to turn on its axis without changing the position of that axis, and in such manner that the lowest part of the demi-spire on which the ball presses becomes elevated, then the ball falls necessarily from that point upon that which succeeds, and which then becomes lowest; and since this second point is more advanced towards the second extremity of the cylinder than the former was, the ball will, by this new descent, be advanced towards that extremity, and so on throughout, in such a manner that it will at length arrive at the second extremity by always descending, the cylinder having its rotatory motion continued. Moreover, the ball by constantly following its tendency to descend, has advanced through a right line equal to the axis of the cylinder, and this distance is horizontal, because the sides of the cylinder are placed horizontally. But if the cylinder had been placed obliquely to the horizon, and we suppose it to be turned on its axis always in the same direction, it is easy to see that if the first quarter of a spire actually descends, the ball will move from the lower end of the spiral tube, and be carried solely by gravity to the lowest point of the first demi-spire; where, as in the preceding case, it will be abandoned by this point, as it is elevated by the rotation, and thrown by its gravity upon that which has taken its place: whence, as the succeeding point is further advanced towards the second extremity of the cylinder, than that which the ball occupied just before, and is consequently more elevated; therefore, the ball, while following its tendency to descend by its gravity, will be always more and more elevated, by virtue of the rotation of the cylinder. It will thus, after a certain number of turns, be advanced from one extremity of the tube to the other, or through the whole length of the cylinder, but it will only be raised through the vertical height, determined by the obliquity of the position of the cylinder.

Instead of the ball, let us now consider water, as entering by the lower extremity of the spiral canal, when immersed in a reservoir; this water descends first in the canal solely by its gravity, but the cylinder being turned, the water moves on the canal to occupy the lowest place, and thus, by the continual rotation, is made to advance farther and farther in the spiral, till at length it is raised to the upper extremity of the canal, where it is expelled.

There is, however, with reference to the computed effect of this machine, an important difference between the water, or any other fluid, and the ball; for the water, by reason of its fluidity, after having descended by its gravity to the lowest point of the demi-spire, rises up on the contrary side to the original level; on which account more than half one of the spires will soon be filled with the fluid. This is an important particular, which, although it need not be considered in a popular illustration, must be attended to in a more accurate exhibition of the theory of this engine. The very little application of this machine to the purposes it is calculated to answer, will not, however, justify our entering upon a more minute examination of its theory in this place; although we must observe, that it seems entitled, in a practical point of view, to greater attention than it now commonly receives. See Gregory's Mechanics, vol ii.

2. The Germans have a similar machine, which they call the water-screw; it consists of a cylinder, with its spiral projections detached from the external cylinder, or coating, within which it revolves, (as shown in fig. 42.) It is evident, that some loss must here be occasioned by the want of perfect contact between the screw and its cover; in general, at least one-third of the water runs back, and the machine cannot be placed at a greater elevation than 30°; it is also very easily clogged by accidental impurities of the water. Yet it has been found to raise more water than the screw of Archimedes, when the lower ends of both are immersed to a considerable depth; so that if the height of the surface of the water to be raised were liable to any great variations, the German water-screw might be preferable to the screw of Archimedes.

VII. *The tympanum, or drum.*

1. This is a great hollow wheel, forming a kind of barrel or drum, consisting of several planks joined together, well calked and pitched, and having a horizontal axis on which it turns; the interior of the drum is divided into eight equal spaces, by as many partitions placed in the direction of the radii; each cell or space has an orifice of about half a foot in the rim of the drum or wheel, so shaped as to facilitate the admission of water; there are also eight hollow channels running contiguous to each other, and parallel to the axle of the wheel, each corresponding to one of the large cells. The water passes out of the cells above mentioned into these channels, and after running along them to a convenient distance, it escapes through orifices into a reservoir placed just under the axle. Thus the water is raised through a vertical space equal to the semi-diameter of the wheel.

When this machine is used for raising water from a running stream, it is moved by means of float boards, which are impelled by the motion of the stream; but when it is employed to raise stagnant water, there is commonly a smaller wheel on the same axle, which is turned by men walking in it, as in the old walking crane. The principal defect of this machine is, that it raises the water in the most disadvantageous situation possible; for the load being always towards the extremity of the radius of the wheel, the arm of the effective lever which answers to it, increases through the whole quadrant the water describes, in passing from the bottom of the wheel to the altitude of the centre; so that the power must act the same as if it were applied to a winch or handle, and therefore cannot act uniformly.

2. In order to remedy, in some measure, the above defect, a different machine has been devised by M. De la Faye, which may be thus illustrated.

When we develope the circumference of a circle so as to form the involute of that curve, all the radii are so many tangents to the circle, and are all likewise respectively perpendicular to the several points of the curve described, the greatest radius being equal to the periphery of the circle evolved.

Therefore, having an axle whose circumference a little exceeds the height through which the water is proposed to be elevated, let the circumference of the axle be evolved, and make a curved canal, whose curve shall coincide throughout exactly with that of the involute just formed: if the farther extremity of this canal be made to enter the water, and the other extremity abut upon the shaft which is turned, then, in the course of the rotation, the water will rise in a vertical direction tangentially to the shaft, and perpendicularly to the canal, in whatever position it may be.

Thus the action of the weight, answering always to the extremity of a horizontal radius, will be as if it acted upon an invariable arm of a lever, and the power which raises the weight will always be the same. And if the radius of the wheel, of which this hollow canal serves as a bent-spoke, be equal to the height that the water is to be raised, and, consequently, equal to the circumfererence of the axle or shaft, the power will be to the load of water reciprocally as the radius of a circle to its circumference, that is, as 1 to 6·2832, or as 1 to 6¼ nearly.

M. Faye, to whom we are indebted for this modification of the ancient tympanum, is of opinion that the machine ought to be composed of four of these canals, but it has frequently been constructed with eight, as shown in (fig. 43). The wheel being turned by the impulsion of the stream upon the float boards, the orifices F, E, D, C, &c. of the curvilinear canals, dip one after another into the water, which runs into them; and as the wheel revolves, the fluid rises in the canals *f*, *e*, *d*, *c*, &c. and runs out in a stream P, from the holes at O; it is received into the trough Q, and conveyed thence in pipes.

By this construction it is obvious that the weight to be raised offers always the same resistance, and that the least possible, while the power is applied in the most advantageous manner of which the circumstances will admit; two important points to be attended to in all machines for this purpose. This machine also raises the water by the shortest way; that is to say, perpendicularly or vertically, in which respect it is preferable to the screw of Archimedes, where the water is carried up an inclined path; each curved canal also of M. Faye's wheel empties all the water it receives each revolution, while the screw of Archimedes yields only a small portion of the water with which it is loaded, frequently not more than one-twentieth part; on which account the latter machine requires an enormous increase of labour when a large quantity is proposed to be raised by it.

Few machines would be preferable to that above described, were it not that it is limited to raising the water only through its radius, or semi-diameter; in consequence of which, when water is to be elevated to a great height, it is not practicable to construct a wheel of sufficient diameter, but where the height falls within any practical limits of construction, it is a very cheap and useful engine.

The reader will observe that the buckets attached to the wheel in the figure above referred to, do not belong to the tympanum, nor to M. Faye's modification of it, but to the noira and the Persian wheel, described in the two following articles.

VIII. *The noria.*

Of the noria.

This machine is commonly employed in Spain; it consists of a vertical wheel, about 20 feet in diameter, on the circumference of which is fixed a number of little boxes, or square buckets, for the purpose of raising the water out of the well, communicating with the canal below, and to empty it into a reservoir above, placed by the side of the wheel; the bucket having a lateral orifice for receiving and discharging the water. The axis of the wheel is embraced by four small beams, crossing each other at right angles, tapering at the extremities, and forming eight little arms. This wheel is near the centre of the horse-walk, contiguous to the vertical axis, into the top of which the horse-beam is fixed; but near the bottom it is embraced by four little beams, forming eight arms, similar to those above described, on the axis of the water wheel. As the mule which they use goes round, the horizontal arms supplying the place of cogs, take hold, each in succession, of those arms which are fixed on the axis of the water wheel, and keep it in rotation.

IX. *Persian wheel.*

Persian wheel.

Fig. 43.

This machine is represented in (fig. 43); the water is raised by means of a stream AB, turning a wheel CDE, according to the order of the letters, with buckets *a*, *a*, *a*, *a*, &c. hung upon the wheel by strong pins *b*, *b*, *b*, *b*, &c. fixed in the side of the rim; which must be made as high as the water is intended to be raised, above the level of that part of the stream in which the wheel is placed. As the wheel turns, the buckets on the right hand go down into the water, where they are filled, and return up full on the left hand, till they come to the top at *k*, where they strike against the end *n*, of the fixed trough M, by which they are overset, and thus empty the water into the trough, whence it may be conveyed in pipes to any place proposed. As each bucket passes over the trough, it falls again into a perpendicular position, and descends empty till it arrives at the water at A, where it is filled as before. On each bucket there is a spring *r*, which, going over the top or crown of the bar *m* (fixed to the vessel M) raises the bottom of the bucket above the level of its mouth, and thus causes it to empty all its water into the said vessel. Sometimes, however, the wheel is made to raise water no higher than its axis. In this case, instead of buckets hung upon it, its spokes C, *d*, *e*, *f*, *g*, *h*, are made of a bent form, and hollow within; these hollows opening into the holes C, D, E, F, in the outside of the wheel, and also into those at O, in the box N upon the axis. So that as the holes C, D, &c. dip into the water, it runs into them; and as the wheel turns, the water rises in the hollow spokes *c*, *d*, &c. and runs out in a sream P, from the holes at O, and falls into the vessel Q, whence it is conveyed in pipes to its destined place.

In order to determine the due relation between the power and the weight, so that this wheel may be capable of producing its greatest effect, the following approximation may be employed. After having fixed the diameter of the wheel, which must be something greater than the height to which the water is to be raised, determine also upon some even number of buckets, to be hung round the periphery of the wheel at equal distances, and mark the positions of their centres of motion in such a manner, that they will stand at corresponding positions in every quarter of the circle; conceive vertical lines to be drawn through the centre of motion of each bucket, in the rising part of the wheel, and they will intersect the horizontal diameter of the wheel, in points at which, if the buckets were hung, they would furnish the same resistance to the moving force, as they do when hanging at their respective places on the rim of the wheel. Suppose, for example, there were eighteen equidistant buckets, then, while eight hung on each side, two would coincide with the vertical diameter. In this case, the resistance arising from all the full buckets, would be the same as if one bucket hung on the prolongation of the horizontal diameter, at the distance of

$$2 \sin 80^\circ + 2 \sin 60^\circ + 2 \sin 40^\circ + 2 \sin 20^\circ;$$

these being the sines to a circle described with the given radius of the wheel.

To know the quantity of water that each bucket should contain, take $\frac{4}{9}$ths of the absolute force of the stream, that is, $\frac{4}{9}$ths of the prism of water, whose base is the surface of one of the float boards, and whose height is equal to that through which the water must fall, to acquire the velocity of the stream; so shall we have the power that should be in equilibrio with the weight of water in the buckets of the rising semicircle. Then say, as the above sum of the sines is to radius, so the power just found is to a fourth term; the half of which will be the weight of water that ought to be contained in one bucket. Lastly, as the velocity of the wheel will be to that of the stream, nearly as 1 to $2\frac{2}{3}$, the number of revolutions it makes, in any determined time, becomes known, and consequently also the quantity of water the wheel will raise in the same time; because we know the capacity of each bucket, and the number of them emptied in every revolution of the wheel.

X. *Multiplying wheel bucket engine.*

Multiplying wheel bucket engine.

1. The first mention we have of this ingenious engine, is made by Schottus, but the first actually erected was by Gironimo Finugio, at Rome, in 1616; and the first in this country was constructed by George Gerves, for Sir John Chester, Bart. at Chichley, in Buckinghamshire.

Fig. 44.

The object of this construction is to raise water from a well, or spring, A (fig. 44) to a reservoir R. In order to effect this, a wheel WW, 6 feet in diameter, is fixed above R, and on the same axis another wheel *ww*, 2 feet in diameter. To the circumference of W is fixed a chain W *x*, to which is hung a small bucket *b*, with a valve in its bottom, and suspended, as seen at *b* (fig. *a*). To the circumference of *w w* is fixed another chain *w w y*, fastened to a rod *y z*, to which is suspended the large bucket B, with a valve in its bottom, as seen above B. This valve is fixed at the end of the arm *o* B, and is kept in its place by the weight *m*, acting at the end of the lever *m o*, whose fulcrum is at *n*, but is raised from its place, by raising the arm *m o*, as shown in (fig. *b*). Let us now suppose the small bucket *b*, to be filled with water, and that in consequence of water being poured into the large bucket B, this bucket descends; the bucket *b* will therefore ascend till it strikes the hook at *s*. This hook, catching the edge of the bucket, turns it aside, as shown in (fig. *a*), and empties it into the reservoir R. By this time the descending bucket B has nearly reached the bottom of the cistern *z* Z. The arm of the lever falling upon the projection M, is raised, as shewn in (fig. *b*). The valve in the bottom is, consequently, also raised, and the water is discharged at B into the cistern Z. The small bucket *b* is now more than a balance for B, in consequence of its acting at the end of a longer lever, and therefore *b* descends to the cistern A, while B ascends to the position it has in the figure. When *b* reaches the cistern A, the lower end of the valve strikes against a fixed obstacle, and is, in consequence, raised out of its place, and admits the water of the cistern into the bucket.

Now, the arm *n o*, of the lever in the large bucket B, striking against the bottom of the valve seen below *a*, in a branch *a*, of the cistern A, raises it, and allows the water from the cistern to run into the bucket B, till the weight of the bucket is sufficient to raise *b* out of the cistern A. As soon as it has received this weight of water, it descends; the valve below *a* falls into its place; the valve in the bottom of *b* also falls into its place, when it rises above the fixed obstacle, and the bucket *b* ascends, as before, to discharge its contents into the reservoir, while the large bucket descends to M, to get rid of its load of waste water.

It is obvious that the load of the chains and rods will be different in different positions of the buckets. When *b* is at R, and B at Z, the bucket B will be loaded with an additional quantity of chain. In order to regulate the weight of the chains in every position, a quadrant Q moves round C as a centre, and a chain *c d* attached to the point *c* of the rod *c z* is fixed to the end *d* of the arc *d* Q: a weight X is also fixed at the end of the radius *d* C. Now when B is down at Z, X will have descended to *p*, and from acting at the end of a shorter lever, will be a less load upon the rod *c z* than when it had the position at X; that is, the additional weight which the bucket B has received from the increase of its chain, is counterbalanced by the diminution of weight occasioned by the descent of CX into the position CP. Desaguliers remarks, that one hogshead of water falling 10 feet, will raise very nearly 1 hogshead 10 feet, or one-fourth of a hogshead 40 feet. See Desagulier's Exp. Phil. vol. ii.

2. This author (Desaguliers) describes also, in the same volume as that above referred to, another very simple machine, for the raising of water, which is as follows:

To one end of a rope is fixed a large bucket, having a valve in its bottom, opening upwards; and to the other is attached a square frame, and the cord is made to pass over two pulleys, each about 15 inches diamemeter (and fixed in a horizontal plane), in such a manner that as the bucket descends, the frame ascends with equal velocity, and the contrary. The frame is made to run freely upon four vertical iron guide rods, passing through holes at its four corners; and when the bucket is filling with water at the well, the frame stands at the horizontal plane, to which the water is to be raised: when the bucket is full, a man steps upon the frame, whose weight, together with that of the frame, exceeding that of the vessel and its contained water, gives an ascending motion to the bucket, and causes the valve in its bottom to close When the bucket is raised to the proper height, a hook fixed there catches into a hasp, at the side of the bucket, turns it over, and causes it to empty its water into a trough, which conveys it to the required place. At this time the man and the frame have arrived at a platform, which prevent their further descent, where the man remains till he finds the bucket above is empty, when he steps from the platform, and runs up a flight of stairs to the place from which he descended. The bucket, in the mean time, being heavier than the frame, descends to the water, and raises the frame to its original position. Thus the work is continued; the man being at rest during his descent, and labouring in his ascent.

Desaguliers employed for this purpose a "tavern-drawer," who weighed 160 lbs., whom he desired to go up and down 40 steps of 6½ inches each (in all about 22 feet) at such a rate as he might continue without fatigue. He went up and down twice in a minute; so that allowing the bucket, with a quarter of a hogshead in it, to weigh 140 lbs., he was able to raise it up, through 22 feet, twice in a minute. This Desaguliers estimates as equivalent to a whole hogshead raised 11 feet in a minute; and rather exceeds what he has assigned as the maximum of human exertion.

XI. *Wirtz's spiral pump, or the Zurich machine.*

When a spiral pipe, consisting of many convolutions, arranged either in a single plane, or in a cylindrical or a conical surface, and revolving round a horizontal axis, is connected at one end by a water-tight joint with an ascending pipe, while the other end receives, during each revolution, nearly equal quantities of air and water, the machine is called a spiral pump. It was invented, about 1746, by Andrew Wirtz, at Zurich, and is said to have been used with great success at Florence and in Russia; it has also been employed in this country by Lord Stanhope; and Dr. Young states, in his Lectures on Natural Philosophy, that he has made trial of it for raising water to a height of 40 feet.

5. This machine is represented in (fig. 45). The end of the pipe is furnished with a spoon, containing as much water as will fill half a coil, which enters the pipe a little before the spoon has arrived at its highest situation, the other half remaining full of air, which communicates the pressure of the column of water to the preceding portion, and in this manner the effect of nearly all the water in the wheel is united, and becomes equivalent to that of the column of water, or of water mixed with air, in the ascending pipe. The air nearest the joint is compressed into a space much smaller than that which it occupied at its entrance, so that where the height is considerable, it becomes advisable to admit a larger portion of air than would naturally fill the half coil; this lessens the quantity of water raised, but it lessens also the force required to turn the machine.

The joint ought to be conical, in order that it may be tightened when it becomes loose, and the pressure ought to be removed from it as much as possible. The loss of power, supposing the machine well constructed, arises only from the friction of the water on the pipe, and the friction of the wheel on its axis; and where a large quantity of water is to be raised to a moderate height, both of these resistances may be rendered very inconsiderable. But when the height is very great, the length of the spiral must be much increased, so that the weight of the pipe becomes extremely cumbersome, and causes a great friction on the axis, as well as a strain on the machinery; thus, for a height of 40 feet, says Dr. Young, I found that the wheel required above 100 feet of pipe, which was three quarters of an inch in diameter; and more than one-half of the pipe being always full of water, we have to overcome the friction of about 80 feet of such a pipe, which will require twenty-four times as much excess of pressure to produce a given velocity, as if there were no friction. The centrifugal force of water in the wheel would also materially impede its ascent if the velocity were considerable, since it would be always possible to turn it so rapidly as to throw the whole water back into the spoon.

XII. *The chain pump.*

Chain pump. 1. The chain pump is now generally made from 12 to 24 feet in length, and consists of two collateral square barrels, and an endless chain of pistons of the same form, fixed at proper distances. The chain is moved round a coarse kind of wheel-work, fixed sometimes at one end, but often at both ends of the machine. The teeth of the wheel-work are so contrived as to receive one-half of the flat pistons, and let them fold in; and they take hold of the links as they rise. A whole row of pistons (which go free of the sides of the barrel by about a quarter of an inch) are always lifting when the pump is at work; and as this machine is generally worked briskly, the pistons, or pallets, bring up a bore full of water in the pump.

Chain pumps are worked sometimes by men turning winches, sometimes by horses, and sometimes by the impulse of a stream of water. They are likewise so contrived, that by the continual folding in of the pistons, stones, dirt, or whatever comes in the way, may be cleared off; they are therefore often used to drain ponds, sewers, and remove foul water, when no other pump could be employed. They are also now frequently used for ship pumps.

Chain pumps are not merely fixed in a vertical position, but are often inclined; and in the latter case they are in a state of the greatest perfection, or raise the most water in a given time, when the breadths of the pallets are equal to their distance from each other, the plane being inclined under an angle of 24° 21′.

Different construction. Fig. 46. 2. It is not unusual for chain pumps to be erected without a barrel to receive the pistons, after the manner represented in (fig. 46). Here the pallets are converted into square boxes, *s*, *s*, &c. which are raised by means of hexagonal axles, each side of the hexagon being equal to the distance from box to box: they descend with their mouths downwards, and thus enter the water.

Paternoster-work. 3. Another contrivance for raising water similar to the chain pump, is an endless rope, with stuffed cushions hung upon it; which, by means of two wheels, or drums, are caused to rise in succession in the same barrel, and to carry water with them. From the resemblance of this apparatus to a string of beads, it is usually called *paternoster-work.*

XIII. *Hair-rope machine.*

Hair-rope machine. 1. Instead of a series of buckets, ropes, or chains, a similar effect is sometimes produced by a simple

Fig. 47.

rope, or a bundle of ropes, passing over a wheel above, and a pulley below, moving with a velocity of about 8 or 10 feet in a second, and drawing a certain quantity of water up by its friction. (See fig. 47.) It is probable that the water commonly ascends with about half the velocity of the rope, and, on this supposition, we might calculate its depth on the rope, by comparing its relative motion with that of a little river; but the rules which serve for calculating the velocity of rivers, do not perfectly agree, as we are informed by Dr. Young, with the results of direct experiments; for the friction required for elevating the quantity raised by such a machine, appears, from calculation, to correspond to a velocity about twice as great as the actual relative velocity. While the water is principally supported by the friction of the rope, its own cohesion is amply sufficient to prevent its wholly falling, or being scattered by any accidental inequality of the motion. Although it may not be absolutely necessary that the rope used for this purpose should be composed of hair, yet for the sake of durability, as well as for increasing the effect, it is much better that it should be formed of that material.

XIV. *The double plunger.*

Double plunger.

Fig. 48.

This is a sort of pump, but its operation is entirely independent of pneumatic pressure, and is well calculated, both from its simplicity and effect, for raising large quantities of water to small heights. It is made by fitting two upright beams, or plungers, of equal thickness throughout, into cavities nearly of the same size, allowing them only room to move without friction, and connecting the plungers by a horizontal beam moving on a pivot, as shown in (fig. 48). The water, being admitted during the ascent of each plunger, through a large valve in the bottom of the cavity, is forced, when the plunger descends, to escape through a second valve in the side of the cavity, and to ascend by a wide pipe to the level of the beam. The plungers ought not to be in any degree tapered, because of the great force which would be unnecessarily consumed in continually throwing out the water with great velocity as they descend, from the interstice formed by their elevation.

This pump may be worked by a labourer walking backwards and forwards, either on the beam, or on a board suspended below it. By means of an apparatus of this kind, described by Dr. Robison, an active man loaded with a weight of 30 lbs. has been able to raise 580 lbs. of water every minute, to a height of $11\frac{1}{2}$ feet, for 10 hours in a day, without fatigue. This is the greatest effect produced by a labourer, that has been correctly stated by any author; it is equivalent to somewhat more than 11 lbs. raised through 10 feet in a second, instead of 10 lbs. which is a fair estimate of the usual force of a man, without any deduction for friction. Dr. Young, Nat. Phil. vol. i.

Hydraulic engines which act in part by the pressure of the air.

As we have not yet treated of pneumatic pressure, we propose to introduce, under this article, such machines only as depend upon the most obvious properties of air considered as a body, that is to say, those whose operations depend upon its actual gravity or elastic force. The pump, and a few other machines, in which the science of pneumatics is more particularly concerned, will be described in the following treatise.

XV. *The siphon.*

Of t siph

We have had occasion, in our examination of the laws of the equilibrium of fluids, to refer to this instrument, and have shown (art. 10), that when water or any other fluid is placed in it, the arc being downwards, it will rise to the same height in both branches, whatever may be their form or dimensions. If these branches are of the same length, and the siphon be inverted, the fluid will still also remain in equilibrio, and be suspended in the tube, the pressure of the air on each extremity being proportional to the weight of water it has to support; provided only that the latter be not greater than the former, that is, as we shall see in the following treatise, provided the length of the equal branches be less than 34 feet. Under this limit the water will be suspended in the siphon while the two extremities maintain their horizontal position.

But if now the siphon be a little inclined to one side, so that the orifice of one end becomes lower than the other, or if the legs be of unequal lengths, which is the same thing, then, the equilibrium being destroyed, the water will descend through the lower orifice and rise in the shorter branch. For the air pressing equally, while the two columns are of unequal weight, motion must necessarily commence where the force is greatest, and, once begun, it will continue till all the water has been discharged at the lower end. If, therefore, the shorter leg be immersed into a vessel of water, the the discharge will continue till the vessel is exhausted, or, at least, till the fluid in it has descended to the extremity of the immersed branch of the siphon. We shall examine the operation of this machine more at length in our treatise of Pneumatics.

XVI. *Hero's fountain.*

Hero fount Fig

The construction of this fountain is as follows: It consists of two vessels KLMN (fig. 49), and OPQR, which are close on all sides. A tube, AB, having its funnel at the top, passes through the uppermost vessel without communicating with it, being soldered into its top and bottom. It passes through the top of the under vessel, where it is also soldered, and reaches almost to its bottom. This tube is open at both ends. There is another open tube, ST, which is soldered into the top of the under vessel, and the bottom of the upper vessel, and reaches almost to its top; these two tubes serve also to support the upper vessel. A third tube, GF, is soldered into the top of the upper vessel, and reaches almost to its bottom; this tube is open at both ends, but the orifice G is very small. Now, suppose the upper vessel filled with water to the height EN, E *e* being its surface, a little below T. Stop the orifice G with the finger, and pour in water at A. This will descend through AB, and compress the air which formerly occupied the whole of the space OPQR, and KL *e* E, into OP *c* C, and KL *e* E; and its elasticity will be in equilibrio with the weight of the column of water, whose base is the surface E *c*, and whose height is A *c*. As this pressure is exerted in every part of the air, it will be exerted on the surface E *e* of the water of the upper vessel; and if the pipe FG were continued upwards, the water would be supported in it to the height *e* H above E *e* equal to A *c*. Therefore if the finger be now taken from off the orifice G, the fluid will spout up through it to the same height as if it had fallen through a tube, whose altitude is H *e*. So long as there is any water in the vessel KLMN, there will be a discharge through the orifice: therefore, the play of the fountain will continue, whilst the water contained in the upper vessel, having spouted out, falls through the pipe AB. The height of the water measured from the basin VAW, to the surface of the water in the lower vessel OPQR, is always equal to the height measured from the top of the jet to the surface of the water in the vessel KLMN. Now, since the surface E *e* is always falling, and the water in the lower vessel always rising, the height of the jet must continually decrease, till it is shorter by the depth of KLMN, which is empty, added to the depth of OPQR, which is always filling; and when the jet is fallen so low, it immediately ceases to play.

XVII. *Irrigating engine.*

This machine is designed to raise water to a great height, for the irrigation of land, in such situations as have the advantage of a small fall. Its principle is nearly the same as the fountain above described, being as follows:

In (fig. 50), *a b* represents the stream of water; *b*, *c*, *c*, the water fall, supposed to be 10 feet; *d*, *e*, are two leaden or iron vessels, containing a certain quantity of water, which may be computed to be about four gallons each; *f*, *g*, *h*, *i*, *k*, *l*, are leaden vessels, each holding about two quarts; *o*, *p*, two cocks, each of which passes through two pipes, opening the one and shutting the other; *q r*, is a water balance, moving on its centre *s*; and by which the two cocks *o*, *p*, are alternately turned; *t u*, and *w x*, are two air pipes of lead, both internally one inch and a quarter in diameter; and *y z*; *y z*; *y z*, are water pipes, each one inch in diameter.

The pipe *b e c*, is always full from the stream *a b*: the small cisterns *g*, *i*, *l*, and the large one *d*, are supposed to have been previously filled with water. The fluid may then be admitted (by turning the cock *o*) through the pipe *c e*, into the large cistern *e*. This air will press the air contained in the cistern *e* up the air pipe *w x*, and will force the fluid out of the cisterns *g*, *i*, *l*, into those marked *h*, *k*, C. At the same time, by opening B, the water and condensed air, which previously existed in the large cistern *d*, and in the smaller ones marked *f*, *h*, *k*, will be discharged at B. After a short time the water balance *q r*, will turn the cocks, and exclude the water, while it opens the opposite ones; the cisterns *f*, *h*, *k*, are emptied in their turns by the condensed air from the cistern *d*, as the water progressively enters the latter from the pipe *b c*.

XVIII. *The Chemnitz fountain, or Hungarian machine.*

1. The Chemnitz fountain is represented in (fig. 51), where A shows the source of water elevated 136 feet above the mouth of the pit. From this there runs down a pipe *b b*, of 4 inches diameter, which enters the top of a copper cylinder B, 8½ feet high, 5 feet in diameter, and 2 inches thick, and reaches within 4 inches of the bottom.

This cylinder has a cock at *f*, and a very large one at *e*. From its top proceeds a pipe *h h*, 2 inches in diameter, which goes 96 feet down the pit, and is inserted into the top of another brass cylinder C, which is 6½ feet high, 4 feet diameter, and 2 inches thick; the latter containing about 83 cubic feet, which is nearly one-half of the capacity of the former, viz. 170 cubic feet. There is another pipe *n n*, of 4 inches diameter, which rises within 4 inches from the bottom of this lower cylinder, is soldered into its top, and rises to the trough which carries off the water from the mouth of the pit. This lower cylinder communicates at the bottom with the water L, which collects in the drains of the mines. A large cock, *k*, serves to admit this water; another cock, *m*, at the top of this cylinder communicates with the external air.

Hungarian machine.

Now, suppose the cock C shut, and all the rest open; the upper cylinder will contain air, and the lower will be filled with water, because it is sunk so deep that its top is below the surface of the mine waters. Shut the cocks *e f*, *g c*, *m k*, and open the cock C: the water of the source A, must run in by the pipe *b b*, and rise in the upper cylinder, compressing the air above it, and along the pipe *h h*; and thus acting on the surface of the water in the lower cylinder. It will, therefore, cause it to rise gradually in the pipe *n n*, where it will always be of such a height that its weight balances the elasticity of the compressed air. Suppose no issue given to the air in the upper cylinder, it would be compressed with one-fifth of its bulk by the column 136 feet high. For, a column of 34 feet nearly balances the ordinary elasticity of the air; therefore, when there is an issue given to it through the pipe *h h*, it will drive the compressed air along this pipe, and expel the water from the lower cylinder. When the upper cylinder is full of water, there will be 34 feet of water expelled from the lower cylinder. If the pipe *n n* had been more than 136 feet long, the water would have risen 136 feet, being then in equilibrio with the water in the feeding pipe *b b*, by the intervention of the elasticity of the air; but no more water would have been expelled from the lower cylinder than what fills this pipe. But the pipe being only 96 feet high, the water will be thrown out at O with a considerable velocity.

If it were not for the great obstruction which water and air must meet with in their passage along pipes, the water would issue at O with a velocity of more than 50 feet per second. Its actual issuing velocity is, however, much less than this; and at last the upper cylinder being full of water, the water would enter the pipe *h h*, and the lower cylinder, and, without displacing the air in it, would rise through the discharging pipe *n* O, and run off to waste. To prevent this, there hangs in the pipe *h h*, at its orifice, a cork ball, or double cone, by a brass wire, which is guided by holes in two cross pieces in that pipe.

When the upper cylinder is filled with water, the cork ball plugs up the orifice *h*, and no water is wasted; the influx from *b b* now stops. But the lower cylinder contains compressed air, which would balance the water in a discharging pipe 136 feet high, whereas *n n* is only 96 feet. Therefore, the water will continue to flow at O till the air is so far expended as to balance only 96 feet of water, that is, till it occupies only half of its ordinary bulk, or one-fourth of the capacity of the upper cylinder, or 42½ cubic feet: therefore, 42½ cubic feet will be expelled, and the efflux at O will cease, the lower cylinder being about half full of water.

Hungarian machine.

2. When the attending workman observes this, he shuts the cock c; at the same time the cock e is opened, and the water issues with great violence, being pressed by the condensed air in the lower cylinder; and it therefore issues with the sum of its own weight and this compression. These gradually decrease together by the efflux of the water and the expansion of the air; but the efflux stops before all the water has flowed out, because there are 42½ feet of the lower cylinder occupied by air. This quantity of water, therefore, nearly, remains in the upper cylinder. The workmen know this, because the discharged water is received first into a vessel of three-fourths the capacity of the upper cylinder. Whenever this is filled, the attendant opens the cock k, by a long rod, which goes down the shaft; this allows the water of the mine to fill the lower cylinder, and the air to get into the upper one, which permits the remaining water to run out of it. Thus every thing is brought into its first condition; and when the attendant sees no more water coming out at d, he shuts the cocks e, m, and opens the cock c, and the operation is repeated.

3. There is a very surprising appearance in the working of this engine. When the efflux O has stopped, if the cock f be opened, the water and air rush out together with prodigious violence, and the drops of water are changed into hail or lumps of ice. It is a sight usually shown to strangers, who are desired to hold their hats to receive the blasts of air; the ice is said to come out frequently with such violence as to pierce the hat like a pistol bullet. This rapid congelation is a remarkable instance of the general fact, that air, by suddenly expanding, generates cold, in consequence of its capacity for heat being increased.

The above account of the procedure in working this machine, shows that the efflux both at O and d becomes very slow near the end. It is therefore found convenient not to wait for the complete discharges, but to turn the cocks when about 30 cubic feet of water have been discharged at O; more work being effected in this way than by waiting for the complete evacuation. A gentleman of great accuracy and knowledge of these subjects made particular observation of the performance of this machine. He remarked that each stroke, as it may be called, occupied about 3 minutes 7½ seconds, and that 32 cubic feet of water were discharged at O, and 66 feet expended at d. The expence, therefore, is 66 feet of water falling 136 feet, and the performance is 32 feet raised 96 feet; the proportion being as

$$66 \times 136 \text{ to } 32 \times 96, \text{ or as } 1 \text{ to } \cdot 3422;$$

that is, nearly as 3 to 1.

This is equal to the performance of the most perfect undershot mill, even when friction and all irregular obstructions are neglected, and is not much inferior to any overshot pump mill that has yet been erected. When we reflect on the great obstructions that water meets with in its passage through long pipes, we may be assured that, by doubling the size of the feeder and the discharger, the performance of the machine will be greatly improved; we do not hesitate to say, that it would be increased at least by one-third. It is true that it will expend more water; but this will not be nearly in the same proportion; for most of the deficiency of the machine arises from the needless velocity of the first efflux from O. The discharging pipe ought to be 110 feet high, and not give sensibly less Again, it ought to be considered how much in the first expence this simple machine must b mill of any kind, which would raise 10 cubic feet high in a minute, and how small the repai need be in comparison with those of a mill. L machine of this kind may be erected where whatever can be put in motion. A small st water, which would not move any kind of whe here raise one-third of its own quantity to th height, working as fast as it is supplied.

For these reasons, we think this engine hig serving the attention of mathematicians and en to bring it to its utmost perfection, and into use. There are situations where it may be ingly useful. Thus, where the tide rises 1 it may be employed for compressing air to eighths of its bulk; and a pipe leading from large vessel, inverted in it, may be used for water in a vessel of one-eighth of its capacity high; or, if this vessel have only one-tenth of pacity of the large one set in the tide way, tw may be led from it; one into the smaller vess the other into an equal vessel 16 feet higher, receives the water from the first. Thus, one-si of the water may be raised 34 feet, and a quantity to a still greater height, and this with of power that can hardly be employed in an way. Machines of this kind are described by Sc Sturmius, Leupold, and other writers, and the not to be forgotten, because opportunities may of making them highly beneficial. Gregory chanics, vol. ii.

XIX. *Boswell's apparatus intended to be applied above, or any similar engine.*

In the above account we have seen that att are constantly employed to turn the cocks, watch the motion of the machine, as was usua working of the first steam engines: the follow paratus invented by Mr. Boswell, is intended t the engine perform all these operations for its has the advantage of greater regularity than ca monly be expected from a labourer or workman.

The means which are employed for effecting indicated in (fig. 52), where the parts commo two machines are marked with the same letter the former figure. By a comparison, therefore, two figures, we easily distinguish the source basin L, of water to be raised, the two recei cylinders B and C, the tubes $b\,b$, $d\,d$, $h\,h$, $n\,n$, &c upper cocks c, g, e, and the lower cock k, in figure, replaced by the valve k in the second.

The principal addition is a vessel V, open which receives, by means of the tube $b\ b\ o\ \phi\ \lambda$ tion of water from the source, designed to op shut the cocks.

The receiver C, which in the first machine co cates with the water to be raised only by the (fig. 51), is, in (fig, 52), entirely plunged into th and is filled by an orifice supplied with a which opens downwards.

The vessel V is filled by the tube $o\ \phi\ \lambda$; the t a cock λ, which opens and shuts alternately, and o, always open, which serves to regulate the ex water. The two cocks λ and c, connected by open and shut at the same time, by the action of the two weights π and R, the one constant, the other variable. R is a vessel that is filled by means of the siphon $t\ t$, which, inserted into the upper vessel V, discharges itself at x into a lower vessel S: this last vessel communicates by means of a cock e, and opens the cock when it descends; it is then discharged by a cock y, which can be more or less open to regulate the discharge.

In (fig. 51) the upper cock g is above the great receiver B, but in (fig. 52), the extremity g, of the tube $h\ h$, is below B, and is shut by a valve which rises or falls by means of the float F, which serves for its support.

In the first figure also, the ascending tube $n\ n$ O has no valve, in the second there is one placed at z, which opens upwards, and the object of which is to retain the water of the column O $n\ n$.

The above being understood, the following is the operation of the machine.

2. Let us suppose the great receiver B filled with air, and that the cock e placed below it is open; the weight of the water to be raised, of which the level is LL, opens the valve k, the receiver C is filled, the air which it contained escapes by the tube $h\ h\ g$, and passes out through the cock e. Now shut this cock, the weight π will descend; the valves λ and c open, and the water of the reservoir A will issue at the same time through the orifice ϕ, and fill the vessel V, and by the extremity b, of the tube $b\ b$, will fill the great receiver B. The air of this receiver presses the water contained in the receiver C, and forces it to open the valve z, the water rising to the level n O. Now the vessel V and cylinder B being full of water, the float F shuts the orifice g of the tube $h\ h$, and the water of the vessel V descends by the siphon $t\ t$, into the vessel R, and shuts the cocks λ and c. By means of the siphon x, the water of the vessel R falls into the vessel S, which descends at the same time as the vessel R, and turns the cock e; and finally the open cock y discharges the superfluous water beyond what is necessary to preserve to S such an excess of weight as is sufficient to keep open the cock e.

While the water of the great receiver B runs through the orifice of the tube $d\ d$, the valve z, pressed by the column of water $z\ n$ shuts itself. The air compressed in the tube $h\ h$, dilates, pressing, notwithstanding, the water of the receiver B, and accelerates its discharge through the tube $d\ d$. At the same time the valve k opens, and the receiver C is filled with water from the reservoir LL. The receiver B, and the vessel V, are emptied at the same time, while the siphon $t\ t$ furnishes more water to the vessel R; this vessel, and the lower one S, losing thus the excess of the weight, which caused the cocks λ and c to shut, the counter weight π re-descends, and opens again the cocks; the receivers B and V being then replenished, the same effects will be repeated. See Mr. Boswell's account of this improvement, in Nicholson's Journal, 4to. vol. i. See also Hachette's Traité Elementaire des Machines.

XX. *Of the centrifugal pump.*

1. The centrifugal pump is a very curious machine, invented by Mr. Erskine, for raising water by means of a centrifugal force combined with the atmospheric pressure. It consists of a large tube of copper, in the form of a cross, which is placed perpendicularly in the water, and rests at the bottom on a pivot. At the upper part of the tube is a horizontal cog wheel, which touches the cogs of another wheel in a vertical position, so that, by help of a double winch, the whole machine is moved round with great velocity. Near the bottom of the perpendicular part of the tube is a valve, opening upwards; and near the two extremities, but on the contrary sides of the arms or cross part of the engine are two other valves, opening outwards. These two valves are, by the assistance of springs, kept shut till the machine is put in motion, when the centrifugal velocity of the water forces them open, and discharges itself into a cistern or reservoir placed there for that purpose. On the upper part of the arms are two holes, which are closed by pieces screwed into the metal of the tube. Before the machine can work, those holes must be opened, and water poured in through them, till the whole tube is filled; by this means all the air will be forced out of the machine, and the water supported in the tube by means of the valve at the bottom.

Fig. 53.

The tube being thus filled with water, and the hole closed by the screw-caps, it is turned round by means of the winch; when the water in the arms, or cross part, acquires a centrifugal force, opens the valves above mentioned, and flies out with a velocity nearly equal to that of the extremities of the said arms. (See fig. 53.)

The theory of this machine will be found under the section Pumps, in our treatise on Pneumatics.

Machines whose operations depend upon the momentum of water.

XXI. *Whitchurst's machine.*

Whitchurst's machine.

1. The momentum of water flowing through a long pipe seems first to have been employed for the purpose of raising small quantities of water to great heights, by Mr. Whitehurst, and described by him in the volume of the Philosophical Transactions for 1775. It has since been carried to a great degree of perfection by Montgolfier, so well known in the history of philosophy in general, and in that of aeronautics in particular. The machine is denominated, by Montgolfier, the *hydraulic ram*; under which name it is described in the following article.

Fig. 54.

The principle of Mr. Whitehurst's construction will be immediately comprehended by referring to (fig. 54); in which A represents the spring or original reservoir, its upper surface corresponding with the horizontal line BC, and with the bottom of the reservoir K. D is the main pipe, an inch and a half in diameter, and nearly 200 yards in length; E is a branch pipe, which, in the particular construction described by the author, was of the same diameter, and intended for the service of the kitchen and offices, situated at least 18 or 20 feet below the surface of the reservoir A; and the cock F was about 16 feet below the same. G represents a valve box, g the valve, H an air vessel, OO the ends of the main pipe inserted into H, and bending downwards, to prevent the air from being driven out when the water is forced into it; W the surface of the water.

Now water discharged from an aperture under a pressure of 16 feet, will be projected with a velocity of 32 feet per second; therefore, such will be the velocity of the water from the cock F: consequently, when a column of water 200 yards in length is thus put into motion, and suddenly stopped by the cock F, its momentum will open the valve *g*, and condense the air in H, as often as water is drawn from F, and the elastic force of the air will raise the water to any required height K.

The reader will perceive immediately, by a comparison of the above machine with the following one, that the principle of operation is exactly the same; the hydraulic ram is doubtless a great improvement, but we cannot consider it in the light of a new machine.

XXII. *Hydraulic ram.*

Hydraulic ram. Fig. 55.

1. The water proceeding from any source is supposed to enter the tube at B (fig. 55) with the velocity due to the height of the fall, and to escape, unless prevented at the orifice C; which orifice may be shut or opened at pleasure by means of a valve.

A reservoir of air, F, is united by a small cylinder *abcd*, to the conduit tube BD; in the middle of the bottom of the receiver F is a circular orifice, to which is adapted a small cylindrical support, of which the extremity E is furnished with a valve; *s* is another valve designed to supply with air the reservoir F, and the space *mn*, which is comprised between the adjutage *abcd* and the cylindrical support. GH is a tube of ascension rising from the point G of the reservoir.

The tube BD, through which the water runs, is called the *body of the ram;* the tube GH, through which the water is raised, is called the *tube of ascension;* the two valves which shut the orifices C and E, receive respectively the denominations of the *stoppage valve,* and the *ascension valve.* These valves are hollow globes, represented at D and *e*, the thickness of the metal being such that they may not weigh more than double the weight of water which they displace. The extremity of the body of the ram C, and the reservoir E, form together what is called the *head of the ram.*

Operation of the ram.

2. The above construction being understood, the operation is as follows: The water running through the orifice C, with a velocity due to the height of the fall, forces the spherical ball or valve D out of its muzzle or bed, and raises it to the orifice C, which it immediately stops; when the water rushing against the other valve *e*, it rises against the orifice E of the reservoir of air F. The water is thus introduced into this reservoir, and into the tube of ascension GH; finally, the ball *e*, losing the velocity communicated to it by the stopping of the orifice C, descends, as well as the ball D, by its own weight into their first positions; the water begins running again at the orifice C; the ball D is again elevated and closes that orifice anew; and thus the effects are continually repeated, in times, which, for the same ram, and the same current, are sensibly equal.

The effect is produced during the stoppage of the orifice C; that is, it commences when the valve D meets the orifice, and ceases when it returns to its first position; in this time four distinct periods or intervals may be traced. In the first the water is running through the orifice C with the velocity due to the fall, and that orifice is closed; in the second, which is much shorter than the first, the compression of the air is effected; in the third, the ascension valve is opened, the air in the reservoir is compressed, the water rises in the ascending tube G, the ascension valve *e* is shut, as is likewise the valve D; in the fourth, the air and metal compressed in the second interval re-acts, the valve D descends from the orifice, and the velocity of the water recommences, and reproduces the like effects.

We have only spoken above of one kind of ram; but the same principles are susceptible of very extensive application: thus, for example, it would be easy to raise other water, besides that which runs through the ram, as the water in mines, pits, &c.

Estim of its

3. To judge of the merits of any hydraulic engine, we must consider its produce, the expence of its erection, and that of keeping it in repair. Now, in every hydraulic engine the force expended is the product of the water which comes from its source, multiplied by the height it falls through before it acts on the machine; the produce being the quantity of water raised in the same time, multiplied by the height to which it is elevated. Applying these principles to the hydraulic ram, it appears from an experiment made upon one at the Polytechnic school, in France, that the expence was to the produce as 100 to 45.

The height of the fall was 1·82 metre, that of the ascending pipe 11·66 metres, the tube of the vertical or active column ·054 metre in diameter, being fixed at the bottom of an oval-shaped vessel: the horizontal or passive column also ·054 metre in diameter; the ascending tube of tin ·002 metre interior diameter, and 11·66 metres elevation; the total length 32·66 metres.

The discharging valve closed from 40 to 42 times per minute, and the water which fell in 10 minutes was 493·7 litres; that which was raised up the ascending tube in the same time 51·8 litres; from which data the ratio above stated is obtained.

In another hydraulic ram, placed by Montgolfier in his garden, the fall which was procured artificially was 7½ feet; the height to which the water was raised, 50 feet; the diameter of the tubes, 2 inches; the water expended in four minutes was 315 litres, that elevated, 30 litres; hence, the expence or force employed is 7½ × 315 = 2362; the useful force 50 × 30 = 1500, which give the ratio of 100 to 64.

In another experiment made upon a ram at Avilly, near Senlis, by M. Turquet, bleacher, the expence was found to be to the produce as 100 to 62. In an experiment on a fourth machine, in which the passive column was 10·4 metres in length, and the height of the ascending pipe nearly seven times that of the head of water, the discharging valve closed 104 times in a minute, and the expence was to the produce as 100 to 57. Taking the mean of these experiments, the expence will be to the produce as 100 to 57; so that an hydraulic ram, placed not in unfavourable circumstances, and executed with care, may be said to employ usefully at least half its force.

XXIII. *Improved hydraulic ram.*

Impr hydra ram.

1. A considerable alteration and improvement has been made in the original ram of Montgolfier, by his son, who has taken out a patent for his construction in this country; the operation is more effective than in that of his father, amounting to about 84 per cent.; at the same time the length of the tube is much less than in the former machine.

These improvements consist, first, in the addition of a pressure of the column of water in the ascending pipe, and by degrees the air will force all the water which was injected through the flap valves, through the said pipe, and cause that quantity of water to issue from the upper extremity of the pipe. The *vis inertia* of the water, which was in motion in the ram, having expended itself by forcing a portion of the water into the air vessel, and making a still greater compression of the contained air, a recoil of the water will take place in the body, arising from the re-action or elasticity of the air contained in the air matrass, as likewise of the metal of which the tube is composed. The flap valves within the vessel being now shut, prevent the return of the water which had been forced into the air vessel, and the recoil of the water in the body towards the open end causes a slight aspiration, the external stop valve descends by its weight, and prevents the water with which it is covered from entering through it; but the air passes through the small pipe leading from the open air to the annulur space above-mentioned, and opens the snifting valve, by which means a very small quantity of air enters the matrass.

During the recoil, the internal stop valve, having nothing to sustain it, falls by its own weight and opens the passage; and as soon as the force of the recoil has expended itself in acting against the column of water contained in the reservoir in the body of the machine, the water begins again to flow in its original direction, and repeats the action above described.

XXIV. *Of the ram siphon.*

Ram siphon.

1. When water is made to pass from any source to a lower place, by means of a common siphon, the column of water which fills the interior of the tube remains continuous; but if the highest point of the siphon communicate with the exterior air, the column will immediately separate into two columns, and be discharged at each extremity of the machine. The intention of the ram siphon, however, is to procure a discharge from the higher point, without the consequences above alluded to. The following construction is due to M. Hachette, of the Polytechnic school, and is described by him in his "Traité des Machines."

Fig. 56.

Let ALCR (fig. 56) represent the siphon which is to carry the water from A to R; on the branch RB is placed the head of the ram, with two valves C and E, of discharge and ascension, and a reservoir of air F; a cock is placed at R, and a valve at K, which opens and shuts by means of a lever, of which the extremity is fixed at L, when the valve is open. In order to charge the siphon, the cock R is shut, as also the valve K, and water is poured into the tube D; the air escaping through the same adjutage, from the vertical and horizontal branches of the siphon. After having shut the orifice D, the cock R and the valve K are opened; the water which runs through the siphon from A to R, shuts the valve C, opens the ascension valve E, and escapes from M in a jet, or is raised in a tube of ascension.

XXV. *Of the Danaide.*

The Danaide.

1. The *Danaide*, a new hydraulic machine, invented by M. Mannoury Dectot, is thus described by MM. Prony and Carnot, in their report to the French Institute.

The model by which this machinist exhibited his experiments, consists principally of a trough, the bottom of which has a hole in its centre: it is cylindrical, nearly as high as it is broad, and made of tin plate. It is fixed to a vertical axis of iron, which passes through the middle of the hole in the bottom, leaving a vacant space all round, through which water escapes as it flows into the trough (from above); this axis turns with the trough upon a pivot, and is fixed above to a collar. The object of the inventor was, that the water flowing into the trough from above, with a certain quantity of *vis viva*, should communicate the whole of it to the solid parts of the machine, so as to be employed afterwards in the producing some useful effects, always excepting the small quantity of force necessary to enable the water to escape by the orifice below. This object he thus obtains: within the trough there is affixed to the axis a drum, likewise, of tin plate, concentric with the trough, and close above and below. This drum, which turns round with the trough, occupies nearly the whole of its capacity; the space between the two not exceeding an inch and a half. A similar space exists also between the bottom of the trough and the drum; it is, however, less than the former, and is divided into several compartments by diaphragms, proceeding from the circumference to the central hole in the bottom of the trough. These diaphragms do not exist between the sides of the drum and the trough, and the compartments at the bottom communicate with this annular space.

The water, which comes from the reservoir above by one or two pipes, makes its way into this annular space between the drum and the trough. The bottoms of these pipes correspond with the level of the water in the trough, and they are directed horizontally, as tangents to the mean circumference between that of the trough and of the drum. The force which the water has acquired by its fall along the pipes causes the machine to move round its axis; and this motion gradually accelerates till the velocity of the water in the space between the trough and the drum equals that of the water from the reservoir; so that the shock of the water from above upon that in the machine becomes imperceptible. Now this circular motion communicates to the water between the trough and the drum a centrifugal force; in consequence of which it presses against the sides of the trough. The centrifugal force acts equally upon the water contained in the compartments at the bottom of the trough, but obviously less and less as the water approaches the centre. The whole water, then, is actuated by two forces, which oppose each other; namely, gravity, and the centrifugal force. The first tends to make the water run out at the orifice at the bottom of the trough; the second tends to drive the water from that hole. To these two forces is joined a third, *friction*, which here acts an important and singular part, since it promotes the efficacy of the machine, while in other machines it always diminishes that efficacy. Here, on the contrary, the effect would be nothing were it not for the friction, which acts in a tangent to the sides of the trough and drum.

By the combination of these three forces, there must result a more or less rapid flow from the orifice at the bottom of the trough, and the less force the water has in its escape, the more it will have employed in moving the machine, and, consequently, in producing the useful effect for which it is destined. The moving power is the weight of the water running in, multiplied by the height of the reservoir from which it flows above the bottom of the trough, and the useful effect is the same product diminished by half the force which the water retains when it issues from the orifice below.

The reporters endeavoured to ascertain, by direct experiment, the amount of its useful effect. They fixed a cord to the axis of the machine, which, by means of pulleys, properly placed, raised a weight as the machine turned round. The result of repeated experiments was, that the effect produced amounted to $\frac{7}{10}$, and sometimes to $\frac{75}{100}$ of the moving cause; an effect which surpasses that of most hydraulic machines.

XXVI. *Hydraulic pendulum.*

1. This machine is intended to produce, by the uniform discharge of a reservoir or source of water, an alternate or vibrating motion, whence it has obtained the name of hydraulic pendulum.

There are several constructions of this machine; that represented in (fig. 57), is due to M. Perrault. It consists of a chest or cask ABCD, moveable on two pivots, and divided into two parts by the partition CD. When the bottom AB of this vessel is horizontal, the water from the source M, falls on C, the middle of the partition; when the machine immediately becomes inclined, the water falls into the part, such as B, elevated above the level of the axis; this part of the vessel increases in weight in proportion as it fills with water, and when it is full, the entire vessel turns on its axis, and the water from the source falls in the part AC, which, in consequence filling up, the weight again causes it to oscillate; and this oscillation will thus continue, and communicate the same to a pendulum bar, or to any other body attached to the pendulum vessel.

2. A different form is given to this machine by M. Boitias. Here there is placed at the lower extremity of a pendulum, a very large float board, moveable on pivots, which turns on the parallel branches of a frame fixed to the pendulum. This float board will assume alternately the vertical and horizontal position. In the first it plunges into the stream, and receives the impulsion of the water; the float board moves with the pendulum, and having reached the lowest point of its oscillation, a counter weight causes it to turn on its pivot, gives it the horizontal position, and commences a new oscillation.

Instead of employing a counter weight to open and shut the float board, there may be employed two cords attached to fixed points, and to the float board. These strings, stretched by the pendulum, will draw the float board at the beginning and end of the oscillation, and cause it to assume successively its vertical and horizontal position.

This hydraulic pendulum, does not, like the common pendulum, oscillate on both sides of the vertical, but simply on one. It moves on towards that side to which it is impelled by the motion of the water, till reaching the highest point of its oscillation, the float board opens, and the whole is brought back by its own weight to its direction of verticallity; in that position the float board again closes the frame, and the impulsion of the water upon it again drives it forward; whereby the motion is rendered alternate, and while the stream is uniform, the oscillations will be so likewise.

Machines for measuring the velocity of water.

XXVII. *Of the bent tube, by M. Pitot.*

Pitot's bent tube.

This machine was first invented by M. Pitot, and is described by him in the Memoirs of the Academy of Sciences, Paris, 1732. It consists of a prism of wood, one of the angles of which is presented to the current. On the posterior face are fixed two tubes of glass, vertical, and parallel to each other, having a graduated scale between them; one of them being bent into a right angle, so that a part of it may pass through the prism horizontally. When this instrument is plunged into a running stream, the general level of the current is shown by the rise of the water in the straight tube, while the height of the water in the bent tube becomes a measure of the force of the stream; the difference between these heights will therefore be the height due to the velocity. With this instrument it is highly necessary (and at the same time difficult in operation) to have it so permanently fixed as to prevent certain small oscillations of the fluids in the two tubes, which necessarily present so many causes of error in our determination. However, independent of this defect, we have seen how much irregularity takes place in the velocities and corresponding heights of fluids; so that, in fact, the instrument answers much better theoretically than in practice. Du Buat found that it could only be trusted to indicate the ratio of different velocities, and that it would not apply to the determination of the actual velocity. He, therefore, suppressed the straight tube, and substituted, instead of the bent glass tube, a tin one, sufficiently large to admit a float for pointing out the height of the water in it, at the same time by a slight change in the immersed end of the tube he prevented, in a great measure, the oscillations of the water mentioned above. See Du Buat Principes d'Hydraulique, tom. ii.

XXVIII. *Of the hydraulic quadrant.*

Hydraulic quadrant.

This instrument is very simple in its construction, as well as in its application. It consists of a quadrant graduated on its arc, and having two threads moving round its centre. One of these strings is shorter than the other, the shorter serving, with a small weight attached to it, as a plumb-line; this hangs in air, while the longer one, having a body specifically heavier than water attached to it, hangs in the water: the quadrant being then so adjusted that the plumb-line answers to the zero of the quadrant, the angle which the other thread forms with this is to be taken as the measure of the velocity of the water.

Bossut has shown that the force is as the tangent of the angle subtended between the threads; so that if u be the velocity when the thread has any position, or forms any angle S, and V the velocity when it has any other position, or when the angle is R, we shall have

$$u : V :: \tan S : \tan R;$$

if, therefore, u be known, V may be determined.

XXIX. *Of the Clepsydra, or water clock.*

Clepsydra.

1. This may be considered as an hydraulic machine, which acts by the momentum of water, although there have been some clepsydræ made with quicksilver. This machine, which has received a variety of forms, and is very ancient; the Egyptians employed it for measuring

not only their time, but also the course of the sun; and the invention of it is attributed to this people.

Ancient clepsydra.

Fig. 58.

According to Vitruvius Pollio, Ctesibius of Alexandria was the first improver of the ancient clepsydra. Its original form has been represented as follows. A (fig. 58) is a reservoir, to the top of which is attached a water pipe (not shown in the figure) to preserve an equal pressure, by carrying off the superfluous water. B is a pipe projecting from the reservoir into the upper part of the drum MN, on the front of which drum the ecliptic circle is marked. ODL is a smaller inner drum, which revolves on a tubed arbor F, and which is represented as drawn out of its place; this small drum has a thorough groove *a b*, varying in breadth all round it, like a hoop, tapering throughout from the broadest part both ways to its opposite point. This is of such a diameter, that the middle of the groove just reaches to, and coincides with, a perforation under the tube B, at the upper part of the great drum; so that as the little drum which carries the diurnal index L, and the nocturnal index O, opposite to the former, is turned round by hand, the variation in the breadth of the groove occasions a corresponding variation in the velocity of the effluent water, by making a larger or smaller aperture, according as the sun's place is more or less advanced in the ecliptic, the largest aperture being when the diurnal index is at the beginning of Capricorn. A little bason, or funnel, attached to the upper part of the fixed tube or hollow arbor F, receives the water in its fall within the drum, and transmits it through the said tube by G into the receiving vessel H, in which floats the piece of cork I. This float is connected, by a chain, with the counterpoise K, after it is folded round the arbor P, which carries the hour hand of the dial plate; consequently, as the water rises in the vessel H, the cork I is raised, and its counterpoise K, at the same time falling, gives motion to the arbor and hour hand, and the hours are longer or shorter according to the breadth of the groove, which is at any time under the perforation of the tube B. By this means the orifice being always adjusted to the time the sun was above and below the horizon, the interval was divided into twelve equal portions, or hours, according to the method of the Egyptians.

Such was, probably, the most ancient form of this instrument: it was afterwards improved, as we have stated above, by Ctesibius: other forms are also described by Beckmann, in his "History of Inventions;" Ozanam, in his Recreations, translated by Hutton; Bion, on Mathematical Instruments; and by various other authors. We shall only select one or two of these, which will be amply sufficient for our present purpose.

Modern clepsydra.
Fig. 59.

2. In fig. 59, ABCD represents an oblong frame of wood, to the upper part of which two cords A *a*, B *b*, are fixed at their upper extremities, and at their lower, to the metallic arbor *a b* of the drum E, which contains distilled water: this water is contained in cells so constructed, that they regulate the velocity with which the drum should descend by the force of gravity from the top to the bottom of the frame, and the ends of the arbor indicate the hours marked on the vertical frame during the time of descent. An observer, unacquainted with the nature of the interior cells of the drum, is surprised to see that, by merely folding the strings round the arbor, its weight does not make it run down rapidly, when mounted to the top of the frame; there being no apparent mechanical impediment to the natural action of gravity.

Fig. 59-a.

In order to explain how this phenomenon is produced, we must refer to fig. 59-*a*, which is a section of the drum at right angles to its arbor: suppose this circular plate to be (as is usual) about six inches in diameter, and that it represent the internal surface of either end of the drum, which may be made of copper, or any other metal not subject to rust; then, if we conceive seven metallic partitions, F *f*, G *g*, H *h*, &c. to be closely soldered at each end of the drum, in the oblique direction indicated in the figure, where the black lines are equi-distant tangents to the small dotted circle of 1½ inch diameter, at the points *f*, *g*, *h*, &c. Then it is evident, that any small quantity of water introduced into the drum, would fall into two, or at most three, of the lower compartments, and would remain there until some external force should alter the position of the drum, supposing, in this case, the cords tied close to the arbor; but it has been stated, that they are wound round the circumference of an arbor that has a sensible diameter, say of one-eighth of an inch; they therefore are removed one-sixteenth of an inch or upwards, taking their thickness into the account, from the centre of the drum, which would also be its centre of gravity, if it were empty; on which account it would, in that case, revolve to the left, in the direction FGH downwards, from the cord being at the remote side of the centre, as represented by NO; but conceive the water to be now and then included, it would be elevated to the right, till its weight became a counterpoise to the gravity of the heavier side of the drum, in which situation all motion would cease, and the drum would remain suspended by the cords, in a state of equilibrio.

Let us now conceive a small hole perforated in the partition pressed upon by the water, near the circumference of the large circle, and also at the points F, G, H, &c. and the consequence will be, that the water will first force its way slowly through the perforation at K, from the more elevated to the lower compartment, which effect will diminish its power as a counterpoise, and give such an advantage to the heavy side of the drum, considered as empty, as will cause a small degree of motion towards the left, and, consequently, carry the water once more towards the right; but the water passes through the perforation of the next partition also at I, and produces again the same effect as has been described with respect to *k*, and will continue to do so at the successive perforations, till all the compartments have been filled and emptied in succession; which kind of motion of the drum, contrary to that of the water, it is now not difficult to conceive, will be pretty regular, provided all the partitions be perforated in exactly the same manner, both in magnitude and position.

The difference in the pressure of the water in cells nearly full, and nearly empty, will occasion some little deviation from regularity; but this will be periodical, and must be allowed for in the hour divisions, which ought to be made by a comparison of the spaces fallen through, with the time indicated by a clock or watch.

About 9 ounces of distilled water is sufficient for a clepsydra six inches in diameter and two inches in depth, and the velocity of the fall may be limited either by varying the quantity of water, or by hanging a small metallic cup, F, to receive weights, by a cord wound round in a direction contrary to the cords of suspension, to act as a counterpoise in aid of the water, if the fall be too rapid; and the contrary, if too slow.

It may be proper farther to observe, that it is absolutely necessary that the arbor should fit the central square hole so as to prevent the escape of water from the drum, otherwise the machine would continue to gain velocity, till at length it would no longer afford a true indication of time.

In some cases a cord *c d*, with a weight P, is made to pass over a pulley fixed to an arbor at the top of the frame, with a noose passing over the axis near *a*, as seen in the same figure, which arbor projecting through a dial-plate, or face, turns round and carries a hand to indicate the hours like a common clock. When this construction is adopted, it is an indispensably requisite, that the circumference of the groove of the pulley be divided exactly as the face of the drum in 12 or 24 hours, according as the dial is divided. This clepsydra it is said goes faster in summer than in winter, which seems to be attributable to the drum being relatively heavier in rarefied than in dense air; for it is scarcely possible that any alteration in the fluidity, as some have supposed, would produce any sensible difference. It has been proposed to have a minute-hand and striking part of a common clock added to a clepsydra of this construction, but we are not aware that it was ever carried into execution.

3. The Hon. Charles Hamilton, in the Philosophical Transactions, vol. xliv. proposes the following form for the construction of a machine of this description.

In (fig. 60) AB, CD, are two similar oblong vessels, attached to a frame of wood, which may be conceived to surround the above figure, which shows only the interior mechanism; *a b*, and *c d* are two columns of wood, so floating in water, that their counterpoises F and G just keep their upper ends level with the surface of the water, by means of connecting chains passing over the pulley *f*, and another hid by the dial-plate; the former of these pullies *f*, has a click, which pushes the racket on the barrel *i*, when the counterpoise F falls, but slips easily over the slopes of the teeth, when the counterpoise rises; the latter pulley has also a similar click, acting, in like manner, with a second racket at the other end of the barrel *i*, which racket is also hid in the drawing; so that whichever of the two counterpoises shall be at any time falling, the barrel *i* will move forward in the same direction, and carry the middle hand along with it on the dial-plate; the hour hand goes round by means of dial-work, as in a clock or watch of the common construction, where a diminution of velocity is effected by means of two wheels and two pinions. The action is thus produced by means of five siphons and two balances.

The water enters with an uniform influx, drawn from a reservoir by a siphon of a small bore, the longer leg of which is seen at J, into the middle of what may be called a horizontal trough, supported like a balance by a fulcrum at K, in such a manner that either end of the balance may be elevated according as the long vessels AB and CD require to be alternately filled; near the top of each of these vessels is inserted a long siphon *l* and *m*, the lower branches of which reach down to two small cylindrical vessels *n* and *o*, which are poised by another balance at the fulcrum *p*; these cylindrical vessels have, in like manner, each a small siphon *q* and *r*; lastly, a silken thread tied to the upper end of the cylinder *n*, is carried up round a small pulley, fixed to the frame at *s*, and is fastened to the end of the trough under it, a similar thread being fastened, in like manner, to the cylinder *o*, and to the end of the trough under the small pulley *t*.

It is obvious now, that when the vessel AB is filled to nearly the head of the siphon *l*, the bore of which is larger than that of the siphon J, the water will be discharged into the cylindric vessel *n*, which consequently will preponderate, and, by means of the silken thread elevate the end of the trough above the horizontal line, and make its opposite end, under the small pulley *t*, to be depressed, which will, therefore, conduct the water into the other long vessel CD; during this action, the counterpoise F rises, and its pulley *f* produces no effect on the racket, by means of the click *h*, sliding over the sloping sides of its teeth; but the counterpoise G falls, and the click of its pulley pushes the second racket forwards, in the direction of the figures of the face, I, II, III, IV, &c.

When CD is nearly full, the long siphon *m*, begins to discharge its water; makes the cylindric vessel *o* preponderate, and again elevates, by means of its silken thread, the end of the thread under the small pulley *t*, and depresses the opposite end to fill the vessel AB again; during which time the click *h*, of the pulley *f*, acts with its racket; and thus the alternate increase and decrease of the water in the two vessels are continued without interruption, as long as the feeding siphon continues to furnish a sufficient supply of pure water.

Authors on Hydrodynamics.

232. We have in this section, given, as far as was consistent with the plan and limits of this treatise, a description of what we consider to be the most useful and important hydraulic engines; but it is obvious that many must necessarily also have been omitted; some in consequence of their connection with other subjects, which will be found in our alphabetical arrangements; others, whose operation is more intimately connected with pneumatic pressure, will be described in our treatise of Pneumatics; and for others of a more limited application, we must beg to refer our readers to the writings of particular authors. The following is a catalogue of the most approved works on the subject of hydraulic machinery.

Descriptio Machinæ Hydraulicæ curiosæ constructa. Joh. Geor. Faudieri. Venet. 1607.

Nouvelle Invention de lever l'Eau plus haut que la Source, avec quelque Machines mouvantes par la Moyen de l'Eau, &c, Par Isaac de Caus. 1657.

Josephi Gregorii a Monte Sacr. Principia Physico-mechanica diversarum Machinarum scu Instrumentorum Pneumatices ac Hydraulices. Venet. 1664.

Nouvelle Machine hydraulique, par Francini. Journ. des Sçav. 1669.

An account of this machine is likewise given in the Architecture hydraulique of Belidor, tom. ii. and in the 2d vol. of Desagulier's Experimental Philosophy; in both which performances many other hydraulic machines are described.

An Undertaking for Raising Water, by Sir Samuel Moreland. Phil. Trans. 1674. No. 102.

An Hydraulick Engine, by ———. Phil. Trans. 1675. No. 108.

A Cheap Pump, by Mr. Conyers. Phil. Trans. 1677. No. 136.

M. de Hautefeuille, Reflexions sur quelques Machines a elever les eaux; avec sa Description d'une nouvelle Pompe, sans Frottement et sans Piston, &c. 1682.

Elevation des Eaux, par toute sorte des Machines, reduite à la Mesure, au Poids, a la Balances, par le Moyen d'un nouveau Piston et Corps de Pompe, et d'un nouveau Mouvement cyclo-elliptique, et rejectant l'Usage de toute Sortes de Manivelles ordinaires. Par le Chevalier Morland. 1685.

A New Way of Raising Water, enigmatically proposed. By Dr. Papin. Phil. Trans. 1685. No. 173. The Solutions, by Dr. Vincent and Mr. R. A. in No. 177.

M. du Torax, Nouvelles Machines pour épuiser l'Eau des Foundations, qui, quoi très simples, font un Effet supprenant. 1695. Journ. des Sçav. 1695. p. 293.

An Engine for Raising Water by the Help of Fire. By Mr. Thomas Savery. Phil. Trans. 1699. No. 253.

D. Papin, Nouvelle Manière pour lever l'Eau par la Force du Feu. A Cassel, 1707.

Memoire pour la Construction d'une Pompe qui fournit continuellement de l'Eau dans le Reservoir. Par M. de la Hire, Mem. Acad. Sci. Paris. 1716.

Description d'une Machine pour Elever des Eaux. Par M. de la Faye, Mem. Acad. Sci. Paris. 1717.

Joh. Jac. Brückmann's und Joh. Hein. Weber's Elementar-maschine, oder universal-mittel bey Allen Wasser Hebungen. Cassel, 1720.

Jacob Leupold, Theatri Machinarium Hydraulicarum. 1724, 1725.

Joh. Frid. Weidler, Tractatus de Machinis Hydraulicis toto Terrarum orbe maximus Marlyensi et Londinensi, &c. 1727. Vide edit. erudit. Lips. 1728.

A Description of the Water Works at London Bridge. By H. Beighton, F.R.S. Phil. Trans, 1731. No. 417.

An Account of a New Engine for Raising Water, in which Horses, or other Animals, draw without any loss of Power (which has never yet been practised); and how the Strokes of the Piston may be made of any Length, to prevent the Loss of Water by the too frequent opening of Valves, &c. By Walter Churchman. Phil. Trans. 1734.

Sur l'Effet d'une Machine hydralique proposée par M. Segner. Par M. Leon. Euler, Mem. Acad. Sci. Berlin. 1750.

Application de la Machine hydralique de M. Segner à toutes Sortes d'Ouvrages, et de ses Avantages sur les autres Machines hydrauliques. Par M. Leon. Euler, Mem. Acad. Sci. Berlin. 1751.

M. Segner's machine is no other than the simple, yet truly ingenious, contrivance known by the name of Barker's Mill, which had been described in the 2d volume of Desaguliers' Philosophy, some years before the German Professor made any pretensions to the [illegible]vention. The theory of it is like- [illegible] at the end of his

Recherches sur une Nouvelle Manière d'Elever de l'Eau proposée par M. de Mour. Par M. L. Euler, Mem. Acad. Berlin. 1751.

Discussion particulière de diverses Manières d'Elever de l'Eau par le Moyen des Pompes. Par M. L. Euler, Mem. Acad. Berlin. 1752.

Maximes pour arranger le plus avantageusement les Machines destinées à Elever de l'Eau par le Moyen des Pompes. Par M. L. Euler, Mem. Acad. Berlin. 1752.

Reflexions sur les Machines hydrauliques. Par M. le Chevalier D'Arcy, Mem. Acad. Sci. Paris. 1754.

Memoir sur les Pompes. Par M. le Chevalier de Borda, Mem. Acad. Sci. Paris. 1768.

Dan. Bernoulli Expositio Theoretica singularis Machinæ Hydraulicæ. Figuri Helvetiorum, exstructea. Nov. Com. Acad. Petrop. 1772.

Abhandlungen von der Wasserschraube. Von D. Scherffer, Priester. Wien, 1774.

Recherches sur les Moyens d'executer sous l'Eau toutes Sortes de Travaux hydrauliques, sans employer aucun Epuisement. Par M. Coulumb. 1779.

Suemund Magnussen Holm, Efterretning om skye Pumpen. Kiöbenhavn, 1779.

Moyen d'augmenter la Vitesse dans le Mouvement de la Vis d'Archimede sur son Axe, tire des Memoires Manuscrits de M. Pingeron, sur les Arts utiles et agreables. Journ. d'Agric. Juin. 1780.

The Theory of the Syphon, plainly and methodically illustrated. 1781. [Richardson.]

Memoria sopra la nuova Tromba funiculare umiliata, dal Can. Carlo Castelli. Milano, 1782.

Dissertation de M. de Parcieux, sur le Moyen d'Elever l'Eau par la Rotation d'une Corde verticale sans, fin Amsterdam et Paris, 1792.

Theorie der Wirzischen spiral Pumpe, erläutert von Heinr. Nicander, Schwed. Abhandl. 1783.

Jac. Bernoulli, Essai sur une Nouvelle Machine hydraulique propre à Elever de l'Eau, et qu'on peut nommer Machine Pitotienne. Nov. Act. Acad. Petrop. 1786.

Ch. Langsdorf's Berechnungen über die vortheilhaeftere benutzung angelegter sammelteiche zur betreibung der Maschinen. Act. Acad. Elect. Mogunt. 1784, 1785.

Nicander's Theorie der spiral Pumpe. 1789.

Nouvelle Architecture hydraulique. Par M. Prony. 1790.

A Short Account of the Invention, Theory, and Practice of Fire Machinery; or, Introduction to the Art of making Machines vulgarly called Steam Engines, in order to extract Water from Mines, convey it to Towns, and Jets d'Eau in Gardens; to procure Waterfalls for Fulling, Hammering, Stamping, Rolling, and Corn Mills. By W. Blakey. 1793. [Egerton.]

Treatise of Mechanics, theoretical, practical, and descriptive. By Olinthus Gregory. 3d edition, 1816.

Traité Elementaire des Machines, par M. Hachette. Paris, 1817.

To these may be added the Transactions of the Society of Arts, the Repertory of Arts, Nicholson's Philosophical Journal, and Tilloch's Philosophical Magazine, in different places.

ceived that in water the column might be so long that its weight would overcome the attraction of the piston, and consequently descend. This length in the case of water was 18 cubits. We see here no indication of Galileo's having a correct idea of the atmospheric pressure, as the cause of the ascent of the water, nor does it seem that his explanation was of that kind commonly attributed to him, viz. that nature's abhorrence of a vacuum only extended to about 34 feet; this would appear to be rather the pump-maker's illustration than the philosopher's. Dr. Young says, that "before Galileo's time, it was generally supposed that water was raised in a sucking-pump, on account of the impossibility of the existence of a vacuum; if, however, a vacuum had been impossible in nature, the water would have followed the piston to all heights, however great; but Galileo found that the limit of its ascent was about 34 feet, and concluded that a column of this height was the measure of the magnitude of the atmospheric pressure."

Torricelli. The author, therefore, attributes this discovery to Galileo, the honour of which is generally said to be due to his pupil, Torricelli: we know not on what data he has founded his assertion; and certainly as far as Galileo's own account of the circumstances may be considered as authority, Dr. Y. does not seem to have done justice to Torricelli, who appears not merely to have "confirmed the hypothesis of his master," but to have been the actual author of the doctrine, viz. that the ascent of the water was owing to the atmospheric pressure.

Galileo undoubtedly taught that the air had weight, and confirmed his doctrine by experiment; and Torricelli, after well considering this circumstance, and probably dissatisfied with Galileo's explanation of the ascent of the water, entered upon the study of the various phenomena connected with the sciences of Hydrodynamics and Pneumatics, in which he was peculiarly successful, and more especially in that part of them which relates to the subject at present under consideration.

Torricellian experiment. It occurred to this author in the year 1643, that whatever might be the cause by which a column of water 34 feet high is sustained above its level, the same force would sustain a column of any other fluid which weighed as much as that column of water on the same base; and hence he concluded, that since quicksilver is about fourteen times as heavy as water, it could not be kept up at a greater height than 29 or 30 inches. He then made the experiment still called after his name. He provided himself with a glass tube, about three feet long, and sealed it hermetically at one end; and having filled it with mercury, he closed it at the open end with his finger, and inverted it in a basin of mercury. Upon withdrawing his finger, he had the satisfaction of seeing the column of mercury descend and settle at the height of between 29 and 30 inches in the tube. He was still, however, not aware that the phenomena was to be attributed to the pressure of the atmosphere, although after a little reflection he fell upon that explanation; and after he had fully convinced himself of the truth of his hypothesis, it is said that, with a feeling of generosity of which we have few examples, he expressed his regret that it had not fallen to the lot of his master to have completed a discovery of which he had first laid the foundation.

Torricelli's experiment was published at Warsaw, in Poland, by Valerianus Magnus, as his own discovery; but from the letters of Roberval, it appears that neither Valerianus, nor Honoratus Fabri, to whom it has been ascribed so early as the year 1641, can justly dispute it with him. The fact appears to be, that Torricelli, before he had tried the experiment himself, had suggested his ideas on the subject to his friend Viviani, who did actually first make the experiment; and it is probable that other philosophers might have been informed of the discovery at this time before the author himself had actually submitted his ideas to practice. The discovery of the weight and elasticity of the air has even been ascribed to Jean Rey, who wrote in 1629, some few years before Galileo; and his fourth and tenth essays have been cited in favour of his claims; but although he was certainly aware that compression augmented the weight of the air, and seems to have believed, with Aristotle and others at a very remote period, that air was heavy; yet the proofs which he alleges were not sufficient to convince those who maintained the contrary doctrine.

Even the Torricellian experiment was for some time insufficient for this purpose, many of the philosophers of that day preferring the most absurd and ridiculous explanations of the phenomenon of the sustained column to the simple and obvious cause assigned by this author; others, however, who could rise above the prejudices of early impressions, hailed the discovery as a most important step in the progress of the physical sciences. Mersenne received an account of the experiment and deduction in 1644, and immediately communicated the news to the philosophers of France; and it was soon repeated in various ways by Pascal, Petit, and others. This gave rise to the ingenious treatise published by the former, at twenty-three years of age, entitled "Experiences Nouvelles touchant la Vuide." Having, after some hesitation, adopted Torricelli's idea, and abandoned the principle *fuga vacui*, he devised several experiments for confirming it. One of these was to make a vacuum above the reservoir of quicksilver, in which case he found that it sunk to the common level; and he then engaged M. Perrier, his brother-in-law, to execute the famous experiment of the Puy de Domme; when it was found that the height of the quicksilver half way up the mountain was considerably less than at the foot of it, and that at the top it had fallen some inches. These facts proved incontestibly that it was the weight of the atmosphere which counterpoised the mercury in the tube and the water in the pump; and all further opposition to the doctrine by the supporters of the old hypothesis fell harmlessly to the ground. Des Cartes, although he cannot be considered as entitled to the honour of this discovery, had, notwithstanding, a very just idea of the power of air for sustaining fluids above their level, as appears by some letters which passed at this time (1647), and some years before, in one of which he lays claim to the idea of the experiment of the Puy de Domme. See "Cartesii Opera," tom. ii. p. 243, 246.

The thermometer, one of the most useful instruments appertaining to the science of Pneumatics, is of earlier date than the barometer, having been invented about the beginning of the seventeenth century. The first idea of the thermometer is generally ascribed to the Academy del Cimento, which flourished at Florence, under the protection of the grand dukes of the house of Medici, and which was the first in Europe that applied to experimental philosophy. It is asserted also that Cornelius Drebbel, of Alcmaer, in North Holland, who lived at the court of James I. king of England, had a share in this invention. But the first description of a thermometer ever published is that of Soloman de Caux, a French engineer, in his book "Des Forces Mouvantes," printed in 1624, in folio, but written, as appears, prior to that period; for the dedication to Louis XIII. is dated 1615, and the privilege granted by that monarch is of 1614.

The invention of the thermometer is founded on the property which all bodies, and particularly fluids, have of dilating by the heat which pervades them. As spirit of wine possesses this property in an eminent degree, this liquid was generally employed in preference to any other; but that described by De Caux acted by the dilatation of air confined in a box, which pressing against water, forced it to rise in a tube; Drebbel's thermometer was of the same kind, but which of these two philosophers were the original inventor of the instrument, is a question difficult to be decided.

The thermometer of the Academy del Cimento consisted of a very narrow glass tube, terminating in a bulb of about one inch diameter, which was filled with spirit of wine, after it had been coloured red, by means of tincture of turnsol, or orchilla weed, in order to render it more visible. It may be easily conceived, that the size of the bulb, being considerable in comparison with that of the tube, as soon as the liquid became in the least dilated, it would be in part forced to pass from the bulb into the tube, where it would necessarily ascend; and on the other hand, that as soon as the liquid became condensed by cold, it would of course descend. It was only necessary to take care that, during very cold weather, the liquor should not entirely descend into the bulb; and, during the greatest degree of heat, that it should not entirely escape from it. Towards the lower part, some degrees of temperature were inscribed, as *cold;* a little lower, *great cold;* towards the middle, *temperate;* and at the top, *heat* and *great heat.*

Such is the construction of the thermometer commonly called the Florentine, and which continued to be used by philosophers for near a century afterwards; but as the indications of the different degrees of heat by it were very vague, and no two instruments being comparable with each other, it was rather a matter of curiosity than of utility. What was required for its perfection was some method of fixing a standard point, to which every thermometer should be reduced, so as to register at that point the same nominal as well as actual temperature. Experiment at length proved, that pure water every where freezes with the same degree of cold; and that it boils (under the same barometrical pressure) with the same degree of heat in all parts of the world; these two points, therefore, furnished precisely what was necessary for the completion of the instrument; from this time the thermometer constructed in one country, or by one artist, might be compared with those of others, and the temperature of the air or of any fluid as accurately known as the time of the day by different clocks.

History of the science.

This is the general principle upon which all thermometers are now constructed; but the interval or space between the boiling and freezing points is very differently divided; their forms also are very different from each other: but as these peculiar constructions belong rather to the history of the thermometer, as a philosophical instrument, than to that of the general science of Pneumatics, we shall reserve our further account of it for its proper place in our alphabetical arrangement.

Barometer applied to the measuring of heights.

The barometer is a Pneumatic instrument which may certainly be said to date its history from the Torricellian experiment; and its applicability to the measuring of heights was clearly indicated in Paschal's experiment of the Puy de Domme. The intention, however, of this, was merely to ascertain whether the height of the mercury was affected by being carried to different altitudes, and it was some time afterwards that theorems and formulæ were invented for the purpose of barometrical measurement; the balance between the mercury and the atmosphere was indeed known, but the value of the weights still remained to be determined. The first thing necessary to be ascertained was the law of the condensation of air under different pressures. Mariotte in France, and Boyle and Townley in England, found from experiment that the density of this fluid was proportional to the compressing weight; but this law is true only when the temperature of the air remains constant; attention, however, was not then paid to this important restriction, which in fact could not be indicated by experiments where the compressed volumes of air differed but little from each other in respect to temperature. The law of compression being otherwise known, Halley made use of it for calculating the decrease of density in the beds of the atmosphere at different altitudes; and thus led to the mathematical formulæ, by means of which the difference of altitude of two places may be calculated from the heights of the mercury in the barometer observed at each of them. Newton, in his Principia, perfected Halley's theory, by showing that regard was to be paid to the diminution of gravity, according as the distance from the surface of the earth increased; but, what is very remarkable in so scrupulous an observer of the laws of nature, he also, as well as Halley, omitted to consider the effect of the variations of heat, and of the progressive decrease of the temperature and density of the different strata of the atmosphere. The barometrical formulæ thus obtained, without the correction which renders them applicable to all temperatures, could only furnish a very imperfect approximation, and therefore philosophers and mathematicians who endeavoured to apply them, found that they succeeded only in a few instances, and that generally the results seemed to be subject to various errors which appeared to follow no uniform law. Hypotheses were therefore formed by some authors for explaining these irregularities, while others maintained that no dependence whatever was to be placed upon such theorems, and that they ought to be wholly excluded from works of science. No person seems to have conjectured the true cause; and the omission is the more remarkable, when we reflect that Bouguer and Lambert, men of such peculiar and opposite talents; the one a most accurate observer and philosopher, and the other a very inventive and acute mathematician, were both much occupied with this instrument and its application.

Halley and Newton.

It was M. Deluc who at last discovered the true source of all these errors and anomalies, by searching in the observations themselves for the correspondence between the temperature of the air and the correction which the general formula required. Numerous experiments on the comparative expansions of air and mercury enabled him to perceive the law that those corrections ought to follow, and the quantity in all cases which should be assigned to them.

Corrections of the barometrical formula.

This remarkable discovery, by giving to the barometrical formula an unexpected accuracy, animated the zeal of philosophers, and observations were multiplied to a great extent. Dr. Maskeleyne undertook to reduce the new formula into English measures, while Playfair added a correction for the variation of gravity in different latitudes. Sir George Shuckburgh, by very exact measures, verified the results of M. Deluc, and gave them a greater degree of precision; General Roy also made an application of it at a great number of places in the progress of his survey: the Alps were levelled by MM. Saussure and Pictet; the Pyrenees by M. Ramond, and the Andes by Humboldt; and the barometer rendered portable, became an indispensible instrument to all well-informed travellers.

Notwithstanding, however, so many successful applications, the theory of barometrical levelling was far from being brought to its most simple terms. M. Deluc had adapted the constant co-efficient of his formula to a certain degree of the thermometer, which he called the *normal temperature*, and which he had fixed from the condition, that, for this temperature, the difference of level became a decimal multiple of the difference of the tabular logarithms of the observed barometrical heights. All the corrections relative to temperature which the formula required, commenced, therefore, according to M. Deluc, at the *normal temperature;* in consequence of which this point of commencement changed whenever the formula was applied to any other measures than French toises. These variations were very inconvenient; and it appeared much more natural to make all the corrections commence at some fixed term, as, for example, the freezing point, which is given by experiment, and common to observers of all countries. This is what Laplace has done in a chapter of his "Mécanique Celeste", in which he has established the requisite formula upon the most simple and accurate data. He determines the correction for temperature relative to the expansion of air, according to the experiments of M. Gay-Lussac; but he has modified his results in such a manner as to take into the account the humidity of the atmosphere; and, what is very fortunate, the sum of this correction and the co-efficient of the expansion of air is just equal to $\frac{4}{1000}$. With respect to the expansion of mercury, Laplace employed the values obtained, in conjunction with Lavoisier, in experiments on the expansion of bodies, of which there unhappily remains only a small number of results. Finally, he determined the general co-efficient of the formula from barometrical observations themselves, by combining for this purpose a great number of experiments made in the Pyrenees by M. Ramond, with a degree of care and an accuracy before unknown in this science. The value of this co-efficient has since been confirmed in a direct manner by the experiments made by M. Arago and M. Biot, on the comparative weight of air and mercury; so that all the elements of the barometrical formula, the research of which has cost travellers so much labour, has been obtained directly, and with great accuracy, without quitting the chemical laboratory. Laplace's formula, founded upon data so exact and so ably combined, coincides with observations better than any other in which these advantages are not united, and the rigorous proofs to which MM. Ramond and Daubusson have submitted it experimentally, have demonstrated its utility. It still remained, however, to render the observations comparable with each other, though made with different barometers; which has also been done by Laplace, who has shown that the different indications of these instruments, in circumstances otherwise equal, are the effect of capillary attraction, and has given tables for correcting this effect.

By Laplace.

The barometrical formula being thus improved, observations with that instrument have been considerably multiplied, and carried to a degree of precision almost incredible; a precision which has already led to the idea of distinguishing every place on the globe (in addition to its latitude and longitude) by its height above the level of the sea, or rather by its distance from the centre of the earth; which corresponds in principle with the determination of the position of a point in absolute space, by means of three rectangular co-ordinates; with this view various tables have been computed, and principles of approximation and compensation invented, highly creditable to their respective authors; several of which will be found investigated at length in our section on barometrical measurements.

4. The next important step in the science of Pneumatics was the invention of the air-pump by Otto de Guericke, the celebrated consul of Magdeburg, who exhibited his first public experiments with it before the emperor and the states of Germany, at the breaking up of the Imperial diet, at Ratisbon, in the year 1654: but his description of the instrument, and of the experiments performed upon it, were first published in his "Experimenta nova Magdeburgia de Vacuo Spatio," in the year 1672.

Like most other important inventions and discoveries, the honour of this has been warmly contested; Dr. Hook and M. Duhamel ascribe the invention of the air-pump to Mr. Boyle; but that ingenious philosopher frankly confesses, that "De Guericke was before-hand with him." In a letter which Boyle wrote to his nephew, Lord Dungarvon, at Paris, about two years after Schottus' book was published, he introduces the acknowledgment of his obligation for the discovery of this useful machine, to what he had heard of it, though he had not perused it, by that well-applied passage of Pliny, "benignum est et plenum ingenui pudoris fateri per quos profeceris." Some attempts, he assures us, he had made upon the same foundation, before he knew any thing of what had been done abroad; but the information he afterwards received from Schottus' "Mechanica Hydraulico Pneumatica," published in 1657, wherein was given an account of De Guericke's experiments, first enabled him to bring his own instrument to maturity. Hence, with some assistance from Dr. Hook, arose a new air-pump, more simple and manageable than the German original, but this noble instrument has since received various modifications and improvements, the most popular of which will be described in the course of the following treatise. We shall therefore merely, as connected with our history, give an explanation of De Guericke's pump, and the first-constructed one by Mr. Boyle, in this place, reserving our further account of its progress for that part of the article above alluded to, after the principles of its operations have been theoretically investigated. De Guericke's machine is exhibited in (plate 1, fig. 1), PNEUMATICS. It consists of an iron three-legged frame, *a b c d f*, supporting a round iron plate *b c*, in the middle of which is inserted a brass syringe *g h*. The upper part of this syringe is furnished with a rim of lead *y* (fig. 1-*a*), and it is fastened below by means of an iron ring *k*, *k*, and three iron arms, *o*, *o*, *o*, to the legs of the frame. Within the rim *y* there is a brass plate *m n* (fig. 1-*b*), encompassed by a ring of leather, and fixed by three screws, which terminate upwards in a small tube *n*, into which the pipe connected with the vessel to be exhausted is inserted as occasion requires. To this, on the lower side, is adapted a valve of leather, through which the air passes into the syringe. In this plate there is also another small valve at *z*, opening upwards, through which it escapes. The plate being covered by a copper vessel, *x*, *x*, intended for containing water. The piston of the syringe *s h* (fig. 1) and (fig. 1-*c*), is connected by a joint at *t* with the iron rod *t u*, which is fastened to the handle *w u u*, and this moves round the pin at *w*, by which it is connected with one of the legs of the frame. In order to prevent the air from entering into the syringe, a copper vessel of water is suspended by hooks to the arms *o*, *o*, *o*, so that the lower part of the syringe at *k*, *k*, and the piston, may be always covered with water when the machine is at work.

The receiver L is a glass sphere, adapted to a brass cap *p p*, which has a pipe with a stop-cock *q r*, and this pipe is fitted to the tube *n* above-mentioned. From this brief description of the machine, its operation may be easily understood. When the piston *s h* is depressed, the air will be expanded in the syringe *g h*, and that of the receiver will descend into it through the valve in the lower surface of the plate *m n*; but when the piston is elevated, and the air is compressed, this valve, shutting upwards, will close the passage to the receiver, and make it escape through the valve *z*, which opens upwards. In order to render the exhaustion more complete, a small exhausting syringe is adapted to the plate, which is represented at *m*.

Much ingenuity is doubtless displayed in the construction of this machine, considering that at this time the elastic property of the air was but little known, but still it had many defects, as was naturally to be expected in a first attempt; amongst others, the labour of working it was very considerable, and its operation very slow; moreover, being worked under water, the number of experiments that might be performed upon it was very limited.

We have said that the description given of this machine of De Guericke, by Schottus, led Mr. Boyle to undertake, with the assistance of Dr. Hook, the construction of a similar machine, in which some of the more prominent defects of the preceding one were avoided. This air-pump is represented in the same plate (fig. 2). It consisted of a spherical receiver A, with a hole at the top, whose diameter BC was about four inches; this was covered with a plate, having a brass rim DE, which was firmly cemented to the ring of glass that surrounded the hole, and to the tapering orifice of the brass rim was adapted a brass stopple FG, ground so exactly as to prevent, as much as possible, the admission of air. In the centre of the cover was a hole, of about half an inch diameter, provided with a socket; to which the brass stopple K was so fitted as to prevent the entrance of the air; the lower part of this stopple was perforated with a hole, and through this passed a string for the convenience of moving to and fro the subjects of experiments. To the neck of the receiver a stop-cock N was fastened, and to the shank of the cock X, a tin plate MTUW was so cemented as to prevent the admission of air. The lower part of the machine consisted of a wooden frame with three legs, *a*, *a*, *a*, and a transverse board *b*, *b*, *b*, on which the pump rested. The cylinder of the pump was cast brass, and it was fitted with a sucker of which one part was covered with shoe leather, so as exactly to fill the cavity of the cylinder; and to this was fastened the other part, which was a thick, narrow plate of iron *e e*, somewhat longer than the cylinder, indented on one edge with narrow teeth, so as to admit the corresponding teeth of a small iron nut, fastened by two staples to the under side of the transverse board *b*, *b*, *b*. On this the cylinder rested, and was turned to and fro by means of the handle *f*. The last part of this cylinder is the valve R, consisting of a hole bored through at the top of the cylinder, somewhat tapering towards the cavity, into which hole a ground peg of brass is fitted, to be taken out or put in at pleasure. In order to prevent more effectually the admission of air, and to prepare the sucker of the pump for motion, a quantity of oil was poured in at the top of the receiver, and also into the cylinder. The operator having fixed the lower shank of the stop-cock into the upper orifice of the cylinder, turns the handle, and thus forces the sucker to the top of it, so that no air may be left in its upper part. Then shutting the valve with the plug, and turning the handle the other way, he draws down the sucker to the bottom of the cylinder, and thus its cavity, into which no air is admitted, will be in an exhausted state. By turning the stop-cock, and opening a passage between the cylinder and the reservoir, the air contained in the one will descend into the other; and this air being prevented from returning, by turning back the key of the stop-cock, will be made to open the valve, and to escape into the external air, by forcing the sucker to the top of the cylinder; thus, alternately moving the sucker upwards and downwards, turning the key, and stopping the valve, as occasion requires, the exhaustion may be carried on to any extent.

We shall not enter more at length upon the construction of the air-pump in this place; we have merely described these two instruments, considering their construction as appertaining as much to the general history of Pneumatics, as to that of the air-pump in particular, for we doubtlessly owe to this invention some of the finest discoveries in the science.

History of the science.

Specific gravity of the atmosphere.

5. At present we have only traced the history of Pneumatics to that point where air was, in the most satisfactory manner, shown to be a heavy body, and even its weight or pressure actually ascertained upon any given area; for it is obvious that the mercury in the tube of Torricelli was exactly equal to the weight of a column of air of equal base, from the surface of the earth to the top of the atmosphere; and it hence appeared, that every square inch of surface was pressed with a weight equal to about 14 or 15 lbs. avoirdupois, but still the weight of any given bulk of air, or, in

other words, its specific gravity; yet remained to be determined. It was soon found that the weight of air is continually varying, according to the different degrees of heat and cold, and the concurrence of other causes. Pascal observed it in France, Descartes in Sweden, in 1650; Mr. Boyle and others in England, in 1656. Some noticed that it was generally heaviest at night, and in winter; and that its variations in this season were the greatest, and still more in northern than in temperate climates; and hence arose the application of this tube to the uses and purposes of a weather-glass. With respect to the weight or specific gravity of atmospheric air, it was variously stated. Ricciolus estimated the weight of air, at a medium, to that of water as 1 to 1000; Mersenne as 1 to 1300, or 1 to 1356; Lana as 1 to 640; Galileo, as 1 to 400. Mr. Boyle, by means of some accurate experiments, stated the ratio to be as 1 to 938, but says, that all things considered, the proportion of 1 to 1000 may be taken as a medium. These determinations, however, which were all made without any reference either to the state of the barometer or thermometer, must be considered as very vague and indefinite. It is, in fact, but lately that the specific gravity of atmospheric air has been ascertained with great precision: Sir George Shuckburgh, by a very accurate experiment, found the ratio to be that of 1 to 836, the barometer indicating at the time 29·27 inches, and Fahrenheit's thermometer 53°; and the comparative gravity of quicksilver to air as 11364·6 to 1. The above reduced to the constant height of the barometer 30 inches, and to 55° of the thermometer, is 1 to 833. Lavoisier makes the specific gravity ·00128, that of water being 1.

Steam employed as a first mover.

6. We have hitherto principally confined our remarks to the progress that has been made in developing the properties of common or atmospheric air; but this fluid forms only one of many which fall within the pale of the science of Pneumatics. Another important fluid, particularly with reference to its practical application as a first mover, is steam; which was first brought into notice by the marquis of Worcester, in his work entitled a "Century of Inventions," published in 1663. Hints, indeed, of the possibility of such a machine had been given a hundred years before by Matthesius, in a collection of sermons entitled *Sarepta*, and at a subsequent period by Brunau; but the marquis of Worcester professes to have carried the project into complete effect, under the denomination of a *fire water-work*. It is thus described: "I have taken a piece of a whole cannon, whereof the end was burst, and filled it three quarters full of water, stopping and screwing up the broken end, as also the touch-hole, and making a constant fire under it: within twenty-four hours it burst, and made a great crack; so that having a way to make my vessels, so that they are strengthened by the force within them, and one to fill after another, I have seen the water run like a constant fountain-stream, forty foot high. One vessel of water rarefied by fire, driveth up forty of cold water; and the man that tends the work is but to turn two cocks, that one vessel of water being consumed, another begins to force and refill with cold water, and so successively, the fire being tended and kept constant; which the self same person may abundantly perform in the interim between the necessity of turning the said cocks." This account, it must be acknowledged, is very obscure, and the machine, not being at the time practically introduced, was soon forgotten; but a similar engine was afterwards, that is to say, a little before the year 1700, constructed by Savary; but whether he were indebted to the account given by the marquis of Worcester, or whether he actually re-invented a similar machine, appears to be doubtful. About the date of 1710, the piston and cylinder was invented by Newcomen, and, with Beighton's apparatus for turning the cocks by its own motion, the engine remained nearly stationary for many years. Within the course of the last half century, however, it has attained an astonishing degree of perfection; but the particulars, we conceive, belong rather to the history of the steam-engine, than to that of the general science of Pneumatics, we shall therefore pursue it no further in this place.

Marquis of Worcester.

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7. We have already, in our historical sketch of Hydrodynamics, had occasion to refer to the theory of the resistance of fluids, and have mentioned various authors who have conducted different courses of experiments, in order to supply such data as are requisite to reduce the computed results to an agreement with experimental deductions. But in that detail we have, as far as possible, confined our account to the case of incompressible fluids, particularly water, reserving for the present article what properly belongs to the resistance of the air, and other elastic fluids.

As far as it respects the common theory of resistance, in which we suppose that each particle after striking the body ceases to be effective, it makes no difference whether the fluid be elastic or non-elastic, our hypothesis is the same in both cases, and consequently our results: but it is obvious that air and water must resist bodies moving in them with very different effects, and probably by different laws; the former approaching much nearer to that state of perfect fluidity, which the theory supposes, than the latter: still, however, we have here occasion for practical data to correct our theoretical deductions, and accordingly many interesting experiments have been undertaken, with a view of supplying such information.

Rol exp on t sist the

The resistance of the air becomes a subject of considerable importance in case of military projectiles, in order to allow for the difference which it causes in the times of their flights, and in the length of their ranges. Before the time of Mr. Robins, it was thought that this resistance to the motion of such heavy bodies, as iron balls, and shells, was too inconsiderable to be regarded, and that the rules and conclusions derived from the usual parabolic theory, were sufficiently correct for the common practice of gunnery. But this author demonstrated, in his "New Principles of Gunnery," that so far from being inconsiderable, it was in reality enormously great, and by no means to be rejected, without causing the greatest errors; the effect being such, that balls and shells, which range in the air at the most only to the distance of two or three miles, would, in a vacuum, range 20 or 30 miles, and even more. At the time to which we now allude, viz. about the year 1740, no correct method had yet been invented for estimating the velocity with which a ball is discharged from a piece of ordnance, and therefore no means could be employed for estimating the resistance of the air. The ranges were known, and by assuming the parabolic theory to be correct, the velocity of projection might be found: but what is stated above shows how excessively erroneous the determinations were that depended upon such an hypothesis.

The first thing to be done was, therefore, to ascertain the velocity of projection, which might be either computed, after determining the explosive power of gunpowder, or be directly deduced from experiment, provided an eligible method could be devised for the latter purpose. The apparatus invented by Mr. Robins for the direct determination of the velocity was the *ballistic pendulum*; we have given a slight description of this instrument (art. 107, DYNAMICS), but it will be treated of more at length under the article GUNNERY. It will be sufficient, in this place, to observe, that by this machine Mr. Robins found the means of ascertaining very accurately the velocity of projection, and hence soon convinced himself of the importance of a due knowledge of the laws of the air's resistance.

To determine the quantity of this resistance in the case of different velocities, the author discharged musket-balls, with various known velocities, against his ballistic pendulum, placed at different distances, and thus discovered, by experiment, the quantity of motion in passing through those spaces, with the several known degrees of velocity. Having thus ascertained the velocity lost or destroyed in passing over a certain space, in a given time (which time is very nearly equal to the quotient of the space divided by the medium velocity between the greatest and least, or between the velocity at the mouth of the gun and at the pendulum), that is, knowing the velocity v, the time t, and the space s, the resisting force is thence easily determined, being equal to $\frac{v\,b}{2\,g\,t}$; where b denotes the weight of the ball, and v the medium velocity above-mentioned; also $g = 16\frac{1}{12}$ feet.

The balls employed by Mr. Robins were leaden ones, of $\frac{1}{12}$ of a pound weight, and $\frac{3}{4}$ of an inch in diameter; to the medium velocity of 1,600 feet per second, he found the constant resistance to be 11 lbs., while, with a velocity of 1,065 feet, it was only $2\frac{4}{5}$ lbs. Now, according to the common theory of resistance, the former of these should have been $4\frac{1}{2}$ lbs. and the latter 2 lbs.; so that, in the former case, the real resistance is more than double that given by the theory, being increased as 9 to 22; whereas, in the latter, the increase is only in the ratio of 5 to 7, or as 9 to 12·6. Mr. Robins invented, likewise, another machine, having a whirling or circular motion, by which he measured the resistance to larger bodies, though with much smaller velocities; a description of which is given in a subsequent article of this treatise.

Euler did not disdain the task of translating and commenting upon the above treatise of Robins; in the course of which he shows that the common doctrine of resistances answers pretty well when the motion is not very swift; but that when it is rapid, it gives the resistance less than it ought to do, for which he assigns the following reasons: 1st. Because, in quick motions, the air does not fill up the space behind the ball fast enough to press on the posterior part to counterbalance the weight of the air on the anterior part. 2dly. The density of the air before the ball, being increased by the quick motion, will press more strongly on the fore part, and will thus resist more than the air in its natural state. He shows that Robins's rule for the resistance will not extend to any greater velocities than 1,670 feet per second; and he has given a new formula, which will extend to much higher velocities: in some other respects also the deductions of this author differ considerably from those of Robins. Mr. Glenie, in his "History of Gunnery," published in 1776, mentions some experiments which gave very different results from those deducible from the formula of our author. Robins had stated, that while the velocity did not exceed 1,100 or 1,200 feet per second, the resistance might be assumed proportional to the square of the velocity; but that when the velocity exceeded 1,200 feet, the absolute resistance would be nearly three times as great as it should be with a comparison of smaller velocities: to which assertions Mr. Glenie replies, that he found, with a much less velocity than that of 1,100 feet per second, the resistance to be considerably greater than that due to the square of the velocity; and that to a velocity greater than 1,200 feet, the resistance was much less than three times that determined by a comparison of less velocities. In fact, a law which steps at once from a ratio of equality to a ratio of 3 to 1, could not but be erroneous, and it is not a little surprising that so accurate a mathematician as Mr. Robins should have been induced to propose such a rule, which was even contradicted by many of his own experiments; for he found that, with a velocity of no more than 400 feet per second, the resistance was greater than that which he proposes to extend to velocities of 1,100 feet.

History of the science.

Dr. Hutton.

The subject of the resistance of air to the motion of military projectiles, begun by Mr. Robins, was continued by Dr. Hutton; this author made use of a much larger apparatus, and employed cannon of larger calibre; his experiments were much more extensive, and his determinations more precise; but, as we shall have occasion to enter at some length upon Dr. Hutton's deductions in the course of the following treatise, we shall dispense with any further notice of it in this place, except in stating, that he found the resistance to be a function of the velocity, expressible in two terms, the one depending on the simple, and the other on the second power of the velocity, with constant co-efficients; viz. denoting the resistance by R, the diameter of the iron ball by d, and the velocity by v, Hutton's formula is

$$r = (\cdot 00002576\,v^2 - \cdot 00388\,v)\frac{d^2}{3\cdot 86},$$

which is general for every velocity, and for an iron ball of any given diameter.

Force of elastic fluids.

8. Another important inquiry connected with the science of Pneumatics is the determination of the force of elasticity in different elastic fluids, and particularly that of fired gunpowder and steam. Mr. Robins, to whom we have so frequently referred above, may be said to be the first who undertook a course of experiments, with a view to this determination, as it relates to the former. He premises that the elasticity of this fluid increases by heat and diminishes by cold, in the same manner as that of air; and that the density of the fluid, and consequently its weight, is the same with an equal bulk of air, having the same elasticity and the same temperature. From these principles, and from the experiments on which they are established, he concludes that the fluid produced by the firing of gun-

powder, is nearly $\frac{3}{10}$ of the weight of the generating powder itself; and that the volume of this air or fluid, when expanded to the rarity of common atmospheric air, is about 244 times the volume of the said generating powder. Count Saluce, in his Miscel. Phil. Mathem. Soc. Prev. Turin, p. 125, makes the proportion as 222 to 1, which he says agrees with the computations of Hauksbee, Amontons, and Belidor.

It follows, therefore, from the preceding deductions of Robins, that any quantity of powder fired in a confined space, which it adequately fills, exerts, at the instant of its explosion, against the sides of the vessel containing it and the bodies it impels before it, a force at least 244 times greater than the elasticity of common air, or, which is the same thing, 244 times greater than the pressure of the atmosphere; and this without considering the great addition arising from the violent degree of heat generated at the instant of explosion. The augmentation due to this latter cause Mr. Robins estimates as somewhat greater than 4 to 1, which, combined with the former ratio, increased for the sake of simplifying to 250, gives about 1,000 times the atmospheric pressure for the elastic force of fired gunpowder at the moment of explosion; and he hence computes that the force or pressure of the flame at that moment is equal to a weight of 14,750 lbs. upon every square inch.

It is to be observed that the ratio of 4 to 1, as arising from the heat, which the author assumes to be the same as the greatest heat of red-hot iron, is taken as a kind of medium heat; and that in the case of large quantities of powder, this medium is greatly exceeded, while it will be lower with smaller charges; and consequently, that in the former case, the pressure will be greater than that due to 1,000 atmospheres, but it will be less than the same when the quantities are small.

Expansive force of steam.

The velocity of expansion of gunpowder, however, and the pressure on which it depends, it must be acknowledged, are yet but imperfectly known, different authors giving very different results. Euler seems to estimate the pressure at about double what results above from Robins's deductions; and D. Bernoulli makes it about ten times as great. Dr. Hutton observes, that the velocity of expansion of the flame of gunpowder, when fired in a piece of artillery without either bullet or other body before it, is prodigiously great, viz. 7,000 feet per second, and upwards, as appears from the experiments of Mr. Robins: but Bernoulli and Euler suspect it to be still greater; and that he (Dr. Hutton) had reason to suppose that it is not less than four times as great at the first moment of explosion. With respect to the expansive force of steam, as we shall have to examine that subject at length under the article STEAM ENGINE, we shall merely state in this place, that the most accurate results which have yet been obtained relative to this inquiry, are those of Mr. Dalton, published in the Manchester Transactions, vol. v., and which have also appeared in Nicholson's Journal, vol. vi. and vii. The author, by a long series of experiments, found the expansive force of steam for all degrees of temperature from 30° of Fahrenheit's thermometer to 212°, estimating it by the number of inches of mercury it was capable of sustaining; and from these experimental deductions he computed the expansive force from 30° to 40° and from 212° to 325°. It appears from these, that at 30° of temperature, the force of the vapour is equal to $\frac{1}{5}$th of an inch of mercury; at 180° it is equal to 15·15 inches; at 212° to 30 inches; and at 325° to 140·70 inches.

9. It now only remains for us to say a few words on one of the most recent inventions connected with the science of Pneumatics, viz. the construction and ascent of air-balloons. Dr. Black had exhibited, in his lectures, a bladder filled with hydrogen gas, and suggested the possibility of forming, in this manner, a mass which, from its small specific gravity, would be capable of floating and rising in the air. Cavallo afterwards pursued the hint, and put it into actual practice, by means of soap balls; and soon after Montgolfier conceived the idea of filling a bag with smoke so as to render the whole lighter than air, and on a sufficiently large scale to enable it to take up very considerable weights. This new invention was eagerly pursued by the philosophers of all countries, and gave rise to that branch of Pneumatics, distinguished by the term Ærostation, or AËRONAUTICS, under which latter designation, in our alphabetical arrangement, will be found an historical sketch of most of the leading facts connected with the gradual progress of this art, from its commencement to the present time; at which it has probably attained the utmost perfection it is capable of receiving.

§ II. *Properties of atmospheric air, with reference to its mechanical effects.*

10. As much of the science of Pneumatics is involved in investigations relative to atmospheric air; and as these investigations are founded upon certain qualities of this fluid, which can only be known from experiment, we shall, previous to reducing the circumstances attending the action of elastic fluids to mathematical computation, give an account of a few of the most popular illustrations and experiments usually employed to show the existence of the different mechanical affections ascribed to the air as a body; such, for instance, as that it is a ponderous fluid; that its pressure is different at different altitudes; that it is elastic, or capable of compression, expansion, &c.

Air is a heavy fluid.

11. First, that the air is a fluid is obvious, because its parts are easily moved, and yield to the smallest inequality of pressure; and that it is ponderable, is clearly shown by the Torricellian experiment mentioned in the preceding chapter; but it may not be amiss to mention a few other instances which are familiar to every one, and which equally prove its ponderous quality.

Thus, for instance, if we shut the nozzle and valve-hole of a pair of bellows, after having expelled the air out of them, we shall find that a considerable force is requisite to separate the boards from each other. This can be attributed to no other cause than the pressure or weight of the atmosphere, which, acting equally upon the upper and lower surfaces externally, without a counterbalancing force inside, operates like a dead weight in keeping them in contact with each other. In like manner, if we stop the end of a syringe, after its piston has been pressed down to the bottom, and then attempt to draw it up again, a considerable force will be found necessary to effect our purpose, depending, as to quantity, upon the diameter of the instrument, being about 14 or 15lbs. to every square inch of the piston. By exerting the requisite force, we may draw up the piston, and may maintain it in its place by continuing to exert the same power; but the moment we cease acting, the piston rushes down and strikes the bottom. This is commonly spoken of under the term *suction*, because we feel something, as it were, drawing the piston down; but it is in reality the weight of the incumbent air pressing it in. The same obtains equally in all positions of the piston, viz. whether it be held vertically, horizontally, or in any inclined position whatever, which shows that air partakes of the general property of fluids; that is to say, that its pressure is exerted equally in all directions. It is to this quality of atmospheric air we must attribute many phenomena of daily observation; as, for example, the great force with which two surfaces that accurately fit cohere to each other; we do not here mean to assert that the cohesion is not frequently greater than can be due to atmospheric pressure, but in many cases this is the principal cause. This fact is familiarly known to glass-grinders and the polishers of marble; in these operations, when the polish is nearly completed, the glass and the tool on which it is ground will adhere to each other with a force requiring more than the strength of a man to separate them; in fact, we shall see hereafter, that every square inch of surface will require a force of 14 or 15lbs. to produce a separation, independently of any other cause than the pressure of the atmosphere; whereas, in a vacuum, the connection is destroyed without any effort.

To the same cause must be ascribed the very strong adhesion of snails, periwinkles, limpets, and other univalve shells, to rocks, &c. The animal forms the rim of its shell so as to fit the shape of the rock to which it intends to cling. It then, by means of some muscular exertion, produces a sort of vacuum, and in this condition it is evident that we must act with the above force of 15lbs. for every square inch of adhering surface, before we can detach it. This will be illustrated if we fill a drinking glass to the brim with water, and having covered it with a piece of thin wet leather, whelm it on a table, and then try to pull it straight up: when it will be found to require a considerable force. In the same manner do the remora, the polypus, the lamprey, and many other animals, adhere with that firmness which we so frequently observe. Those animals also which possess the power of walking in opposition to gravity, as flies, for example, on the ceiling of a room, or on the smooth perpendicular surface of a looking-glass, owe their facility of support to a peculiar construction of the feet, which enables them to produce a vacuum, by mere muscular operation. Naturalists were long perplexed with this phenomenon, and it is but very lately that Sir E. Home, by means of a very accurate examination of the foot of a peculiar sort of lizard, the *lacerta gecko* of Java (which possesses the same power), has been able to discover the cause. See Phil. Trans. for 1816.

Boys are constantly seen amusing themselves in pulling large stones from the pavement by means of a circle of stiff wetted leather fastened to a string, where the adhesion must still be attributed to the atmospheric pressure; and to the same cause is owing the firmness with which bivalve shell-fish appear to close themselves. We are apt to think the muscular power of an oyster very prodigious, on account of the force which is required to open it; but if we grind off a part of the shell so as to make a hole in it, though without injuring the fish in the smallest degree, it opens with great ease. Many other instances might be mentioned of a similar kind, but the above are sufficient to show the important part which atmospheric pressure performs in the production of many common and interesting natural phenomena.

Properties of air.

Barometer.

12. With respect to the Torricellian experiment, we may be allowed to say a few words, by way of further illustration. If a glass tube, 32 or 33 inches long, and closed at one end, be filled with quicksilver, and the open end, covered with the finger, be inverted, and immersed in a vessel of the same fluid, the mercury, when the finger is removed, will subside (as we have already stated in our historical sketch) to a certain height, somewhere between 31 and 28 inches from the surface of the external mercury; depending, as to the actual quantity, upon the weight of the atmosphere, which is different at different times. The state of the air may therefore always be ascertained by means of this instrument, and it is in consequence of this application of the Torricellian tube, that it has received its name of *barometer*, from βαροσ, *weight*, and μετρον, *measure*. The weight of the column of quicksilver, whose base is equal to the lower orifice of the tube, and altitude equal to that of the fluid above the surface of that in the vessel or reservoir, is equal to the weight of the column of air extending to the top of the atmosphere. And since the weight of the column of quicksilver is, *cæteris paribus*, as its altitude, it follows that the weight of the air is proportional to the altitude of the mercury in the barometer. The height at which the mercury is sustained above the surface of that in the reservoir is called the standard altitude, and will be the same in any number of tubes, whatever be their bore or their position, provided the tube be not so very slender as to expose the mercury to a sensible alteration from the capillary attraction.

The method above indicated enables us to determine very accurately, at any time, the weight of a column of the atmosphere on any given surface; for the specific gravity of mercury being known, we may compute directly the weight of it as due to any given base and observed altitude, and the same will be the weight due to a column of air on the same base and of indefinite altitude. In this way it has been determined, that the pressure of the atmosphere on every square inch of surface is, at a medium, about equal to 15lbs. avoirdupois. If, instead of having one end of the tube hermetically sealed, as we have supposed above, it be closely covered with a piece of bladder, the mercury will still stand at the same height, but on piercing the bladder with a needle, so as to admit the air, it will immediately fall; for, in this case, the weight of the air presses upon the mercury in the tube, and the weight of the two together must obviously preponderate over the contrary pressure and destroy the equilibrium.

To the same cause is to be attributed the well-known fact, that a cask will not run by the cock unless a hole be opened in some other part. If the cask be not quite full, indeed, some liquor will be discharged, but it will stop, after a time, till an opening be made, as above-stated. For the same reason we commonly see a small hole made in the lid of a tea-pot, for if that were not done, and the lid were to fit very accurately, we should, for want of the atmospheric pressure inside, be unable to pour out the tea through the spout. It seems principally owing to a like cause that a frost immediately occasions a scantiness of water in our fountains and wells. This is erroneously accounted for by supposing that the water freezes far below the surface of the earth. But this is a great mistake; the most intense frost of a Siberian winter would not freeze the ground two feet deep, but a very moderate frost will consolidate the whole surface of a country, and make it impervious to the air, especially if the frost have been preceded by rain, which has soaked the surface. When this happens, the water which was filtering through the ground is stopped, and held suspended in its capillary tubes by the upward pressure of the air, and a scarcity of supply is the necessary consequence. A thaw, by melting the superficial ice, again opens the communication with the atmosphere, and the flowing of the water is renewed.

Air is an elastic fluid.

13. That air is an elastic fluid, and capable of compression and expansion, may be shown by a great variety of methods, particularly with the assistance of the air-pump; and, even independently of this machine, it may be rendered evident by numberless experiments; as, for example, by squeezing in the hand a blown bladder, in which case we find an obvious resistance from the included air, and, upon taking off the pressure, the compressed parts immediately restore themselves to their former round figure. The elastic and compressible quality of common air may also be shown by inverting an open glass vessel, and immersing it in a fluid: in the first instance, viz. when the orifice of the vessel first comes in contact with the water, the whole internal space of the vessel will be occupied by the air, but on forcing it downwards to a greater depth, the pressure of the fluid will act upon the surface of the air in contact with it, and force it to occupy a less and less space, as the vessel is more and more immersed. This shows the air to be compressible; and if the power be removed that caused the vessel to descend, and which has kept it at any given depth, it will re-ascend with considerable force; which demonstrates the air to be elastic and expansible; for had it been incompressible, the water could not have entered; and had it been merely compressible, without possessing elasticity, it would not have re-ascended with the force above-described.

Other experiments are sometimes cited to prove the above quality of atmospheric air, viz. by the great expansion of a small quantity of air in a bladder, apparently nearly empty, when the air is removed from the external parts, by placing it under the receiver of an air-pump. This experiment is sometimes performed by placing the flaccid bladder in a small box, and laying a proper weight on the lid, when, on exhausting the air externally, the lid will be raised up by the expansion of that in the bladder.

Every particle of air possesses this property, and tends to expansion, and thus counteracts an equal tendency of the ambient particles; and if the resistance of any of these happen to be weakened, it immediately diffuses itself to an immense extent. Hence it is that thin glass bubbles, and bladders filled with air and exactly closed, being included in the exhausted receiver of an air-pump, burst by the force of the included air.

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The pressure of the atmosphere varies at different altitudes.

14. This fact was first established by the celebrated experiment of the *Puy de Domme*, mentioned in our historical chapter, and may be made otherwise obvious by merely carrying a bladder, with a small portion of air in it, to the top of a high mountain, where it will be found distended, although at the bottom it was perfectly flaccid; this change is obviously occasioned by the external air pressing with less force at the top than at the bottom of the mountain, which enables the included air to expand itself and distend the bladder. This proof is sufficient for our present purpose; we shall, in a subsequent chapter, enter more at length upon this subject, in treating of the measuring heights by the barometer.

Limits of condensation and rarefaction.

Lin con tion rar

15. The limits of condensation and rarefaction of air, by artificial powers, are not ascertained. Certain experiments of General Roy, detailed in vol. lxvii. of the Phil. Trans., render it probable that the particles of air may be so far removed from each other, by the diminution of pressure, as to lose a very considerable portion of their elastic force. It also appears that the elastic force of common air is proportionally greater than when its density is considerably augmented or diminished, by an addition to, or a subtraction from, the weight with which it is usually loaded; a fact which contradicts the experiments of Boyle, Mariotte, and others. These experiments also further show that the elasticity of moist air is greatly superior to that of dry air; in some cases the total expansion of the former was quadruple that of the latter.

Ela pow tain long

Philosophers have doubted whether this elastic power of air were capable of being entirely destroyed. Mr. Boyle made several experiments with a view to discover how long air, brought to the greatest degree of expansion to which he could reduce it, would maintain its spring, and could never discern any sensible diminution. Desaguliers found that after air had been enclosed for half a year in a wind-gun, it had lost none of its original elasticity; and Roberval, after preserving it in the same manner for sixteen years, found that its expansive projectile force was the same as if it had been recently condensed. Hawksbee, however, from a later experiment, concludes, that the spring of the air may be so disturbed by a violent pressure as to require some time to return to its natural tone: and Dr. Hales inferred, from a number of experiments, that the elasticity of the air is capable of being impaired and diminished by a variety of causes, so as to be actually destroyed and reduced to a fixed state: hence he concludes, that elasticity is not an essential and immutable property of the particles of air; and that the atmosphere is a chaos, consisting not only of elastic, but also of nonelastic air particles, which copiously float in it. This philosopher found that when dry wood was put into a strong vessel, which it almost filled, and the remainder was filled with water, the swelling of the wood occasioned by its imbibing the water, condensed the air of his gage into one-thousandth of its original bulk. He found also that peas, treated in the same manner, generated elastic air, which, pressing on the air in the gage, condensed it into a fifteen-hundredth part of its bulk. This is the greatest condensation that has been ascertained with precision, although, in other experiments, it has certainly been carried much farther; but the precise degree could not be ascertained.

16. The elasticity of the air exerts its force equally in all directions, and when released from the force which compresses it, it assumes a spherical figure in the interstices of the body which contains it. By exhausting the air from liquors placed under the receiver of an air pump, the bubbles which gradually arise, and are enlarged in size, retain their round figure. Such are also the bubbles discharged from a plate of metal immersed in a fluid under the same circumstances. On this account large glass globes always assume a spherical shape by blowing air through an iron tube into a piece of melted glass at the end of the tube. How far air has a power to dilate itself when all external pressure is removed is uncertain, although we know that it possesses this power in a very high degree. In several experiments made by Mr. Boyle, it dilated into 9 times its former space, then into 31, then into 60, and lastly into 150: but it was afterwards dilated into 8,000 times its first space, and ultimately into 13,679 times its original bulk; and this without the assistance of fire. (See Boyle's Works by Birch, vol. i. and vol. iii.)

It appears, therefore, from what has been above stated, that the air we breathe near the surface of the earth is compressed by the weight of the superincumbent column of air into at least the 13,000th part of the space it would occupy in vacuo; but if the same air be condensed by art to the greatest degree hitherto effected, the ratio of the spaces occupied by the same quantity of air in its greatest state of rarefaction, to the same under its highest degree of condensation, will be as 550,000 to 1.

Law of condensation and elasticity.

17. The weight or pressure of the air, it is obvious, has no dependence on its elasticity; and would be the same whether it possessed the latter property or not; but being elastic, it is necessarily effected by the pressure, which reduces it into such a space, as that the elasticity which re-acts against the compressing weight is equal to that weight. Indeed, the law of this elasticity is, that it increases as the density of the air increases, and the density increases as the force increases with which it is pressed, at least within certain limits, and there must necessarily be a balance between the action and re-action; that is to say, the gravity of the air, which tends to compress it, and the elasticity of the air, which has a tendency to expand it, must be equal; hence the elasticity of the air, not very different from its natural state, being as the density, will of course be inversely as the space it occupies.

Consequently, since the elasticity increases or diminishes as the density increases or diminishes; that is to say, as the distance between the particles diminishes or increases, it is indifferent whether the air be compressed and retained in such space by the weight of the atmosphere, or by any other means; for it will, in either case, have a tendency to expand with the same force. Therefore, if air near the earth be confined in a vesssel shut on all sides, so as to cut off all communication with the external air, the pressure of the enclosed air will be equal to that of the atmosphere. Accordingly we find mercury sustained to the same height by the elastic force of air enclosed in a glass vessel, as by the whole atmospheric pressure.

Properties of air.

Experiments.
Fig. 3.

In order to prove experimentally the truth of the preceding deductions, let a cylindrical tube BC, (fig. 3) open at one end B, be filled with mercury to an altitude equal to BD, before inversion, and after the immersion of B in a bason of mercury, the air, which before occupied a space equal to CD, is dilated through a larger space at CA; and if BN be the altitude due to the atmospheric pressure, the mercury will be depressed from N to A. Now the elastic force of air in its natural state, or occupying a space equal to CD, is to the elastic force of air occupying the space BD, as the columns of water they are capable of supporting, that is, as BN to AN; but experiment shows that BN : AN :: AC : CD, consequently the elasticities are to each other as AC : CD.

The same may be otherwise shown by a bent tube, as follows: Take a long crooked glass tube (fig. 4), equally wide throughout, or at least in the part BD; open at A, but close at the other end B; pour in a little quicksilver at A, just to cover the bottom to the bend at C *d*, to stop the communication between the external air and that contained in BD. Then pour in more quicksilver, and mark the corresponding height at which it stands in the two legs, and suppose when it rises to H in the open leg AC, that it ascends to E in the close one, reducing the volume of the included air from its natural bulk BD to the contracted space BE, by the pressure of the column H *e*; and when the quicksilver stands at I and K, in the open leg, let it rise to F and G in the other, reducing the air to the respective spaces BF, BG, by the weight of the columns I *f*, K *g*. Then it will be found, that the condensation and elasticities are as the compressing weights or columns of quicksilver and the atmosphere together. Hence, if the natural bulk of the air BD, be compressed into the spaces $\frac{3}{4}$, $\frac{2}{4}$, $\frac{1}{4}$ of BD, or as the numbers 3, 2, 1, then the atmosphere, together with the corresponding columns H *e*, I *f*, K *g*, are also found to be in the same proportion reciprocally; viz. as $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{1}$, or as the numbers 2, 3, 6. And then $He = \frac{1}{3}A$, $If = A$, and $Kg = 3A$, where A denotes the weight of the atmosphere, which shows that the condensations are directly as the compressing forces; and the elasticities are in the same ratio, since the columns in AC are sustained by the elasticities in BD.

The above law not general.

18. The experiments to which we have above alluded, however, cannot be carried to a sufficient extent to ascertain the exact law between the compression and the space occupied; and it is well known, that although it may be admitted as a general principle that the density of air is proportional to the force by which it is compressed, as we have stated to be deducible from the above experiments, yet, in the case of air condensed to any considerable degree, the rule will not obtain. When air is very forcibly condensed, so as to reduce it to a small part of its ordinary bulk, it makes a greater resistance, and requires a stronger

force to compress it than is conformable with the preceding principle.

Sulzer's experiments.

Thus Sulzer, who compressed air into $\frac{1}{8}$th of its original volume, found a considerable deviation from the above law. The results of his experiments are exhibited in the following table; where the first column shows the densities, and the second marks the corresponding elasticities.

TABLE *of Sulzer's experiments on compressed air.*

1st Set.		2d Set.		3d Set.	
Density.	Elasticity.	Density.	Elasticity.	Density	Elasticity.
1·000	1·000	1·000	1·000	1·000	1·000
1·100	1·093	1·236	1·224	1·091	1·076
1·222	1·211	1·294	1·288	1·200	1·183
1·375	1·284	1·375	1·332	1·333	1·303
1·571	1·599	1·466	1·417	1·500	1·472
1·692	1·669	1·571	1·515	1·714	1·659
1·833	1·796	1·692	1·647	......	......
2·000	1·958	2·000	1·964	2·000	1·900
2·288	2·130	......	......	......	......
2·444	2·375	2·444	2·392	2·400	2·241
3·143	2·936	3·143	3·078	3·000	2·793
3·666	3·391	3·666	3·575	......	......
4·000	3·706	......	......	4·000	3·631
4·444	4·035	4·444	4·320	......	......
4·888	4·438	......	......	......	......
5·500	4·922	5·500	5·096	......	......
5·882	5·552	......	......	6·000	5·297
......	......	7·333	6·694	......	......
......	......	......	......	8·000	6·835

Robison's experiments.

Other experiments for the same purpose were instituted by professor Robison on dry and moist air; the results of which may be seen below.

TABLE *of Professor Robison's experiments.*

Dry air.		Moist air.		Air impregnated with vapours of camphire.	
Density.	Elasticity.	Density.	Elasticity.	Density.	Elasticity.
0·000	1·000	1·000	1·000	1·000	1·000
2·000	1·957	2·000	1·920	2·000	1·909
3·000	2·848	3·000	2·839	3·000	2·845
4·000	3·737	4·000	3·726	4·000	3·718
5·500	4·930	5·500	5·000	5·500	5·104
6·300	5·342	6·000	5·452	6·000	5·463
7·620	6·490	7·620	6·775	7·620	6·812

Results.

The conclusion to be drawn from these experiments is manifest; it appears, in the clearest manner, that the elasticities do not increase so fast as the densities; this is the case in both series of experiments; but it is more particularly so in the latter. The second set in Dr. Robison's experiments were the results obtained from very damp air in a warm morning in summer, and in these the elasticities are very nearly proportional to the densities *plus*, the same small constant quantity, 0·11 The deviation from this rule lies principally between the densities 1 and 1·5, within which limits we have very nearly $D = E^{1·0017}$, or log. D = 1·0017 log. E. As the state of the air between these limits is nearer to the consitution of atmospheric air than the former, the above law may be more properly applied in cases where atmospheric air is concerned, than that which makes D + 0·11 vary as E; that is, in such cases as the admeasurement of heights and depths by the barometer, &c.

The air employed in forming the third set of results was strongly impregnated with the vapours of camphire: in this case the elasticity varies more nearly at the densities than in either of the other two; what deviation there is, is in favour of the elasticity; that is, the elasticities increase rather faster than the densities.

Expansion of air by heat.

19. This property of common air is rendered obvious by means of a very simple experiment. If, for example, we tie a bladder very close with some air in it, and lay it before a fire, as it experiences the heat, the bladder will become more and more distended, and will at last burst, if the heat be continued long enough, and provided there were a sufficient quantity of air in the bladder at first. If now the bladder be removed from the fire, as it cools it will contract again as before, and assume its original figure and volume. It is on this principle that the first fire-balloons of Montgolfier were made to ascend. See AËRONAUTICS. Another experiment, which shows the rarefaction of air by heat, and its return afterwards to its former bulk, is as follows: Take a wine glass that may be completely covered by the palm of the hand; light a small piece of paper, and put it while burning into the glass, by which means the contained air is greatly expanded: now, while the paper is still light, cover the mouth of the glass with the hand, the fire is immediately extinguished, and the air is soon cooled; when the external air pressing with a greater elastic force than the air inside, in consequence of the rarefaction of the latter, the hand will be pressed with very considerable force into the glass, and it is with some difficulty that it is afterwards removed.

20. We have spoken above only of the general fact, that air is rarefied by heat and condensed by cold; but it is important for many philosophical inquiries to know the degree of expansion corresponding to different temperatures, which is, however, very difficult to be ascertained; the expansion being different under the same temperature at different times, owing to the variation in density, coldness, humidity, &c. It appears from the very numerous experiments of General Roy (Phil. Trans. vol. lxvii.), that 1,000 parts of air, of the density of the common atmosphere at 0° of heat, become 1·484·21 at 212°; viz. the air is expanded 484·21 of those parts by the application of 212° of heat.

It is also shown by the same philosopher, that 1,000 parts, loaded with $2\frac{1}{4}$ atmospheres, are expanded by 434 of those parts, with the same degree of heat; that 1,000 parts pressed only with $\frac{4}{5}$ths of an atmosphere, are expanded nearly 484 of those parts by 212° of heat; and lastly, that when the pressure is reduced to $\frac{1}{5}$th of an atmosphere, the expansion is only 141 parts, at 180° of heat; viz. from the freezing to the boiling point.

From these and similar experiments General Roy and, in the same manner, if h be taken to denote the height of the column corresponding to the pressure p, we shall have $p = 2ghd$; D and d, in both cases, denoting the corresponding densities. Substituting now these values for P and p, in our first two equations, they become

$$h\,\mathrm{D} = \mathrm{H}\,d \text{ and } h\,m = \mathrm{HM}.$$

From which it appears, that two columns of elastic fluids will balance each other, when their weights are equal, or when their heights are reciprocally as their densities. We have found above,

$$p = \frac{a\,\mathrm{M}}{m}\,\mathrm{P} = \frac{a\,d}{\mathrm{D}}\,\mathrm{P}. \qquad (2)$$

But this supposes that the fluid under both pressures is the same; or, that it is such, that under the pressure P the density is D. If, instead of this, the fluid under the pressure p be such that, with a pressure P the density would be δ, while the other fluid remains as before, we should then have

$$p = \frac{a\,\mathrm{M}}{m}\,2g\,\delta\,\mathrm{H} = \frac{a\,d}{\mathrm{D}}\,2g\,\delta\,\mathrm{H}. \qquad (3)$$

For, in this case, we should have $\mathrm{P} = 2g\,\mathrm{H}\,\delta$, which substituted for P, in equations (2), produces those above.

Consequences of the laws of repulsion.

24. If the nature of an elastic fluid be such, that its particles repel each other with equal forces at equal distances, and with unequal forces when the distances are unequal, and such a fluid be in a state of quiescence, all its particles must be situated at equal distances from each other, or, which is the same, its density must be uniform. For, if the distances of any two of its particles from an intermediate particle were unequal, their repulsive forces would be unequal, and motion would ensue; the fluid, therefore, would not be quiescent, consequently, when it is quiescent, its density is uniform. The converse of this proposition will also obtain, viz. if an elastic fluid, or any portion of an elastic fluid of uniform density, be equally pressed on all sides, it will be quiescent.

Under what law of repulsion a fluid is elastic.

25. If the particles of a fluid repel each other with forces which are in a less ratio than the square of their distances, whether direct or inverse, that fluid will be elastic. For, denoting the interval between any two contiguous particles by I, and the force with which they repel each other at that distance by F, then the repulsive force on any surface will vary as $\frac{\mathrm{F}^2}{\mathrm{I}^2}$; that is, as the number of particles in that surface, which will be inversely as the square of the linear distance of the particles from each other; if, therefore, F vary in any inverse or any direct ratio less than the duplicate ratio of I, the whole force will vary in some inverse ratio of the distance of the particles, and the fluid will therefore necessarily be elastic.

Denoting, as above, the density of an elastic fluid by D, the linear distance of the particles, which we have called I, will obviously vary as $\frac{1}{\mathrm{D}^{\frac{1}{3}}}$; that is, reciprocally as the cube root of the density; and the number of particles in any given surface is evidently as the square of the same, that is as $\frac{1}{\mathrm{D}^{\frac{2}{3}}}$. If, therefore, the repulsion of the particles be supposed to vary inversely, as the nth power of their distance, the whole repulsion on any given surface will vary, as

$$\left(\frac{1}{\mathrm{D}^{\frac{1}{3}}}\right)^n \times \frac{1}{\mathrm{D}^{\frac{2}{3}}} = \frac{1}{\mathrm{D}^{\frac{n+2}{3}}};$$

whence, if the particles of an elastic fluid repel each other with forces varying inversely as the nth power of their distance, and C be taken to denote the compressing force on any given surface, the compression C will vary as that power of the density whose index is $\frac{n+2}{3}$; that is, C will vary as $\mathrm{D}^{\frac{n+2}{3}}$.

And hence again, conversely, when the compression varies as $\mathrm{D}^{\frac{n+2}{3}}$, it follows that the repulsion of the particles is inversely as the nth power of their distance.

26. We have seen that in common atmospheric air the density varies as the compressing force very nearly; consequently, in this fluid, n must be very nearly equal to 1; that is, in common air the particles repel each other with forces that are inversely as their distances.

Since C varies as $\mathrm{D}^{\frac{n+2}{3}}$, it follows that D varies as $\mathrm{C}^{\frac{3}{n+2}}$; therefore, in incompressible fluids whose density is the same under all degrees of compression, n must be infinite; and if water be in a slight degree compressible, so that C remains, not exactly, but nearly constant under all degrees of pression, it will follow that the particles of this fluid must repel each other with forces varying inversely as a very high power of their distance.

27. We call a *homogeneous atmosphere* a supposititious column of air of uniform density, equal to that of the atmosphere at the surface of the earth. If we assume H to denote the height of such a homogeneous atmosphere; D its uniform density; h the height of a column of quicksilver of equal base and weight; and d the density of the latter fluid; we shall have

$$\mathrm{HD} = h\,d, \text{ or } \mathrm{H} = \frac{h\,d}{\mathrm{D}}.$$

Now it will be seen in our table of specific gravities (STATICS), that the density of air, at a medium, is to that of mercury as $1\frac{2}{9}$ to 13600. Assuming, therefore, h, or the height of the barometrical column = 30 inches, or $2\frac{1}{2}$ feet, we shall have

$$\mathrm{H} = \frac{2\frac{1}{2} \times 13600}{1\frac{2}{9}} = 27818 \text{ feet} = 5{\cdot}268 \text{ miles.}$$

If, therefore, the density of the air were every where the same as at the surface of the earth, its height would be equal to rather more than $5\frac{1}{4}$ miles.

28. We know, however, that the atmosphere is not of the same density at all altitudes; but that, on the contrary, the density varies at the compressing force, and consequently decreases from the surface upwards; on which decrease is founded the principle of the method for ascertaining altitudes by the barometer. If we suppose gravity to be the same at all heights, the law by which the density and elasticity of the atmosphere vary is such, that while the heights increase in an arithmetical progression, the density decreases in a geometrical progression, as we shall show in the following article; but if we suppose that gravity varies as the nth power of

the distance from the centre of the earth, and that the compressing force varies as the density, we shall have a more complicated relation between the successive densities at different altitudes; but, at the same time, it will be more general, because, by given different values to n, different results will be obtained, amongst which we shall find that stated above, when we suppose the action of gravity to be constant.

29. Let x represent any variable distance from the surface of the earth, the radius of the latter being denoted by unity; d the density of the air at the distance x; and H the height of the homogeneous atmosphere. Now, since by hypothesis the compressing force varies as the density, the fluxion of the former will vary as the fluxion of the latter, while at any distance x, the fluxion of the compressing force must vary as the force of gravity, the density, and the fluxion of the altitude conjointly; so that the fluxion of the compressing force, will be to that of the density, in the constant ratio of $x^n d \dot{x}$ to $-\dot{d}$ (the latter fluxion having the negative sign, because the density decreases, while the altitude increases), consequently, since H denotes the height of a homogeneous atmosphere, it will also represent the compressing force at the surface of the earth, and we shall have

$$\mathrm{H} : 1 :: x^n d \dot{x} : -\dot{d};$$

whence
$$n^n \dot{x} = -\mathrm{H}\frac{\dot{d}}{d};$$

and taking the fluents, we obtain

$$\frac{x^{n+1}}{n+1} = -\mathrm{H} \text{ hyp. log. } d + \mathrm{C}.$$

The correction C may be found by considering that when $x = 1$, $d = 1$; whence $\mathrm{C} = \frac{1}{n+1}$, and the correct fluent is $\frac{x^{n+1}}{n+1} = \frac{1}{n+1} - \mathrm{H} \text{ hyp. log } d$;

or,
$$\frac{1 - x^{n+1}}{n+1} = \mathrm{H} \text{ hyp. log. } d,$$

which is a general equation, expressing the relation between the altitude and density.

If we suppose $n = o$, or the force of gravity, to be constant, as we may do, without any sensible error, for all altitudes to which we can attain, the above equation reduces to $1 - x = \mathrm{H}$ hyp. log. d; consequently, while x, or the altitude, increases in arithmetical progression, $1 - x$, will decrease according to the same law, as well as the hyperbolic logarithm of d; but while the logarithms of quantities decrease in arithmetical progression, the quantities themselves decrease in geometrical progression: therefore, while the altitudes to which we ascend increase according to any arithmetical series, the densities will decrease according to a certain geometrical series, which remarkable relation forms the foundation of the method of determining altitudes by means of the barometer.

30. If we suppose, as is really the case, that the force of gravity decreases as the square of the distance from the centre of the earth increases, that is, if we suppose $n = -2$, then

$$\frac{1 - x^{n+1}}{n+1} = \mathrm{H} \text{ hyp. log. } d,$$

becomes

Equilibrium of elastic fluids.

$$\frac{1 - x^{-1}}{-1} = \mathrm{H} \text{ hyp. log. } d, \text{ or } \frac{1}{x} - 1 = \mathrm{H} \text{ hyp. log. } d;$$

so that while x increases in harmonic progression, $\frac{1}{x}$ will decrease in arithmetical progression, and consequently hyp. log. d will decrease still according to the same geometrical law. That is, according to this supposition, while the altitudes increase in harmonic progression, the densities will decrease in a geometrical progression.

Since the common logarithms of numbers are to the hyperbolic logarithms of the same numbers in a constant ratio, viz. as ·43429448 to 1, it follows that the same law is still observed in the common logarithms of the densities.

Equations.

31. If, according to the hypothesis of equal gravity, A and a denote two different altitudes, and d and D the corresponding densities, we shall have

$$\mathrm{A} = -\mathrm{C} \log. d,$$
$$a = -\mathrm{C} \log. \mathrm{D};$$

and, consequently, by subtraction,

$$\mathrm{A} - a = \mathrm{C}\,(\log. \mathrm{D} - \log. d),$$

or
$$\mathrm{A} - a = \mathrm{C} \log. \frac{\mathrm{D}}{d}.$$

That is, generally, the difference in the altitudes of two places, is proportional to the difference of the logarithms of the densities at those places; the quantity C being a constant co-efficient, which remains to be determined.

The preceding deductions illustrated.

32. The above deductions are frequently illustrated by means of the logarithmic curve, as follows:

Let ARQ (fig. 5) represent a section of a terrestrial hemisphere, by a plane through its centre O, *m* OAM a vertical line, AE a horizontal line drawn through A, a point on the earth's surface. Let this line AE denote the density of the air at A, and take DH to AE as the density at D to the density at A, and draw DH parallel to AE; then it is obvious, that if a logarithmic curve EHN be drawn, having AM for its axis, and passing through the points E and H, the density of the air at any other point C, in the vertical AM, will be represented by CG, the ordinate to the curve at that point; for it is a known property of this curve, that if portions AB, AC, AD of its axis be taken in arithmetical progression, the ordinates AE, BF, CG, &c. will be in geometrical progression.

We know also, that if tangents be drawn to this curve at any points, as E and H, the subtangents AK, DS will be constant, and will denote the modulus of the system of logarithms which the particular curve represents; and another property equally well known is, that the infinitely extended area AENM is equal to the rectangle AELK of the ordinate at A and the subtangent; and in like manner, the area MDHN above DH is equal to the rectangle SD × DH, or KA × DH; so that the area lying above, or beyond any ordinate, is proportional to that ordinate. These geometrical properties of this curve are precisely analogous to the analytical deductions we have obtained, with reference to the altitudes and varying density of the atmosphere, upon the supposition of the uniform action of gravity. For example, the area MBFN may represent the whole quantity of the elastic fluid which is above B; for BF is the density at B, and BC is the depth or thickness of the stratum between B and C;

wherefore, BFCG will be as the quantity of air in that stratum. In like manner, CGHD will be as the quantity of fluid in the stratum, whose depth is CD; and the same of their sums, or of the whole area MBFN.

Again, as each ordinate is proportional to the area above it, so each density and the quantity of air in each stratum is proportional to the quantity of air above it; and as the whole area AENM is equal to the rectangle AELK, so all the air of variable density above A might be contained in a column KA of the same base, if, instead of being compressed with its whole weight, it were without weight, and compressed by an external force, equal to the pressure of the air at the surface of the earth. In this case it would be of the uniform density AE, which it has at the surface, and would therefore constitute what we have denominated the homogeneous atmosphere. It follows, therefore, that the height of the homogenous atmosphere is the subtangent of the curve, whose ordinates are as the density of the air at the different altitudes, on the supposition of equal gravity. This curve has hence been denominated by Dr. Halley, who first treated of it, the *atmospheric logarithmic;* and its subtangents, or the heights of the homogeneous atmosphere, is, as we have stated, above the modulus of the system of logarithms to which the curve corresponds.

Height of the homogeneous atmosphere otherwise determined.

33. The height of the homogenous atmosphere we have seen may be readily determined by means of the barometer, that is, by saying, as the density of the air at the surface is to the density of mercury, so is the height of the latter to that of the former. We might also find the same from what is given above, as follows: When the mercury and the air are both of the same temperature, viz. about 32° of Fahrenheit's thermometer, the barometer on the sea shore is supposed to stand at 30 inches; and if we take the latter instrument to an altitude of 936 feet, the temperature being the same, the mercury will have descended $\frac{1}{30}$th part of its original column, viz. to 29 inches. And since in all logarithmic curves, having equal altitudes, the portions of the axes intercepted between the corresponding pairs of ordinates are proportional to the subtangents; and the subtangent of the curve, corresponding to our common logarithms, is ·43429448, &c.; also the difference of the logs. of 30 and 29, which is the portion of the axes intercepted between the ordinates 30 and 29, being ·0147233, we have ·0147233 : ·43429448 :: 935 : 27600 feet, or $5\frac{1}{4}$ miles nearly, the same result as before. It must be acknowledged, however, that our preceding determination of this altitude (art. 27) has the advantage of much greater simplicity.

Relation between the altitude and density.

34. We shall conclude this part of our inquiry, by showing the application of the above principles to the determination of altitudes, the densities being supposed known, or to the finding of the densities when the altitudes are given; premising, however, that it is only with a view to a general illustration, and that it must not be considered as actually applicable to the determination in question, for various observations have shown, that the above general principles require very considerable modifications, on account of the difference in low and elevated situations, the latitude of the place and many other circumstances, which will be minutely investigated in a subsequent chapter.

We have seen (art. 31) that, generally,

$$A - a = C \log. \frac{D}{d},$$

where C denotes a constant co-efficient, which, in the article above referred to, we left to be determined.

Let us, in order to this determination, take the density at the surface, and at one foot above it; the former, when the barometer stands at $29\frac{1}{2}$ inches, and Fahrenheit's thermometer at 31°, will be as follows:

$$a = 0 : D = 26057,$$
$$A = 1 ; d = 26056,$$

the densities being estimated by the height of the homogeneous atmosphere above them; that is, at the surface the density will be proportional to a pressure of 26057 feet, and 1 foot above, the pressure being only 26056, the density will be proportionally decreased.

We have, therefore,

$$A - a = 1 = C \log. \frac{26057}{26056};$$

or, making 26057 = n, the above becomes

$$1 = C \log. \frac{n}{n-1}.$$

But log. $\frac{n}{n-1} = M \left\{ \frac{1}{n} + \frac{1}{n^2} + \frac{1}{3n^3} + \&c. \right\}$

And, as in the present case, n is a great number, all the terms past the first may be neglected; and since also M = ·43429448, &c. in the common system of logarithms, we shall obtain

$$1 = C \times \frac{\cdot 43429448}{26057}.$$

Whence again by division,

$$C = \frac{26057}{\cdot 43429448} = 60000, \text{ very nearly;}$$

therefore the above general formula becomes

$$A - a = 60000 \times \log. \frac{D}{d}.$$

Or, since by the supposition $a = o$, the altitude above the actual surface will be expressed by

$$A = 60000 \times \log. \frac{D}{d};$$

the quantities D and d, as employed in the above investigation, denote the numerical values of the atmospherical pressure; but as the mercurial columns are always proportional to those pressures, we may substitute M and m for the heights of the mercurial columns at those altitudes, and we shall then have

$$A - a = 60000 \times \log. \frac{M}{m};$$

that is, to find the difference in the altitudes of any two places, we have only to multiply the difference between the logarithms of the heights of the mercurial columns by the constant co-efficient 60000, and the product will give the difference of altitude between the two stations in English feet.

Den diff altit

35. By transposing the formula

$$A - a = C \log. \frac{M}{m}, \text{ or } A - a = C (\log. M - \log. m)$$

we have C log. m = C log. M − A + a;

responding variations in the weather. The causes which influence the one, produce also a similar effect on the other; therefore, if the former were known, the latter might be ascertained. The immediate causes may probably be reduced to the two following, viz. an emission of latent heat from the vapours of the atmosphere, or of electric fluid from these or from the earth. Both these causes are observed to produce the same effect with the solar heat in tropical climates, that is, they both rarefy the air, by blending with it, or setting loose a lighter fluid, which did not previously act with such power in any particular place. It is not here our business to pursue these inquiries: as far as they relate to the present subject we are only concerned with the effect, the causes of which will be considered more at length in another part of this work.

Height of the atmosphere.

39. The actual height of the atmosphere has been made the subject of particular investigation. We have seen (art. 27), that if this fluid were, like water, incompressible, or every where of the same density as at the surface of the earth, its height must then be about 5¼ miles, in order to produce the pressure actually indicated by the barometer. But, in consequence of the elastic property of atmospheric air, it expands and contracts according to the pressure it has to sustain; and, consequently, the height to which it actually extends must very far exceed that above assigned to an homogeneous atmosphere: in fact, if the earth were an isolated body in space, we could imagine no limit to the atmosphere; but as this is not the case, we must doubtless arrive at a point where the particles of this fluid would be equally attracted by the body nearest to our earth. The moon is this body; and if our atmosphere for a moment exceeded in height the distance of the point of equal attraction between the earth and moon, as the former revolved on its axis, all the particles beyond this point, and nearer to the moon, would be attracted towards that body; the terrestrial atmosphere cannot therefore, permanently, be higher than the point of equal attraction between the earth and moon: whether it actually attain this altitude, it is, perhaps, impossible to say; but that it cannot permanently exceed it is obvious from what is stated above. We have already observed (art. 35), that at the distance of 500 miles above the surface of the earth the air becomes so extremely rarefied, that one cubic inch of the air we breathe would fill a sphere equal in diameter to the orbit of Saturn; consequently, to whatever extent the atmosphere might be expanded, it would become, at a comparatively small distance, so attenuated, as to be utterly imperceptible in its effects as a resisting medium; and if the atmosphere of the planets resemble that of the earth, they must have ultimately such a degree of rarity as to offer no sensible resistance to the motion of those bodies round the sun, for many ages; supposing even that the boundaries of them were not limited by their respective secondary or nearest planets.

§ IV. *Principle and operation of the air-pump.*

Air-pump.

40. We have already, in our historical chapter, traced the first invention of this instrument as far as we considered it connected with the history of the general science of Pneumatics; and we shall, in a subsequent section, enter more minutely into the various modifications it has undergone in the hands of different philosophers, and describe one or two such instruments of the most improved constructions; in this place we shall only endeavour to give a simple and general idea of its operation.

Constru tion of th air-pump.

General principle.

This consists in expelling a portion of air out of a vessel, which is so secured as to prevent any of the external air rushing into its place, and consequently producing a certain degree of rarefaction within; another portion being again expelled, the air becomes further rarefied, and thus, by repeated operations, such a state of rarity is produced as to be nearly equivalent to an absolute vacuum; though it is obvious that it is impossible ever to arrive completely at it, because, at each operation, only a part of the remaining air is excluded. The basis, or essential part of the air-pump, is a metallic tube, answering to the barrel of a common pump; having a valve at the bottom, opening upwards, and a moveable piston, or embolus, answering to the sucker of a pump, furnished also with a valve opening upwards, the whole being supplied with a proper vessel as a recipient, or receiver.

Fig. 6.

Thus, let B (fig. 6) represent the section of a perfectly uniform metallic barrel, communicating with the glass receiver A, by means of the tube *t t*; let P be a piston attached to the rod R, and fitting the barrel air-tight; *v* and *v'* two valves opening upwards, the one adapted to the hole in the piston, and the other to the end of the tube *t t*. Now the receiver A being placed upon its support CD, in such a manner as to prevent any communication with the external air, by setting it on wet leather collars, or, which is better, upon a surface well ground and level; the operation is carried on as follows:

Operat

When the piston rod is forced down, the air in the barrel is compressed in the first instance, which shuts the lower valve *v'*, and forces open the upper valve in the piston, whereby all the air in the barrel escapes; then by drawing back the piston the incumbent atmosphere shuts the upper valve *v*, and a vacuum would be left behind it, but that the air in the receiver, pressing against the lower valve *v'*, and finding there no resistance, rushes into the barrel, and the whole is filled with air as before. But it is now so much more rare, as the contents of the barrel and receiver exceed that of the receiver alone. The operation being again repeated, another barrel of air escapes, and when the piston is again lifted up, the air rushing into the barrel as before, produces a further rarefaction; and thus, by a number of repetitions, the receiver is finally nearly exhausted of air; but it is obvious, as we have stated above, that it can never be completely so, as the portion of air each time expelled is only a certain part of the whole quantity contained in the vessel previous to each stroke of the piston.

Degree exhau

41. The above must be considered simply as an illustration of the operation of this machine, and not as a practical construction, which we shall explain in a subsequent article; we shall here only examine the degree of exhaustion and rarefaction that will be produced by any number of strokes, the proportion of the capacities of the receiver and barrel being given. If we suppose no vapour to arise in the receiver from moisture or other cause, it is obvious, from what is above stated, that every stroke of the piston expels a portion of air equal to the capacity of the barrel, which bearing a constant ratio to the capacity of the receiver and barrel together, it follows that the air

forced out at every stroke is a certain fractional part of the whole before the stroke; the exhaustion will, therefore, go on in a geometrical progression, the ratio of which is the same as that which the sum of the receiver and barrel together bears to that of the receiver alone: and this ratio of exhaustion will be followed till the elasticity of the included air is so far diminished by its rarefaction as to render it too weak to force open the valve of the piston.

If, therefore, we denote the capacity of the receiver by r, and that of the barrel by b; and consequently the sum of the two together by $b + r$; calling also the original density d, we shall have

$b + r : r :: d : \frac{r}{b+r} d$, the density after 1 stroke;

$b + r : r :: \frac{r}{b+r} d : \frac{r^2}{(b+r)^2} d$, the density after 2 strokes;

$b + r : r :: \frac{r^2}{(b+r)^2} d : \frac{r^3}{(b+r)^3} d$, the density after 3 strokes, &c. &c.;

$b + r : r :: \frac{r^{n-1}}{(b+r)^{n-1}} d : \frac{r^n}{(b+r)^n} d$, the density after n strokes. Denoting, therefore, the density after n strokes by δ, we shall obtain the equation

$$\delta = \frac{r^n}{(r+b)^n} d, \text{ or } \frac{\delta}{d} = \frac{r^n}{(r+b)^n}.$$

Suppose, for example, the ratio of the capacities of the receiver and barrel to be as 4 to 1, and let it be required to determine the rarefaction produced in the receiver after 10 strokes; the density in the first instance being assumed as unity. Here we shall have $r = 4$, $b = 1$, $n = 10$, and $d = 1$; whence

$$\delta = \frac{4^{10}}{(4+1)^{10}} = \cdot 8^{10} = \cdot 10737, \&c;$$

that is to say, the rarefaction will be about ten times greater than in the first instance, or the density ten times less.

Our equation

$$\frac{\delta}{d} = \frac{r^n}{(r+b)^n}$$

may be put under the form

$$\log. \delta - \log. d = n \log. r - n \log. (r + b);$$

or, denoting the primitive density by 1,

$$\log. \delta = n \{\log. r - \log. (r + b)\},$$

whence

$$n = \frac{\log. \delta}{\log. r - \log. (r + b)}.$$

From which the number of strokes may be found that are requisite for producing any proposed degree of rarefaction.

As δ in the above equation expresses the density, and is therefore necessarily fractional, let its reciprocal, viz. $\frac{1}{\delta} = N$, where N is an integer expressing the rarefaction; then, since

$$\log. \delta = - \log. N,$$

the above equation may be written

$$n = \frac{\log. N}{\log. (r + b) - \log. r}.$$

Another form under which our equation

$$\frac{\delta}{d} = \frac{r^n}{(r+b)^n}$$

may be put, is as follows, viz.

$$\frac{r}{r+b} = \sqrt[n]{\frac{\delta}{d}}, \text{ or } \log. \frac{r}{r+b} = \frac{\log. \delta - \log. d}{n}.$$

Consequently from this, we may, when the number of strokes n, and the densities δ and d are given, determine the ratio between the receiver alone, and the receiver and barrel together; and hence the ratio of the two latter. It is seldom, however, that we have occasion for the equation in this form.

We shall not enter more particularly into a description of the present improved construction of the air-pump till we treat of Pneumatic machines generally at the conclusion of this treatise.

Construction of pumps for raising water.

§ V. *Of the construction of pumps for raising water.*

42. The term *pump* is generally applied to a hydraulic machine for raising water by means of the pressure of the atmosphere. Of these there are many different kinds, a few of the most approved of which will be described at the end of this treatise when we treat of Pneumatic machines; at present we shall only speak of the three fundamental pumps, from a combination of which all the other sorts owe their principles of construction and operation. These are the *sucking-pump*, the *lifting-pump*, and the *forcing-pump*, to which we may also add the *centrifugal-pump*.

Of the sucking-pump.

Description of the sucking-pump. Fig. 7.

43. The common sucking-pump is represented in (fig. 7); it is composed of two vertical tubes AMNC, ABDC, which have the same axis, and are united together in AC. The first, which is placed partly in the water of a well, or other reservoir, is called the *suction-tube*, and the other the *body of the pump*. In AC is a diaphragm, pierced with holes, and covered by a valve E, which opens upwards. In the body of the pump is a piston, which is made to ascend and descend alternately, by means of the rod Z attached to a lever. The head of this piston is pierced in the direction of its axis with a hole, which is covered by a valve F, also opening upwards. The play of the piston we will suppose to be limited by KG, that is to say, the piston being depressed, its lower surface is supposed to be in the horizontal plane GH, and being lifted up, the same base rises to the plane of KI. Practically, the inferior base of the piston, in its lowest position, ought to come as near as possible to the valve E. We may observe, that of the two valves E and F, which open and shut alternately in the same manner, the first E always occupies the same place, and is for this reason called the *fixed valve*, and the second the *moveable valve*, because it ascends and descends with the piston.

The pressure of the air, we have seen in the preceding section, is subject to certain variations; but, as far as relates to our present inquiry, we shall consider it as constant, and equal in quantity to the pressure of a column of water of 34 feet, which corresponds to about 30 inches of mercury.

Operation explained.

44. In order to explain the operation of this pump, let us suppose, in the first instance, the base of the piston to be in GH; then the air comprised in the space MC, that also in AH, and the natural air, or the atmosphere, in the place where the machine is acting, have the same density and the same elastic force. The two valves E and F being supposed to permit, by the facility with which they open, a free communication between the air of the tubes and that of the atmosphere, after which they naturally close with their own weight. If now we raise the piston from GH to KI, the valve F will remain shut by means of its own weight, and by the pressure of the air above it. The air in the space GC, MC, being now relieved of the pressure to which it was exposed, will expand; and the first, by its power of expansion, will force open the valve E, and the whole space between KI and MN will be again filled with air of one common density, the degree of which, and of its consequent elasticity, will be to that before the first stroke of the piston as the space GHMN to the space KLMN. In this state, therefore, the elasticity of this air can no longer keep in equilibrio the pressure of the air on the external surface of the water MN of the reservoir; consequently, the latter pressure will force a portion of water into the tube MNAC, in which it will rise to a certain height; let us suppose the quantity of its ascent to be M x for this first movement of the piston.

Now, the height M x is such, that the weight of the column of water M u, added to the weakened elastic force of the air in the space KI xu, and to the weight of the valve E, is in equilibrio with the pressure of the external air. The piston being now at KI, the valve E, by its own weight, falls and isolates the air comprised in the space C x, and the column of water MN xu remains suspended at the same height M x Let us now depress the piston from KI to GH; the elasticity of the air contained in the space AI, acting in every direction, shuts the valve E, and opens the moveable valve F, by means of which all the air comprised between KI and GH again escapes, and mixes with the external atmosphere. The piston being now in GH, and the valve F being shut by its own weight, if we again elevate the piston from GH to KI, the valve F will still remain shut, the valve E will open, and the water will again rise another certain quantity, as xy, in the sucking-tube, and so on for every ascent and descent of the piston. It is obvious, therefore, that after a certain number of strokes of the piston, the water will enter the body of the pump, and will ultimately escape either over the top of the tube at BD, or through the lateral tube O, fixed in the body of the pump; and this discharge will obviously remain as long as we continue the alternate motion of the piston, provided the source of water be not exhausted.

Limits of ascent.

45. We may observe, that, even neglecting the weight of the valve E, and supposing that we could produce as perfect a vacuum in the tube ACMN as in the tube of a barometer, the height AM ought still to be less than 34 feet, that is, less than a column of water whose pressure is the same with that of the atmosphere in the same place; because, otherwise, it will never pass into the body of the pump. But this condition being once fulfilled, the height LV, of the surface BD, of the water in the body of the pump, above the surface MN, of the water of the reservoir, may be greater than that of a column of water equal to the atmospheric pressure; which enables us to place the spout O, arbitrarily, at any height; but in this case it is necessary that the rod Z of the piston move in the body of the pump ABCD; at the same time, however, the height to which the water is to be raised, beyond that which appertains to the atmospheric pressure, ought not to be very great, because it will produce a considerable strain upon the fulcrum of the lever, as well as on the rod and piston itself.

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46. It is obvious, from the above description, that the discharge from a pump of this kind is not made in a continuous and uniform stream; it is only while the piston is rising that any water issues; when it descends, there is a suspension of the operation, at least, according to the above construction; but in actual practice a cistern is placed in the body of the pump, opposite to the discharging spout; this cistern receives the water quicker than it can be discharged at O; and it therefore continues to run while the piston is descending; by this means the stream is rendered, if not uniform, at least continuous, provided the strokes of the piston be repeated with sufficient rapidity. According to either of these constructions, the quantity of water discharged, in a given time, is obviously equal to as many times the volume of the tube KIHG as the piston has been elevated during that time, because every ascent of the piston brings up with it all the water contained in that space.

Force
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47. In order to determine the force which must be, at any given time, employed to overcome the resistance which the piston meets with in its ascent, let us denote by h the height LV of the surface of the water in the body of the pump above the level MN of that in the reservoir; a^2 the area of the circular section GH; and f the effort of the water on the head of the piston, or the force of its tendency to cause it to descend. Now, in whatever part hg of its course the piston may be in its ascent, it will constantly be opposed by a force $\mathrm{F} = a^2 \times \mathrm{LV}$, or $a^2 \times h$; for let VS be the height of the column of water equivalent to the pressure of the atmosphere, and let us suppose that the piston, in ascending, is arrived at any position gh, which answers to the height rV; then it is obvious, in the first place, that the piston is acted upon downwards by the pressure of the atmosphere, which is equal in intensity to $a^2 \times \mathrm{VS}$; and by the pressure of the column of water Dg, the effort of which is equal to $a^2 \times r\mathrm{L}$: so that, in fact, the whole tendency of the piston to descend is expressed by

$$a^2 (\mathrm{VS} + r\mathrm{L}).$$

In the next place, the piston is operated upon upwards by the pressure of the air on the external surface of the water MN of the reservoir, equal in its effect to $a^2 \times \mathrm{VS}$, which effort is in part destroyed by the weight of the column of water having for its base the circle gh, or GH, and for its height rV; so that the entire action on the piston, upwards, will be

$$a^2 (\mathrm{VS} - r\mathrm{V});$$

consequently, we shall have

$$\mathrm{F} = a^2 (\mathrm{VS} + r\mathrm{L}) - a^2 (\mathrm{VS} - r\mathrm{V}), \text{ or}$$

$$\mathrm{F} = a^2 (r\mathrm{L} + r\mathrm{V}) = a^2 \times \mathrm{LV} = a^2 \times h;$$

that is, the piston, throughout its whole ascent, will be opposed by an effort equal to the weight of a column of water having the same base as the piston, and for its altitude a height equal to that of the surface of the water in the body of the pump above that in the reservoir.

Therefore, in order to produce the upward motion of the piston, we must employ a force which is equal to the effort above determined, plus the weight of the piston and rod, and plus the resistance which the piston may experience in consequence of the friction against the internal bore of the tube.

With respect to the descent of the piston, this will be nearly accomplished by means of its own weight, as in this case there is no other resistance to be overcome than the friction and a slight impact against the water.

Of the lifting-pump.

48. A pump of this kind is shown in (fig. 8), the body of it, ABCD, is, in this case, inserted into the water of the reservoir, of which we may suppose the surface to be MN, the piston enters below, and lifts the water upwards; the piston-rod Z is firmly fixed to a moveable frame, TYZ, which is made to ascend and descend alternately by means of the usual lever or pump-handle. The head of the piston is pierced with a hole, which is covered by a valve F, opening upwards; and at AC, a little below the surface MN of the reservoir is a diaphragm, or plug, pierced with a hole, which is covered by another valve, E, also opening upwards. The body of the pump is united in AC with the ascending tube ACV, through which the water is to be raised.

In order to explain the operation of this pump, let us suppose, in the first place, that KI is the lower limit of the motion of the base or end of the piston; then, when the body of the pump is filled with water, and this water is on a level with that in the reservoir, the two valves, E and F, permit, by the facility with which they open, the free communication of the water in the reservoir with that in the body of the pump, after which the valves shut by their own weight. If now we raise the piston from KI to GH, which latter is the limit upwards of its course, the lower valve F will remain shut, the valve E will open, and the water contained in the space KH will be carried above GH, and pass into the ascending tube; moreover, while the piston is ascending, it is followed by the water which enters from the reservoir into the body of the pump. The piston being then depressed from GH to KI, the valve F is opened, and the valve E is shut, and by this means the water, which was at the first stroke carried above GH, is prevented from returning. Elevate now the piston a second time, the valve F will be shut as before, and the valve E opened, by which means the water is again raised in the ascending tube ACV; and by thus repeating the operation, it is obvious that the water may be raised to any height at pleasure. In this pump, as in the one described in the last article, the elevation of the water is not continuous and uniform, but intermitting: its discharge, however, may, as in the former case, be regulated by means of a cistern properly proportioned and adjusted. The discharge obtained by a pump of this description, for every stroke of the piston, is equal to the capacity of the cylinder GHKI; and the pressure upon the piston is equal to the weight of a column of water whose base is the same as that of the piston, and whose height is equal to the distance between the surface of water in the reservoir and in the ascending tube.

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The power, therefore, which is requisite to be applied, must exceed this pressure by a quantity equal to the weight of the piston-rod and frame, plus the resistance arising from the friction of the piston against the sides of the pump-barrel. With respect to the return of the piston, it will have a tendency to descend by its own weight, but it will be resisted by friction, and by a slight impact against the water.

Of the forcing-pump.

Forcing-pump.

49. This machine unites, in some measure, the properties of both those above described, and may be represented under two different forms, as in (figs. 9 and 10.) In the former it operates by suction as the piston ascends, and it forces in its descent. In the second, on the contrary, the atmospheric pressure takes place when the piston descends, and it forces in its ascent. We shall describe each of these constructions separately.

First construction. Fig. 9.

Referring to (fig. 9), we may observe, in the first place, that the pump consists of a suction-tube AMNC, partly immersed in the water MN of the reservoir; and of the pump-barrel, or body of the pump ABDC, in which the piston P works, as in the common sucking-pump above described, and of the tube of ascension CQV. At CA and QR are two valves, which open upwards, and the play of the piston we shall suppose to be in the space GK, but its head is solid, having no hole through it, as in the two preceding cases. It is obvious now, that by making the piston ascend and descend alternately, the water will first be raised up in the suction-tube, and in the body of the pump, in the same manner as in the sucking-pump, at least with this difference only, that instead of escaping, as in the latter, through the hole in the head of the piston, it is in this case forced through the valve F, and expelled at the opening of the ascending tube; in other respects the alternate motion of the two valves E and F is exactly the same in both cases.

After a few strokes of the piston, all the air will be discharged, and the void filled with the water which passes up through the suction-tube. This being supposed to have obtained, and the piston being forced down, it presses the water contained in the space KIGH through the valve F into the ascending tube, which latter valve prevents it from returning into the body of the pump. The space which would thus be left void is filled by the water rushing up through the suction-tube, in consequence of the pressure of the atmosphere, and follows the piston in its ascent to KI, when a second descending motion produces again the same effect; and, consequently, a repetition of those alternate motions will raise the water to any required height.

Second construction. Fig. 10.

50. In (fig. 10) the piston is also solid; and by making it descend, the water is raised successively in the suction-tube AMNC; and after a certain number of strokes it arrives at the ascending tube QRV, where it is then successively raised by the upward motion of the piston. The valves E and F act here the same as in (fig. 9); and the discharge in a given time is computed, in both cases, by the number of strokes of the piston, and the quantity it discharges at each.

Force to be applied. First construction. Fig. 9.

51. Let us now determine the force that must be applied at any instant in either of the above machines, in order to produce the requisite motion of the piston.

In the first place, in (fig. 9), let us suppose that the head of the piston is in the position *g h*, which corresponds to the vertical height *g* M, above the water in the reservoir. Let MS be the height of a column of water equivalent to the pressure of the atmosphere, and ML the height to which the water is raised. Assume a^2 to denote the area of the circle *g h*, and call $g\,M = h$; $g\,L = H$; the weight of the piston and its appendages $= P$; X the force requisite to push the piston upwards, that is, during the suction, abstracting from the friction; and Y the force employed to produce a motion in the same downwards, that is, while the forcing takes place.

These suppositions being made, and conceiving the piston now in *g h*, and that it is ascending, and consequently the valve F shut, it is obvious that

$$X = P + a^2.\,SM - (a^2.\,SM - a^2.\,g\,M), \text{ or,}$$
$$X = P + a^2.\,g\,M = P + a^2\,h.$$

Again, supposing the piston to be arrived at the same point in its descent, in which case the valve E will be shut, we shall have

$$Y = a^2.\,SM + a^2.\,g\,L - (a^2.\,SM + P), \text{ or,}$$
$$Y = a^2.\,g\,L - P = a^2\,H - P;$$

consequently, the sum of the two forces is

$$X + Y = a^2\,h + a^2\,H = a^2\,(h + H) = a^2.\,ML.$$

Hence it appears that the effort requisite to be employed to produce the desired effect, is equal to the weight of a column of water, which shall have for its base the head of the piston, and its height equal to the distance between the point of delivery and the surface of the water in the reservoir. But the advantage in this species of pump is, that the effort is divided into two parts; the one being opposed to the suction, and the other to the forcing; whereas, in the sucking and lifting-pumps, the whole effort is made at once during the time the water is elevated.

Force to be applied. Second construction. Fig. 10.

We shall find a similar result in the pump (fig. 10). For denoting the area of the circle *g h* again by a^2, and calling $r\,M = h$, $r\,L = H$, P the weight of the piston and appendages as before; also, still denoting by X and Y the two separate forces to overcome the suction and forcing, we shall have, independent of the friction,

$$X = a^2\,h - P,$$
$$Y = a^2\,H + P;$$

consequently,

$$X + Y = a^2\,(h + H) = a^2.\,ML,$$

the same result as before.

Forces ought to be uniform.

52. There is one point deserving of remark in the operation of this machine, which is, that as it is always desirable to have the applied force uniform, we ought, as far as possible, to have X = Y, which may be done by equating their values as above determined; that is, by writing, in case of (fig. 9),

$$P + a^2.\,h = a^2\,H - P,$$

which gives

$$P = \tfrac{1}{2}\,a^2\,(H - h);$$

and, in (fig. 10), by making

$$a^2.\,h - P = a^2\,H + P;$$

whence we have

$$P = \tfrac{1}{2}\,a^2\,(h - H).$$

In the former figure, therefore, the piston ought to be made to play in such a manner that the point *g*, at which the whole height ML is divided, may gi greater than *g*M; and in (fig. 10), *r*M ought greater than *r*L; and in both cases the division be such that the difference in the weights of th columns of water be equal to double the weight piston and its appendages; the latter being sup given.

If the position of the piston be given by the of the construction, or by the circumstances which the machine is employed, then the weigh should be determined by means of the above equa and if the pump be so situated that the water always at the same level, the load of the piston be so contrived that it may be increased or dimi as circumstances may require; in all these assuming the middle point between the highe lowest position of the piston as its mean place, for the point *g*.

53. We have already remarked, that, accord the construction of the three preceding machin exhibited in the figures, the discharge must nece be intermitting, because it is only during one two alternate motions of the piston that the w elevated. If, for example, the water be raised the piston is raised, there is no discharge, or, at a very trifling one, while the piston descends; as in (fig. 9), the water be discharged when the pi depressed, there is a suspension during its ele This inconvenience, however, may be in a grea sure avoided, by supplying the ascending tube cistern, or small reservoir; in which case the wa in the first instance, thrown into this cistern, a continue to flow during the noneffective motion piston; this cistern is an usual appendage to all of common construction. Another method of pro an uniform discharge is by means of an air- attached to any convenient part of the tube of sion: the air in this vessel is, in the first insta the same density as the external air; but aft water is raised in the tube, it distributes itself in vessel, cuts off the communication between t contained in it and the external air, and the for thus compressed into a less space; then, as s the piston descends, or during its noneffective m the elasticity of the air begins to operate; the is expelled from the air-vessel, and continues to and flow in an uniform stream.

54. A great deal depends as to the most ef operation of a pump upon the position of the valve. In a sucking-pump of the usual constr where the fixed valve E (fig. 7) is situated at th tion of the suction-pipe with the body of the the water is successively raised, and will, after tain number of strokes of the handle or lever, en body of the pump, provided the suction-tube not exceed 34 feet, and that the air be complet hausted; consequently, in a pump of this kin position of the fixed valve is the most advanta there is, however, still this inconvenience, th leather with which the valve is commonly cover comes dry when the pump has been for some ti active, in consequence of which the valve E w exactly close. To prevent this inconvenien suction-tube is commonly made less than wo otherwise necessary. In some pumps of this ki

fixed valve is even placed at, or a little below, the surface of the water MN, in order to keep the valve always moist, and which, of course, entirely prevents the inconvenience in question: in others there are two fixed valves, one in MN, and another in AC, by which means the water is retained during the inaction of the machine to the height AC, and the valve E kept constantly moist, the same as if it had been situated at the surface of the water in the reservoir. In this case the two valves act at the same time, both opening and closing together.

t of water.

55. Having thus described the fundamental principles of the construction and operation of pumps, we shall now bestow a few pages in investigating the rate of ascent of the water and other circumstances connected with the theory of these machines, in which we shall assume as a principle, that in the sucking-pump, after every elevation of the piston to KI, there will be an equilibrium between the pressure of the atmosphere on the surface of the water in the reservoir MN, and the elastic force of the rarefied air contained between KI, and the surface of the column of water in the ascending tube, plus the weight of that column of water.

In order to establish clearly the conditions of this equilibrium between those three forces, we shall reduce them to the same species, by representing the two former by columns of water of the requisite heights. The first of these columns, which depends upon the pressure of the atmosphere, is constant, or at least we shall consider it as such; but the second, which depends upon the degree of rarity of the included air in the tube AN, will be variable.

This being supposed, it is clear that we shall have an equilibrium between our three forces, when the height due to the pressure of the atmosphere is equal to the sum of the height due to the pressure of the interior air, and the height of column in the suction of tube above the general surface of the water in the reservoir MN.

56. Let it now be required to determine the successive elevations of the water in the suction-tube NA, supposing the fixed valve E to be situated in AC, that is, immediately at the junction of the tube with the body of the pump.

We shall assume that in consequence of the first stroke of the piston, that is, its first ascent from GH to KI, the water is raised from MN to $x\,u$ in the suction-tube; and let us conceive the piston to remain for an instant in that position, then the pressure of the atmosphere will counterbalance the pressure of the column of water M u, and the elastic force of the dilated air in the space K u.

Let us denote

The height AM of the suction-tube by	a.
The height of the play of the piston KG	b.
The radius of the pump-barrel AD	R.
The radius of the suction-tube AN	r.
The ratio of the circumference to diameter	π.
The height due to the pressure of the atmosphere	h.
The height M x of the column M u	x.
The height due to the pressure of the dilated air in the space K $x\,u$ I	y.

We shall then have for our equation of equilibrium,

$$x + y = h. \qquad (1)$$

The height due to the elastic force of the air before suction, that is, while the included air occupied the space AN, being denoted by h, and this air being now expanded through the whole space K u, the height y due to its elastic force, after suction, will be found to be

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$$y = h \times \frac{\text{AN}}{\text{K}u}.$$

Now the space AN, or the content of the cylinder AMNC $= \pi r^2 a$; and the space Ku, or the sum of the two cylinders KACI, and A $x\,u$ C $=$

$$\pi \text{R}^2 b + \pi r^2 (a - x),$$

whence we obtain a second equation,

$$y = \frac{h r^2 a}{\text{R}^2 b + r^2 (a - x)}. \qquad (2)$$

Substitute this value of y in equation (1) and we shall have

$$h = x + \frac{h r^2 a}{\text{R}^2 b + r^2 (a - x)};$$

or

$$h \text{R}^2 b + h r^2 (a - x) = \text{R}^2 b x + r^2 (a x - x^2) + h r^2 a,$$

which, transposed and reduced by writing,

$$\frac{\text{R}^2}{r^2} = k, \text{ and } h + a + k b = p,$$

becomes $x^2 - p x = - k h b$;

$$\left.\begin{aligned} \text{wherefore, } x &= \frac{p \pm \sqrt{(p^2 - 4 k h b)}}{2} \\ \text{and} \quad y &= \frac{2 h - p \mp \sqrt{(p^2 - 4 k h b)}}{2} \end{aligned}\right\}. \qquad (3)$$

Ascent due to the first stroke.

We have here two values both for x and y, but it is obvious that only one in each will apply, and this one in both equations must be that answering to the lower sign, for it is evident that both x and y must be less than h, and if in the equation for x, we use the upper sign +, we should have $x > h$, or

$$\frac{p + \sqrt{(p^2 - 4 k h b)}}{2} > h;$$

that is, substituting for p,

$$\sqrt{\{(h + a + k b)^2 - 4 k h b\}} > (h - a - k b)$$

for squaring both sides of the latter equation, and reducing, we have $4 a h > 0$.

If, on the contrary, the lower sign be employed, we shall find that the value of x is less than h, and the corresponding value of y, that is, the quantity

$$\frac{h - a - k b + \sqrt{\{(h - a - k b)^2 - 4 k h b\}}}{2}$$

also less than h.

Consequently, the two values of x and y, which answer the conditions of the question, are

$$\left.\begin{aligned} x &= \frac{h - \sqrt{(p^2 - 4 k h b)}}{2} \\ y &= \frac{2 h - p + \sqrt{(p^2 - 4 k h b)}}{2} \end{aligned}\right\}. \qquad (4)$$

These are the values of x and y, which answer to the first stroke of the piston, but knowing these we shall readily find the analogous values corresponding to the second, third, &c. strokes.

General equations.

57. Let M y represent the height of the water in the suction-tube after the second ascent of the piston, and call this height x', and the height due to the elastic force of the air y'; then we shall have as before,

$$h = x' + y'. \qquad (5)$$

Now the
that after the second
the space K z, supposing the w
roke to rise to yz. We shall therefore have

$$y' = \frac{Au}{Kz},$$

or,

$$y' = \frac{y r^2 (a - x)}{R^2 b + r^2 (a - x')}; \qquad (6)$$

that is, agreeably to our preceding substitutions,

$$y' = \frac{y(a - x)}{a - x' + kb}.$$

Comparing equations (5) and (6), we obtain

$$h = x' + \frac{y(a - x)}{a - x' + kb};$$

whence we deduce, in the same manner as before,

$$x' = \frac{p - \sqrt{\{p^2 - 4khb - 4x(h + a - x)\}}}{2};$$

$$y' = \frac{2h - p + \sqrt{\{p^2 - 4khb - 4x(h + a - x)\}}}{2};$$

observing here the same conditions of the signs as in the preceding case.

We may now, from these expressions, deduce analogous ones for the ascent of the water after the third stroke of the piston, in the same manner, as the last were drawn from those which preceded them; wherefore, without repeating the actual operations, we may infer that we shall have

$$x'' = \frac{p - \sqrt{\{p^2 - 4khb - 4x'(h + a - x')\}}}{2};$$

$$y'' = \frac{2h - p + \sqrt{\{p^2 - 4khb - 4x'(h + a - x')\}}}{2}.$$

And in the same way,

$$x''' = \frac{p - \sqrt{\{p^2 - 4khb - 4x''(h + a - x'')\}}}{2};$$

$$y''' = \frac{2h - p + \sqrt{\{p^2 - 4khb - 4x''(h + a - x'')\}}}{2};$$

whence, generally denoting by $x^{(n)}$ the height of the water after $(n + 1)$ strokes of the piston, by $x(n-1)$ the height after n strokes, and by y^n the corresponding value of y, we shall have

$$x^{(n)} = \frac{p - \sqrt{\{p^2 - 4khb - 4x^{(n-1)}(h + a - x^{(n-1)})\}}}{2};$$

$$y^{(n)} = h - \tfrac{1}{2}pp + \frac{\sqrt{\{p^2 - 4khb - 4x^{(n-1)}(h + a - x^{n-1})\}}}{2}.$$

Height after n strokes.

58. By means of these equations and the preceding ones, we may determine the height to which the water will ascend after any given number of alternate motions of the piston, and the elastic force of the air contained in the suction-tube; we shall thus also know the elevation due to each particular stroke, the differences of those elevations, and the successive differences in the elastic force of the remaining air; for having computed x, x', x'', &c. y, y', &c. we shall of course have also $x' - x$, $x'' - x'$, &c. $y' - y$, $y'' - y'$, &c.

provided
by the preceding for
have $x^{(n)} = x^{(n-1)}$; and our general valu
may be written

$$x^{(n-1)} = \frac{p - \sqrt{\{p^2 - 4khb - 4x^{(n-1)}(h + a - x^{(n-1)})\}}}{2}$$

from which we deduce $x^{(n-1)} = h$, the greatest possible height of the column of water in the tube; therefore, if the suction-tube AM do not exceed h, or the height due to the atmospheric pressure, the water will continue to ascend in it after every stroke of the piston, till it at length passes into the body of the pump; but if the tube AM have an altitude greater than h, the water will continue for an indefinite time to ascend without ever arriving at a height which it cannot surpass; that is, without ever actually attaining an altitude equal to h; for in the latter case we can never produce an actual vacuum, no more than we can in the air-pump; and since we have always

$$h = x^{(n)} + y^{(n)},$$

and as, in the case we have supposed, $y^{(n)}$ can never become zero, it is obvious that we can never have $x^{(n)} = h$; since, however, the successive values of y^n, $y^{(n+1)}$, &c. continually diminishes, the corresponding values of x, viz. $x^{(n)}$, $x^{(n+1)}$ will continually increase: but, as the former can never become zero, the latter will never attain to the value of h, although it will approximate to it indefinitely near: the common maximum, therefore, that water will ascend in the suction-tube to a height equal to that of a column of water equivalent to the pressure of the atmosphere, is not strictly true; that height is a limit to which it constantly approaches, but which it can only obtain after an infinite time.

59. In the preceding part of this investigation we have made abstraction of the weight of the valve E, but we may introduce it, when requisite, by assuming a certain column of water as equal to the additional pressure which must be overcome. Suppose the height of this column to be m, then our first equation will be

$$h = x + y + m,$$

or

$$h - m = x + y.$$

If, therefore, we write $h - m = h'$, we have again

$$h' = x + y,$$

and it will only be requisite to substitute this value of h' instead of h in the foregoing equations, in order to render them applicable to this case.

60. Every thing remaining as in the preceding investigation, except that the fixed valve is removed to MN, let it be required to determine the law and conditions of the motion of the ascending fluid.

Let Mx, My, &c. be the successive heights to which the water ascends in the suction-tube AN, then employing still the same substitutions as in (art. 56), it is e dent that, after the first stroke of the piston, we shall have the two equations

$$x + y = h,$$

$$y = \frac{h\,a}{k\,b + a - x};$$

whence we deduce

$$x = \frac{p - \surd\,(p^2 - 4\,k\,h\,b)}{2},$$

$$y = \frac{2\,h - p + \surd\,(p^2 - 4\,k\,h\,b)}{2}.$$

These equations are the same as those which answer to the first stroke of the piston in the preceding article; but the analogous equations which express the height and reduced pressure after the second, third, &c. strokes of the piston, have not the same form as in the former case.

In order to obtain these, we may in the first place observe, that at the commencement of the second aspiration, the air compressed in the space A u, is the natural air; whence, after the second aspiration, this being spread through the space K z, the height due to the elastic force of the same, will be

$$y' = \frac{h \times \mathrm{A}\,u}{\mathrm{K}\,z};$$

or,

$$y' = \frac{h\,(a - x)}{k\,b + a - x}.$$

And we have still $x' + y' = h$; whence, by equating and reducing

$$x' = \frac{p - \surd\,\{p^2 - 4\,k\,h\,b - 4\,h\,x\}}{2},$$

$$y' = \frac{2\,h - p + \surd\,\{p^2 - 4\,k\,h\,b - 4\,h\,x\}}{2}.$$

In the same manner, observing that x'' and y'' are derived from x' and y', as these latter are from x and y, we shall find

$$x'' = \frac{p - \surd\,\{p^2 - 4\,k\,h\,b - 4\,h\,x'\}}{2},$$

$$y'' = \frac{2\,h - p + \surd\,\{p^2 - 4\,k\,h\,b - 4\,h\,x'\}}{2};$$

and, generally,

$$x^{(n)} = \frac{p - \surd\,\{p^2 - 4\,k\,h\,b - 4\,h\,x^{n-1}\}}{2};$$

$$y^{(n)} = \frac{2\,h - p + \surd\,\{p^2 - 4\,k\,h\,b - 4\,h\,x^{n-1}\}}{2}.$$

Now, supposing the water to gain no further ascension after $n + 1$ strokes of the piston, we shall have $x^{(n)} = x^{(n-1)}$, and consequently

$$x^{(n-1)} = \frac{p - \surd\,\{p^2 - 4\,k\,h\,b - 4\,h\,x^{n-1}\}}{2};$$

whence we have

$$x^{(n-1)} = \frac{a + k\,b \pm \surd\,\{(a + k\,b)^2 - 4\,k\,h\,b\}}{2}.$$

Wherefore, in consequence of the double sign which appertains to the radical, it appears that there are two heights where the water may stop in its ascent, supposing the quantity under the radical to be a real quantity, a condition which requires either, that

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$$(a + k\,b)^2 = 4\,h\,k\,b,$$

or that

$$(a + k\,b)^2 \; 7 \; 4\,h\,k\,b.$$

In the first instance, as the radical vanishes, the two values of $x^{(n-1)}$ are equal to each other, or which is the same, there will be but one point where the water will remain in equilibrio; which height is expressed by the equation

$$x^{(n-1)} = \frac{a + k\,b}{2}.$$

In the second case, that is, when $(a + k\,b)^2 \; 7 \; 4\,h\,k\,b$, there are two such points, which will be determined from the preceding equations. When the quantity under the radical is imaginary, that is, when $(a + k\,b)^2 \angle 4\,k\,h\,b$, then the water will not stop, but will after a time enter the body of the pump.

Examples.

61. As an example, let us assume h, after deducting for the weight of the valve = 32 feet; a, or the height of the suction-tube = 20 feet; b, or the length of the play of the piston = 4 feet; and let k, or the ratio of $\mathrm{R}^2 : r^2 = 1$, that is, let the pump-barrel and suction-tube have equal diameters, then we shall find

$$x^{\overline{(n-1)}} = \frac{20 + 4 \pm \surd\,(24^2 - 4 \cdot 32 \cdot 4)}{2};$$

or,

$$x^{(n-1)} = 12 \pm 4 = 16 \text{ or } 8.$$

As a second example, let h = 32 feet, a = 25 feet, b = 2 feet, and k = 4; then we shall find the two values of $x^{(n-1)}$ to be

$$x^{(n-1)} = \frac{33 \pm \surd\,65}{2},$$

and the water will stop in two places as before; but if all the other quantities remain the same, except k, which we shall suppose to be equal to b, there will be no stoppage, the radical becoming, in this case, imaginary; or, if we suppose k to remain as at first = 4, and make b = 4, there will also be no stoppage.

It follows, therefore, that all the other quantities being given, b may always be so assumed that the action of the pump may be effectual, it is for this purpose only necessary to take b, so that $(a + k\,b)^2 \angle 4\,h\,k\,b$.

If $k = 1$, and we write $a + b = c$, the greatest height of the piston above the surface of the water in the reservoir, we must have

$$b \; 7 \; \frac{c^2}{4h}$$

in order to ensure an effective operation of the machine.

The same geometrically. Fig. 7.

62. The same may be shown geometrically, as follows: Suppose, as above, the valve E to be situated at the surface of the water in MN, the tube to be of uniform bore, and VL to be the height due to the pressure of the atmosphere, *minus* the weight of the piston; that is, let LV = 32 feet. Conceive the water to be raised by working to $t\,s$, then the weight of the column of water t N, together with the elasticity of the air above it, exactly balances the column whose height is LV. But the elasticity of the air in the space I t (KI being the highest, and GH the lowest position of

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the piston) is proportional to $LV \times \frac{Hs}{Is}$, and consequently in the place where the limit obtains, and the water rises no further, we shall have

$$LV = sN + LV \times \frac{Hs}{Is};$$

transposing Ns, we have

$$LV - sN = LY = LV \frac{Hs}{Is};$$

whence, $Is : Hs :: LV : LY$;

or, by the division of ratios,

$$Is - Hs (= Lr') : Is :: LV - LY = VY : LV;$$

consequently, $Lr' \times LV = Is \times VY$.

Hence, we see that, if KM, the height of the piston in its highest position above the surface of the water in the reservoir, and KG, the length of the play of the piston, be given, there is a certain determined height M s, to which the water can be raised by the difference of the pressures of the exterior and interior air; for, LV is to be considered as a constant quantity, and of course when KG is given, $Is \times VY$ is given likewise.

To ensure, therefore, the delivery of water by the pump, the stroke must be such that the rectangle $Lr' \times LV$ may be greater than any rectangle that can be made of the parts of IN, that is, greater than the square of $\frac{1}{2}$ IN.

Hence, therefore, we may deduce the following practical maxim, viz. No sucking-pump can raise water effectually unless the play of the piston, in feet, be greater than the square of the greatest height of the piston, divided by four times the height due to the atmospheric pressure, *the fixed valve being applied at the surface of the reservoir.*

When the fixed valve is at the junction of the pump-barrel.

The circumstances above investigated do not occur when the fixed valve E is situated at the junction of the suction-tube and pump-barrel; we therefore see the defect of placing that valve at the surface MN, although some advantage may be supposed to be derived from its being kept always moist when so posited: but if precautions be taken to keep the valve E moist in its upper situation, it must be considered a more favourable position than when it is placed below, or at the surface of the water.

Forcing-pump. Fig. 9.

63. The forcing-pump is subject to no stoppage when the valve E is placed in the water of the reservoir; but if it be placed above the surface, as in rs (fig. 9), or at such a height st, that the interior space AC sr is greater than the volume of water that the piston raises in its motion from KI to GH; then it may happen that the water cannot arrive at rs, that is, it may stop at some intermediate height. For let us conceive that by the first elevation of the piston the water ascends to uz; at this moment the air comprised between the surface of the water and rs is the natural air which is dilated and weakened when the piston descends, and so on, alternately, during the entire play of the piston.

When the latter descends, the column of water A z is operated upon downwards by its own weight and by the elastic force of the air contained in the space us; so that the resultant or the sum of the two forces may be in equilibrio with the pressure of the external air on the surface MN of the reservoir. Now, since the column of water A z ascends and descends with the piston, if it happen (which is evidently possible) that from one stroke of the piston to the following, the descent of the column A z be equal to its ascent, it can never reach rs; and there will, therefore, in this case, be a stoppage in the action of this machine. The position of this point is computed, as also the successive elevations of the water, precisely in the same manner as in the case of the sucking-pump; it is, therefore, unnecessary to repeat the operation; and the same calculus will also apply to the circumstance attending the ascent and stoppage of the water in the forcing-pump, whether the suction take place with the ascent or descent of the piston.

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64. In all the preceding cases we have supposed the several machines to be worked simply by the operation of one lever applied to a single piston-rod, but frequently, when a great supply or discharge is wanted, a more powerful moving force, and a greater number of pistons are employed, we propose, therefore, before concluding this section, to give the solution of the following problem, which may be of some practical utility, viz.

To determine the circumstances attending the elevation of water in the ascending tube of a sucking-pump, when the operation depends upon the action of a triple lever.

Fig.

Let AMNC (fig. 10-a) represent the ascending tube of such a machine, with three equal pump-barrels, in which are situated three pistons moved by means of the three levers, RV, TY, SX; these levers receiving their motion by means of three other levers, or three chains, RB, TK, SH, the extremities of which, B, K, H, form the summits of an equilateral triangle B, K, H, which is made to turn circularly about its centre O. The wheel BHK may be moved by any agent, as, for instance, by a current of water, which causes it to turn in the direction BKDH. The position and operation of the valves in each pump-barrel are supposed to be the same as in the common sucking-pump. As to the fulcrums on which the levers turn, they may be situated in any point, according as circumstances require, or render desirable; but for the sake of simplicity, in the following investigation, we shall suppose them to bisect each its respective lever; we shall also assume that the lever of each piston is equal to the diameter of the circle described by BHK.

Now, whatever may be the initial situation of the wheel, it is evident that the air comprised in the suction-tube, and in the body of the pump, is the same as that of the atmosphere. Afterwards, when we cause the wheel to turn, the air contained in the suction-tube, and in the pump-barrel, whose piston ascends, is rarefied or dilated, while any piston which descends, gives entrance to the exterior air into the space comprised between the moveable and fixed valve; so that in fact, when a piston ascends, its moveable valve is shut, and the fixed valve is open, and the contrary when it descends.

Let us suppose, in the first instance, the pivot B of the lever RB to be situated at the upper extremity of the vertical diameter BD of the wheel, and that consequently the horizontal line which joins the other two pivots K, H, divides the radius OD into two equal parts. At this instant, the valve F is in its lowest position, and may be supposed in contact with the fixed valve E; and the other two moveable valves F′

and F″ are each distant from their respective fixed valves E′ and E″ by a quantity = BG.

Let the points Q and M be the middle of the arcs BK, BH, and draw the right line QM. The circumference of the wheel will thus be divided into six equal parts BQ, QK, KD, DH, HM, MB, and we shall therefore have BL = DG.

As the pivots B, K, H rise and fall by the motion of the wheel, they will produce a contrary motion in the piston-rods; and it is obvious, that after one-sixth of a revolution, B will have passed to Q, K to D, and H to M; the piston corresponding to B will be raised a quantity equal to BL; the piston corresponding to K, already raised a quantity equal to BG, will be *further* elevated by the quantity GD; and finally, the corresponding piston to H, which at first was raised a quantity equal to BG, will now be depressed through a space equal to GL.

The interior air being thus dilated to a certain extent, the water will ascend in the suction-tube to some height, which let us denote by M r. Since the dilation of the interior air takes place, as we have seen, only during the ascent of a piston, it follows that if we denote by π the constant number 3·14159, &c. viz. the semi-circumference of a circle whose radius is unity; by m the radius of the suction-tube; by n the radius of each of the pump-barrels; it will follow that the portion of air which in the first instant occupied a space represented by

$$\pi m^2 \times \text{AM} + \pi n^2 \times \text{BG},$$

will occupy, at the end of one-sixth of a revolution, a space equal to

$$\pi m^2 \times \text{A}r + \pi n^2 \times (\text{BD} + \text{BL}).$$

In this second state, the elastic force of the air thus dilated, *plus* the weight of the column of water r N, ought to be in equilibrio with the atmospheric pressure; and we may therefore determine, as in (art. 54), the height M r to which the water will ascend after the wheel has made one-sixth of a revolution. We may also determine, in the same manner, the height M t of the water after the second sixth part of a revolution; by considering, that as soon as the pivot K, now in D, passes this point, the portion of dilated air which still remains in the interior of the pump is reduced from the quantity

$$\pi m^2 \times \text{A}r + \pi n^2 \times \text{BL}$$

to

$$\pi m^2 \times \text{A}t + \pi n^2 \times \text{BG},$$

when the point K arrives at H, and so on.

Hence, generally, let us suppose that in the first instance the three pivots are placed in the points b, k, h: draw through the vertical BD the perpendiculars bf, kg, hi, and let us endeavour to compute the height M u to which the water will be raised in the suction-tube, while b passes to Q, k to D, and h to M.

In this interval of time the portion of the natural interior air which occupied a space represented by

$$\pi m^2 \times \text{AM} + \pi n^2 (\text{B}f + \text{B}g),$$

now fills a space equal to

$$\pi m^2 \times \text{A}u + \pi n^2 (\text{BD} + \text{BL}).$$

The moment the point k passes the point D, the portion of the interior air is reduced to the quantity

$$\pi m^2 \times \text{A}u + \pi n^2 \times \text{BL};$$

and when the point k arrives at H, its volume becomes

$$\pi m^2 \times \text{A}z + \pi n^2 \times \text{BG},$$

the height M z being then that of the water in the suction-tube, and so on.

Construction of pumps for raising water.

It would not be difficult to express, analytically, the progressive ascensions of the water in the suction-tube, computed on the above principles; and thence to assure ourselves whether it will or will not pass into the body of the pump; but the detail of the operation would carry us too far; we leave, therefore, the operation to be supplied by the reader, who will readily extend a similar calculus to any greater number of levers.

Theory of the centrifugal-pump.

Centrifugal-pump.

65. We have already described the construction and operation of the centrifugal-pump (art. 231, No. xx), HYDRODYNAMICS; we shall here, therefore, only attend to the theory of it, as far as it relates to the quantity of water which it is capable of discharging in a given time, the limits of its operation, &c.

Fig. 11.

In order to this, let a = the length of the arm LI (fig. 11), in feet, l = the length of the leg KL also in feet, t the time of a revolution in seconds, $2g$ = 32¼ feet, the measure of the force of gravity, and π = 3·14159, the semi-circumference of a circle whose radius is unity.

Then, since the centrifugal force is as the velocity generated in an unit of time, we shall have the centrifugal force of a particle $\dot{x}$ of the fluid, at the distance x from L, equal to

$$\dot{x}\left(\frac{2\pi x}{t}\right)^2 \div x = \frac{4\pi^2 x \dot{x}}{t^2};$$

the fluent of which is $\frac{2\pi^2 x^2}{t^2}$, which is the centrifugal motive force of the column x; this, when $x = a$, becomes $\frac{2\pi^2 a^2}{t^2}$, for the whole centrifugal motive force of the water in the arm LI. Now the pressure of a column whose length is a, will, *cæteris paribus*, vary as $2ga$: hence we shall have

$$2ga : a :: \frac{2\pi^2 a^2}{t^2} : \frac{\pi^2 a^2}{g t^2};$$

which latter term is equal to the length of a column of water whose pressure is equivalent to the centrifugal force. If from this we deduct the altitude l of the vertical leg, the remainder,

$$\frac{\pi^2 a^2}{g t^2} - l,$$

will be the length of a column whose action would expel the water from the orifice I, with the same force as that occasioned by the whirling motion; and the column moved will be $a + l$. This will also denote the space through which the column is accelerated.

The circumstances, therefore, of the case under consideration are the same as if a constant head of water, whose height $= \frac{\pi^2 a^2}{g t^2} - l$, impelled a column of water horizontally at the bottom of the depth $a + l$; hence the accelerative force is

$$\phi = \left(\frac{\pi^2 a^2}{g t^2} - l\right) \div (a + l),$$

and the space $s = a + l$, so that the velocity generated will be

$$v = 2\sqrt{gs\phi} = 2\sqrt{\left\{g\left(\frac{\pi^2 a^2}{g t^2} - l\right)\right\}},$$

or $v = 8{\cdot}0208 \times \sqrt{\left(\frac{\pi^2 a^2}{g t^2} - l\right)}$.

This result, it will be observed, is independent of the contraction of the vein, therefore, to reduce it to practical purposes, the co-efficient 8·0208 must be multiplied by the co-efficient for the contraction of the orifice, viz. $\frac{5}{8}$, or ·62; in which case the equation will become

$$v = 5\sqrt{\left(\frac{\pi^2 a^2}{g t^2} - l\right)}, \text{ very nearly.}$$

Limits the same as in the sucking-pump.

66. It is obvious that l must never exceed 34 feet, or the height due to the pressure of the atmosphere, because a greater column cannot be supported by that pressure; so that the limits of the operation of this machine are the same as for those of the sucking-pump. Besides which, the time of the revolution of the arm has its limits; between which only, the pump is effective: it is therefore necessary to carry the investigation a little further.

When the centrifugal force is barely equal to the weight to be raised, the expression under the radical vanishes, and $v = o$, or the water is stationary in the pump. In this case we have evidently

$$\frac{\pi^2 a^2}{g t^2} = l;$$

from which we obtain

$$t = \pi a \sqrt{\frac{1}{g l}} = {\cdot}78336 \sqrt{\frac{a^2}{l}},$$

for the time of a revolution in seconds, when no work is done.

If the machine be effectual, the velocity of the stream can never exceed the velocity which a heavy body would acquire in falling through a height equal to the difference between the height due to the pressure of the atmosphere and the height of the vertical leg; for a greater velocity would cause a vacuity in the machine that would not be supplied by the pressure of the atmosphere. Now, by the laws of falling bodies, the velocity acquired in falling through the height $(34 - l)$, is $v = 2\sqrt{(34 - l)g}$;

and, from what is shown above, the velocity of the efflux is

$$v = \sqrt{\left\{\frac{\pi^2 a^2}{g t^2} - l\right\}};$$

whence, making these values equal, we have

$$\frac{\pi^2 a^2}{g t^2} - l = 34 - l;$$

from which we find

$$t = \pi a \sqrt{\frac{1}{34 g}} = {\cdot}78336 \frac{a}{\sqrt{34}}.$$

If $l = 34$, then the last value of t will be the same as the preceding, and no discharge can take place, agreeably to the observations at the beginning of this article. In all other cases, having found the time of revolution when the water is stationary, we have only to diminish that time in the ratio of $\sqrt{34}$ to $\sqrt{l}$, and we shall have the time of a revolution, when the work done is the greatest possible. Gregory's "Mechanics," vol. i.

§ VI. *On the measurement of heights by the barometer.*

67. We have already explained the fundamental principles of this admeasurement, but there are many minute particulars which it is necessary to introduce beyond what we have hitherto considered, in order to render the results of the observations and calculations sufficiently accurate for any practical determination. We propose now to enter upon this subject at some length, and to examine it with all the detail its importance and utility demands.

Let us conceive a vertical tube, filled with air, extending from the surface of the earth to the limits of the atmosphere; also, for the purpose of simplifying the problem, let us suppose, in the first place, that this column is composed of perfectly dry air, and that it is of the same temperature throughout; let us assume also, that there is no decrease of gravity in consequence of the greater elevation, or that gravity acts uniformly through the whole extent. On these suppositions let us examine the state of equilibrium of such a column.

68. First, it is evident that each particle will be compressed by the weight of all those above it; and as the air in consequence of its elasticity is condensed proportionally to the compressing force, its density will decrease from the bottom to the top by insensible gradations. In order to discover the law of this decrease, conceive the column to be divided into an indefinite number of very thin beds, or strata, so that the sensible density may be the same throughout the height of each, varying only from one to another. Then, if the barometer be carried successively into each of these beds at different distances from the centre of the earth, there will be a certain ratio between these distances, which may be represented by x', x'', x''', &c., and the elevation of the mercury H', H'', H''', &c.; and it is this ratio which is required to be determined.

With a view to this determination, it may be remarked, that the thickness of the first bed, or stratum, is expressed by $x'' - x'$, and the depression of the mercury in raising it through this stratum is $H' - H''$. Consequently, at that elevation, a column of air of the height $x'' - x'$ weighs as much as a column of mercury whose height is $H' - H''$, the base of each being the same. Thus the density of that stratum, compared with that of the mercury, is as $\frac{H' - H''}{x'' - x}$ to 1, the densities being reciprocally as the masses when the weights are equal: but this ratio between the density of the stratum and that of mercury may also be obtained in a different manner, as follows:

69. At an equal temperature, the density of each stratum is proportional to the pressure it sustains, that is, to the weight of the superior beds; and since all the beds are supposed to have the same temperature, the pressure sustained by each is proportional to the height of the mercury in the barometer. Thus, on the hypothesis above assumed, the density of the different beds may be represented by

$$CH', CH'', CH''', \&c.$$

C being a constant co-efficient common to every stratum in the whole column.

In this manner two expressions for the density of the first stratum are obtained, viz.

$$\frac{H'-H''}{x''-x} \text{ and } CH';$$

and by putting these equal to each other, we have

$$CH' = \frac{H'-H''}{x''-x'};$$

whence

$$H'' = H'\{1 - C(x''-x')\}.$$

The same relation will obviously obtain in passing from the second stratum to the third, from the third to the fourth, and so on in succession, at least on the suppositions that have been admitted; whence, therefore, we have the following equations:

$$\begin{aligned} H'' &= H'\{1 - C(x''-x')\}, \\ H''' &= H''\{1 - C(x'''-x'')\}, \\ H^{iv} &= H'''\{1 - C(x^{iv}-x''')\}, \\ H^{v} &= H^{iv}\{1 - C(x^{v}-x^{iv})\}, \\ \&c. &= \&c. \end{aligned}$$

Or, if we represent the thickness of each stratum by D, which is supposed to be always the same, we shall have

$$\begin{aligned} H'' &= H'\ (1-CD), \\ H''' &= H''\ (1-CD), \\ H^{iv} &= H'''\ (1-CD), \\ H^{v} &= H^{iv}\ (1-CD), \\ \&c. &= \&c. \end{aligned}$$

Or, substituting for H″, H‴, &c. in the second members of these equations, they become

$$\begin{aligned} H'' &= H'\ (1-CD), \\ H''' &= H'\ (1-CD)^2, \\ H^{iv} &= H'\ (1-CD)^3, \\ H^{v} &= H'\ (1-CD)^4, \\ \&c. &= \&c. \end{aligned}$$

70. We have, therefore, between the differences of level and the depressions of the mercury, the following relations:

$$\left.\begin{aligned} x''-x' &= D \\ x'''-x' &= 2D \\ x^{iv}-x' &= 3D \\ x^{v}-x' &= 4D \\ \&c. &= \&c. \end{aligned}\right\} \text{ and } \left\{\begin{aligned} \frac{H''}{H'} &= (1-CD), \\ \frac{H'''}{H'} &= (1-CD)^2, \\ \frac{H^{iv}}{H'} &= (1-CD)^3, \\ \frac{H^{v}}{H'} &= (1-CD)^4, \\ \&c. &= \&c. \end{aligned}\right.$$

Now the quantity 1 — CD is necessarily a fraction; for C and D are both positive; let, therefore, C be what it may, D may always be taken so small that the product CD may be a fraction, and therefore 1 — CD a positive fraction, and consequently the powers of it will decrease as the indices increase. Hence, then, we arrive at the same conclusion as in (art. 29), viz. *That when the altitudes above the first station increase in arithmetical progression, the heights of the mercury in the barometer decrease in a geometrical progression.*

71. We have been enabled to arrive at this result by supposing that the density is the same for each stratum throughout its whole depth, but we have not assigned to these strata any given and determinate thickness; if we had, the hypothesis itself would have been erroneous, because the density varies at every height; but as it appears that the conclusion is wholly independent of any particular dimensions of the strata, we may infer that it is true when they are indefinitely thin; and the same may be otherwise shown, as follows:

Measurement of heights by the barometer.

If the rank of any term in the two preceding series be denoted by the *n*th, and its value be found, which may be done by the second series, by means of logarithms, we find

$$n = \frac{x^{(n+1)} - x'}{D},\ n = \frac{(\log.\ H' - \log.\ H^{(n+1)})}{\log.\ (1-CD)};$$

whence

$$x^{(n+1)} - x' = -\frac{D\ (\log.\ H' - \log.\ H^{(n+1)})}{\log.\ (1-CD)}.$$

Now $x^{(n+1)} - x'$ = the difference of level of any two stations, which, for the sake of simplifying, may be denoted by X. If also $H^{(n+1)}$, the height of the mercury at the higher station be called *h*, and H′ the height of it at the lower station be called H, we shall have

$$X = \frac{-D}{\log.\ (1-CD)} \times (\log.\ H - \log.\ h).$$

Now we know that

$$\log.\ (1-CD) = -\frac{1}{M}\left\{CD + \frac{C^2D^2}{2} + \frac{C^3D^3}{3} + \frac{C^4D^4}{4} +,\ \&c.\right\},$$

M denoting the modulus of the common table of logarithms, or M = 2·30258509, &c.

We have, therefore, rigorously,

$$X = \frac{+M}{C + \frac{1}{2}C^2D + \frac{1}{3}C^3D^2 + \frac{1}{4}C^4D^3 +,\&c.} \times (\log. H - \log. h)$$

Since, also, by the supposition, D, which denotes the thickness of the several strata, is very small, and may in fact be considered as zero, the preceding formula becomes

$$X = \frac{M}{C}(\log.\ H - \log.\ h);$$

that is to say, *the difference of level between any two places is found by multiplying the difference between the logs. of the two mercurial columns by a constant co-efficient, which remains to be determined.* This result still agrees with our deductions (art. 29, et seq.).

Co-efficient determined.

72. In order to determine this constant multiplier, let us represent the density of the air under the pressure H by δ, that of the mercury being unity, and we shall have, agreeably to the preceding suppositions, δ = CH, H being the height of the mercury in the barometer; consequently, we have $C = \frac{\delta}{H}$; and it therefore only remains to determine, by a well-conducted and extensive series of observations, the values of δ corresponding to the different values of H. This, at least, is all that would be required, if there were actually that uniform relation between the density and altitude which we have supposed, and if the former of these quantities were independent of different degrees of temperature; and, moreover, if the action of gravity were constant at all places and at all heights. It was by supposing all these to obtain, that we deduced our constant co-efficient in the articles above referred to;

but not one of these assumptions ought to be admitted when we wish to apply the barometrical measurement to actual practice.

Modification of the preceding hypothesis.

73. In the first place, the ratio between the densities of the air and mercury under a given atmospheric pressure, is not the same in all places. We shall suppose that δ denote the density of the air under a given pressure, as for example under 29·920896 inches, or ·831136 of an English yard, corresponding to ·76 of a metre. But according as the intensity of gravity is greater or less, the column of mercury having always ·831136 of a yard for its altitude, its weight will be greater or less, and consequently the air sustaining this pressure will be proportionally compressed. From experiments it has been ascertained, that calling the force of gravity in the parallel of 45° of latitude $= 1$, the gravity under any other parallel of latitude ψ will be expressed by $1 - 0{\cdot}002837 \cos 2\,\psi$;
and the density δ being proportional to the force of gravity, will vary in the same ratio, that is, by calling it δ under a pressure H, where gravity is unity, viz. at 45°, it will become for any other latitude, under a column of mercury of the same height,

$$\delta\,(1 - 0{\cdot}002837 \cos 2\,\psi.)$$

As depending on the latitude.

The co-efficient C, which expresses the ratio of the density to the height of the barometrical column, ought therefore to vary in the same proportion, and consequently it will become

$$C\,(1 - 0{\cdot}002837 \cos 2\,\psi),$$

which being substituted in the value of X, found above, gives

$$X = \frac{M}{C(-\,0{\cdot}002837 \cos 2\,\psi)} \times \log.\left(\frac{H}{h}\right);$$

it will therefore be sufficient to find the co-efficient

$$\frac{M}{C\,(1 - 0{\cdot}002837 \cos 2\,\psi)}$$

by experiment for any given latitude; for then, since $2\,\psi$ will be known, $\frac{M}{C}$ may be determined, and the formula will become applicable to all latitudes.

If we convert the above co-efficient into a series by division, we shall find

$$\frac{M}{C} \times \frac{1}{(1 - 0{\cdot}002837 \cos 2\,\psi)} =$$

$$\frac{M}{C} \times (1 + 0{\cdot}002837 \cos 2\,\psi + 0{\cdot}00000804857 \cos^2 2\,\psi + \&c.),$$

and retaining only the first two terms of the development, the third, fourth, &c. being too small to effect the result, the above formula may be more conveniently expressed, as follows:

$$X = \frac{M}{C}\,(1 + 0{\cdot}002837 \cos 2\,\psi) \log.\left(\frac{H}{h}\right).$$

Other causes.

74. It has hitherto been considered that the value of the co-efficient C or $\frac{\delta}{H}$ is the same in all the strata of the column; but this is not actually the case, there being several causes which have the effect of varying this ratio. The principal of these is the difference in the temperature of the strata. The elasticity of air is, for example, increased by heat, so that with a less density it may sustain an equal column of mercury,
which, therefore, likewise causes a variation i
$\frac{\delta}{H}$, or of the co-efficient C.

This ratio also varies with the quantity o
vapour that is suspended in the different str
column which weighs less than dry air of
elastic force; so that its presence in the
beds of air, renders them susceptible of sustai
a less density, a column of mercury of equ
Lastly, the decrease of gravity, arising from t
elevation, is another cause of change; for
quence of this decrease, a column of mer
length of which is H, weighs less as it is mo
from the centre of the earth; and, conseque
weighs less, the beds of air into which it is c
less compressed. Thus, the ratio of their
the length of the mercurial column, or $\frac{\delta}{H}$
the same for these beds as for those below t

We propose now to investigate the nume
of the influence of these various causes on t
cient C.

75. In the first place, let g', g'', g''', &c. d
successive decreasing actions of gravity in t
strata of the atmosphere, the weights of the c
mercury as depending upon these actions
noted by H′, H″, H‴, and which are prop
them; consequently, if all other circumsta
the same, the densities of the beds of air com
these columns would also have the same p
Therefore, the ratio $\frac{\delta}{H}$ ought to vary from o
to another, proportionally to the gravity g.

76. If now we consider the influence of th
ture, we shall find, that in consequence of
a mass of air, of which the magnitude is 1
gree of a given scale, will be a certain quanti
at any higher degree of it; for example, a m
magnitude is 1 at 32° of Fahrenheit's the
will be $1 + {\cdot}0020833\,t$ at t degrees above t
point, the barometrical pressure remaining
And the densities of this mass under a cons
sure are reciprocally as the spaces it occupi
quently, if its density at 32° be 1, its den
$+\ t$ degrees, will be

$$\frac{1}{1 + {\cdot}0020833\,t},$$

the pressure remaining the same; therefore,
$\frac{\delta}{H}$ varies in the different beds proportiona
former fraction.

77. The next point to be considered is the
of the aqueous vapour. Now, according t
periments of Saussure and Watt, the weig
vapour is to that of air as 10 to 14, when the
tures and elastic forces are the same; that
the air and vapour having the same tempera
tain equal columns of mercury. Hence, the
of this vapour in the columns of air renders
cifically lighter, without diminishing their ela

In order to obtain an expression for th
this effect, let H denote the barometrical
which a bed of air supports; and call F t
force of the aqueous vapour contained in i

of the whole column of mercury H; let F be the part of it which is due to the elasticity of the aqueous vapour, and consequently H — F that part due to the pressure of the dry air, F and H being both estimated in the same measure. The French estimate both in metres, but it will be more convenient for the reader if we express them in some English measure; as for example, in inches, feet, or yards; and as it is wholly indifferent which, we shall adopt the latter.

The total weight of the column may therefore be considered as composed of two parts; viz. of a certain quantity of vapour, the elastic force of which is F, and of a certain quantity of dry air, of which the elasticity is H — F; let p be the whole weight of the column, supposing it entirely composed of dry air, under the pressure H, then the weight of the same column of dry air, under the pressure H — F will be $p \times \frac{H-F}{H}$; and the weight of the same, under the pressure F, will be $p \times \frac{F}{H}$; so that if this volume, remaining always the same under the pressure F, were composed entirely of aqueous vapour, its weight would be $\frac{10}{14}$ths of the preceding, that is, it would be $p \times \frac{10\,F}{14\,H}$.

Now it has been ascertained from numerous accurate experiments, that in a mixture of vapour and air in a state of permanent equilibrium, the two fluids are uniformly expanded throughout the whole space which they occupy. Thus the weight of the mixture, in the preceding proportions, will be equal to the sum of the weights of air and vapour which occupy the given space under the pressures H — F and F; that is, the weight will be

$$p.\frac{(H-F)}{H} + \frac{10\,p\,F}{14\,H} = \frac{H-\frac{2}{7}F}{H}\,p.$$

But before the introduction of the vapour, the weight of the same volume of dry air, sustaining the same pressure H, was denoted by p; and the density being proportional to the weights, if δ represent the density of a stratum in the dry state, its density in the humid state will be

$$\delta.\frac{H-\frac{2}{7}F}{H} = \delta\left(1-\frac{2}{7}\frac{F}{H}\right),$$

the pressure being the same.

Whence it appears that the existence of aqueous vapour in the beds of air, causes the ratio $\frac{\delta}{H}$, or the co-efficient C, to vary proportionally to $\left(1-\frac{2}{7}\frac{F}{H}\right)$.

78. Introducing now the three sources of variation, or the expressions for the several causes of change above investigated, we shall find for the most general expression of the co-efficient,

$$C = \frac{A\,g\left(l-\frac{2}{7}\frac{F}{H}\right)}{1+\cdot0020833\,t},$$

where A denotes a constant quantity common to all the strata, and t the number of degrees of Fahrenheit's thermometer above the freezing point. There only remains now to substitute in this expression for g F, H and t their respective values, relative to the different beds.

Measurement of heights by the barometer.

79. With respect to the value of g, it is known, that in receding from the centre of the earth, the force of gravity is reciprocally as the square of the distance. Now the distances of the different beds from the centre have been denoted by x', x'', x''', &c.; therefore, calling g', g'', g''' the corresponding intensities of gravity, we shall have

Value of g.

$$g' = g';\; g'' = \frac{g'\,x'^2}{x''^2},\; g''' = \frac{g'\,x''^2}{x'''^2}, \&c.$$

80. The values of g', g'', g''', &c. being thus established, let us next endeavour to determine, in a like general manner, the value of F; this, we may observe, is always very small under the circumstances in which barometrical measurements are conducted. Laplace, in his "Mecanique Celeste," has calculated the values of F for the point of extreme saturation, according to experiments conducted by Dalton, and finds that at

Value of F.

32° of Fahrenheit's therm. F = ·00560142 yards,
86° of the same, F = ·0346562 yards,

and between these limits, which are nearly those of barometrical observation, the increase of F may be represented with sufficient accuracy by

$$F = (\cdot00560142 + \cdot00052547\,t),$$

t denoting still the number of degrees above the freezing point of Fahrenheit's thermometer. This formula, as we have hinted above, is not strictly accurate, but it is sufficiently so for our purpose, its influence being very small on the observed heights. But before it is made applicable to the atmosphere it will still require a further modification.

This relates to the point of extreme saturation, which very rarely takes place in atmospheric air, and consequently the value of F, as determined from the above formula, will be generally in excess. Unfortunately we cannot determine any thing certain relative to the quantity of aqueous vapour suspended in the atmosphere. It is variable, and differs considerably from one time to another; and even from one strata to another there is frequently a considerable variation. But, independent of these extraordinary circumstances, there is great reason to believe that we shall approach most frequently to nature, by avoiding the extremes, which will be best done by assuming for F half the value which answers to the state of maximum humidity; that is, by making

$$F = \cdot00280071 + \cdot0002674\,t. \qquad (a)$$

When we substitute this value in the expression for the co-efficient C, it must be multiplied by the variable factor $\frac{2}{7}\frac{F}{H}$; but in consequence of the smallness of this correction, and the little difference in the values of H, in the extent of the columns of air which are usually measured, it will be sufficient to put for H, the constant value ·831136 yards, answering to the mean pressure at the level of the sea; by which means this term of our correction will be diminished for the upper strata, which is consonant to the observed laws of nature, the humidity of these strata generally decreasing as the heights increase, sometimes the most elevated possessing a great degree of dryness.

The same modified.

81. By adopting this simplification, we shall have

$$1-\frac{2\,\text{F}}{7\,\text{H}}=1-\frac{2}{7\times\cdot 831136}\times$$
$$\{\cdot 00280071+\cdot 00026274\,t\}$$
$$=1-\cdot 0009628-\cdot 0000904\,t.$$

This expression, without sensible error, may be reduced to

$$(1-\cdot 0009628)\,(1-\cdot 0000904\,t);$$

which gives

$$\text{C}=\frac{\text{A}g\,(1-\cdot 0009628)\,(1-\cdot 0000904\,t)}{1+\cdot 0020833\,t}.$$

In this way C contains a constant factor, common to all the strata; and the other factor, depending upon t, which is still found in the numerator, may be united with that which arises from the temperature. For the co-efficient ·0000904 being very small, we may, without any sensible error, substitute

$$\frac{1}{1+\cdot 0000904\,t},\text{ instead of }1-\cdot 0000904\,t\,;$$

in which case we shall have, in the denominator of the product,

$$(1+\cdot 0000904\,t)\,(1+\cdot 0020833\,t).$$

In performing the multiplication, the product

$$(\cdot 0000904\,t)\times(\cdot 0020833\,t)$$

may be neglected: the denominator is, therefore, ultimately reduced to

$$1+\cdot 0021737\,t.$$

We may further change this co-efficient of t into ·0022, without danger of any sensible error; by which means the co-efficient C is more simply expressed as follows, viz.

$$\text{C}=\frac{\text{A}\,(1-\cdot 0009628)\,g}{(1+\cdot 0022\,t)},$$

where, it must still be remembered, that t indicates the number of degrees above the freezing point.

General investigation.

82. Let us now search, according to the above general principle, the ratio of the heights of the barometer, with the elevation of the strata in which it is observed; in order to effect this, we must employ again the same chain of reasoning as in the simple hypothesis investigated in the beginning of this section.

In considering the first stratum, it has been remarked, that a bed of air, the depth or thickness of which is $x''-x'$, weighs as much as a column of mercury of the same base, and of which the height is $\text{H}'-\text{H}''$ (art. 69); whence it has been seen that

$$\frac{\text{H}'-\text{H}''}{x''-x'}$$

is the ratio of the densities of air and mercury in such stratum. The same consideration is also applicable to the actual case, except that, as gravity is supposed now to vary from one stratum to another, the intensity of this force upon the column of the mercury H″, which takes place in the second stratum, is different from that which solicits the column of mercury H′ in the first.

In order to express the weight of the first stratum

in parts of the column of mercury H′, it will be

lumn H″ to what it would be

multiply by $\frac{g''}{g'}$ the ratio of gravity

We shall thus have

$$\text{H}'-\frac{\text{H}''g''}{g'}$$

for the diminution of the barometrical pressure in the extent of the first stratum of air, the thickness of which will still be $x''-x'$, as before. The ratio of the densities of air and mercury in this statum will, therefore, be equal to

$$\frac{\text{H}'-\dfrac{\text{H}''g''}{g'}}{x''-x'}=\frac{\text{H}'\,g'-\text{H}''\,g''}{g'\,(x''-x')};$$

which ratio may still be expressed by C′ H′, representing by C′ the value of the co-efficient C under consideration: then, by equating the two values, and denoting, as in (art. 69), the thickness of the stratum by D, we shall have

$$x^{v}-x'=\text{D};\ \text{C}'\,\text{H}'=\frac{\text{H}'g'-\text{H}''g''}{g'\,(x''-x')};$$

whence we derive

$$\text{H}''\,g''=\text{H}'\,g'\,\{1-\text{C}'\,(x''-x')\}.$$

The passage from the second stratum to the third, from the third to the fourth, and so on, will all give similar equations; viz. we shall have

$$x''-x'=\text{D},\ x'''-x''=\text{D},\ x^{\text{iv}}-x'''=\text{D},\ \text{\&c. and}$$
$$\text{H}''\,g''=\text{H}'\ g'\ \{1-\text{C}'\ (x''-x')\},$$
$$\text{H}'''\,g'''=\text{H}''\ g''\ \{1-\text{C}''\ (x'''-x'')\},$$
$$\text{H}^{\text{iv}}\,g^{\text{iv}}=\text{H}'''\ g'''\ \{1-\text{C}'''\ (x^{\text{iv}}-x''')\},$$
$$\text{\&c.}=\text{\&c.}-\text{\&c.}$$

$$\text{H}^{(n+1)}\,g^{(n+1)}=$$
$$\text{H}^{(n)}\,g^{(n)}\{1-\text{C}^{(n)}(x^{(n+1)}-x^{(n)})\}.$$

83. By performing successively, as in (art. 70), the several eliminations, and substituting D for $x''-x'$, $x'''-x''$, &c. the above nth term becomes

$$\text{H}^{(n+1)}\,g^{(n+1)}=\text{H}'g'(1-\text{C}'\text{D})(1-\text{C}''\text{D})(1-$$
$$\text{C}'''\ \text{D})\ \text{\&c.}\ (1-\text{C}^{(n)}\text{D}).$$

We have here, in the second member of this equation, as many factors as there are strata, the same as in (art. 70), but in that case they are all equal to each other, whereas in the one at present under consideration they are all different, on account of the variable co-efficient C.

If we put the last equation into logarithms, we shall have, after dividing by H′ g',

$$\log.\frac{\text{H}^{(n+1)}}{\text{H}'}+\log.\frac{g^{(n+1)}}{g'}=\log.\,(1-\text{C}'\text{D})+$$
$$\log.(1-\text{C}''\text{D})+\log.(1-\text{C}'''\text{D})+\log.(1-\text{C}^{\text{iv}}\text{D})+\text{\&c.}$$
$$\text{Or, }\log.\frac{\text{H}'}{\text{H}^{(n+1)}}+\log.\frac{g'}{g^{(n+1)}}=-\log.\,(1-\text{C}'\text{D})$$
$$-\log.(1-\text{C}''\text{D})-\log.(1-\text{C}'''\text{D})-\log.(1-\text{C}^{\text{iv}}\text{D})-\text{\&c}$$

In this series, as we have no explicit value of n, we cannot pursue the same principles of elimination as in (art. 71), but we may still follow the same method calculation; and by taking the values of $x^{(n+1)}$ —

heights by the baromet

$$x^{(n+1)} - x' = \frac{-nD\left\{\log.\frac{H'}{H^{(n+1)}} + \log.\frac{g'}{g^{(n+1)}}\right\}}{\log.(1-C'D) + \log.(1-C''D) +, \&c. \log.(1-C^{(n)}D)}$$

Now, by developing the logarithms, dividing by D, and then supposing D to be indefinitely small, or equal to zero, precisely the same as in (art. 71), we shall obtain

$$x^{(n+1)} - x' = Mn\frac{\log.\frac{H'}{H^{n+1}} + \log.\frac{g'}{g^{n+1}}}{C' + C'' + C''' +, \&c.\ C^{(n)}},$$

M being the modulus of the common system of logarithms.

84. This formula is analogous to that in (art. 71); but here we have the sum of all the C's in the denominator, and they are all different from each other; while in the formula referred to, they are all equal, and their sum equal to *n* C. This *n*, and that in the numerator, cancel each other, and, consequently, that letter disappears in the former equation, but is necessarily retained in the above.

Let us, for the sake of simplifying, denote the sum of all the C's by ΣC′; we may also denote the altitude, or the difference $x^{(n+1)} - x'$ by X, as before; and, moreover, substitute *h* for $H^{(n+1)}$, and H for H′: by this means the above expression is reduced to

$$X = Mn\frac{\log.\frac{H}{h} + \log.\frac{g'}{g^{(n+1)}}}{\Sigma C'}.$$

85. It remains now to find the value of the fraction $\frac{g'}{g^{(n+1)}}$, which is the ratio of gravity in the two stations. Here, since the force of gravity is reciprocally as the square of the distance from the centre of the earth, we shall have

$$\frac{g'}{g^{(n+1)}} = \frac{(x^{(n+1)})^2}{x'^2};$$

or, since $x^{(n+1)} = x' + X$, we have

$$\log.\frac{g'}{g^{(n+1)}} = 2\log.\frac{x^{n+1}}{x'} = 2\log.\left(1 + \frac{X}{x'}\right).$$

Now *x′* being the distance of the lower station from the centre of the earth, we may, without any sensible error, assume it equal to the mean radius of the earth, viz. = 6962074 yards, and denoting it by *a*, we shall have, with sufficient accuracy,

$$\log.\frac{g'}{g^{(n+1)}} = 2\log.\left(1 + \frac{X}{a}\right);$$

consequently,

$$X = \frac{Mn\left\{\log.\frac{H}{h} + 2\log.\left(1 + \frac{X}{a}\right)\right.}{\Sigma C'}.$$

86. Before we investigate the value of ΣC′, we may apply to each of the co-efficients C′, C″, C‴, &c. the correction respecting the variation of gravity for different latitudes, which, as we have seen (art. 73), consists in multiplying each of them by the factor (1 − ·002837 cos 2 ψ), ψ denoting the latitude; and as this factor is common to all the co-efficients, because all the points of the column of air are situated in the same vertical, and may therefore be considered as in the same latitude; whence the corrected sum will be, for any latitude, ψ

Measurement of heights by the barometer.

$$(1 - \cdot002837 \cos 2\psi)\ \Sigma C';$$

and by transferring this correction into the numerator, by developing it into a series, as in (art. 73), we shall have $X = \frac{Mn(1 + \cdot002837 \cos 2\psi)}{\Sigma C'} \times$

$$\left\{\log.\frac{H}{h} + 2\log.\left(1 + \frac{X}{a}\right)\right\}.$$

87. In order now to find the value of ΣC′, it must be observed, that from the general expression for the co-efficient C (art. 81), it is evident that

Value of ΣC′ determined.

$$\Sigma C' = A(1 - \cdot00009628) \times$$

$$\left\{\frac{g'}{1 + \cdot0022t} + \frac{g''}{1 + \cdot0022t} + \frac{g'''}{1 + \cdot0022t} +, \&c.\right\}$$

it is, therefore, the sum of this latter series which is required; and to accomplish this summation in a perfect manner, it would be necessary to know the law of the decrease, which is subject to many irregularities; but, generally, at small heights, such as those at which barometrical observations are made, it is a very slowly decreasing arithmetical progression; we shall therefore deviate but little from the truth, if we assume all the temperatures equal to each other, and to the mean temperature between the two extreme strata, that is, by assuming the mean temperature $= \frac{t' + t^{(n+1)}}{2}$.

This supposition will increase the temperatures of the upper strata, while it diminishes those in the lower ones, and thus produce a sort of compensation, which, as we have said above, cannot be far from the truth.

By this means the factor, which is dependent upon the temperature, becomes common to all the strata, and the above expression for the sum ΣC′ becomes

$$\Sigma C' = \frac{A(1 - 0009628)}{1 + \left(\frac{T+t}{2}\right)\cdot0022} \times$$

$$g'\left\{1 + \frac{g''}{g'} + \frac{g'''}{g'} + \frac{g^{iv}}{g'} +, \&c.\right\},$$

in which *t′* of the former expression is here denoted by T, and $t^{(n+1)}$ by *t*, each of these denoting the number of degrees above freezing; a notation analagous to that employed for the mercurial columns H and *h*.

Again, we have seen (art. 79) that

$$\frac{g''}{g'} = \frac{x'^2}{x''^2},\ \frac{g'''}{g'} = \frac{x'^2}{x'''^2}, \&c.$$

Or, since the difference between any two consecutive strata is D, we may write

$$\frac{g''}{g'} = \frac{x'^2}{(x'+D)^2};\ \frac{g'''}{g''} = \frac{x'^2}{(x'+2D)^2};\ \frac{g^{iv}}{g'''} = \frac{x'^2}{(x'+3D)^2}, \&c,$$

The second members of these equations may be further reduced by division, and retaining in each resulting series only the first two terms, which will be sufficiently exact for the purpose we have in view, there will result

$$\frac{g''}{g'} = \frac{x'^2}{(x' + D)^2} = 1 - \frac{2\,D}{x'},$$
$$\frac{g'''}{g'} = \frac{x'^2}{(x' + 2D)^2} = 1 - \frac{4\,D}{x'},$$
$$\frac{g^{iv}}{g'} = \frac{x'^2}{(x' + 3D)^2} = 1 - \frac{6\,D}{x'},$$
$$\text{\&c.} = \quad \text{\&c.} \quad = \quad \text{\&c.};$$

whence the required sum

$$\frac{g''}{g'} + \frac{g'''}{g'} + \frac{g^{iv}}{g'} +, \text{\&c.}\ \frac{g^{(n+1)}}{g'}$$

becomes

$$n - \frac{2}{x'}(D + 2\,D + 3\,D +, \text{\&c.}\ n\,D).$$

The part included between the parenthesis obviously constitutes an arithmetical progression, of which the sum is

$$\frac{(1 + n)\,n\,D}{2};$$

the term n D, being evidently the difference of the level of the two stations, is therefore equal to X; consequently,

$$\frac{g''}{g} + \frac{g'''}{g'} + \frac{g^{iv}}{g'} +, \text{\&c.} = n - \frac{2}{x'}\left(\frac{X(n+1)}{2}\right) = n\left\{1 - \left(1 + \frac{1}{n}\right)\frac{X}{x'}\right\}.$$

We see here again the factor n, which remained in the numerator of the expression for the difference of level. By substituting this value in the expression for $\Sigma C'$, it will be sufficient to take, as in (art. 85), instead of x', the mean radius of the earth; moreover, as the number of strata is, by the supposition, indefinitely great, we may neglect the term in which $\frac{1}{n}$ enters: in fact, as we ultimately make $D = o$, n must necessarily be infinite; we shall thus have

$$\Sigma C' = \frac{A\,(1 - \cdot0009628)\,g'\,n\left(1 - \frac{X}{a}\right)}{1 + \left(\frac{T + t}{2}\right)\cdot0022}.$$

Or, since $\cdot0022 = \frac{1}{900}$ nearly, we may write

$$\Sigma C' = \frac{A\,(1 - \cdot0009628)\,g'\,n\left(1 - \frac{X}{a}\right)}{1 + \frac{T + t}{900}}.$$

General equation for the altitude.

88. The value of $\Sigma C'$ being thus determined, it may be substituted into the general value of X (art.), and we shall find that the letter n in the numerator and denominator will disappear, being common to both, consequently,

$$X = \frac{M\,(1 + \cdot002837\cos 2\,\psi)}{A\,g'\,(1 - \cdot0009628)\left(1 - \frac{X}{a}\right)} \times \left\{1 + \frac{T + t}{900}\right\} \times \left\{\log.\frac{H}{h} + 2\log.\left(1 + \frac{X}{a}\right)\right\}.$$

The factor $1 - \frac{X}{a}$, found in the denominator of our first fraction, may be transferred into the numerator by division; that is, by writing

$$\frac{1}{1 - \frac{X}{a}} = 1 + \frac{X}{a} +, \text{\&c.}$$

and retaining only the first two terms, c power only of $\frac{X}{a}$, which will be sufficiently cases; by this means we obtain

$$X = \frac{M}{A\,g'(1 - \cdot0009628)} \times \Big\{ (+ \cdot002837$$
$$\left(1 + \frac{T+t}{900}\right) \times \left(\log.\frac{H}{h} + 2\log.\left(1 + \frac{X}{a}\right)\right)\Big($$

This equation contains X in both its men therefore appears not perfectly resolved; bu remarked that X, in the second member, is a, which is always extremely great with res there is, therefore, no necessity in calcula terms to know X very accurately, but only n may, therefore, divide the calculation into first finding X approximatively by neglec terms in which it enters on the second side introducing this approximate value in order actual value of it; at least this second ope bring us to a result differing extremely littl truth.

89. In order that we may be able to formula which has been above obtained, it determine the constant co-efficient A; and b to (art. 78), where it was first introduced, that by calling δ the ratio of the densities and mercury under the pressure H, and th ture t, in a place whose latitude is ψ, and t g, we have generally

$$\delta = \frac{A\,(1 - \cdot002837\cos 2\,\psi)\,g\,H}{1 + \cdot0020833\,t}$$

The most simple method of determining of weighing very exactly known quantities mercury under a determinate pressure and te and at a place the latitude and height of known.

These experiments have been performe with great accuracy by Biot and Arago; fr sult of which it appears, that at the tem melting ice, and under a pressure of ·76 ·831136 English yard, or 29·9209 inches,

$$\delta = \frac{1}{10463};$$

whence

$$A = \frac{1}{10463\,g\,(1 - 002837\cos 2\,\psi)\cdot8}$$

ψ here denoting the latitude of Paris; cons the modulus of the common system of loga 2·30258509, be denoted by M, the co-effic barometrical formula, or $\frac{M}{A\,g'}$, will becom

$$\frac{M}{A\,g'} = 10463(1 - \cdot002837\cos 2\,\psi) \times \cdot83$$

In this formula ψ, or the latitude of the c at Paris is 48° 50′ 14″. The cosine of t expressed numerically, and introduced int ceding equation, gives us

$$\frac{M}{A\,g'} = 20031\cdot27435\,\frac{g}{g'};$$

and, consequently,

$$\frac{M}{A\,g'\,(1-{\cdot}0009628)} = 20050{\cdot}5654\,\frac{g}{g'}.$$

Let r represent the height of the lower station above the level of the sea, then $a+r$ will be its distance from the centre of the earth (a denoting the radius). Now M. Biot estimates the height of the Paris observatory as 65·6 yards above the level of the sea, and, consequently, its distance from the centre of the earth is $a+65{\cdot}6$, or $a+66$, and therefore the ratio of the gravities is very nearly

$$\frac{g}{g'} = \frac{(a+r)^2}{(a+66)^2},$$

which is reducible to

$$\left(1-\frac{131{\cdot}2}{a}\right)\left(1+\frac{2r}{a}\right),$$

by developing the two squares and limiting ourselves to the first powers of $\frac{66}{a}$ and $\frac{r}{a}$.

The first factor, $1-\frac{131{\cdot}2}{a}$, may be converted into numbers, by taking $a = 6962074$ yards, the number already assumed; this will reduce the co-efficient to

$$\frac{M}{A\,g'} = 20050{\cdot}1826\left(1+\frac{2r}{a}\right).$$

90. This co-efficient may be otherwise determined, *à priori*, by comparing observations of the barometer with differences of level measured trigonometrically. A great number of very accurate observations, made by M. Ramond, after this manner, have given 20052·2496 for the value of the co-efficient, which M. Biot and M. Arago found, from the weights of air and mercury, to be 20050·1826; an agreement which proves, in the most satisfactory manner, the accuracy of the formula and that of the data on which it is founded.

91. There might even be obtained, from this agreement, a confirmation of the decrease of gravity in a vertical line; for if that decrease were disregarded, the barometrical observations of M. Ramond would give 20114·5848 for the co-efficient, instead of 20052·2496, or 20050·1826, as determined from the comparative weights of air and mercury. This difference cannot be attributed to the value which we assigned to the humidity of the air, that value being rather too great than too little; besides, even the difference would not disappear by supposing the air in a state of extreme humidity, because this supposition, by doubling the correction, which has already been made with this view, would only cause 19·2911 to be added to 20050·1826, which would be only 20069·4737, still much inferior to 20114·5848. It must, therefore, be acknowledged, that the decrease of gravity, though but inconsiderable in the limits within which barometrical observations are made, becomes, notwithstanding, very sensible; and the agreements of the results, when this decrease is taken into account, demonstrates its reality.

92. The inequality of temperature in the highest and lowest beds of a column of air which is measured, is communicated to the barometer employed, and requires a reduction on this account of the observed heights. In fact, mercury, like other bodies, is condensed by cold and expanded by heat. This variation from the freezing point to 212° of Fahrenheit's thermometer, is uniform, according to the experiments of Gay-Lussac, being equal to $\frac{1}{9742}$ for each degree of that thermometer, conformably to the deductions of MM. Lavoisier and Laplace, which agree also with certain experiments published in the Philosophical Transactions.

Measurement of heights by the barometer.

Thus, when the height of the barometer is observed in the colder station, the column of mercury which is condensed ought to appear a little shorter than if it had been observed at the warmer, which is generally the lower. To bring those heights to the same terms, the length of the mercurial column at the upper, or, at least, at the colder station, must be increased in consequence of the different temperatures of the mercury, and proportionally to the condensation which ought to result from it; that is, if the observed length be h', there must be taken

$$h = h'\left(1+\frac{T-t}{9742}\right),$$

T and t denoting the number of degrees above freezing.

Attached thermometer.

93. This, again, is upon a supposition that the mercury of the barometer has the same temperature as the surrounding air, but this is not always the case, the temperature of the mercury and the surrounding medium being frequently very different. If at each station where this circumstance is met with, we wished to wait till the barometer had acquired the same temperature as the ambient air, we should be obliged to wait several hours before we should be able to make the observation, these changes taking place very slowly; therefore, to avoid this inconvenience, the temperature of the mercury in the barometer is measured by means of a small thermometer adapted to the former instrument, and the degree of heat indicated by this thermometer at the two stations, is that which must be used in reducing the barometrical columns to the same temperature. This is called the *attached thermometer*.

Let us, for example, suppose that it marks T° (degrees) at the lower station, and $t°$ at the upper; also, that the observed length of the mercurial column observed at this last station is h', then we shall have to make

$$h = h'\left(1+\frac{T-t}{9742}\right).$$

General resulting formula.

94. Whence, resuming the foregoing considerations, the definite formula for the measure of heights, by barometrical observations, according to the experiments above indicated, will be

$$X = 20050{\cdot}1826\,(1+{\cdot}002837\cos 2\psi)\times\left\{\left(1+\frac{2r}{a}\right)\left(1+\frac{T+t}{900}\right)\left(1+\frac{X}{a}\right)\left(\log.\frac{H}{h}+2\log.\left(1+\frac{X}{a}\right)\right)\right\}.$$

Where ψ is the latitude of the place, H and h the height of the barometrical columns at the upper and lower stations; T and t, the number of degrees of Fahrenheit's scale above 32°, that is, above the freezing point; r the height of the lower station above the level of the sea; and a the mean radius of the earth expressed in the same linear measure as r. The co-efficient 20050·1826 gives the result in English yards; we may, therefore, express a and r also in yards; if we were to employ 60150·5478. we should have our result in feet, and a and r might then also be taken in feet; this is, however, perfectly optional, all that is necessary is, that r and a, in the second factor, be taken in the same measure, and X and a in the third and fourth.

Practical remarks.

95. By means of the preceding formula, the difference of level of any two places may be very accurately determined from barometrical observations; but these observations must be made with great care, and with good instruments, without which great errors may be committed. Calm weather, and about the middle of the day should be chosen for the observations as often as possible. One observer should be fixed at each station, with instruments previously compared. Each of them should observe, at the time appointed, the height of the barometer; noting at the same instant the state of the thermometer attached to the barometer, in order to have the temperature of the mercury; observing also that of a very sensible detached thermometer in the shade, for the purpose of ascertaining the temperature of the air. These observations should be repeated every quarter of an hour (the watches having been well regulated by each other), until a certain number of observations have been obtained; that is, not less than ten or twelve. Then the two observers meet and compare again their instruments, in order to ascertain whether they have experienced any change from accident or otherwise. If they are found to correspond accurately, the mean of the observations made at each station is to be taken, and the difference of the level calculated with these means. If the operation have been performed with all the precautions above described, the result will only be susceptible of very small errors, arising from accidental irregularities of pressure and temperature of the atmospheric strata; errors which may be made to disappear by their reciprocal compensation, by repeating the observations on different days, and taking an arithmetical mean between all the results. By thus uniting five or six series of corresponding observations, made with good thermometers, and with a barometer, mounted with a nonius, which gives at least $\frac{1}{250}$th of an inch, we may answer for our results to two or three yards in the greatest heights.

If, by a long series of observations, made at the same place, the mean height of the barometer and the mean temperature of the atmosphere be determined, the formula will give the height of that place above the level of the sea, or any other determinate point. For this we must also have the mean height of the barometer and the thermometer at this second point, and with these mean results calculate by the formula in the same manner as we should do for two stations where we had corresponding observations. This supposes that the mean temperature at the surface of the earth always remains constant, as well as the height of the barometer at each place. It is possible that these elements may experience some variations; but the invention of the barometer and thermometer being of too modern a date to determine this, we may, at least without sensible errors, regard them as constant during an interval of some years.

For performing the calculation, the mean height of the barometer at the level of the sea must be known. According to the experiments of Sir George Shuckburgh, which are regarded as very accurate, it is 30·035 inches, in the latitude of 50°, the mean temperature of the air being 55° of Fahrenheit's meter, or 12°·2 of the centigrade. These dat therefore considered as established, when th pressure and temperature are known from r observation for any given place, the height of th above the level of the sea becomes determinal may be given as a third co-ordinate in fixing th situation of it with reference to the centre of th The latitude and longitude only define the ve which a place is situated, but the introductior above element fixes its absolute position, and i fore, a great desideratum in physical geograph

96. In order to facilitate, as far as possible, complishment of this object, M. Biot, in the of his "Physical Astronomy," has reduced ceding equation for the height into the form of whereby the difference in the level of two plac a slight reduction of the barometrical pressur station, may be found by simple inspection.

We propose to exhibit to our readers a sh cimen of this table, after first illustrating the p on which the reduction is founded.

Our resulting formula (art. 94) may be simp writing

$$N = 20050·1826 \left(1 + \frac{2r}{a}\right)\{1 + ·002837 \text{ cc}$$

$$\left(1 + \frac{T+t}{900}\right);$$

and then

$$X = N \left\{ \log. \frac{H}{h} + 2 \log. \left(1 + \frac{X}{a}\right) \right\} \times \left(1 +\right.$$

According to this formula, the number answ the height will be in yards; and T and t, the t tures, are supposed to be those of Fahrenheit's 32°, the actual degrees shown by the instrumen 32° + T and 32° + t; but if we, after the m of M. Biot, obtain the answer in metres, and our temperatures by the centigrade thermome must change our co-efficient 20050·1826 into and our divisor of T + t from 900 to $\frac{2}{1000}$; adopt these measures, and therefore write

$$N = 18334\left(1 + \frac{r}{a}\right)\{1 + ·002837 \cos 2\psi\}$$

$$\left(1 + \frac{2(T+t)}{1000}\right);$$

and

$$X = N \left\{ \log. \frac{H}{h} + 2 \log. \left(1 + \frac{X}{a}\right) \right\} \times \left(1 +\right.$$

in which T and t are now the actual number of indicated by the centigrade thermometer.

As X is contained in each member of the equation, our first object must be to transfer th to one side only, which may be done by expan $\log. \left(1 + \frac{X}{a}\right)$, and limiting the series to term only, which is $\frac{X}{Ma}$; M being the modulu system; this substitution being made, and th multiplication performed, with the other factor $\Big($

we have

$$X = N \log. \frac{H}{h} + \frac{N}{a} \left\{ \log. \frac{H}{h} + \frac{2}{M} \right\}$$

whence

$$X = \frac{N \log. \frac{H}{h}}{1 - \frac{N}{a}\left(\log. \frac{H}{h} + \frac{2}{M}\right)}.$$

The denominator of the latter fraction is nearly equal to unity; the co-efficient $\frac{N}{a}$ being itself a very small fraction, never exceeding $\frac{1}{347}$, and the other factor will never exceed unity within the limits in which observations are usually made; we know, for example, that $\frac{2}{M} = \cdot 868583$, and the other part, $\log. \frac{H}{h}$, which is variable, will not exceed ·1026623, even for an altitude of 2000 metres; that is, for the values $H = \cdot 76$, and $h = \cdot 600$. If, therefore, we were to neglect this term entirely, it would produce no very sensible error, but it may be better to preserve it by giving to it the mean value derived from the preceding calculation; for the error which may result from it when h is less than ·600, will always be extremely small, and that which may arise when h is greater, will be diminished by the smallness of $\log. \frac{H}{h}$ in the numerator. By this means the second term in the denominator becomes constant; for being so small we may calculate it with the constant part of the co-efficient N, and then its value (by using $\frac{N}{a} = \frac{1}{347}$) becomes

$$\frac{18334 \times \cdot 9712453}{6366198} = \cdot 0028061;$$

the denominator of our fraction is, therofore,

$$1 - \cdot 0028061,$$

which being introduced, the expression becomes

$$X = N(1 + \cdot 0028061) \log. \frac{H}{h}.$$

97. With respect to the value of N, it will be necessary to assume a mean value for r, which M. Biot thinks may be called 1200 metres, this being, perhaps, nearly the mean height above the level of the sea, at which travellers most commonly make their observations in our climates. The introduction of this number gives

$$X = 18393(1 + \cdot 002837 \cos 2\psi)\left(1 + \frac{2(T+t)}{1000}\right) \log. \frac{H}{h}.$$

This formula has all the exactitude which we can hope to attain to in barometrical measurements. When compared with the rigorous formula above given, it appears that an error of little more than 4 yards will be made in the height of Chimbaraço, which is about 6500 yards above the level of the ocean.

98. There is, however, still in this formula a factor which is inconvenient for reducing it into a tabular form, viz. that part of it which relates to the latitude of the place; but here we may observe, that this formula may be separated into two parts, by writing

$$X = 18393\left(1 + \frac{2(T+t)}{1000} \log. \frac{H}{h}\right) +$$

$$18393 \times \cdot 002837 \cos 2\psi \left(1 + \frac{2(T+t)}{1000}\right) \log. \frac{H}{h};$$

of which the latter will always be to the former as

$$\cdot 002837 \cos 2\psi : 1.$$

This proportion may therefore be computed to all the varieties of latitudes, and the corresponding correction made in the first member; this latter only, therefore, need be reduced into a tabulated form. In the latitude of 45°, $\cos 2\psi = 0$, and the fraction vanishes; when ψ 7 45°, the $\cos 2\psi$ is negative, and the fraction must therefore be subducted from the height; but when it is less than 45°, $2 \cos \psi$ is positive, and the proportional part must be made addative.

Measurement of heights by the barometer.

Table of correction for the latitude.

99. The following table shows the influence of the latitude for every 5°, the fraction indicated being the proportional part of the calculated height which must be added to, or subtracted from, the result given either by the tables, or by actual logarithmic computation:

Latitude.	Correction.
0°	$+ \frac{1}{352}$ of calculated height.
5°	$+ \frac{1}{358}$.
10°	$+ \frac{1}{375}$.
15°	$+ \frac{1}{407}$.
20°	$+ \frac{1}{460}$.
25°	$+ \frac{1}{548}$.
30°	$+ \frac{1}{705}$.
35°	$+ \frac{1}{1030}$.
40°	$+ \frac{1}{2030}$.
45°	0.
50°	$- \frac{1}{2030}$.
55°	$- \frac{1}{1030}$.
60°	$- \frac{1}{705}$.
65°	$- \frac{1}{548}$.
70°	$- \frac{1}{460}$.
75°	$- \frac{1}{407}$.
80°	$- \frac{1}{375}$.
85°	$- \frac{1}{358}$.
90°	$- \frac{1}{352}$.

Final reduction.

100. This correction being thus provided for, we have only to attend to that part which is independent of the latitude, that is to

$$X = 18393\left(1 + \frac{2(T+t)}{1000}\right) \log. \frac{H}{h}.$$

This again may be written

$$X = 18393\left(1 + \frac{2(T+t)}{1000}\right)(\log. H - \log. h.);$$

$$\text{or } X = 18393\left(1 + \frac{2(T+t)}{1000}\right) \log. H$$
$$- 18393\left(1 + \frac{2(T+t)}{1000}\right) \log. h.$$

Or the same may be otherwise expressed by

$$X = 18393\left(1 + \frac{2(T+t)}{1000}\right) \log. \frac{\cdot 76}{h}$$
$$+ 18393\left(1 + \frac{2(T+t)}{1000}\right) \log. \frac{\cdot 76}{H}.$$

Where the two terms being now of the same form, may be both given in one table, and the sum of the two, or their difference, when H exceeds ·76, will be the height sought. In the following table the value of $T + t$ is computed for every second degree, from 12° to 42°, and the value of H from ·765 to 600.

We have therefore only to look out the values of H, and h in the first vertical column of the table, and that of $T + t$ in the upper horizontal line, then the sum or differences of the two resulting tabular values will be the height sought.

101. There still remains one other correction, which does not, however, in any respect affect the form of

our general equation; this is for the attached thermometer, and it must be made before we attempt to enter the table, or in any other manner to make our calculation.

Correction for the attached thermometer.

Mercury expands $\frac{1}{5412}$ part for every degree of the centigrade thermometer, or $\frac{1}{9742}$ part for every degree of Fahrenheit's: we must, therefore, in the first instance, reduce our barometrical columns to the same temperature, by correcting the height of the upper column by a quantity equal to

$$\frac{T - t}{5412} \text{ for the centigrade; or}$$

$$\frac{T - t}{9742} \text{ for Fahrenheit's.}$$

These are generally addative, the upper station being, in most cases, the lowest temperature; if the contrary should ever occur, the correction is subtractive.

One example will be amply sufficient to illustrate the use and application of the table.

Example of the use of the table.

102. Let us suppose the following observations to have been made:

Latitude 50°.

	Height of barometer.	Detached thermometer.	Attached thermometer.
Lower station	·75 metre	18°	18°
Upper station	·68 metre	8°	8°

The difference of the attached thermometer being 10°, we have $\frac{\cdot 68 \times 10}{5412} = \cdot 00125$ addative for the correction of the barometrical column at the upper station; it therefore becomes ·68125; whence

$$T + t = 26°, \; H = \cdot 75, \text{ and } h = \cdot 68125.$$

By the table $\left\{\begin{array}{l} 26°, \text{when } h = \cdot 681, \text{ gives } 922\cdot 3 \text{ metres.} \\ 26°, \text{when } H = \cdot 75, \text{ gives } 111\cdot 3 \text{ metres.} \end{array}\right.$

$$\begin{array}{lr} \text{Difference} & 811\cdot 0 \\ \text{Correct for latitude } \frac{1}{2030} & \cdot 4 \textit{ nearly.} \\ \hline \text{The height sought} & 810\cdot 6 \text{ metres.} \end{array}$$

Remark.

103. It is obvious, from this example, that the tables are well calculated to facilitate the computation of the difference of level between any two stations, at least while the heights of the barometrical columns are found within its limits; and even when they are beyond it, a certain modification may be made to render them still applicable; but as the logarithmic operation is not very difficult, and as the table itself includes all those heights which are likely to become an object of measurement in this country, we do not think it necessary to enter into any further explanation, particularly as the table is not given in English measures.

We shall conclude this article by a reduction of the preceding formula to English feet and inches, showing the logarithmic operation in one or two examples.

Formula reduced to English measure.

104. The formula in French metres, answering to the centigrade thermometer, abstracting from that part which depends upon the latitude, is

$$18393 \left(1 + \frac{2\,(T + t)}{1000}\right) \log. \frac{H}{h}.$$

In this formula $\frac{H}{h}$ may be taken in any measure; let us suppose English inches, and the co-efficient 18393 will give the answer in metres; in order, therefore, to reduce the result to feet, we must multiply it by the number of feet in a metre, namely, by 3·281, whence it will become 60347. Again, if T and t be taken according to the scale of Fahrenheit, since the number of degrees of this scale are to those of the centigrade as 9 to 5, within any proposed limits, we much change $\frac{2}{1000}$ into $\frac{1}{900}$; whence we shall have, in English feet,

$$X = 60347 \left(1 + \frac{T + t}{900}\right) \log. \frac{H}{h};$$

in which expression it is to be observed that T and t denote the number of degrees indicated above the freezing point of Fahrenheit's thermometer. It is to be observed also, that the correction for the barometrical column in degrees of Fahrenheit's scale will be expressed by $\frac{T - t}{9742}$, instead of $\frac{T - t}{5412}$.

105. Let us conclude this article by the solution of an example in which the data are given in English measures.

Humboldt made observations on the mountain of Quindiù, in the kingdom of New Granada, which, when reduced to English measure, were as follow:

Latitude 5°.

Observations.	Height of barometer.	Detached thermometer.	Attached thermometer.
Upper station	2·0713 in.	65° ·75	68°
At the level of the Pacific ocean at the same time	30·036	77° ·54	77° ·54

First, the difference of attached thermometers is 9° ·54, whence $\frac{9\cdot 54}{9742} = \cdot 00098$ (addative), therefore

$$\left.\begin{array}{l} h = 20\cdot 07228 \\ H = 30\cdot 036 \end{array}\right\} \begin{array}{l} t = 77°\cdot 54 - 32° = 45°\cdot 54 \\ T = 65°\cdot 75 - 32° = \underline{33°\cdot 75} \end{array}$$

$$T + t = 79°\cdot 29$$

$$\text{and} \quad 1 + \frac{T + t}{900} = 1 + \frac{79\cdot 29}{900} = 1\cdot 0881.$$

Again,

$$\begin{array}{llcl} \log.\ 30\cdot 036 & = 1\cdot 4776421 & & \\ \log.\ 20\cdot 07228 & = \underline{1\cdot 3025966} & & \\ \quad \log. \text{ of} & \quad \cdot 1750495 & = & -\ 1\cdot 2431373 \\ \quad \log. \text{ of} & \quad 60347 & = & 4\cdot 7806557 \\ \quad \log. \text{ of} & \quad 1\cdot 0881 & = & \underline{0\cdot 0366688} \\ \text{Height} & \quad 11492 & & 4\cdot 0604618 \\ \left.\begin{array}{l}\text{Correction} \\ \text{for latitude}\end{array}\right\} & \quad \underline{32} & & \\ & \quad 11524 & \text{feet correct altitude.} & \end{array}$$

It may not be amiss to observe here that some little variation will appear in the results of different solutions when reduced to our measures, according to the ratio which is assumed between the English foot and the French metre: we have used the ratio of 1 : 3281, as given in Barlow's "Mathematical Tables." The more recent determination of Captain Kater is 1 : 3·2809.

TABLE *for facilitating the computations of heights by barometrical measurement.*

Height of the barometer in metres.	Sum of the temperatures of the air at the two stations, or the values of T + t in degrees of the centigrade thermometer.																Difference for 1°.
	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	
765	53·6	53·8	54·0	54·2	55·5	54·7	54·9	55·1	55·3	55·5	55·7	55·9	56·1	56·3	56·5	56·7	0·10
764	42·9	43·1	43·2	43·4	43·6	43·8	43·9	44·1	44·3	44·4	44·6	44·7	44·9	45·0	45·2	45·4	0·08
763	32·2	32·4	32·5	32·6	32·7	32·9	33·0	33·1	33·2	33·4	33·5	33·6	33·7	33·8	33·9	34·0	0·06
762	21·5	21·6	21·7	21·7	21·8	21·9	22·0	22·1	22·2	22·2	22·3	22·4	22·4	22·5	22·6	22·7	0·04
761	10·8	10·8	10·8	10·9	10·9	11·0	11·0	11·0	11·1	11·1	11·2	11·2	11·2	11·3	11·3	11·3	0·02
760	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0·00
759	10·8	10·8	10·9	10·9	10·9	11·0	11·0	11·1	11·1	11·1	11·2	11·2	11·2	11·3	11·3	11·3	0·02
758	21·6	21·7	21·7	21·8	21·9	22·0	22·0	22·1	22·2	22·3	22·4	22·4	22·4	22·5	22·6	22·7	0·04
757	32·4	32·4	32·6	32·7	32·8	33·0	33·0	33·1	33·3	33·4	33·5	33·6	33·7	33·8	33·9	34·0	0·06
756	43·3	43·4	43·6	43·7	43·8	44·0	44·2	44·3	44·5	44·7	44·8	44·8	44·9	45·0	45·2	45·4	0·08
755	54·0	54·2	54·4	54·5	54·7	54·9	55·0	55·2	55·4	55·5	55·7	55·9	56·1	56·3	56·5	56·7	0·10
754	64·8	65·1	65·4	65·6	65·8	66·0	66·2	66·4	66·6	66·9	67·2	67·4	67·6	67·9	68·1	68·3	0·12
753	75·7	76·0	76·3	76·6	76·9	77·2	77·5	77·8	78·1	78·4	78·6	78·8	79·1	79·3	79·7	79 9	0·14
752	86·6	86·9	87·2	87·6	87·9	88·2	88·6	88·9	89·2	89·6	89·9	90·2	90·5	90·9	91·2	91·6	0·16
751	97·4	97·8	98·2	98·4	99·0	99·3	99·7	100·1	100·5	101·0	101·2	101·6	102·0	102·4	102·8	103·2	0·18
750	108·3	108·8	109·2	109·6	110·0	110·5	110·9	111·3	111·7	112·1	112·6	113·0	113·5	113·9	114·4	114·8	0·21
749	119·3	119·7	120·2	120·7	121·1	121·6	122·1	122·5	123·0	123·4	123·9	124·5	125·0	125·4	126·0	126·4	0·23
748	130·2	130·7	131·2	131·7	132·2	132·7	133·2	133·7	134·2	134·7	135·3	135·9	136·5	137·0	137·5	138 0	0·25
747	141·1	141·7	142·2	142·8	143·3	143·9	144·4	145·0	145·5	146·1	146·6	147·4	148·0	148·5	149·1	149·7	0·28
746	152·1	152·7	153·3	153·9	154·5	155·1	155·6	156·2	156·8	157·4	158·0	158·8	159·5	160·1	160·7	161·3	0·30
745	163·1	163·7	164·3	165·0	165·6	166·2	166·9	167·5	168·1	168·8	169·4	170·3	171·0	171·6	172·3	172·9	0·32
744	174·0	174·7	175·4	176·1	176·7	177·4	178·1	178·8	179·5	180·1	180·8	181·8	182·4	183·1	183·8	184·5	0·34
743	185·1	185·8	186·5	187·2	187·9	188·6	189·3	190·1	190·8	191·5	192·2	193·2	193·9	194·7	195·4	196·1	0·37
742	196·1	196·8	197·6	198·4	199·1	199·9	200·7	201·4	202·2	202·9	203·7	204·7	205·4	206·2	207·0	207·8	0·39
741	207·1	207·9	208·7	209·5	210·3	211·1	211·9	212·7	213·6	214·4	215·2	216·1	216·9	217·8	218·5	219·4	0·41
740	218·1	219·0	219·9	220·7	221·5	222·4	223·2	224·1	224·9	225·8	226·6	227·6	228·4	229·3	230·1	231·0	0·43
739	229·2	230·1	231·0	231·9	232·7	233·6	234·4	235·3	236·1	237·0	237·8	239·2	240·0	241·0	241·8	242·8	0·45
738	240·3	241·2	242·2	243·1	244·6	245·0	245·9	246·8	247·8	248·7	249·7	250·8	251·7	252·6	253·5	254·5	0·47
737	251·4	252·4	253·3	254·3	255·3	256·3	257·3	258·2	259·2	260·2	261·2	262·4	263·3	264·3	265·3	266·3	0·50
736	262·5	263·5	264·5	265·5	266·6	267·6	268·6	269·6	270·7	271·7	272·7	274·0	275·0	276·0	277·0	278·0	0·52
735	273·6	274·7	275·8	276·8	277·9	278·9	280·0	281·1	282·1	283·2	284·3	285·6	286·6	287·7	288·7	289·8	0·54
734	284·7	285·9	287·0	288·1	289·2	290·3	291·4	292·5	293·6	294·7	295·8	297·1	298·2	299·3	300·4	301·6	0·56
733	295·9	297·0	298·2	299·3	300·5	301·7	302·8	304·0	305·1	306·3	307·4	308·7	309·9	311·0	312·1	313·3	0·58
732	307·1	308·3	310·5	310·6	311·8	313·0	314·2	315·4	316·6	317·8	319·0	320·3	321·5	322·7	323·9	325·1	0·60
731	318·3	319·5	320·7	322·0	323·2	324·4	325·7	326·9	328·2	329·4	330·7	331·9	333·2	334·3	335·6	336·8	0·62
730	329·5	330·7	332·0	333·3	334·6	335·8	337·1	338·4	339·7	341·0	343·3	343·5	344·8	346·0	347·3	348·6	0·64
729	340·8	342·1	343·3	344·6	345·9	347·3	348·6	349·9	351·3	352·6	353·9	355 3	356·6	357·8	359·2	360·5	0·66
728	352·0	353·2	354·6	356·0	357·4	358·7	360·1	361·5	362·9	364·2	365·6	367·0	368·4	369·7	371·1	372·4	0·68
727	363·2	364·6	366·0	367·4	368·8	370·2	371·6	373·0	374·4	375·9	377·3	378·8	380·2	381·5	382·9	384·4	0·71
726	374·4	375·9	377·3	378·7	380·2	381·7	383.1	384·6	386·1	387·5	389·0	390·5	392·0	393·4	394·8	396·3	0·73

Measurement of heights by the barometer.

TABLE—*continued*.

Height of the barometer in metres.	Sum of the temperatures of the air at the two stations, or the values of $T + t$ in degrees of the centigrade thermometer.																Difference for 1°.
	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	
725	385·7	387·2	388·7	390·2	391·7	393·2	394·7	396·2	397·7	399·2	400·7	402·3	403·8	405·2	406·7	408·2	0·75
724	397·0	398·5	400·1	401·6	403·1	404·7	406·2	407·8	409·3	410·9	412·4	414·0	415·5	417·0	418·6	420·1	0·77
723	408·2	409·8	411·4	413·0	414·6	416·2	417·8	419·4	421·0	422·6	424·2	425·8	427·3	428·9	430·5	432·0	0·79
722	419·6	421·2	422·8	424·5	426·1	427·7	429·4	431·0	432·7	434·3	435·9	437·5	439·1	440·7	442·3	444·0	0·82
721	430·9	432·6	434·3	435·9	437·6	439·3	441·0	442·7	444·4	446·1	447·7	449·3	450·9	452·6	454·2	455·9	0·84
720	442·3	444·0	445·7	447·4	449·1	450·8	452·6	454·3	456·0	457·7	459·5	461·0	462·7	464·4	466·1	467·8	0·86
719	453·6	455·4	457·2	458·9	460·7	462·5	464·3	466·0	467·8	469·6	471·3	473·0	474·7	476·4	478·2	479·9	0·88
718	465·0	466·8	468·7	470·5	472·3	474·1	475·9	477·7	479·5	481·3	483·2	484·9	486·7	488·5	490·3	492·1	0·90
717	476·4	478·3	480·1	482·0	483·8	485·7	487·6	489·4	491·3	493·1	495·0	496·9	498·7	500·5	502·4	504·2	0·93
716	487·8	489·7	491·6	493·5	495·4	497·4	499·3	501·2	503·1	505·0	506·9	508·8	510·7	512·6	514·5	516·3	0·95
715	499·3	501·2	503·2	505·1	507·1	509·0	511·0	512·9	514·9	516·8	518·8	520·8	522·7	524·6	526·6	528·5	0·97
714	510·7	512·7	514·7	516·7	518·7	520·7	522·7	524·7	526·7	528·6	530·6	532·7	534·7	536·6	538·7	540·6	1·00
713	522·2	524·3	526·3	528·3	530·3	532·4	534·4	536·4	538·5	540·5	542·6	544·7	546·7	548·7	550·7	552·7	1·02
712	533·7	535·7	537·8	539·9	542·0	544·1	546 2	548·2	550·3	552·4	554·5	556·6	558·7	560·7	562·8	564·8	1·04
711	545·1	547·3	549·4	551·5	553·7	555·8	557·9	560·0	562·2	564·3	566·4	568·6	570·7	572·7	574·9	577·0	1·06
710	556·7	558·9	561·0	563·2	565·3	567·5	569·7	571·9	574·0	576·2	578·4	580·5	582·7	584·8	587·0	589·1	1·09
709	568·3	570·5	572·7	574·8	577·1	579·3	581·5	583·7	585·9	588·2	590·4	592·6	594·8	597·0	599·2	601·4	1·11
708	579·7	582·0	584·3	586·5	588·8	591·0	593·3	595·6	597·8	600·1	602·4	604·7	607·0	609·2	611·5	613·7	1·13
707	591·3	593·6	595·9	598·2	600·5	602·9	605·1	607·5	609·8	612·1	614·4	616·8	619·1	621·4	623·7	626·0	1·16
706	602·9	605·2	607·6	609·9	612·3	614·6	617·0	619·4	621·7	624·1	626·4	628·9	631·3	633·6	636·0	638·3	1·18
705	614·4	616·8	619·2	621·6	624·0	626·4	628·8	631·2	633·6	636·0	638·4	641·0	643·4	645·9	648·2	650·6	1·20
704	626·1	628·6	631·0	633·4	635·9	638·3	640·7	643·2	645·6	648·1	650·5	653·1	655·5	658·1	660·4	662·8	1·22
703	637·7	640·2	642·7	645·2	647·7	650·2	652·6	655·1	657·6	660·1	662·6	665·2	667·7	670·3	672·7	675·1	1·25
702	649·3	651·9	654·4	656·9	659·5	662·0	664·6	667·1	669·7	672·2	674·7	677·3	679·8	682·5	684·9	687·4	1·27
701	661·0	663·6	666·2	668·7	671·3	673·9	676·5	679·1	681·7	684·2	686·8	689·4	692·0	694·7	697·2	699·7	1·29
700	672·7	675·3	677·9	680·6	683·2	685·8	688·4	691·1	693·7	696·3	699·0	701·5	704·1	706·9	709·4	712·0	1·31
699	684·5	687·1	689·8	692·5	695·2	697·8	700·5	703·2	705·8	708·5	711·2	713·8	716·4	719·3	721·8	724·5	1·33
698	696·2	698·9	701·6	704·4	707·1	709·8	712·5	715·3	718·0	720·7	723·5	726·1	728·8	731·6	734·2	736·9	1·35
697	708·0	710·7	713·5	716·3	719·1	721·8	724·6	727·4	730·1	732·9	735·7	738·3	741·1	744·0	746·7	749·4	1·38
696	719·8	722·5	725·3	728·2	731·0	733·8	736·6	739·5	742·3	745·1	747·9	750·6	753·4	756·3	759·1	761·9	1·40
695	731·6	734·4	737·2	740·1	743·0	745·8	748·7	751·6	754·4	757·3	760·2	762·9	765·8	768·7	771·5	774·4	1·42
694	743·3	746·2	749·1	752·0	754·9	757·8	760·7	763·6	766·5	769·4	772·4	775·2	778·1	781·1	783·9	786·8	1·44
693	755·1	758·0	760·9	763·9	766·9	769·8	772·8	775·7	778·7	781·6	784·6	787·5	790·4	793·4	796·3	799·3	1·47
692	766·9	769·8	772·8	775·8	778·8	781·8	784·8	787·8	790·8	793·8	796·8	799·7	802·7	805·8	808·8	812·8	1·49
691	778·6	781·6	784·6	787·7	790·8	793·8	796·9	799·9	803·0	806·0	809·1	812·0	815·1	818·1	821·2	824·2	1·51
690	790·4	793·4	796·5	799·6	802·7	805·8	808·9	812·0	815·1	818·2	821·3	824·3	827·4	830·5	833·6	836·7	1·54
689	802·3	805·4	808·5	811·7	814·8	818·0	821·1	824·3	827·4	830·6	833·7	836·8	839·9	843·1	846·2	849·3	1·56
688	814·3	817·4	820·6	823·8	827·0	830·1	833·3	836·5	839·7	842·9	846·1	849·2	852·4	855·6	858·8	862·0	1·58
687	826·2	829·4	832·6	835·8	839·1	842·3	845·6	848·8	852·0	855·3	858·5	861·7	864·9	868·2	871·4	874·6	1·61
[illegible]	838·1	841·[illegible]	844·6	847·9	851·2	854·5	857·8	861·0	864·3	867·6	870·9	874·1	877·4	880·7	884·0	887·3	1·63

Measurement of heights by the barometer.

685	850·1	853·4	856·7	860·0	863·4	866·7	870·0	873·3	876·7	880·0	883·3	886·6	889·9	893·3	896·6	899·9	1·65
684	862·0	865·3	868·7	872·1	875·5	878·8	882·2	885·6	889·0	892·4	895·7	899·1	902·4	905·8	909·1	912·5	1·68
683	873·9	877·3	880·7	884·2	887·6	891·0	894·4	897·8	901·3	904·7	908·1	911·5	914·9	918·4	921·7	925·2	1·70
682	885·8	889·3	892·7	896·2	899·7	903·2	906·7	910·1	913·6	917·1	920·5	924·0	927·4	930·9	934·3	937·8	1·73
681	897·8	901·3	904·8	908·3	911·9	915·3	918·9	922·3	925·9	929·4	932·9	936·4	939·9	943·5	946·9	950·5	1·75
680	909·7	913·3	916·8	920·4	924·0	927·5	931·1	934·6	938·2	941·8	945·3	948·9	952·4	956·0	959·5	963·1	1·78
679	921·8	925·5	929·0	932·7	936·3	939·9	943·5	947·1	950·7	954·3	957·9	961·5	965·1	968·7	972·3	975·9	1·80
678	934·0	937·6	941·2	944·9	948·6	952·2	955·9	959·5	963·2	966·9	970·5	974·2	977·8	981·5	985·1	988·8	1·83
677	946·1	949·8	953·5	957·2	960·9	964·6	968·3	972·0	975·7	979·4	983·1	986·8	990·5	994·2	997·8	1001·6	1·85
676	958·2	962·0	965·7	969·4	973·2	976·9	980·7	984·4	988·2	992·0	995·7	999·5	1003·2	1006·9	1010·6	1014·4	1·87
675	970·4	974·2	977·9	981·7	985·5	989·3	993·1	996·9	1000·7	1004·5	1008·3	1012·1	1015·9	1019·7	1023·4	1027·3	1·89
674	982·5	986·3	990·1	994·0	997·8	1001·7	1005·5	1009·3	1013·2	1017·0	1020·8	1024·7	1028·5	1032·4	1036·2	1040·1	1·92
673	994·6	998·5	1002·3	1006·2	1010·1	1014·0	1017·9	1021·8	1025·7	1029·6	1033·4	1037·4	1041·2	1045·1	1049·0	1052·9	1·95
672	1006·7	1010·7	1014·6	1018·5	1022·4	1026·4	1030·3	1034·2	1038·2	1042·1	1046·0	1050·0	1053·9	1057·8	1061·7	1065·7	1·97
671	1018·9	1022·8	1026·8	1030·7	1034·7	1038·7	1042·7	1046·7	1050·7	1054·7	1058·6	1062·7	1066·6	1070·6	1074·5	1078·6	1·99
670	1031·0	1035·0	1039·0	1043·0	1047·0	1051·1	1055·1	1059·1	1063·2	1067·2	1071·2	1075·3	1079·3	1083·3	1087·3	1091·4	2·01
669	1043·3	1047·4	1051·4	1055·5	1059·5	1063·6	1067·7	1071·7	1075·9	1079·9	1084·0	1088·1	1092·2	1096·2	1100·3	1104·4	2·03
668	1055·6	1059·7	1063·8	1067·9	1072·0	1076·2	1080·3	1084·4	1088·6	1092·7	1096·8	1101·0	1105·0	1109·2	1113·3	1117·4	2·05
667	1067·9	1072·1	1076·2	1080·4	1084·5	1088·7	1092·9	1097·0	1101·2	1105·4	1109·6	1113·8	1117·9	1122·1	1126·2	1130·5	2·08
666	1080·2	1084·4	1088·6	1092·8	1097·0	1101·3	1105·5	1109·7	1113·9	1118·1	1122·4	1126·6	1130·8	1135·0	1139·2	1143·5	2·11
665	1092·5	1096·8	1101·0	1105·3	1109·5	1113·8	1118·1	1122·3	1126·6	1130·9	1135·2	1139·4	1143·7	1147·9	1152·2	1156·5	2·13
664	1104·8	1109·1	1113·4	1117·7	1122·0	1126·3	1130·6	1134·9	1139·3	1143·6	1147·9	1152·3	1156·5	1160·8	1165·2	1169·5	2·16
663	1117·1	1121·5	1125·8	1130·2	1134·5	1138·9	1143·2	1147·6	1152·0	1156·3	1160·7	1165·1	1169·4	1173·8	1178·2	1182·5	2·18
662	1129·4	1133·8	1138·2	1142·6	1147·0	1151·4	1155·8	1160·2	1164·6	1169·0	1173·5	1177·9	1182·3	1186·7	1191·1	1195·6	2·20
661	1141·7	1146·2	1150·6	1155·1	1159·5	1164·0	1168·4	1172·9	1177·3	1181·8	1186·3	1190·7	1195·1	1199·7	1204·1	1208·6	2·22
660	1154·0	1158·5	1163·0	1167·5	1172·0	1176·5	1181·0	1185·5	1190·0	1194·5	1199·1	1203·6	1208·0	1212·6	1217·1	1221·6	2·25
659	1166·5	1171·0	1175·6	1180·1	1184·7	1189·2	1193·8	1198·3	1202·9	1207·4	1212·1	1216·6	1221·1	1225·7	1230·3	1234·8	2·27
658	1179·0	1183·6	1188·2	1192·8	1197·4	1202·0	1206·6	1211·2	1215·8	1220·4	1225·0	1229·6	1234·2	1238·8	1243·4	1248·0	2·30
657	1191·4	1196·1	1200·7	1205·1	1210·0	1214·7	1219·3	1224·0	1228·6	1233·3	1238·0	1242·7	1247·2	1252·0	1256·6	1261·3	2·32
656	1203·9	1208·6	1213·3	1218·0	1222·7	1227·4	1232·1	1236·8	1241·5	1246·2	1251·0	1255·7	1260·3	1265·1	1269·8	1274·5	2·34
655	1216·4	1221·2	1225·9	1230·7	1235·4	1240·2	1244·9	1249·7	1254·4	1259·2	1264·0	1268·7	1273·4	1278·2	1283·0	1287·7	2·37
654	1228·9	1233·7	1238·5	1243·3	1248·1	1252·9	1257·7	1262·5	1267·3	1272·1	1276·9	1281·7	1286·5	1291·3	1296·1	1300·9	2·39
653	1241·4	1246·2	1251·1	1255·9	1260·8	1265·6	1270·5	1275·3	1280·2	1285·0	1289·9	1294·7	1299·6	1304·4	1309·3	1314·1	2·42
652	1253·8	1258·7	1263·6	1268·5	1273·4	1278·3	1283·2	1288·1	1293·0	1297·9	1302·9	1307·8	1312·6	1317·6	1322·5	1327·4	2·45
651	1266·3	1271·3	1276·2	1281·2	1286·1	1291·1	1296·0	1301·0	1305·9	1310·9	1315·8	1320·8	1325·7	1330·7	1335·6	1340·6	2·47
650	1278·9	1283·8	1288·8	1293·8	1298·8	1303·8	1308·8	1313·8	1318·8	1323·8	1328·8	1333·8	1338·8	1343·8	1348·8	1353·8	2·50
649	1291·6	1296·5	1301·6	1306·6	1311·7	1316·7	1321·8	1326·8	1331·9	1336·9	1342·0	1347·0	1352·1	1357·1	1362·2	1367·2	2·52
648	1304·3	1309·3	1314·4	1319·5	1324·6	1329·7	1334·8	1339·9	1345·0	1350·1	1355·2	1360·3	1365·4	1370·4	1375·5	1380·6	2·55
647	1316·9	1322·0	1327·1	1332·3	1337·4	1342·6	1347·7	1352·9	1358·0	1363·2	1368·3	1373·5	1378·6	1383·8	1388·9	1394·1	2·57
646	1329·6	1334·8	1339·9	1345·1	1350·3	1355·5	1360·7	1365·9	1371·1	1376·3	1381·5	1386·7	1391·9	1397·1	1402·2	1407·5	2·60
645	1342·3	1347·5	1352·7	1358·0	1363·2	1368·6	1373·7	1379·0	1384·2	1389·5	1394·7	1400·0	1405·2	1410·4	1415·6	1420·9	2·62
644	1355·0	1360·2	1365·5	1370·8	1376·1	1381·4	1386·7	1392·0	1397·3	1402·6	1407·9	1413·2	1418·5	1423·7	1429·0	1434·3	2·65
643	1367·7	1373·0	1378·3	1383·6	1389·0	1394·3	1399·7	1405·0	1410·4	1415·7	1421·1	1426·4	1431·8	1437·0	1442·3	1447·7	2·67
642	1380·3	1385·7	1391·0	1396·4	1401·8	1407·2	1412·6	1418·0	1423·4	1428·8	1434·2	1439·6	1445·0	1450·4	1455·7	1461·2	2·70
641	1393·0	1398·5	1403·8	1409·3	1414·7	1420·2	1425·6	1431·1	1436·5	1442·0	1447·4	1452·9	1458·3	1463·7	1469·0	1474·6	2·72
640	1405·7	1411·2	1416·6	1422·1	1427·6	1433·1	1438·6	1444·1	1449·6	1455·1	1460·6	1466·1	1471·6	1477·0	1482·4	1488·0	2·75

TABLE—*continued.*

Height of the barometer in metres.	Sum of the temperatures of the air at the two stations, or the values of T + *t* in degrees of the centigrade thermometer.																Difference for 1°.
	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	
639	1418·6	1424·1	1429·6	1435·1	1440·7	1446·2	1451·8	1457·3	1462·9	1468·4	1474·0	1479·5	1485·8	1490·5	1496·0	1501·6	2·77
638	1431·4	1437·1	1442·6	1448·2	1453·8	1459·4	1465·0	1470·6	1476·2	1481·8	1487·4	1493·0	1498·6	1504·1	1509·6	1515·3	2·80
637	1444·3	1450·0	1455·6	1461·2	1466·9	1472·5	1478·2	1483·8	1489·5	1495·1	1500·7	1506·4	1512·0	1517·6	1523·2	1528·9	2·82
636	1457·2	1462·9	1468·6	1474·3	1480·0	1485·7	1491·4	1497·1	1502·8	1508·4	1514·1	1519·8	1525·5	1531·2	1536·8	1542·6	2·85
635	1470·1	1475·9	1481·6	1487·3	1493·1	1498·8	1504·6	1510·3	1515·1	1521·8	1527·5	1533·3	1539·0	1544·7	1550·4	1556·2	2·87
634	1482·9	1488·8	1494·5	1500·3	1507·1	1511·9	1517·7	1523·5	1529·3	1535·1	1540·9	1546·7	1552·5	1558·2	1564·0	1569·8	2·90
633	1495·8	1501·7	1507·5	1513·4	1519·3	1525·1	1530·9	1536·8	1542·6	1548·4	1554·3	1560·1	1566·0	1571·8	1577·6	1583·5	2·92
632	1508·7	1514·6	1520·5	1526·4	1532·3	1538·2	1544·1	1550·0	1555·9	1561·7	1567·6	1573·5	1579·4	1585·3	1591·2	1597·1	2·95
631	1521·5	1527·6	1533·5	1539·5	1545·4	1551·4	1557·3	1563·3	1569·2	1575·1	1581·0	1587·0	1592·9	1598·9	1604·8	1610·8	2·97
630	1534·4	1540·5	1546·5	1552·5	1558·5	1564·5	1570·5	1576·5	1582·5	1588·4	1594·4	1600·4	1606·4	1612·4	1618·4	1624·4	3·00
629	1547·5	1553·6	1559·7	1565·7	1571·8	1577·8	1583·9	1589·9	1596·0	1602·0	1608·0	1614·1	1620·1	1626·2	1632·2	1638·3	3·02
628	1560·6	1566·8	1572·9	1579·0	1585·1	1591·2	1597·3	1603·4	1609·5	1615·5	1621·6	1627·7	1633·8	1639·9	1646·0	1652·1	3·05
627	1573·7	1579·9	1586·1	1592·2	1598·4	1604·5	1610·7	1616·8	1623·0	1629·1	1635·2	1641·5	1647·5	1653·7	1659·8	1666·0	3·07
626	1586·8	1593·1	1599·3	1605·5	1611·7	1617·9	1624·1	1630·3	1636·5	1642·6	1648·8	1655·0	1661·2	1667·4	1573·6	1679·8	3·10
625	1599·9	1606·2	1612·5	1618·7	1625·0	1631·2	1637·5	1643·7	1650·0	1656·2	1662·4	1668·7	1674·9	1681·2	1687·4	1693·7	3·12
624	1613·0	1619·3	1625·6	1631·9	1638·2	1644·5	1650·8	1657·1	1663·4	1669·7	1676·0	1682·3	1688·6	1694·9	1701·2	1707·5	3·15
623	1626·1	1632·5	1638·8	1645·2	1651·5	1657·9	1664·2	1670·6	1676·9	1683·3	1689·6	1696·0	1702·3	1708·7	1715·0	1721·4	3·17
622	1639·2	1645·6	1652·0	1658·4	1664·8	1671·2	1677·6	1684·0	1690·4	1696·8	1703·2	1709·6	1716·0	1722·4	1728·8	1735·2	3·20
621	1652·3	1658·8	1665·2	1671·7	1678·1	1684·6	1691·0	1697·5	1703·9	1710·4	1716·8	1723·3	1729·7	1736·2	1742·6	1749·1	3·22
620	1665·4	1671·9	1678·4	1684·9	1691·4	1697·9	1704·4	1710·9	1717·4	1723·9	1730·4	1736·9	1743·4	1749·9	1756·4	1762·9	3·25
619	1678·7	1685·2	1691·8	1698·4	1704·9	1711·5	1718·0	1724·6	1731·1	1737·7	1744·2	1750·8	1757·3	1763·9	1770·4	1777·0	3·27
618	1692·0	1698·6	1705·2	1711·8	1718·4	1725·0	1731·6	1738·2	1744·8	1751·4	1758·0	1764·7	1771·3	1777·9	1784·5	1791·1	3·30
617	1715·3	1711·9	1718·6	1725·3	1731·9	1738·6	1745·2	1751·9	1758·6	1765·2	1771·9	1778·5	1785·2	1791·8	1798·5	1805·2	3·32
616	1718·6	1725·3	1732·0	1738·7	1745·4	1752·1	1758·8	1765·5	1772·3	1779·0	1785·7	1792·4	1799·1	1805·8	1812·5	1819·3	3·35
615	1731·9	1738·6	1745·4	1752·2	1755·9	1765·7	1772·5	1779·2	1786·0	1792·8	1799·5	1806·3	1813·1	1819·8	1826·6	1833·4	3·37
614	1745·1	1751·9	1758·7	1765·6	1772·4	1779·3	1786·1	1792·9	1799·7	1806·4	1813·3	1820·2	1827·0	1833·8	1840·6	1847·4	3·40
613	1758·4	1765·3	1772·1	1779·1	1785·9	1792·8	1799·7	1806·5	1813·4	1820·3	1827·1	1834·1	1840·9	1847·8	1854·6	1861·5	3·42
612	1771·7	1778·6	1785·5	1792·5	1799·4	1806·4	1813·3	1820·2	1827·2	1834·1	1841·0	1847·9	1854·8	1861·7	1868·6	1875·6	3·45
611	1785·0	1792·0	1798·9	1806·0	1812·9	1819·9	1826·9	1833·8	1840·9	1847·8	1854·8	1861·8	1868·8	1875·7	1882·7	1889·7	3·48
610	1798·3	1805·3	1812·3	1819·4	1826·4	1833·5	1840·5	1847·5	1854·6	1861·6	1868·6	1875·7	1882·7	1889·7	1896·7	1903·8	3·51
609	1811·8	1818·9	1825·9	1833·1	1840·1	1847·3	1854·3	1861·4	1868·5	1875·6	1882·7	1889·8	1896·9	1903·9	1911·0	1918·1	3·54
608	1825·3	1832·5	1839·6	1846·8	1853·9	1861·1	1868·2	1875·3	1882·5	1889·6	1896·7	1903·9	1911·0	1918·1	1925·2	1932·4	3·56
607	1838·9	1846·0	1853·2	1860·4	1867·6	1874·8	1882·0	1889·2	1896·4	1903·6	1910·8	1918·0	1925·2	1932·3	1939·5	1946·7	3·59
606	1852·4	1859·6	1866·9	1874·1	1881·4	1888·6	1895·9	1903·1	1910·4	1917·6	1924·8	1932·1	1939·3	1946·5	1953·7	1961·0	3·61
605	1865·9	1873·2	1880·5	1887·8	1895·1	1902·4	1909·7	1917·0	1924·3	1931·6	1938·9	1946·2	1953·5	1960·7	1968·0	1975·4	3·64
604	1879·4	1886·8	1894·1	1901·5	1908·8	1916·2	1923·5	1931·8	1938·3	1945·5	1952·9	1960·3	1967·6	1974·9	1982·3	1989·7	3·66
603	1892·9	1900·4	1907·8	1915·2	1922·6	1930·0	1937·4	1944·7	1952·2	1259·5	1967·0	1974·4	1981·8	1989·2	1996·5	2004·0	3·69
602	1906·5	1913·9	1921·4	1928·8	1936·3	1943·7	1951·2	1958·6	1966·1	1973·5	1981·0	1988·5	1995·9	2003·4	2010·8	2018·3	3·72
601	1920·0	1927·5	1935·1	1942·5	1950·1	1957·5	1965·1	1972·5	1980·1	1987·5	1995·1	2002·6	2010·1	2017·6	2025·0	2032·6	3·75
[illegible]	[illegible]	[illegible]	1948·7	1956·2	1963·8	1971·3	1978·9	1986·4	1994·0	2001·5	2009·1	2016·7	2024·2	2031·8	2039·3	2046·9	3·77

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equal to a column of water whose height is 33¼; which gives for the velocity of an issuing stream of water under this pressure of $46\frac{12}{25}$ feet, instead of 45, the number derived by Dr. Papin. If, therefore, we use this numerical determination, and multiply $46\frac{12}{25}$ by 29, we shall have for the velocity in question, $46\frac{12}{25} \times 29 =$ 1348 feet, a result agreeing pretty accurately with that in the preceding article. If we assume the pressure of the atmosphere as equal to that due to a column of water 33 feet high, the velocity will become 1332 feet, and Dr. Robison thinks that this number might be assumed as a standard in Pneumatics, as 16 and 32, or $16\frac{1}{12}$ and $32\frac{1}{6}$ are, in general mechanics, for expressing the action of gravity at the surface of the earth.

Additional pressure does not increase the velocity.

It may be important to remark, in concluding this article, because it appears at first sight somewhat paradoxical, that no artificial pressure can be employed to increase the velocity with which air issues into a void; for, supposing any degree of pressure to have been artificially produced, the density of the fluid will be increased in the same proportion. The density, therefore, varying as the pressure, the velocity will be the same under all circumstances, agreeably to the principle stated in the commencement of this investigation.

The velocity of air rushing into a rarer medium.

109. Let us now investigate the circumstances relating to air rushing into a medium rarer than itself.

Whatever may be the density of this air, its elasticity, which follows the proportion of its density, will balance a proportional part of the pressure of the atmosphere, and it is the excess of this part only which is the moving force. Let D denote the natural density of the air, and ∂ the density of the air contained in the vessel into which it is supposed to run, and let P be the pressure of the atmosphere, and therefore equal to the force which impels it into a void; let also ϕ represent the force with which this rarer air would run into a vacuum; then we shall have

$$\mathrm{D} : \partial :: \mathrm{P} : \phi;$$

consequently
$$\phi = \frac{\partial \mathrm{P}}{\mathrm{D}}.$$

Now the moving force, in the present instance, is

$$\mathrm{P} - \phi, \text{ or } \mathrm{P} - \frac{\partial \mathrm{P}}{\mathrm{D}} = \frac{\mathrm{PD} - \partial \mathrm{P}}{\mathrm{D}}.$$

Again, let V denote the velocity with which we have determined air to rush into a void, and v the required velocity, that is, the velocity with which it rushes into this rarefied air. Then, since we have seen that the velocities are as the square roots of the pressures, every thing else being the same; that is, since our expression for the velocity (art. 61, HYDRAULICS) is

$$v = 2\sqrt{g\,h}, \text{ or } v^2 = 4\,g\,h;$$

where h may be supposed to represent the pressure, we shall have

$$\mathrm{P} : \mathrm{P} - \frac{\partial \mathrm{P}}{\mathrm{D}} :: \mathrm{V}^2 : v^2;$$

consequently
$$v^2 = \mathrm{V}^2\left(1 - \frac{\partial}{\mathrm{D}}\right);$$

or
$$v = \mathrm{V}\sqrt{\left(1 - \frac{\partial}{\mathrm{D}}\right)} = \mathrm{V}\sqrt{\left(\frac{\mathrm{D} - \partial}{\mathrm{D}}\right)}.$$

Supposing, therefore, the rarefied air to have one-fourth of the density of common air, the velocity with which the atmospheric air would rush into such a medium would be expressed by

$$v = \mathrm{V}\sqrt{\left(\frac{4-1}{4}\right)} = \tfrac{1}{2}\,\mathrm{V}\sqrt{3}.$$

We do not here consider the resistance which the air of the atmosphere will meet with from the inertia of that in the vessel, which it must displace in its motion.

It appears from the above, that there will be a continual current into the vessel, while ∂ is less than D; but as soon as this equality takes place, the motion will subside; since, in this case, we shall have $1 - \frac{\partial}{\mathrm{D}} = 0$.

We see, also, that as the vessel fills and ∂ increase, the expression $1 - \frac{\partial}{\mathrm{D}}$ diminishes, and consequently the velocity of influx, till it ceases entirely in the extreme case of $\partial = \mathrm{D}$.

110. Let us now endeavour to determine the time in which the air of the atmosphere will flow into a given vessel, or void space, before the air in the vessel attains a given density.

Let D denote the mean density of the air, and ∂ that to which the air in the vessel is to arrive; let also H be the height of a homogeneous atmosphere, or the height through which a heavy body must fall to acquire the velocity above denoted by V. Assume C, to represent the capacity of the vessel in cubic feet; A, the area of the section of the orifice; and t the required time in seconds.

The quantity of air contained in any vessel depends upon the density of the former and the capacity of the latter, conjointly; we may therefore express the quantity of the air which would fill the vessel by CD when the air is in the ordinary state, that is, when its density is D, and by $\mathrm{C}\,\partial$ when its density is $= \partial$. Now, in order to obtain the rate at which it fills, we must take the fluxion of $\mathrm{C}\,\partial$, which is $\mathrm{C}\,\dot{\partial}$; C being a constant quantity, and ∂ the variable or flowing quantity.

The velocity of influx, in the first instant, we have seen (art. 107) is

$$\mathrm{V} = 2\sqrt{g\,\mathrm{H}};$$

or, assuming $g = 16$, instead of $16\frac{1}{12}$, we have

$$\mathrm{V} = 8\sqrt{\mathrm{H}};$$

and when the air has attained the density ∂, that is, at the end of the time t, the velocity will be

$$\mathrm{V}\sqrt{\left(1 - \frac{\partial}{\mathrm{D}}\right)}, \text{ or } 8\sqrt{\mathrm{H}} \times \sqrt{\left(1 - \frac{\partial}{\mathrm{D}}\right)} \text{ (art. 109).}$$

The rate of influx, therefore (which may be conceived as measured by the little mass of air which will enter during the time $\dot{t}$ with this velocity), will be

$$\frac{8\sqrt{\mathrm{H}} \times \sqrt{(\mathrm{D} - \partial)} \times \mathrm{DA}\,\dot{t}}{\sqrt{\mathrm{D}}};$$

or, which is the same, by

$$8\sqrt{\mathrm{H}} \times \sqrt{(\mathrm{D} - \partial)}\sqrt{\mathrm{D}} \times \mathrm{A}\,\dot{t},$$

that is, by multiplying the velocity by the orifice an density.

We have thus two values of the rate of influx, an by equating them we shall have an equation from whic the fluents may be obtained, and whence we may de duce the value of the time t in seconds. These expre sions are $C\,\dot{\delta}$, and that last obtained in terms of $\dot{t}$, wherefore, making

$$8\sqrt{H}\sqrt{D}\sqrt{(D-\delta)} \times A\,\dot{t} = C\,\dot{\delta},$$

we obtain

$$\dot{t} = \frac{C}{8\,A\,\sqrt{HD}} \times \frac{\dot{\delta}}{\sqrt{(D-\delta)}};$$

the fluent of which gives

$$t = \frac{-C}{4\,A\,\sqrt{HD}} \times \sqrt{(D-\delta)} + \text{correction.}$$

Or, denoting the correction by B,

$$t = \frac{-C}{4\,A\,\sqrt{HD}} \times \{\sqrt{(D-\delta)} + B\}.$$

We may determine the value of the correction by observing, that when δ is zero, that is, when $\sqrt{(D-\delta)} = \sqrt{D}$, t is then zero also; we must, therefore, have $B = \sqrt{D}$, and the correct fluent will be

$$t = \frac{C}{4\,A\,\sqrt{HD}} \times \{\sqrt{D} - \sqrt{(D-\delta)}\}.$$

This expresses the time necessary to bring the air in the vessel to the density δ; whence, in order to find the time in which the vessel will be completely filled, we must make $\delta = D$, and the expression becomes

$$t = \frac{C}{4\,A\,\sqrt{HD}} \times \sqrt{D} = \frac{C}{4\,A\,\sqrt{H}}.$$

111. Let us illustrate these formulæ by applying them in the solution of a few numerical problems.

Suppose the capacity of the vessel to be 8 cubic feet, or nearly a wine hogshead, and that the orifice by which the ordinary air enters (whose density D we may call 1), to be a square inch, or $\frac{1}{144}$th part of a square foot. Then we shall have $H = 27818$, $C = 8$, $A = \frac{1}{144}$, and $D = 1$; whence

$$t = \frac{C}{4\,A\,\sqrt{H}} = \frac{8 \times 144}{4\sqrt{27818}} = \frac{1152}{668} = 1{\cdot}7245$$

seconds, for the time of completely filling the vessel. As the time is reciprocally as the area of the orifice, it is obvious, that if we suppose the area of the orifice to be only $\frac{1}{10}$th of a square inch, the time will be 17·245 seconds; and if we suppose it to be square, and its side only $\frac{1}{10}$th of an inch, and consequently its area equal to $\frac{1}{100}$th of an inch, the time will be 172·45 seconds, or 3 minutes nearly.

It will be remarked, that in the above determination we have not considered the contraction of the vein, consequently the time will be greater than that above determined theoretically, in the ratio of the actual to the contracted orifice. That a contraction takes place in this case as well as in the discharge of a vessel of water we cannot doubt, but whether the quantity of that contraction be the same in both cases, is perhaps very doubtful; supposing it to be the same, the actual experimental time of filling will be to that determined above in about the ratio of $1 : \frac{5}{8}$, or as 8 to 5. See (art. 179) HYDRAULICS.

This supposes the air to enter through an orifice in a thin plate; if it pass through a short additional tube, and the same law be supposed to obtain here as in the discharge of water, the velocity will only be diminished in the ratio of 16 to 13, and, consequently, the time of filling will be to that theoretically determined as 16 to 13. See the article above referred to.

Motion of elastic fluids.

This equality in the contraction of the issuing vein of water and air under similar circumstances is assumed by writers on Pneumatics, but we think it extremely doubtful; it must be acknowledged, at the same time, to be very difficult to submit to the test of experiment.

Let us, as a second example, suppose all the numbers to remain the same as in the preceding case, to find the time requisite to bring the density of the air in the vessel to $\frac{3}{4}$ths of that of the ordinary air. Here we must employ the general formula

$$t = \frac{C}{4\,A\,\sqrt{H}} \times \{\sqrt{D} - \sqrt{(D-\delta)}\}.$$

in which we have merely to assume $\delta = \frac{3}{4}$, and, consequently, $D - \delta = \frac{1}{4}$, or $\sqrt{(D-\delta)} = \frac{1}{2}$; this also gives $\sqrt{D} - \sqrt{(D-\delta)} = \frac{1}{2}$; and, since the value of the co-efficient $\frac{C}{4\,A\,\sqrt{H}}$, will still remain as before, it appears that we shall have, in this case,

$$t = \frac{C}{8\,A\,\sqrt{H}} = {\cdot}86225 \text{ second};$$

that is, it will be exactly one-half of that determined in the preceding example.

Velocity when a moving force is employed.

112. Let us now conceive the air in the vessel to be compressed by a weight acting on the cover, supposed to be moveable down the vessel, to determine the velocity of the air expelled at the aperture.

Fig. 12.

Conceive the proposed vessel to be denoted by ABCD (fig. 12), and C to be the aperture; and let D denote the pressure of the external air due to the pressure P. Now, the immediate effect of the external pressure is to compress the air in the vessel, and to give it another density; let this additional pressure be called p, and the consequent density of the air in the vessel d, then we shall have, from the nature of elastic fluids,

$$P : P + p :: D : d,$$

or

$$d = \frac{D(P+p)}{P},$$

and

$$p = P \times \frac{d-D}{D}.$$

Now, because the pressure which expels the air is the difference between the force which compresses the air in the vessel and that which compresses the external air, the expelling force is p; and because the quantities of motion are as the forces which similarly produce them, we shall have

$$P : P \times \frac{d-D}{D} :: MV : mv,$$

where M and m express the quantities of matter expelled, V the velocity with which air rushes into a vacuum, and v the velocity required. But because the quantities of air which issue from the same orifice in an instant are as the densities and velocities jointly, we shall have

$$MV : mv :: DV^2 : dv^2;$$

consequently,

$$P : P \times \frac{d-D}{D} : DV^2 : dv^2;$$

and hence we deduce

$$v = V\sqrt{\frac{d-D}{d}}.$$

Another expression for the velocity may be obtained without computing the densities: for since

$$P : P + p :: D : d,$$

we have

$$d = \frac{D(P+p)}{P};$$

and consequently

$$d - D = \frac{D(P+p)}{P} - D = \frac{Dp}{P};$$

whence

$$\frac{d-D}{d} = \frac{Dp}{P} \times \frac{P}{D(P+p)} = \frac{p}{P+p};$$

wherefore we ultimately obtain

$$v = V\sqrt{\frac{p}{P+p}},$$

a very simple and convenient expression.

Velocity as depending upon elasticity.

113. We have hitherto considered the motion of the air as produced by its weight only; let us now investigate the effects due to its elasticity.

Let ABCD (fig. 12) be a vessel containing air of any density D; in this case the air is in a state of compression, and if the compressing force be removed, it will expand, and its elasticity will diminish with the density; the measure of the elasticity being estimated by the compressing force. The force which keeps common air in its ordinary density, is the weight of the atmosphere, which we have seen, is the same as the weight of a column of water 33½ feet high. If, therefore, we suppose that this air, instead of being confined by the top of the vessel, is pressed down with a moveable piston, carrying a column of water 33½ feet in height, its elasticity will balance this as it balances the pressure of the atmosphere; and as it is a fluid, and presses equally in all directions, it will press upon any little portion of the vessel by its elasticity, in the same manner as when loaded with this column. It follows, therefore, that if this small portion of the vessel be removed, and a passage thus made into the void, the air will begin to issue out with the same velocity as it would do when impelled by its weight only, or with the velocity acquired in falling through a homogeneous atmosphere. But the moment any air has entered into what, in the first place, was void space, that is, the moment any air has been discharged from the vessel containing it, the elasticity of the remaining air is diminished, and consequently also the expelling force. But the matter to be moved is also diminished in exactly the same proportion, so that there is still the same ratio existing between the force and the body, and consequently the velocity will always be the same whatever may be the degree of rarefaction; which is an analogous conclusion to that which obtained with respect to condensation (art. 108).

It appears, therefore, that whatever may be the degree, either of compression or condensation, to which air may be brought by any artificial means, it will always rush into a void with one and the same velocity during the first instant of its issuing, and this velocity would continue till the exhaustion was complete, if it were possible to destroy the air as soon as discharged, so that the vacuum could be constantly preserved.

This may be otherwise illustrated as follows: Let P denote the pressure of the atmosphere, which being the measure of the initial elasticity, is equal to the expelling force in the first instant. Let D be the first density, and V the initial velocity; assume d to denote the density after any time t, and v for the contemporaneous velocity. Then it is obvious that, at the end of this time, the expelling force ϕ will be

$$\phi = \frac{Pd}{D},$$

for

$$D : d :: P : \phi = \frac{Pd}{D}.$$

Now these forces are proportional to the quantities of motion which they produce, and the quantities of motion are proportional to the quantities of matter or masses M, m, and to the velocities V, v, jointly; that is,

$$P : \frac{Pd}{D} :: MV : mv.$$

But the quantities of matter which escape through a given orifice are as the densities and velocities jointly; that is,

$$M : m :: DV : dv;$$

hence

$$P : \frac{Pd}{D} :: DV^2 : dv^2;$$

consequently

$$P \times dv^2 = Pd \times V^2;$$

and therefore

$$V^2 = v^2,\ \text{or}\ V = v;$$

that is, the velocity is the same, under all degrees of compression or rarefaction, as deduced from the preceding investigation.

114. Let us now suppose that the quantity of air which is discharged into a void in any given time t were required, and consequently the density of the remaining air after that time.

In order to this determination we must obtain a general expression for the rate of efflux; now, from what is stated above, it is obvious that during the time $\dot{t}$ (the density being 1) the quantity discharged is expressed by

$$8A\sqrt{H} \times \dot{t},$$

because the velocity $V = 8\sqrt{H}$ is constant, and consequently, when the density is d, the quantity will be

$$8Ad\sqrt{H} \times \dot{t}.$$

At the commencement of the efflux the quantity of air was CD, C being the capacity of the vessel, and D the density of the air; and when the air has acquired the density d, the quantity will be C d, and quantity discharged

$$CD - Cd;$$

whence, what has run out during the time $\dot{t}$ must be the fluxion of CD − C d, which is $-C\dot{d}$, d being the only variable contained in the expression.

We have thus two values of the fluxion of the discharge, viz. $8Ad\sqrt{H} \times \dot{t}$ and $C\dot{d}$, we may therefore equate them, and thence deduce the value of d, and consequently also of CD − C d the quantity discharged; that is, we shall have

$$8Ad\sqrt{H} \times \dot{t} = -C\dot{d};$$

whence

$$\dot{t} = \frac{-C\dot{d}}{8Ad\sqrt{H}} = \frac{C}{8A\sqrt{H}} \times -\frac{\dot{d}}{d};$$

taking the fluents

$$t = \frac{C}{8A\sqrt{H}} \times (-\text{hyp. log. } d + \text{correction}).$$

This fluent must be taken so that t may be zero when $d = D$.

Consequently, the correction must be hyp. log. D, and the corrected expression becomes

$$t = \frac{C}{8A\sqrt{H}} \times (\text{hyp. log. } D - \text{hyp. log. } d);$$

or,

$$t = \frac{C}{8A\sqrt{H}} \times \text{hyp. log. } \frac{D}{d}.$$

When $d = 0$, $\frac{D}{d}$, and hyp. log. $\frac{D}{d}$ are both infinite, whence we learn that it will require an infinite time for the vessel to exhaust itself.

By transposing the first of the above equations we have

$$\text{hyp. log. } D - \text{hyp. log. } d = \frac{8At\sqrt{H}}{C};$$

wherefore

$$\text{hyp. log. } d = \text{hyp. log. } D - \frac{8At\sqrt{H}}{C};$$

from which the density, after any proposed time, may be determined, and consequently the quantity discharged, this being, as we have seen, expressed by

$$CD - Cd.$$

115. In the above investigation the discharge is supposed to be made into a void: let us now inquire into the circumstances of the discharge when it is made into a medium of given density, supposed to remain constant during the efflux, as we may suppose to be the case when a vessel containing condensed air emits it into the open atmosphere.

Let the initial density of the air in the vessel be δ, and that of the atmosphere D. Then it is obvious that the expelling force is

$$P - \frac{PD}{\delta};$$

and that, after the time t, it is

$$\frac{Pd}{\delta} - \frac{PD}{\delta}.$$

We have therefore

$$P - \frac{PD}{\delta} : \frac{PD}{\delta} - \frac{PD}{\delta} :: MV : mv : \delta V^2 : dv^2;$$

or $\quad \delta - D : d - D :: \delta V^2 : dv^2;$

wherefore $\quad dv^2(\delta - D) = \delta V^2 (d - D);$

consequently $\quad v = V\sqrt{\frac{\delta(d - D)}{d(\delta - D)}}.$

It appears, from this expression, that when $d = D$ the numerator vanishes, and therefore the motion will terminate, as is otherwise obvious; and when $\delta = D$ there can be no efflux.

Motion of elastic fluids.

Relation between the time and density.

116. In order to find the relation between the time and the density, let H, as before, be the height requisite to produce the velocity V. Then the height producing the velocity of efflux v, will be

$$H + \frac{\delta(d - D)}{d(\delta - D)};$$

and the portion of air which flows out during the time $\dot{t}$, will be

$$8Ad\sqrt{\left\{H \times \frac{\delta(d - D)}{d(\delta - D)}\right\}}\,\dot{t}.$$

Considering it again as the fluxion of the quantity in the vessel after the time t, when the density is d, and the quantity $CD - Cd$, it is $-C\dot{d}$, we have therefore

$$8Ad\sqrt{\left\{H \times \frac{\delta(d - D)}{d(\delta - D)}\right\}}\,\dot{t} = -C\dot{d};$$

and, consequently,

$$\dot{t} = \frac{C\sqrt{(\delta - D)}}{8A\sqrt{H\delta}} \times \frac{-\dot{d}}{\sqrt{(d^2 - Dd)}}.$$

The fluent of this corrected, so as to make $t = 0$ when $d = \delta$, is

$$t = \frac{C\sqrt{(\delta - D)}}{8A\sqrt{H\delta}} \times \text{hyp. log.} \left\{\frac{\delta - \frac{1}{2}D + \sqrt{(\delta^2 - D\delta)}}{d - \frac{1}{2}D + \sqrt{(d^2 - Dd)}}\right\},$$

and the time of completing the efflux when $d = D$, is

$$t = \frac{C\sqrt{(\delta - D)}}{8A\sqrt{H\delta}} \times \text{hyp. log.} \left\{\frac{\delta - \frac{1}{2}D\sqrt{(\delta^2 - D\delta)}}{\frac{1}{2}D}\right\}.$$

Air discharged out of one vessel into another.

117. Hitherto we have considered the vacuum into which the air is discharged to remain perfect, and the medium, when we have taken this case, has also been supposed to remain of the same density; but when air is discharged from one vessel to another, this hypothesis can be no longer employed; for, in this case, as the air in one vessel becomes rarer by the efflux, the other will become more dense by the influx; let us, therefore, in concluding this part of our inquiry, examine the circumstances attending the discharge of air under these circumstances.

Fig. 13.

Let there be two vessels, ABCD, CFGH (fig. 13), containing air of different densities, and communicating by the orifice C; then there will be a current from the vessel containing the denser air, into that in which medium is rarer; let us suppose from ABCD into CFGH.

Let P denote the initial elastic force of the air in the former of these, Q its density, and V its velocity; and assume D for the density of the air in the other vessel. Again, after the time t, let the density of the air in ABCD be q, its velocity v, and the density of the air in CFGA be δ.

Then the expelling force from ABCD will be, in the first instant,

$$P - \frac{PD}{Q},$$

and at the end of the time t it will be

$$\frac{Pq}{Q} - \frac{P\delta}{Q};$$

we have, therefore, the following analogy:

$$P - \frac{PD}{Q} : \frac{Pq}{Q} - \frac{P\delta}{Q} :: QV^2 : qv^2;$$

Pneumatics. or

$$Q - D : q - \delta : QV^2 : qv^2;$$

consequently

$$v = V\sqrt{\frac{Q(q - \delta)}{q(Q - D)}};$$

the motion, therefore, will cease when $\delta = q$.

Let us assume the capacity of the first vessel to be A, and that of the second B; then we shall have this second equation,

$$AQ + BD = Aq + B\delta;$$

for the quantity of air in the case we are investigating will obviously remain constant

This latter equation gives

$$\delta = \frac{A(Q - q) + BD}{B};$$

and substituting this for δ in the foregoing value of v, that equation will become

$$v = V \times \sqrt{\frac{Q\{B(q - D) - A(Q - q)\}}{qB(Q - D)}};$$

which give the relation between the velocity v and the density q.

In order to determine the relation between the time and the density, that is, to find the density q due to any time, let us simplify our expression by writing $Q(B + A) = m$, $BDQ + BQ^2 = n$, $BQ - BD = r$, and $\frac{n}{m} = s$.

Then proceeding the same as above, and denoting, for distinction sake, the area of the orifice by A′, we shall have

$$8A'\sqrt{H} \times \frac{\sqrt{(mq - n)}}{\sqrt{rq}} \times q\dot{t}$$

for the fluxion of the quantity taken with reference to the time and velocity. And the same, taken with reference to the quantity, after the time t, in the first vessel, that is with reference to $(AQ - Aq)$, will be $-A\dot{q}$; whence we shall have

$$8A'\sqrt{H} \times \frac{\sqrt{(mq - n)}}{\sqrt{rq}} \times q\dot{t} = -A\dot{q};$$

or, which is the same,

$$\dot{t} = \frac{A\sqrt{r}}{8A'\sqrt{H}m} \times \frac{\dot{q}}{\sqrt{(q^2 - sq)}}.$$

The fluent of which must be taken so that when $t = 0$, $q = Q$; this gives, for the complete fluent,

$$t = \frac{A\sqrt{r}}{8A'\sqrt{H}m} \times \text{hyp. log.}\left\{\frac{Q - \frac{1}{2}s + \sqrt{(Q^3 s)}}{q - \frac{1}{2}s + \sqrt{(q^3 s)}}\right\}.$$

Remarks. 118. Many of the deductions obtained in this section bear a certain analogy to those deduced in our treatise on Hydraulics, respecting the flowing of water and discharges of vessels under the various circumstances there supposed. We have seen, in the course of our investigations on the latter subject, that however necessary our theoretical principles might be, in order to arrive at correct practical conclusions, yet, that they would have been employed to little or no purpose without the aid of experiments, in order to modify and adjust them to the different cases that may happen. The discrepancy between the theory and practice doubtless arises from various circumstances being omitted to be considered in the former, which actually obtain in the latter, from our imperfect knowledge of the nature of fluidity, and the erroneous hypotheses which are employed in consequence. Experiments, by giving the actual results under various circumstances, supply, in some measure, these defects, and enable us to modify our theoretical expressions, in such a manner as to produce corresponding results, and having thus produced the required coincidences, in all cases where we are able to form the comparisons, we infer, and with great propriety, that in any other cases, not very different from the above, the same formulæ will apply, and thus, by blending experiment with theory, our knowledge is rendered subservient to various useful and practical purposes of life.

There is little doubt that modifications of a similar nature, but different probably in quantity, become also necessary in order to render our preceding deductions actually applicable to such cases as may result respecting the discharge of air under different circumstances; but experiments on this medium are much more difficult to execute and observe than those respecting water and other heavy fluids, and our practical pneumatical knowledge is therefore much more limited than in the cases alluded to. The only experiments, indeed, having any decided reference to the present subject, that we are acquainted with, are those made by Mr. Banks, and published by that gentleman in his work on the power of machines, the most important of which are as follow:

119. Referring to (fig. 14), A is a cask of known capacity, into the top of which is screwed an aperture a, of known area, the tube T d, recurved at d, is soldered or screwed into the top of the said cask. The hole a is stopped, and water poured into the tube T till it is full; at which time a quantity of water will have passed out of the tube at d, and condense the air in the cask till its spring is equal to the weight of the water in the tube. At this time a cock placed over the tube T, sufficiently large to supply water as fast as it can descend in the vessel A, must be opened, to keep the tube constantly filled; for this purpose one person must attend it, and another must open the aperture a, which needs only to be closed by the finger, and he must measure the seconds, from the moment that the finger is removed, till the water flies out at the jet. Hence, from knowing the capacity of the vessel, and the area of the jet, the velocity may be obtained. If the tube T d should be continued nearly to the bottom of A, while A was filling with water, the length of the compressing column would be gradually diminished, and of consequence the pressure would be constantly changing: to avoid any irregularity this might produce, the open end of the tube is as near the bottom of the cask as is consistent with a free passage for the water.

EXPERIMENTS.

1. The vessel contained 15 lbs. 6 oz. of water, from which we find its capacity 425·088 cubic inches. The area of the aperture a, through which is discharged ·0046 of an inch.

1. The altitude of T above the cask 30 inches.
Time of expelling the air 33 seconds.
This result was obtained from the mean of several trials.

2. The altitude of T, six feet.
The time of expelling the air 21·3 seconds.
This result is also the mean of several trials.

In the first experiment, 425·088, the capacity of the cask, being divided by ·0046, the area of the hole, gives 92410·4 inches, for the length of the stream driven out in 33 seconds. Hence

$$\frac{92410 \cdot 4}{12 \times 33} = 233 \cdot 3 \text{ feet},$$

the velocity per second.

From the second experiment we deduce, by a similar process, 361·6 feet, for the velocity per second, and to show the correspondence of this with the first, say, as

$$\sqrt{30} : \sqrt{6 \times 12} :: 233 \cdot 3 : 361 \cdot 8,$$

differing from the former by only ·2, or $\frac{1}{5}$th of a foot.

If we were to compute these velocities theoretically, we should have

$$\sqrt{33\tfrac{1}{4}} : \sqrt{2\tfrac{1}{2}} :: 1339 : 367.$$

This is too great, but if we suppose the contraction in this case to be the same as in the discharge of water, through an orifice in a thin plate, we must reduce the above in the ratio of 8 to 5, which gives for the velocity, under a constant pressure of 30 inches, 230 feet per second, which agrees very nearly with the above practical determination.

Again, $\sqrt{33\tfrac{1}{4}} : \sqrt{6} : 1339 : 568 \cdot 8,$

and $568 \cdot 8 \times \frac{5}{8} = 355$ feet,

the computed velocity in the second experiment.

As far, therefore, as the preceding experiments can be depended upon, it appears that the contraction of air, in passing through an orifice in a thin plate, is nearly the same as that of water; still we must again repeat, that the great dissimilarity of the two fluids leads us to suspect the accuracy of the deductions. A well-conducted course of experiments directed to this inquiry, would be highly interesting; although, perhaps, it would be a subject of curiosity, rather than of much practical utility, for it fortunately happens, that such determinations are not frequently necessary in the usual applications of the science of Pneumatics.

led re- on.

120. The cases of greatest utility are those where air is expelled from a vessel by an external force, as in the working of bellows, either of the ordinary construction, or as consisting of a cylinder fitted with a moveable piston. This last case deserves some attention, and we shall see that the investigation is very simple and direct. Suppose, for example, a machine of this kind to be represented in (fig. 12), AD being considered as a piston moving downwards with the uniform velocity u, and let the area of the piston be n times the area of the orifice; then the velocity of efflux arising from the motion of the piston will be $n\ u$, and if to this we add the velocity V due to the elasticity of the air, we shall have $V + n\ u$ for the whole velocity of efflux.

This is the principal object of inquiry as far as any practical application is concerned; if we were, as a matter of curiosity, to examine the density of the air in the vessel after any given time, we should still find the problem, although so simple as far as it regards the above determination, extremely intricate and unmanageable, even on the supposition of an uniform motion of the piston. At the beginning of the motion the air in the vessel will be of the usual density, but as the piston descends faster than the air can escape, it will be both expelled and compressed, and its density will vary according to a law very difficult to express, and the same will be the case with respect to its resistance or re-action on the piston. On account of this irregular density of the air, a piston which moves uniformly by means of an external force, will never produce an uniform blast by successive strokes. The best method of obtaining uniformity is to employ the external force only for lifting up the piston, and to let it descend by its own weight; in this way it will sink down in the first instants rapidly, compressing the air till its density and corresponding elasticity exactly balance the weight of the piston, after which it will descend equably, and the blast will be uniform.

Motion of elastic fluids.

Motion of air through long pipes.

121. We have seen, in our treatise on Hydraulics, that the discharge of water through long pipes differs very essentially from the discharge through a simple orifice, and we have reported various experiments from which the retardation, under certain limits, may be determined. A similar effect is observed in the passage of air through tubes of great length; but whether we can with safety infer the laws of resistance and retardation to be the same here as in the former case, appears to be exceedingly doubtful. We know that the motion of water is greatly impeded by all contractions of its passage; these requiring an acceleration of velocity where the passage is narrowest, render an increase of pressure to force them through also necessary, which pressure ought to be proportional to the square of the velocity. Thus, if a machine working a pump cause it to give a certain number of strokes in a minute, it will deliver a determinate quantity of water in that time; but should it happen that the passage is contracted to one-half, in any part of the machine, a thing which frequently occurs at the valves, the water must move through this contraction with twice the velocity that it has in the rest of the passage, which requires four times the force to be exerted on the piston; and a similar effect will be produced if, instead of a contraction, a widening of the pipe have place in any part of its length; in fact, any irregularities of form will produce a greater or less retardation of velocity, or a greater force to produce the same velocity as in a pipe of uniform dimensions.

The same thing takes place in the motion of air, and therefore all contractions and dilations ought to be as carefully avoided in the transmission of air as in the case of water.

Failure of certain machines.

122. When the bore of the pipe is perfectly uniform, a retardation will still take place, and that greater or less in proportion to the length, diameter of the tube through which the air passes, and the force with which it is impelled. By not attending to this circumstance, engineers of the first reputation have been prodigiously disappointed in their expectations of the quantities of air which will be delivered by long pipes. Its extreme mobility and lightness hindered them from suspecting that it should suffer any sensible retardation. The ingenious Dr. Papin proposed the following as the most effectual method of transferring the action of a moving power to a great distance. Suppose that it were required to raise water out of a mine by a water-machine, and that there were no fall of water nearer than a mile; he proposed to employ this water to drive a piston which should compress the air in a cylinder communicating by a long pipe, with another cylinder at the mouth of the mine; and this second cylinder to be supplied with a piston whose rod was to give motion to the pumps at the mine. He expected that, as soon as the piston at the water-machine had compressed the air

sufficiently, it should cause that in the cylinder, at the mine, to force up its piston, and thus work the pumps. Dr. Hook made many objections to the method, when laid before the Royal Society, and it was much debated there; but at this time Dynamics was in its infancy, and very little understood. Notwithstanding the great reputation of the projector of this machine, he could not bring his scheme into use in England; but afterwards, in France and in Germany, where he settled, he found some persons of fortune to employ him in this project; and he erected several considerable machines in Auvergne and Westphalia for draining mines; but so far from their being effective, they would not even begin to move. He attributed this failure to the quantity of air in the pipe of communication, which must be condensed before it can condense the air in the cylinder. This, indeed, is true, and he should have thought of it earlier; he therefore diminished the size of the pipe, and made his water-machine exhaust instead of condensing, and had no doubt but that the immense velocity with which air rushes into a vacuum, would make a rapid and effectual communication of power; but he was equally disappointed in this construction, for the machine at the mine stood still as before.

Long after this a very enlightened engineer attempted a much more plausible thing of the same description at an iron foundery in Wales. He erected a machine at a powerful water-fall, which works a set of cylinder bellows; the blow-pipe of which was conducted to the distance of a mile and a half, where it was applied to a blast furnace; but notwithstanding every care to make the conducting-pipe exactly air-tight, of great size, and as smooth as possible, it would hardly blow out a candle. The failure was ascribed to the impossibility of making the pipe perfectly air-tight; but what was surprising, about ten minutes elapsed after the action of the piston in the bellows, before the least wind could be perceived at the end of the pipe; whereas the engineer expected an interval of six seconds only.

These facts show the extreme inadequacy of mere theoretical deductions to guide us in any actual practical application, and the great necessity for multiplied observations and experiments in order to render our knowledge of any real utility in mechanical constructions.

Operation of bellows.

123. In the operation of bellows, of whatever construction, the pipe through which the air passes is not of great length, and our theory may be employed to more advantage in determining their effect, than in most other cases, while it is fortunate that this is almost the only one of real utility. The questions which present themselves to the engineer in connection with this inquiry, are to determine the proper size of the bellows, the requisite load on the board or piston, what size of tube will give the blast which the service requires, and what force must be employed to give them the necessary degree of motion. We shall accomplish these purposes by considering the efflux of the compressed air through the tube.

That we may proportion every thing to the power employed, we must recollect, that if the piston of a cylinder employed for expelling air be pressed down with any force p, it must be considered as superadded to the atmospheric pressure P on the same piston, in order that we may compare the velocity v of efflux with the known velocity V with which air rushes into a void. From the principles illustrated in the preceding part of this section (art. 112), it appears that this velocity is $v = V \times \sqrt{\frac{p}{P+p}}$, where P is the pressure of the atmosphere on the piston, and p the additional load laid on it. This velocity is expressed in feet per second; and, when multiplied by the area of the orifice (also expressed in square feet), it will give us the cubical feet of condensed air expelled in a second; but the bellows are always to be filled again with common air, and therefore we want to know the quantity of common air which will be expelled; for it is this which determines the number of strokes which must be made in a minute, in order that the proper supply may be obtained. Therefore we must recollect, that the quantity expelled from a given orifice with a given velocity, is in the proportion of the density; and that when D is the density of common air produced by the pressure P, the density d produced by the pressure $P + p$, is $D \times \frac{P+p}{P}$; or if D be made 1, we have $d = \frac{P+p}{P}$.

Hence, calling the area of the orifice expressed in square feet A, and the quantity of common air, or the cubic feet expelled in a second Q, we have

$$Q = V \times A \times \sqrt{\frac{p}{P+p}} \times \frac{P+p}{P}.$$

It will be sufficiently exact for all practical purposes to suppose P to be 15 pounds on every square inch of the piston: and p is then conveniently expressed by the pounds of additional load on every square inch; we may also take $V = 1339$ feet, the velocity with which air rushes into a vacuum, as determined in (art. 107).

As the orifice through which the air is expelled is generally very small, never exceeding three inches in diameter, it will be more convenient to express it in square inches; which being the $\frac{1}{144}$ of a square foot, we shall have the cubic feet of common air expelled in a second, or $Q = \frac{1339}{144} A \sqrt{\frac{p}{P+p}} \times \frac{P+p}{P}$; or, $9{\cdot}3\,A \sqrt{\frac{p}{P+p}} \times \frac{P+p}{P}$; which seems to be as simple an expression as we can obtain.

This will, perhaps, be illustrated by taking an example in numbers. Let the area of the piston be four square feet, and the area of the round hole through which the air is expelled two inches, its diameter being 1·6, and let the load on the piston be 1728 pounds; this is three pounds on every square inch. We have $P = 15, p = 3, P + p = 18$, and $A = 2$; therefore we shall have $Q = 2 \times 9{\cdot}3 \times \sqrt{\frac{3}{18}} \times \frac{18}{15}$, $= 9{\cdot}056$ cubic feet of common air expelled in a second. This will, however, be diminished about one-third by the contraction of the jet; and therefore the supply will not exceed six cubic feet per second. Supposing, therefore, the blowing-machine to be a cylinder or prism of this dimension in its section, the piston so loaded would (after having compressed the air) descend about 15 inches in a second. It would first sink $\frac{1}{6}$ of the whole length of the cylinder pretty suddenly, till it had reduced the air to the density $\frac{18}{15}$, and would then descend uniformly at the above rate, expelling six cubic feet of common air in a second.

The computation is made much in the same way for bellows of the common form, with this additional circumstance, that as the loaded board moves round a hinge at one end, the pressure of the load must be calculated accordingly. The computation, however, becomes a little intricate, when the form of the loaded board is not rectangular; it is almost useless when the bellows have flexible sides, either like smith's or organ bellows, because the change of figure, during their motion, makes continual variation on the compressing powers. It is therefore chiefly with respect to the wooden or iron bellows, of which the board or piston slides down between the sides, that the above calculation is of service.

The propriety, however, of this piece of information is evident: we do not know precisely the quantity of air necessary for animating a furnace; but this calculation tells us what force must be employed for expelling the air that may be thought necessary. If we have fixed on the strength of the blast, and the diameter of the cylinder, we learn the weight with which the piston must be loaded; the length of the cylinder determines its capacity, and the above calculation tells the expence per second; hence we have the time of the piston's coming to the bottom. This gives us the number of strokes per minute: the load must be lifted up by the machine this number of times, making the time of ascent precisely equal to that of descent; otherwise the machine will either catch and stop the descent of the piston, or allow it to lie inactive for awhile after each stroke. These circumstances determine the labour to be performed by the machine, and it must be constructed accordingly.

In machines which force the piston or bellows-board with a certain determinate motion, different from that due to their own weight, the computation, as we have observed before, becomes extremely intricate; we may, however, approximate to the force required by employing the following considerations:

Every time the piston is drawn up, a certain space of the cylinder is filled again with air of the common density, and this is expelled during the descent of the piston. A certain number of feet of common air is therefore expelled during this time with a velocity, perhaps constantly varying; but there is doubtless a medium velocity with which it might have been uniformly expelled, and a pressure corresponding to that velocity. To find this, divide the area of the piston by the area of the blast-hole multiplied by ⅝ (assuming this fraction for the contraction of the vein), and multiply the length of strokes performed in a second by the quotient arising from this division, and the product is the medium velocity of the air of natural density. Then find, by the proper rules, the height through which a body must fall to acquire this velocity, and this will be the height of a homogeneous atmosphere which would expel it with the same velocity. The weight of this column is the least force that can be exerted by the engine; it is too small to overcome the resistance in the middle of the stroke, it is too great for the end of the stroke, and much too great for the beginning. But if the machine be turned by a very heavy water-wheel, this will act as a regulator, accumulating in itself the superfluous force during the two favourable positions of the crank, and exerting it by its *vis insitu* during the time of its greatest effort. A force not greatly exceeding the weight of this column of air will, therefore, suffice. On the other hand, if the strength of the blast be determined, which is the general state of the problem, this will give the degree of condensation of the air, and the load on the square inch of the piston, or the mean force which the machine must exert on it.

Motion of elastic fluids.

Banks' experiments on the blast of bellows.

124. Mr. Banks, in his Treatise on the Power of Machines, describes certain experiments performed by him for determining the velocity with which air is driven out of bellows, of any form or size, and loaded with unknown weights, and without knowing the area of the pipe or aperture through which it is discharged.

Apparatus. Fig. 15.

Referring to (fig. 15), B is a tube of brass or iron, to which a bottom may be screwed or soldered, but not with soft solder; AC is a tube made fast in that bottom, and reaching nearly to the top. The upper plate L is fixed with a screw, or with screws and leather; DE is a pretty strong glass tube, the bore of which need not be more than ⅛th of an inch. When the instrument is to be used it is filled with water to within an inch of the top, and the end A may be stuck fast into a hole made to receive it in the upper board of a pair of common smith's bellows; then blowing gently, the pressure will act upon the surface of the water contained in the instrument, and cause it to rise in the tube DE, which is open at top, and by the altitude of the water we know by means of the following table the velocity with which the air leaves the bellows.

When the air requires to be much compressed, as in iron-furnaces, the water would flow out at the top of the tube, except it were 5 feet or more in length; in this case quicksilver may be used instead of water, and the tube may be shorter; one foot, for instance, would be long enough for the strongest blast, that is, for the greatest compression of air in the most powerful bellows.

TABLE. *Of the velocity communicated to air by water, &c.*

Altitude of mercury.	Altitude of water.	Velocity of air, in feet, per second.	Altitude of mercury.	Altitude of water.	Velocity of air, in feet, per second.
Inches.	Inches.		Inches.	Feet.	
·077	1	42	8·3	9	442
·15	2	57	10·0	12	510
·23	3	73	13·0	14	550
·30	4	85	14·0	16	589
·38	5	95			
·46	6	104	Feet.	Feet.	
·53	7	112	1·30	17	608
·61	8	123	1·38	18	622
·69	9	128	1·46	19	638
·77	10	134	1·53	20	658½
·84	11	141	1·61	21	672
·92	12	147½	1·70	22	690⅗
			1·77	23	704
1·15	15	164	1·84	24	719
1·38	18	180	1·92	25	736
1·61	21	194	2·00	26	748
			2·07	27	763
1·84	24	208	2·15	28	779
2·80	36	251·8	2·23	29	789
3·70	48	294	2·30	30	803
4·15	54	312	2·38	31	818
4·61	60	329	2·46	32	832
5·53	72	360	2·53	33	845

By means of this table, and the instrument above described, it will be easy to compare the velocities communicated to air by different bellows, whether we employ mercury or water. In small velocities water will be the most convenient, as the scale of variation will be larger; but when the pressure is equal to 4 or 5 feet of water, on account of the length of the tube, quicksilver will be the best fluid to employ; the instrument or gage may be connected with any pair of bellows or tube, where the air is condensed; a hole may be bored through the iron, near the *tewel*, or nosel of the bellows, as a proper place to try the condensation.

Let us suppose, for example, it were observed that the mercury was raised 5¼ inches, then we should learn from an inspection in the above table, that the velocity of the air was 360 feet per second, or about 245 miles per hour; and in the same manner any other elevation of the mercury or water will point out the velocity of the effluent air.

A different construction. Fig. 16.

125. Instead of a long tube for a strong blast, Mr. Banks proposes the following adjustment, viz. to seal the end E (fig. 16). According to this plan, a tube 12 inches long will be amply sufficient, to which a scale may be adapted.

Let us suppose it filled with water to DD, and the top L screwed tight; then whatever quantity of air may be forced through AC into the gage B, it will press upon the water, and cause it to rise in the tube DE, till the condensed air is of the same density as that in B: that is, if the density in B be doubled, that in DE will be doubled also, or the water will have filled half the tube. In short, the density in B, and the upper part of the tube DE, will always be the same; and in the latter it will be reciprocally as the length; that is, as the length of the tube is to the pressure of the atmosphere, so is the length of the column of compressed air to the force which compresses it; which force being thus known, the dependent velocity may be determined as in the former case.

In the foregoing articles we have spoken only of the theory of the operation of bellows and blowing machines; for their construction we must refer the reader to their respective heads in our alphabetical arrangement.

§ VIII. *On the motion, impulse, and resistance of air.*

Of the velocity of the wind.

126. One of the first and most obvious effects of the motion of air, is the natural production of wind, the causes of which have given rise to much philosophical inquiry and discussion. It is, however, not our province, in the present article, to investigate causes, but to estimate effects; we shall therefore refer the reader to the article WIND, in our alphabetical arrangement, for an account of the philosophical arguments and theories respecting the causes of the production of this important natural agent, and shall here confine ourselves to a statement of its actual operation, and of the effects it is capable of producing.

It is a matter of considerable importance to mechanical science, as well as an interesting subject of philosophical inquiry, to determine the velocity of the wind; but no good and unexceptional method has been contrived for this purpose.

The space passed over by the shadow of a clo
given time has been proposed as one of the be
this is extremely fallacious. In the first place it
tain, that if even we admit the cloud to have th
city of the air in which it is carried along, this
an exact measure of the current on the surface
earth; we may be almost certain that it is grea
air, like all other fluids, is retarded by the sid
bottom of the channel in which it moves. But
next place, it is very gratuitous to suppose t
velocity of the cloud is the same as the velocit
stratum of air between the cloud and the earth;
it is pretty certain that it is not. It has been
by Dr. Hutton of Edinburgh, that clouds are
formed when two parcels of air of different te
tures mix together, each sustaining a proper q
of vapour in the state of chemical solution,
know that different strata of air will flow in d
directions for a long time; a cloud, therefore,
acted upon by two distinct currents, and cannot
means of forming a correct estimate of the vel
either.

The cloud being formed on the contiguous su
two different currents, it is pretty obvious, that
of these should come from the east and the oth
the west, with equal velocities, the cloud form
tween them will have no motion at all; and sho
come from the east and the other from the no
cloud will move from the north-east, but it will n
the velocity of either of the principal current
certain measure of the wind is therefore to be f
the motion of the clouds; a better estimate
made by thick smoke from a furnace; but t
of all seems to be the velocity with which a ba
carried from one place to another at a consi
distance.

We have some few partial observations of thi
thus Lunardi in one of his voyages passed
miles in an hour, although at the time of his
from Edinburgh it was quite a calm, and conti
during his whole voyage; Garnerin, also, in his
from London to Colchester, passed over a
estimated at 60 miles, in three quarters of a
which gives for the velocity of the wind, at th
80 miles per hour, or 1⅓ mile per minute.

That the currents of air, which we call win
with such a velocity as this is therefore obvio
we have no means still of determining the actu
city at the surface of the earth, nor the natur
gale which is produced by such a velocity of th
because it is the velocity of the upper currents
thus determined, of the violence of which the
has no means of judging, as he is moving in th
fectly easy, and the calm at the surface (as
pened in both these cases) equally prevents the
trial spectator from judging of the effect such a
would produce.

127. Mr. Smeaton, in the 59th volume of th
sophical Transactions, has given the velocit
wind corresponding to the usual denomination
language. These were communicated to him
Rouse, and are founded upon great practical
ledge and experience.

Motion, impulse, and resistance of air.

TABLE. *Of the different velocities and forces of the wind, according to their common appellations. By J. Smeaton.*

Velocity of the wind.		Perpendicular force on a square foot in lbs. avoirdupois.	Common appellations of the wind.
Miles per hour.	Feet per second.		
1	1·47	·005	Hardly perceptible.
2	2·93	·020	Just perceptible.
3	4·40	·044	
4	5·87	·079	Gentle pleasant gale.
5	7·33	·123	
10	14·67	·492	Pleasant brisk gale.
15	22·00	1·107	
20	29·34	1·968	Very brisk gale.
25	36·67	3·075	
30	44·01	4·429	High winds.
35	51·34	6·027	
40	58·68	7·873	Very high winds.
45	66·01	9·963	
50	73·35	12·000	A storm or tempest.
60	88·02	17·715	A great storm.
80	117·36	31·490	A hurricane.
100	146·70	49·200	A hurricane that tears up trees by their roots, and carries buildings, &c. before it.

128. There have been various instruments invented for measuring the velocity and the power of the wind on a given surface, some few of which it will be proper to describe in this article. They are generally denominated *anemometers* or *wind-gages*.

1. The first instrument of this description seems to have been invented by Wolfius, in 1708, and is described by him in his "Areometry," in 1709, and afterwards in two or three other works of this author; but as considerable improvements have since been made upon the construction proposed by Wolfius, we think it unnecessary to enter here into a description of his machine.

One of the most ingenious and convenient methods for measuring the force of the wind is to employ its pressure in supporting a column of water, in the same way as M. Pitot measures the velocity of a current of water, a description of which machine is given in page 293, HYDRAULICS.

Lind's d-gage. 2. An instrument of this kind was, we believe, first proposed by Dr. Lind, and is described by him in the Phil. Trans. vol. lxxv. a brief account of which is as follows. Referring to (fig. 17), AB, CD, are two glass tubes, which should not be less than 8 or 9 inches long, the bore of each being about ·4 of an inch in diameter; they are connected together by a small bent glass tube *a b*, only $\frac{1}{10}$th of an inch in diameter, to check the undulations of the water, caused by a sudden gust of wind. On the upper end of the leg AB is fitted a thin metal tube, having its end open to receive the wind blowing horizontally into it. The two tubes, or rather the two branches of the tube, are connected to a steel spindle KL by slips of brass near the top and bottom, by the sockets of which, at *e* and *f*, the whole instrument turns easily about the spindle, which is fixed into a block by a screw in its bottom, by the wind blowing into the orifice at F. When the instrument is used, a quantity of water is poured in, till the tubes are about half full; then exposing the instrument to the wind, this, by blowing in at the orifice F, forces the water down lower in the tube AB, and raises it so much higher in the other tube; and the difference between the heights of the surfaces of the water in the two tubes, estimated by a scale of inches and parts HI, placed by the side of them, will be the height of a column of water, whose pressure is equal to the force or momentum of the wind, blowing or striking against an equal base.

A cubic foot of water weighs 1000 ounces, or $62\frac{1}{2}$ pounds, the twelfth part of which is $5\frac{1}{5}$ lbs. nearly; therefore, for every inch the surface of the water is raised, the force of the wind will be equal to so many times $5\frac{1}{5}$ lbs. on a square foot. Thus, suppose the water to stand 3 inches higher in one tube than in the other, then $3 \times 5\frac{1}{5} = 15\frac{3}{5}$ lbs. will be equal to the pressure or force of the wind on the surface of a square foot.

Dr. Hutton's results. 129. This instrument of Dr. Lind measures only the force or momentum of the wind, and not its velocity; but Dr. C. Hutton, in his Mathematical Dictionary, proposes a method of deducing the velocity by means of the above machine and certain experiments performed by himself at the Royal Military Academy, Woolwich. From these experiments it appears, that a plane surface of a square foot suffers a resistance of 12 ounces from the wind, when blowing with a velocity of 20 feet per second, and that the force is nearly as the square of the velocity. Hence, then, taking the force at $15\frac{3}{5}$ lbs. found above, for the force of the wind, when it sustains 3 inches of water, and taking the square roots of the forces, we shall have

$$\surd 12 : \surd 15\tfrac{3}{5} :: 20 : 22\tfrac{4}{5};$$

that is, $22\frac{4}{5}$ feet per second, or $15\frac{1}{2}$ miles per hour, is the rate at which wind blows when the distance between the surfaces of the two columns of water, in the instrument above described is three inches. And further, as the said height is as the force, and the force as the square of the velocity, we shall have the force and velocity corresponding to the several heights of the water in the one tube above that in the other, as in the following table:

TABLE. *Of the corresponding height of water, the force on a square foot, and the velocity of wind.*

Height of the water.	Force of wind on a square foot.	Velocity per hour.
$0\frac{1}{4}$ inches.	.... lbs.	18·0 miles.
$0\frac{1}{2}$	....	25·6
1	5·2	36·0
2	10·4	50·8
3	15·6	62·0
4	20·8	72·0
5	26·0	80·4
6	31.25	88·0
7	36·5	95·2
8	41·7	101·6
9	46·9	108·0
10	52·1	113·6
11	57·3	119·2
12	62·5	124·0

These results differ in some respects from those given above from Mr. Smeaton's paper; but, perhaps, the agreement is as near as is to be expected from two different observers, considering that the best means of obtaining the requisite data cannot but be considered as extremely deficient in point of accuracy. We have seen in the article above referred to in Hydraulics, that experiments made with Pitot's tube, which Dr. Lind's instrument strongly resembles, requires various corrections before we can conclude the velocity of the current from it, and similar corrections are in all probability equally necessary in the present case; but it would require multiplied experiments in order to form a correct measure of the various modifications that may be requisite, in order to reduce the operation of the machine to accuracy and uniformity.

Regnier's anemometer.

130. It would be inconsistent with the nature of this work to describe the various constructions that have been proposed under the denomination of anemometers, or wind-gages, we shall therefore content ourselves with a short account of the following, as described by M. Regnier:

Figs. 18 and 19.

Figures 18 and 19 exhibit two views of this machine, from which the nature of its operation will be as well understood as from any verbal description that we can give; GH is a surface of given dimensions, which is exposed to the direct impulse of the wind; this surface, as shown in the first figure, is attached to the stem EB in the second. AF is a spring which is forced back by the operation of the wind, the power of which spring may be previously ascertained, and the effect registered on the graduated dial-plate KI (fig. 18). The index is fixed to the wheel C, shown in the second figure, and which is made to revolve by means of the rectilinear motion of the principal stem EB. This appears to be by far the simplest construction for an anemometer of any which has been yet invented. See "Bulletin de la Société d'Encouragement, &c." No. 150.

Resistance of the air.

131. The next important subject for investigation, connected with this branch of Pneumatics, is the absolute resistance which a body of given figure and dimensions experiences in passing through the air with a given velocity, and the actual effect produced upon a body either at rest or in motion by the action of the air or wind. This is, indeed a particular case of the resistance of fluids, a subject which we have already had occasion to treat of partially in our treatise on Hydrodynamics; that is, we have there examined the subject as far as relates to the percussion and resistance of incompressible fluids, more especially water, and have detailed a great variety of experiments performed by different authors, with a view of obtaining the requisite data, but we reserved those which relate to the resistance of the air for the present section, in consequence of their obvious connection with the science of Pneumatics. The common theory of resistances treated of in the article above referred to, being general for all descriptions of fluids, we shall not repeat it in this place; but shall pass immediately to an account of the most satisfactory and best-conducted experiments that have been made on the resistance of the air, and of the results that have been deduced from them.

Experiments on this subject are by no means numerous; at least such experiments as can be depended upon for the foundation of any practical application. The first that have any claim to this character, are those published by Robins, in 1742, in his Treatise on Gunnery. They were repeated, with some additions and variations, by Borda, and some account of them published in the Memoirs of the Academy of Paris, in 1763. We find also an account of similar experiments by Mr. Edgeworth, in the Philosophical Transactions, vol. lxxiii.; but the most complete of any hitherto made are those detailed by Dr. C. Hutton, in the 3d volume of his Mathematical Tracts, 1812.

Robins' experiments.

132. The principal objects which Mr. Robins seems to have had in view in carrying on the following experiments, were to establish the two following propositions:

1. "That the resistance of the air to a 12 lb. iron bullet, moving with a velocity of 25 feet per second, is not less than $\frac{1}{2}$ an ounce avoirdupois."

2. "That the resistance of the air within certain limits, is nearly in the duplicate ratio of the velocity of the resisted body."

These propositions, the author observes, are neither unknown nor doubtful; but yet as they are the basis of some other assertions, which have been hitherto contested on the one hand and denied on the other, it may be proper to evince their veracity by more unquestionable and simpler methods than have been hitherto practised. In order to effect this, Mr. Robins caused a machine to be constructed, which he called a *whirling machine*, represented in (fig. 20), and the description of which is given by the author as follows: "BCDE is a brass barrel, moveable on its axis, and so adjusted by means of friction-wheels, which are not represented in the figure, as to have no friction worth attending to. The frame in which this barrel is fixed is so placed that its axis may be perpendicular to the horizon. The axis itself is continued above the other plate of the frame, and has fastened on it a light hollow cone AFG; from the lower part of this cone there is extended a long arm of wood GH, which is very thin, and cut feather-edged, and at its extremity there is a contrivance for fixing on the body, whose resistance is to be investigated, as here, the globe P; and to prevent the arm GH from swaying out of its horizontal position by the weight of the annexed body P, there is a brace AH, of fine wire fastened to the top of the cone, which supports the end of the arm."

"Round the barrel BCDE there is wound a fine silk line, the turns of which appear in the figure, and after this line has taken a sufficient number of turns, it is conducted horizontally to the pulley L, over which it is passed, and then a proper weight M is hung to its extremity. If this weight be left at liberty, it is obvious that it will descend by its own gravity, and by its descent will cause the barrel BCDE to revolve, together with the arm GH, and the body P fastened to it. While the resistance on the arm GH and the body P is less than the weight M, that weight will accelerate its motion, and thereby the motion of GH and P will increase, and consequently their resistance will increase, till at last this resistance and the weight M becomes nearly equal to each other. The motion with which M descends, and with which the body P revolves, will not sensibly differ from an equable one. Whence it is not difficult to conceive that, by proper observations made with this machine, the resistance of the body P may be determined."

133. "The most natural method of proceeding in this investigation is as follows: Let the machine have first acquired its equable motion (which, as will hereafter

appear, will be usually attained in five or six turns from the beginning), and then let it be observed, by counting a number of turns, what time is taken up by one revolution of the body P; then taking off the body P and the weight M, let it be examined what smaller weight will make the arm GH revolve in the same time as when P was fixed to it; this smaller weight being taken from M, the remainder is obviously equal in effort to the resistance of the revolving body P; and this remainder being reduced in the ratio of the length of the arm to the semidiameter of the barrel, will then become equal to the absolute quantity of the resistance. And as the time of one revolution is known, and consequently the velocity of the revolving body, there is thereby discovered the absolute quantity of the resistance to the given body P, moving with a given degree of celerity."

"And note, that to avoid all exceptions, I have generally chosen when the body P was removed, to fix in its stead a thin piece of lead, of the same weight, placed horizontally; so that the weight which was to turn round the arm GH without the body P, did also carry round this piece of lead. This I did lest it should be objected that the body P retarded the weight M by its quantity of matter, as well as by its resistance. But mathematicians will easily allow, that there is no necessity for this precaution.

"The measures of the parts of this machine were as follow:

	Inches.
"The diameter of the barrel BCD, and of the silk string wound round it, was................	2·06
"The length of the arm GH, measured from the axis to the surface of the globe P, was...	49·5
"The body P, the globe made use of, was of pasteboard, its surface very neatly coated with marble paper, and was not much distant from the size of a 12 lb. shot, being in diameter..	4·5
"So that the radius of the circle described by the centre of the globe, was	51·75

"When this globe was fixed at the end of the arm, and a weight of half a pound was hung at the end of the string at M, it was examined how soon the motion of the descending weight M, and of the revolving globe P would become equable as to sense; and with this view, three revolutions being suffered to elapse, it was found, that

The next	10	were performed in......	27¾″
	20	in less than..............	55
	30	in..........................	82½
So that the 1st	10	were performed in......	27¾
2d	10	in	27¼
3d	10	in	27½

These experiments sufficiently evince, that even with half a pound, the smallest weight hereafter used, the motion of the machine was sufficiently equable after the first three revolutions.

"Now, to prove the two fore-mentioned propositions, the following experiments were made; the times marked down being observed by several stop-watches, which rarely differed half a second from each other.

"The fore-mentioned globe being fixed at the end of the arm, there was hung on, in the situation M, a weight of 3¼ lb. and 10 revolutions being suffered to elapse, the succeeding 20 were performed in 21½″.

Motion, impulse, and resistance of air.

"The fore-mentioned globe being fixed at the end of the arm, there was hung on, in the situation M, a weight of 3¼ lbs. and 10 revolutions being suffered to elapse, the succeeding 20 were performed in 21½″.

"Then the globe being taken off, and a thin plate of lead, equal to it in weight, placed in its room, it was found, that instead of 3½lb. a weight of 1lb would make it revolve in less time than it did before, it performed 20 revolutions, after 10 were elapsed, in the space of 19 seconds.

"Hence it follows, that from the 3¼lb. first hung on, there is less than 1 lb. to be deducted for the resistance on the arm, and consequently the resistance on the globe itself is not less than the effort of 2½ lb. in the situation M; and it appearing from the former measures, that the radius of the barrel is nearly $\frac{1}{50}$ part of the radius of the circle described by the centre of the globe; it follows, that the absolute resistance of the globe, when it revolves 20 times in 21½″ (which, if computed by the measures given above, comes out a velocity of about 25½ feet in a second), it follows, I say, that the resistance of the globe in this case is not less than the $\frac{1}{40}$ part of 2¼lb. or than the $\frac{1}{50}$ part of 36 ounces; and this being considerably more than half an ounce, and the globe being nearly the size of a 12-pounder, it irrefragably confirms our first proposition, '*That the resistance of the air to a* 12*lb. iron bullet, moving with a velocity of* 25 *feet in a second, is not less than half an ounce avoirdupois.*'

"The next experiments were made with a view of examining the second proposition. And, for this purpose, there were successively hung on, in the situation M, weights in the proportion of the numbers, 1, 4, 9, 16; and letting 10 revolutions first elapse, the observations on these were as follow:

With 1 lb.	the globe went	20 turns in	54½″
	that is, it went	10 turns in	27¼
With 2 lb.	it went..........	20 turns in	27½
With 4½ lb.	it went..........	30 turns in	27½
With 8 lb.	it went..........	40 turns in	27½

"So that it appears, that to the resistance of the proportions of the numbers 1, 4, 9, 16, there correspond velocities of the resisted body, in the proportion of the numbers 1, 2, 3, 4; which proves with great nicety, within the limits of these experiments, the second proposition, '*That the resistance of the air is nearly in the duplicate proportion of the velocity; 9 times as much when it moves with three times the velocity, and so on.*'"

Dr. C. Hutton's experiments.

Dr. Hutton's experiments.

134. The experiments thus ingeniously proposed and commenced by Mr. Robins, were prosecuted with great perseverance and ability by Dr. Hutton, with the same machine, or, if it were not the identical one, it was, at least, one made by the same accurate workman, Mr. Ellicot, and precisely answering the above description; it is still preserved, in perfect order, in the model-room of the Royal Military Academy, Woolwich.

Dimensions of the balls and apparatus.

Dr. Hutton commenced his experiments by ordering three hemispheres of pasteboard, of different sizes, to be very accurately made, with a view of determining the difference in the resistances between the plane surface and the convex side of the hemisphere; the weight and dimensions of which were as follow:

Pneumatics.

	No. 1.	No. 2.	No. 3.
Weights	2 oz. 5 drs.	2 oz. 11 drs.	4 oz. 3 drs.
Diameters	4·75 inches,	5·1 inches,	6·375 inches.
Area of the circle	17·7 inches, or $\frac{1}{8}$ foot	21·2 inches, or $\frac{5}{34}$ foot	32 inches, or $\frac{2}{9}$ foot.

The radius of the barrel, about which the string was wound, was found to be very accurately 1·044 inch, and the radii of the circles, described by the several hemispheres, were determined to be as follow:

	No. 1. In.	No. 2. In.	No. 3. In.
Radii	52·42	52·74	53·34
Circumferences	329·36	335·38	335·41

And, consequently, the ratio between the radius of the barrel and the radii of the above circles were

50·2	50·5	51·1

These data being established, the following experiments were performed.

Experiments with a hemisphere.

The hemisphere, No. 3, was fitted on, and actuated by a small weight of only $1\frac{3}{4}$ oz. or 1 oz. 12 drs.; that is, this weight was suspended from the end M of the thread, opposing first the convex side to the air, and then the flat side; when the number of rounds and corresponding times, were observed as follow, both in this and all the other experiments, by stop-watches, and by a clock of peculiar construction, made for the purpose.

Experiments.—*First series.*

Plane side foremost.

No. of turns.	Time.	Difference.
1	0′ 27″	—
2	0 47	20
3	1 6	19
4	0 24	18
5	0 42	18
6	2 0	18
7	0 18	18
8	0 36	18
9	0 54	18
10	3 $12\frac{1}{2}$	$18\frac{1}{2}$
11	0 $30\frac{1}{2}$	18
12	0 48	$17\frac{1}{2}$
13	4 5	17
14	0 $22\frac{1}{2}$	$17\frac{1}{2}$

The mean of all these gives for the time of one turn 18″, or $\frac{1}{18}$th of a revolution per second.

Convex side foremost.

No. of turns.	Time.	Difference.
3	0′ 59″	—
6	1 37	38
9	2 12	35
12	2 47	35
15	3 22	35

The mean of these, gives for the difference due to 1 turn $11\frac{2}{3}$″, or $\frac{3}{35}$ths of a turn to 1 second.

Note.—That the resistance of the arm is still wanted to be deducted from these resistances, and then to be reduced from the circumference of the barrel to the centre of the hemisphere.

In the above table, the 1st column shows the number of turns made by the arm; the 2d, the corresponding times elapsed from the beginning; and the 3d, shows the differences of those times, answering to differences of the turns in the 1st column.

135. A thin piece of lead, of the same weight as the hemisphere, was now fitted on the arm, and it was found that it required about 10 drs. to overcome the friction and *vis inertia*, and to give a small motion to the machine. With 12 drs. the average of the turns was from $11\frac{1}{2}$ to 12 seconds each, being nearly the same velocity that the convex side of the hemisphere had in the above trials; but, that with a weight of 24 drs. the average time of one turn of the arm was about 6″. It appeared, therefore, that each turn was made in about 12″ with 12 drs., and in about 6″ with 24 drs.; whence it was inferred that a weight of 8 drs. would carry the arm round in about 18″, being the time of revolution with the plane of the circle foremost.

The weight in the former instance, we have seen, was 1 oz. 12 drs., or 28 drs.; the difference, therefore, between 28 drs. and 12 drs., viz. 16 drs. or 1 oz. is the part of the weight overcome by, and equal to, the resistance of the air, measured at the circumference of the barrel, against the convex side of the large hemisphere, moving with a velocity of 2·394 per second; and 28 — 8 = 20 ozs. is the difference in the second case; that is, with the plane side moving forward, when the velocity is 1·552 feet per second; and, each of these divided by 51·1 (the ratio between the radii of the barrel and of the circle described by the centre of the hemisphere) will give the actual resistance acting at the centre of the body.

Weights were next tried for producing motion in the hemisphere, with the plane side foremost, in order to give it the same velocity as in the first table, when the convex side was foremost; and the mean of the several results was 1 turn in $11\frac{2}{3}$ seconds, with a weight of 2 ozs. 12 drs. In each trial the motion became uniform after the first two or three turns, after which it was permitted to go 12 times round, and the time noted down at every round. The difference of these times commonly agreed with the mean time to half a second, or less; but the mean time itself was very accurately obtained by dividing the whole of the observed times of the equable rounds by the number of them, from which was determined the mean result above stated, viz. 1 round in $11\frac{2}{3}$ seconds.

It appears, therefore, that with a velocity of 1 turn in $11\frac{2}{3}$ seconds, the arm and plane side of the body were resisted with a weight of 2 ozs. 12 drs.; the arm and convex side of the body with a weight of 1 oz. 12 drs.; and the arm, with a thin piece of lead of equal weight to the hemisphere, with 12 drs.; the velocities in the three cases being as nearly equal as possible. Subtracting, therefore, 12 drs. from each of the preceding resistances, we find the resistance of the air in the first case, viz. with the plane side foremost, equal to 2 ounces; and, in the latter, viz. with the convex side foremost, 1 ounce. The one being just double the other, which agrees with our theoretical deduction, page 259.

These resistances, however, it will be observed, are those which take place at the circumference of the barrel; they must each be divided by the ratio between the radii of this barrel and that of the circle described, in order to obtain the actual resistance at the centre of the body: this ratio, we have seen, is 51·1; wherefore, $\frac{2}{51 \cdot 1} = \cdot 039$ oz. for the absolute resistance of the flat surface; and $\frac{1}{51 \cdot 1} = \cdot 019\frac{1}{2}$ ozs. for the convex surface, each moving perpendicularly against the medium with a velocity of one turn in $11\frac{2}{3}$ seconds, or at the rate of 2·394 feet per second; the area of the circle in the first case being 32 inches, or $\frac{2}{9}$ths of a square foot.

EXPERIMENT.—*Second series.*

136. Various weights were now tried in order to bring the same hemisphere, with the flat side foremost, and the arm without it, to revolve each in 10 seconds. Then again the same body, both sides foremost, and the arm alone, all in 8 seconds each turn. The following are the particulars of these experiments:

For 10 seconds.					For 8 seconds.			
Weights.	Plane side.		Arm only.		Weights.	Plane side.	Convex side.	Arm only.
	12 turns.	1 turn.	12 turns.	1 turn.				
oz. dr.					oz. dr.			
3 0	140″	$10\frac{5}{6}$	—	—	4 10	8″	—	—
2 2	126″	$10\frac{1}{3}$	—	—	6 0	7″	—	—
3 4	123″	$10\frac{1}{4}$	—	—	8 0	6″	—	—
3 $5\frac{1}{2}$	120″	10	—	—	2 2	...	10″	—
1 0	...	...	108	9	2 14	...	8″	—
0 14	...	...	132	11	1 3	...	...	8
0 15	...	...	120	10	1 8	...	...	6

Now, by deducting each weight without the hemisphere, from the corresponding weight with it, there remains the weight due to the resistances of the two faces, as below:

For 10 *seconds.*

	ozs.	drs.	ozs.	drs.
From	3	$5\frac{1}{2}$	2	2
Take	0	15	0	15
Remains	2	$6\frac{1}{2}$	1	3

For 8 *seconds.*

	ozs.	drs.	ozs.	drs.
From	4	10	2	14
Take	1	3	1	3
Remains	3	7	1	11

Where in both cases the resistance to the plane side is very nearly double that opposed to the convex, as required by the theory, page 259.

137. The following may be considered as the results of the experiments concluding the second series.

The number of turns in each experiment was 15, but the medium time of one revolution, deduced from the 12 last turns only, are entered in the following table, with the weights which produced them.

NOTE.—*The convex surface in the following experiments was foremost.*

Weight.	Time of 1 turn.	Weight.	Time of 1 turn.	Weight.	Time of 1 turn.	Weight.	Time of 1 turn.
oz. drs.	sec.	oz. drs.	sec.	oz. drs.	sec.	oz. drs.	sec.
4 10	$5\frac{5}{6}$	6 0	$4\frac{11}{12}$	7 12	$4\frac{1}{6}$	12 7	$3\frac{5}{24}$
4 9	$5\frac{3}{4}$	6 5	$4\frac{3}{4}$	8 1	$4\frac{1}{12}$	12 10	$3\frac{1}{4}$
4 11	$5\frac{5}{6}$	7 0	$4\frac{1}{2}$	11 5	$3\frac{3}{8}$	13 0	$3\frac{1}{6}$
5 12	5	7 8	$4\frac{1}{4}$	12 0	$3\frac{7}{24}$	—	—
8 5	$4\frac{1}{12}$	9 11	$3\frac{9}{12}$	13 12	$3\frac{1}{12}$	15 8	$2\frac{10}{12}$
8 8	$4\frac{1}{24}$	10 1	$3\frac{8}{12}$	14 0	3	16 0	$2\frac{9}{12}$

TABLE—*continued.*

Motion, impulse, and resistance of air.

Weight.	Time of 1 turn.	Weight.	Time of 1 turn.	Weight.	Time of 1 turn.	Weight.	Time of 1 turn.
oz. drs.	sec.	oz. drs.	sec.	oz. drs.	sec.	oz. drs.	sec.
8 12	4	10 8	$3\frac{7}{12}$	14 5	3	16 8	$2\frac{9}{12}$
9 0	$3\frac{11}{12}$	11 0	$3\frac{6}{12}$	14 10	$2\frac{11}{12}$	17 0	$2\frac{9}{12}$
9 6	$3\frac{10}{12}$	13 8	$3\frac{1}{12}$	15 0	$2\frac{11}{12}$	17 8	$2\frac{8}{12}$
18 0	2·67	28 0	2·12	38 0	1·79	58 0	1·46
19 0	2·58	29 0	2·10	40 0	1·75	60 0	1·42
20 0	2·50	30 0	2·00	42 0	1·75	64 0	1·33
21 0	2·42	31 0	2·00	44 0	1·67	68 0	1·33
22 0	2·42	32 0	2·00	46 0	1·58	72 0	1·29
23 0	2·33	33 0	1·92	48 0	1·54	76 0	1·25
24 0	2·25	33 0	1·92	50 0	1·54	80 0	1·21
25 0	2·25	35 0	1·83	52 0	1·50	84 0	1·17
26 0	2·25	36 0	1·83	54 0	1·50	88 0	1·17
27 0	2·17	37 0	1·83	56 0	1·50	92 0	1·13

138. The following experiments were carried on in a manner precisely similar to the above, except that the *flat side of the hemisphere was now presented to the resistance of the air.*

Plane surface foremost.

Weight.	Time of 1 turn.	Weight.	Time of 1 turn.	Weight.	Time of 1 turn.	Weight.	Time of 1 turn.
oz. drs.	sec.	oz. drs.	sec.	oz. drs.	sec.	oz. drs.	sec.
2 10	13·1	2 12	12·1	3 0	11·4	3 6	10·5
2 $10\frac{1}{2}$	12·7	2 13	11·9	3 1	11·2	3 8	10·3
2 11	12·5	2 14	11·8	3 2	10·9	3 10	10·0
2 $11\frac{1}{2}$	12·3	2 15	11·6	3 4	10·7	3 12	9·6
3 14	9·3	4 12	8·3	5 10	7·5	6 8	6·7
4 0	9·1	4 14	8·2	5 12	7·4	6 12	6·6
4 2	9·0	5 0	8·1	5 14	7·2	7 0	6·5
4 4	8·9	5 2	7·9	6 0	7·1	7 4	6·3
4 6	8·7	5 4	7·8	6 2	7·0	7 8	6·1
4 8	8·6	5 6	7·7	6 4	6·9	7 12	6·0
4 10	8·5	5 8	7·6	6 6	6·8	8 0	6·0
8 4	6·0	12 0	4·8	21 0	3·6	44 0	2·45
8 8	5·9	13 0	4·6	22 0	3·5	48 0	2·35
8 12	5·8	14 0	4·5	24 0	3·3	52 0	2·25
9 0	5·7	15 0	4·3	26 0	3·2	56 0	2·15
9 8	5·5	16 0	4·1	28 0	3·1	60 0	2·05
10 0	5·4	17 0	4·0	30 0	3·0	64 0	2·00
10 8	5·2	18 0	3·9	32 0	2·9	68 0	1·90
11 0	5·1	19 0	3·8	36 0	2·7	—	—
11 8	5·0	20 0	3·7	40 0	2·55	—	—
72 0	1·90	80 0	1·80	100 0	1·65	124 0	1·45
74 0	1·85	84 0	1·75	108 0	1·60	132 0	1·40
76 0	1·85	92 0	1·70	116 0	1·50	140 0	1·40

139. Considering the above as a sufficient number of results with the flat side foremost, the following experiments were performed upon the simple arm of the machine, the hemisphere being removed, and a piece of thin lead of equal weight put in its place.

Experiments with the lead.

The following are the results of these experiments:

Experiments on the simple arm of the machine.

Weight.		Time of 1 turn.	Weight.		Time of 1 turn.	Weight.		Time of 1 turn.	Weight.		Time of 1 turn.
oz.	drs.	sec.	oz.	drs.	sec.	oz.	drs	sec.	oz.	drs.	sec.
1	0	19·0	1	6	7·5	1	8	6·5	1	10	5·5
1	2	11·0	1	7	7·0	1	9	6·0	1	12	5·0
1	4	9·0	—		—	—		—	—		—
1	14	5·0	4	0	2·9	8	8	1·9	22	0	1·1
2	0	4·7	4	8	2·75	9	0	1·8	24	0	1·1
2	4	4·4	5	0	2·6	10	0	1·75	26	0	1·1
2	8	4·0	5	8	2·45	11	0	1·65	28	0	1·0
2	12	3·7	6	0	2·3	12	0	1·55	30	0	1·0
3	0	3·5	6	8	2·2	14	0	1·45	32	0	1·0
3	4	3·3	7	0	2·1	16	0	1·35	34	0	0·9
3	8	3·1	7	8	2·0	18	0	1·25	—		—
3	12	3·0	8	0	1·95	20	0	1·20	—		—

General table of results.

140. The author, considering now that he had obtained a sufficient series of experiments, both with the convex and plane surface foremost, as also with the simple arm of the machine, or with that arm loaded to an equal weight by a thin piece of lead; he deduces from them the following general table of results, in which the 1st column shows the several rates of velocity, from 3 to 20 feet per second; the next three columns contain the corresponding weights acting on the axis, the radius of which was 1·044 for both sides of the body, and for the arm without the body. The next three columns show the same quantities, but a little modified, in order to reduce them to a greater uniformity than was obtained from the actual experiments, a change, however, which only amounts to a little correction of the fractional parts. The two next columns show the difference in the three former, or the actual resistance opposed to the motion of the body at the circumference of the barrel, or axis; and the last contains the ratio of those resistances according as the plane or convex surface is presented to the action of the air.

Velocity per sec.	Experimental weights.			Corrected weights.			Real Resistance.		Ratio.
	Flat.	Convex.	Arm.	Flat.	Convex.	Arm.	Flat.	Convex.	
	oz.	oz.	oz.	oz.	oz.	oz.	oz.	oz.	
3	3·8	2·2	1·2	3.8	2·3	1·2	2·6	1·1	2·37
4	6·1	3·4	1·4	6·2	3·4	1·4	4·8	2·0	2·40
5	9·3	4·9	1·7	9·2	4·9	1·7	7·5	3·2	2·35
6	12·7	6·6	2·0	12·8	6·7	2·0	10·8	4·7	2·30
7	17	8·7	2·5	17·0	8·7	2·4	14·6	6·3	2·31
8	22	11·0	3·0	21·9	11·0	2·9	19·0	8·1	2·35
9	28	13·3	3·5	27·6	13·5	3·5	24·1	10·0	2·41
10	34	15·8	4·2	34·0	16·2	4·2	29·8	12·0	2·48
11	40	19·	5·2	41·0	19·2	5·0	36·0	14·2	2·53
12	48	23	6·0	48·7	22·6	5·9	42·8	16·7	2·56
13	56	27	6·8	57·1	26·4	6·8	50·3	19·6	2·57
14	64	31	7·8	66·2	30·6	7·8	58·4	22·8	2·56
15	73	35	8·6	76·0	35·1	8·9	67·1	26·2	2·56
16	84	40	10·	86·6	40·0	10·0	76·6	30·0	2·55
17	101	45	11	98·2	45·3	11·2	87·0	34·1	2·55
18	112	50	12	111·0	51·0	12·4	98·6	38·6	2·55
19	121	57	14	125·0	57·2	13·7	111·3	43·5	2·56
20	140	64	15	140·0	64·0	15·0	125·0	49·0	2·55

141. The reader will observe, that the resistances shown in the above table are those applied at the circumference of the barrel; and that to find the actual resistance in any case, when applied to the centre of the body, the corresponding tabular number must be divided by 51·1, the ratio of the radii of the two circles.

It appears from the last column of the above table, that the resistance to the plane side of the body is to the resistance on the convex surface, at a medium as 2·48 to 1, or as 2½ to 1 nearly, which exceeds considerably the ratio 2 to 1 assigned by the common theory of resistances, and which was found to obtain in the first series.

It also appears from the two columns for both the plane and convex surfaces, that the resistances increase in a ratio a little higher than that of the squares of the velocities. The exact determination of the index of the power of the velocity will be examined in a subsequent article.

142. These experiments being completed, the author next endeavoured to ascertain, whether a different figure hindmost, as well as foremost, would make any difference, and if any, what the amount of that difference might be. The hemisphere, cone, and cylinder, each with the plane surface foremost, have three different figures hindmost, and the contrary. Also the sphere and hemisphere with the convex surface foremost, have different figures behind, and it was the examination of the circumstances attending these several cases, which the author proposed to himself in the following series of experiments:

EXPERIMENTS.—*Third series.*

Short cylinder; diameter 6·375 inches, weight 5 ozs. 1 dr.; the time of revolution deduced from a mean of the last 30 out of 33 revolutions.

Weight.		Time.	Weight.		Time.	Weight.		Time.	Weight.		Time.
oz.	dr.	sec.	oz.	dr.	sec.	oz.	dr.	sec.	oz.	dr.	sec.
2	0	15·10	3	4	10·57	3	12	9·50	4	12	8·23
2	2	14·14	3	6	10·32	4	0	9·07	5	0	7·93
3	0	11·13	3	8	9·90	4	4	8·83	5	4	7·72
3	2	10·85	3	10	9·73	4	8	8·50	5	8	7·47
5	12	7·23	9	0	5·60	20	0	3·67	64	0	1·97
6	0	7·05	10	0	5·32	22	0	3·48	72	0	1·88
6	4	6·87	11	0	5·02	26	0	3·20	80	0	1·80
6	8	6·67	12	0	4·78	30	0	2·97	90	0	1·67
7	0	6·47	13	0	4·58	34	0	2·77	100	0	1·60
7	8	6·23	14	0	4·38	40	0	2·53	110	0	1·53
8	0	6·00	16	0	4·10	48	0	2·30	120	0	1·50
8	8	5·90	18	0	3·87	56	0	2·10	Thread broke.		

Experiments with the globe; diameter 6·375 inches, weight 8 ozs. 14 drs.

Weight.		Time.	Weight.		Time.	Weight.		Time.	Weight.		Time.
oz.	dr.	sec.	oz.	dr.	sec	oz.	dr.	sec.	oz.	dr.	sec.
2	0	14·5	12	0	3·35	30	0	2·07	80	0	1·22
3	0	8·55	14	0	3·1	34	0	1·95	90	0	1·15
4	0	6·75	16	0	2·85	40	0	1·8	100	0	1·07
5	0	5·75	18	0	2·7	48	0	1·65	110	0	1·02
6	0	5·1	20	0	2·6	56	0	1·5	112	0	0·97
8	0	4·3	22	0	2·45	64	0	1·4	—		—
10	0	3·8	26	0	2·22	72	0	1·3	—		—

143. In remarking on the results of the above experiments, the author observes, that the globe required the 1st weight, or 2 ozs. to put it barely in motion, and that a less weight would not stir it, which he considers to be attributable partly to the greater weight of the globe, and the length of string between the axis of the machine and the weight; the difference, therefore, observable with the small weights between the globe and cylinder, is not to be attributed to the form of the after-part of the body; in fact, those differences diminish as the weights increase, so as to become, at length, almost insensible; whence the author concludes that, under circumstances, in all other respects the same, the hinder part of the body produces no sensible effect.

144. Experiments on the cone; diameter 6·375 inches, height 6·625 inches, weight 8 ozs. 6 drs.

Vertex foremost.

Weight.		Time.	Weight.		Time.	Weight.		Time.	Weight.		Time.
oz.	dr.	sec.	oz.	dr.	sec.	oz.	dr.	sec.	oz.	dr.	sec.
2	0	13·50	8	0	4·42	18	0	2·80	48	0	1·65
3	0	8·65	9	0	4·15	20	0	2·67	56	0	1·52
4	0	6·95	10	0	3·87	24	0	2·40	64	0	1·42
5	0	5·97	12	0	3·57	28	0	2·17	72	0	1·35
6	0	5·30	14	0	3·25	32	0	2·07	80	0	1·30
7	0	4·77	16	0	3·00	40	0	1·85	100	0	1·17

By comparing these times with those in the preceding table, it appears that they are a very little greater, and consequently that the resistance to the convex surface of the cone is nearly equal to, but rather greater than that on the globe, or hemisphere, the diameters being the same.

In the following experiments the base of the cone was foremost:

Weight.		Time.	Weight.		Time.	Weight.		Time.	Weight.		Time.
oz.	dr.	sec.	oz.	dr.	sec.	oz.	dr.	sec.	oz.	dr.	
12	0	4·95	24	0	3·40	48	0	2.37	72	0	1·93
16	0	4·25	30	0	3·05	56	0	2·20	80	0	1·85
20	0	3·77	40	0	2·60	64	0	2·02	100	0	1·65

These times are rather greater than those of the cylinder, but the cone was rather the heavier.

As far as these experiments go, therefore, we cannot infer that the hinder part of the body produces any very sensible effect.

EXPERIMENTS.—*Fourth series.*

145. The author's next object was to ascertain whether the resistances to two similar bodies were as the surfaces, all other things being the same; and with this view experiments were made upon the smaller hemisphere, No. 1, the base of which was to that of No. 3, before employed, as 9 to 16; and the radius of the circle described by its centre, to that of the circle described by the latter, was as 52·42 to 53·34; which gives for the compound ratio of the area and path nearly 21 to 38.

Motion, impulse, and resistance of air.

On the hemisphere.

Hemisphere flat side foremost.

Weight.		Time.	Weight.		Time.	Weight.		Time.	Weight.		Time.
oz.	dr.	sec.	oz.	dr.	sec.	oz.	dr.	sec.	oz.	dr.	sec.
1	0	21·80	7	0	4·67	20	0	2·65	56	0	1·57
2	0	10·82	8	0	4·35	24	0	2·45	64	0	1·45
3	0	7·70	10	0	3·90	28	0	2·25	72	0	1·40
4	0	6·47	12	0	3·50	32	0	2·10	80	0	1·32
5	0	5·67	14	0	3·25	40	0	1·82	—		—
6	0	5·15	16	0	3·00	48	0	1·67	—		—

Experiments with the lead.

146. The following experiments were made after the hemisphere was removed, and its weight supplied by a thin strip of lead:

Weight.		Time.	Weight.		Time.	Weight.		Time.	Weight.		Time.
oz.	dr.	sec.	oz.	dr.	sec.	oz.	dr.	sec.	oz.	dr.	sec.
1	0	8·10	2	4	3·55	4	0	2·40	8	0	1·60
1	4	6·00	2	8	3·30	4	8	2·25	9	0	1·50
1	8	4·95	2	12	3·10	5	0	2·10	10	0	1·40
1	12	4·35	3	0	2·90	6	0	1·90	11	0	1·30
2	0	3·80	3	8	2·60	7	0	1·70	12	0	1·25

The experiments which follow were made upon the simple arm loaded with a thin piece of lead, whose weight was 8 ozs. 10 drs., viz. only 4 drs. more than the cone, and 4 less than the whole globe, so as nearly to agree with both.

Weight.		Time.	Weight.		Time.	Weight.		Time.	Weight.		Time.
oz.	dr.	sec.	oz.	dr.	sec.	oz.	dr.	sec.	oz.	dr.	sec.
1	8	9·0	2	8	4·2	4	0	2·8	7	0	2·0
1	10	7·0	2	10	4·0	4	4	2·7	8	0	1·8
1	12	6·4	2	13	3·8	4	8	2·65	9	0	1·7
1	14	5·9	3	0	3·6	4	12	2·6	10	0	1·6
2	0	5·3	3	3	3·4	5	0	2·5	11	0	1·5
2	2	4·9	3	6	3·2	5	8	2·4	12	0	1·4
2	4	4·6	3	9	3·0	6	9	2·2½	13	0	1·35
2	6	4·4	3	12	2·9	6	8	2·1	14	0	1·3

Replaced the preceding weight of 8 ozs. 10 drs. by a less weight, viz. of 4 ozs. 3 drs.

Weight.		Time.	Weight.		Time.	Weight.		Time.	Weight.		Time.
oz.	dr.	sec.	oz.	dr.	sec.	oz.	dr.	sec.	oz.	dr.	sec.
2	0	4·6	3	8	3·0	7	0	1·9	12	0	1·3
2	4	4·2	4	0	2·7	8	0	1·75	13	0	1·25
2	8	3·85	4	8	2·5	9	0	1·6	14	0	1·2
1	12	3·5	5		2·3	10	0	1·5	—		—
3	0	3·3	6		2·1	11	0	1·4	—		—
1	4	10·8	1	6	8·4	1	8	7·0	1	12	5·8
1	5	9·8	1	7	7·6	1	10	6·4	1	14	5·3

General table of results.

147. The experiments being thus varied and extended, the author next proceeds to arrange them in an uniform series, in order to draw from them the results and determinations which he had in view in first undertaking this laborious but interesting task.

TABLE *of regular actuating weights and velocities, the latter reduced to feet per second, from* 3 *to* 20, *and the actual experimental results modified by the addition and subtraction of small fractional parts.*

Velocity per second.	Cone, weight 8 oz. 6 drs.		Whole globe, weight 8 oz. 14 drs.	Cylinder weight 5 oz. 1 dr.	Large hemisphere, weight 4 oz. 3 drs.		Small hemisphere plane foremost, weight 2 oz. 5 dr.	Lead, weight 8 oz. 10 drs.	Lead, weight 4 oz. 3 drs.	Lead, weight 2 oz. 5 drs.
	Vertex foremost.	Base foremost.			Convex.	Plane.				
Ft.										
3	2·9	4·7	2·8	3·5	2·2	3·8	2·4	1·4	1·3	0·9
4	4·0	6·9	3·9	6·0	3·4	6·1	3·6	1·6	1·5	1·1
5	5·4	9·9	5·2	9·0	4·9	9·2	5·1	1·9	1·8	1·4
6	7·3	13·7	7·1	12·5	6·6	12·8	7·1	2·2	2·0	1·7
7	9·5	18·0	9·2	16·8	8·7	17·0	9·5	2·6	2·4	1·9
8	12·5	22·9	11·3	21·8	11·0	22	12·0	3·1	2·8	2·3
9	15·1	29·0	14·0	27·6	13·3	28	15·1	3·5	3·3	2·7
10	18·0	35·6	16·7	33·5	15·8	34	18·3	4·0	3·8	3·1
11	21·8	41·7	20·7	39·7	19·0	40	22·0	4·4	4·4	3·6
12	25·2	49·9	24·0	47·0	23	48	26·4	5·8	5·1	4·1
13	28·8	58·2	28·0	54	27	56	30·7	6·2	5·7	4·7
14	34·5	65·8	32·0	62	31	64	34·9	7·0	6·5	5·5
15	39·3	78·0	37·2	72	35	73	38·6	7·7	7·2	6·2
16	44·0	90	42·7	84	40	84	43·7	8·5	8·0	6·8
17	48·6	101	48·5	95	45	101	50·4	9·5	8·6	7·6
18	54·2	110	53·3	109	50	112	57·3	10·5	9·4	8·5
19	60·0	126	58·4	125	57	121	62·7	11·3	10·3	9·3
20	66·3	...	64·0	...	64	140	72·0	12·0	11·0	10·0

Deductions from the above table.

148. If now we subtract from the resistances given in the above table, for the different bodies, the resistance due to the arm and lead nearest corresponding in weight, the remainders will show the actual resistance of the several bodies under the particular circumstances in which they are placed; these remainders show the resistances as applied at the circumference of the cylinder, or barrel, but by dividing each of them by 51·1, which expresses the ratio between the radius of the barrel, and that of the circle described, we shall have the actual resistance as applied to the centre of each body.

These reductions are all performed, and the results exhibited in the following table; besides which, the two last columns show the index of the power of the velocity which corresponds with the several resistances, and the ratio of the resistances experienced by the hemisphere, according as the flat or convex surface was presented to the action of the medium.

The indices of the powers of the velocities are found as follow. Taking, for example, the whole globe, and the velocities 10 and 11, the resistances to which are ·255 and ·310, and assuming the index $= x$, we shall have

$$10^x : 11^x :: 255 : 310;$$

consequently $$\frac{11^x}{10^x} = \frac{310}{255},$$

and $$x \log. 1 \cdot 1 = \log. 310 - \log. 255;$$

whence $$x = \frac{\log. 310 - \log. 255}{\log. 1 \cdot 1},$$

and so on for the rest.

TABLE *of the medium true resistances applied to the centre of each body, in ounces, &c.*

Velocity per second.	Cone.		Whole globe.	Cylinder.	Hemisphere.		Small hemisphere.	Index of the power of the velocity.	Ratio of the base to the convexity.
	Vertex.	Base.			Plane.	Convex.			
feet.	oz.	oz.	oz.	oz.	oz.	oz.	oz.		
3	·028	·064	·027	·050	·051	·020	·028	……	2·36
4	·048	·109	·047	·090	·096	·039	·048	……	2·33
5	·071	·162	·068	·143	·148	·063	·072	……	2·38
6	·098	·225	·094	·205	·211	·092	·103	……	2·30
7	·129	·298	·125	·278	·284	·123	·141	……	2·30
8	·168	·382	·162	·360	·368	·160	·184	……	2·30
9	·211	·478	·205	·456	·464	·199	·233	……	2·32
10	·260	·587	·255	·565	·573	·242	·287	……	2·36
11	·315	·712	·310	·688	·698	·297	·349	2·049	2·40
12	·376	·850	·370	·826	·836	·347	·418	2·042	2·41
13	·440	1·000	·435	·979	·988	·409	·492	2·036	2·42
14	·512	1·166	·505	1·145	1·154	·478	·573	2·031	2·42
15	·589	1·346	·581	1·327	1·336	·552	·661	2·031	2·42
16	·673	1·546	·663	1·526	1·538	·634	·754	2·033	2·43
17	·762	1·763	·752	1·745	1·757	·722	·853	2·038	2·43
18	·858	2·002	·848	1·986	1·998	·818	·959	2·044	2·44
19	·959	2·260	·949	2·246	2·258	·922	1·073	2·047	2·45
20	1·069	2·540	1·057	2·528	2·542	1·033	1·196	2·051	2·46
Proportional Nos.	126	291	124	285	288	119	140	2·040	—

Note.—The numbers in the bottom row denote the proportional resistances of the several bodies corresponding to the mean velocity of 10 feet per second.

The general height of the barometer, during the experiments, was 30·1 inches, and the thermometer, at a mean, was 62° Fahrenheit's; the ratio of the weight of air to water was, therefore, 1 to 840, or the weight of a cubic foot of air was 1⅕ oz. avoirdupois.

All the bodies, except the small hemisphere, had their greatest circular section 32 inches, and the side of the cone was inclined to its axis at an angle of 25° 24′.

149. From the above table of resistances we may draw the following practical inferences.

1. That the resistance is nearly as the surface, but that it increases a little above that proportion in the larger surfaces.

2. The resistance to the same surface, with different velocities, is in these slow motions nearly as the square of the velocity, but gradually increasing more and more above that proportion as the velocities increase.

3. The round ends and sharp ends of bodies suffer less resistance than the flat or plane ends of the same diameter: thus, the cylinder, and the plane end of the hemisphere, and cone, experience more resistance than their convexities, but the sharper end has not always the smaller resistance; for the convexity of the hemisphere has less resistance than the pointed end of the cone.

4. When the hinder parts of bodies are of different forms, the resistances are slightly different, although the fore parts be exactly alike and equal; this difference becomes less and less sensible as the velocities are more considerable, and vanish when those velocities are very great.

5. The resistance on the base of the hemisphere, is to that on the convexity, or on the whole sphere, nearly as 2⅖ to 1, or as 12 to 5, instead of 2 to 1, as assigned by the theory; the actual resistance, also, in both cases, is about ⅕ part more than that given from theoretical investigation.

6. The resistance on the base of the cone, is to that on its convexity, as 2·3 to 1, or as 23 to 10; the angle formed between the axis and side being 25° 42′. The ratio between the radius and sine of the angle is very nearly the same; so that in this instance the resistance is nearly in the direct ratio of the radius to the sine of the angle of incidence, the transverse section being the same.

7. Hence we may determine the altitude of a column of air, whose pressure shall be equal to the resistance of a body moving with any given velocity; for,

Let $a =$ the area of the section of the body, similar to any of those in the table, perpendicular to the direction of motion,
$r =$ the resistance to the velocity in the table,
$x =$ the altitude sought, of a column of air, whose base is a, and pressure r; then will
$a x =$ the content of the column in feet, and
$1\frac{1}{5} a x = \frac{6}{5} a x$, its weight in ounces; consequently,

$$r = \frac{6}{5} a x, \text{ and } x = \frac{5}{6} \times \frac{r}{a},$$

is the altitude sought in feet; that is, ⅚ths of the quotient arising from dividing the resistance due to the given velocity by the area of its transverse section, is equal to the altitude of a column of air, whose pressure is equal to the resistance.

8. Again, if it were required to find with what velocity any flat surface must be moved, so as to experience a resistance equal to a given pressure; as, for

example, equal to that of the atmosphere, we should be able to determine it as follows:

The resistance on the whole circle, whose area is 32 inches, or $\frac{2}{9}$ of a foot, is, by the table ·051 oz. when the velocity is 3 feet per second; and, as the resistance is nearly as the square of the velocity, it will be but one-ninth of this, or ·00566 oz. with a velocity of 1 foot; but, $2\frac{1}{2} \times 13600 \times \frac{2}{9} = 7555\frac{5}{9}$ oz. is the whole pressure of the atmosphere: whence,

$$\sqrt{\cdot 00566} : \sqrt{7555\tfrac{5}{9}} :: 1 : 1162 \text{ feet},$$

nearly the velocity sought; which is nearly equal to the velocity with which air rushes into a vacuum.

9. We may hence also infer the great resistances opposed to the motion of military projectiles; for, by the foregoing tables it appears, that a globe of $6\frac{3}{8}$ inches diameter, which is equal in size to an iron ball weighing 36 lbs. moving with a velocity of only 16 feet per second, meets with a resistance equal to the pressure of $\frac{2}{3}$ of an ounce; and, therefore, computing only according to the square of the velocity, the least resistance that such a ball would meet with, when moving with a velocity of 1600 feet per second, would be equal to $\frac{2}{3}$ of an ounce multiplied by 10000, the square of the ratio of the two velocities. Now,

$$\frac{2}{3} \times \frac{10000}{16} = 417 \text{ lbs.}$$

Moreover, this is independent of the pressure of the atmosphere itself on the forepart of the ball, which would be 487 lbs. more; for, the velocity of the ball being greater than that with which air rushes into a vacuum, it follows that the hinder part of the body is left unsupported by any counter pressure. The whole resistance to the motion, with such a velocity, would therefore be more than 900 lbs.

The author, to whom we have been indebted for the preceding experiments and deductions, concludes this section of his valuable work, by the following observations.

"On a review of the whole of the premises, we find that the resistance of the air, as determined from the foregoing experiments, differ very widely, both in respect to its quantity on all figures, and in regard to the proportions of its action on oblique surfaces, from the same actions and resistances, as assigned by the most plausible and imposing theories, which have hitherto been delivered and confided in by philosophers. Hence it may be concluded, that all the speculative theories on the resistance of the air, hitherto laid down, are very erroneous, and that it is from experiments only, carefully and skilfully executed, that a rational hope can be grounded, of deducing and establishing a true and useful theory of the action of forces, so intimately connected with the numerous and important concerns in human life."

EXPERIMENTS.—*Fifth series.*

Experiments.—Fifth series.

150. Some peculiarity being observable in the results of the preceding experiments on the cone and hemisphere: these being found, practically, to be nearly equal, while the theory required the ratio between them to be as 2 to 5, moreover, the resistance having been found to vary as the sine of incidence, instead of the square of the sine, some further experiments were thought necessary to examine these cases wit greater accuracy. With this view, the author pr himself with a thin rectangular plate of brass to the arm of the machine; its weight being 11 ozs. or 11$_2{}^1{}_T$ ozs. and its dimensions 8 inches by 4 i and, consequently, its area 32 inches, or $\frac{2}{9}$ of a the same as the area of the base of the cone, cy and hemisphere, in the foregoing experiments. plate was adapted for fitting on the end of the both directions: viz. in the direction both of its and breadth, in order to try the effect of the sam face in both positions. It was also contrived in way, that it might be inclined in any degree direction of the motion, in order to try the resis at all angles of inclination.

When fitted on with its length in the direction arm, the distance of its centre from the axis of was $52\frac{1}{4}$ inches, and the distance was still the when fitted on the other way.

151. *Experiments on the plate in both position tuating weight 2 lbs.*

Angles of inclination.	Time of one revolution, plane lengthwise.	Time of one revol plane breadthw
	seconds.	seconds.
0°	1·00	1·00
5	1·17	1·17
10	1·34	1·35
20	1·67	1·69
30	1·98	2·00
40	2·27	2·30
45	2·40	—
50	2·54	2·57
60	2·78	2·80
70	2·94	2·93
80	3·00	3·00
90	3·02	3·03

The 1st column of the above table shows the degrees of inclination at which the plate was the direction of its motion; the 2d column mean time of one revolution, as deduced from a m of the whole time of revolving, corresponding different angles of inclination in the 1st colum plate having in this case its length in the direc the arm. The 3d column shows the same whe breadth of the piece was placed in the direction arm.

Note.—The area of the plane, as above stated $\frac{2}{9}$ths of a foot; the radius of the circle, described centre, $53\frac{3}{4}$ inches; and its circumference 237·7 i or 28·4 feet. The ratio of the radius of the axis machine to that of the circle described, 51·1, as

By comparing the times in the two last colu appears, that the differences are very small, an be attributed to errors of observations, rather t any effect due to the position of the plate.

152. Similar experiments being now perform the simple arm, loaded with a weight equal to the plate, and with different actuating weigh following table of results was deduced:

TABLE *of results.*

Inclination of the plane.	Time of one revolution.	Actuating weight,		Difference of resistance.	The same reduced to the extremity of arm.	Velocity per second.
		with plane.	without plane.			
		oz.	oz.	oz.	oz.	
0°	1·00	32	32·0	0·0	·000	28·1
5	1·17	32	24·8	7·2	·140	24·1
10	1·35	32	19·5	12·5	·243	20·8
20	1·68	32	15·5	16·5	·320	16·8
30	1·99	32	12·7	19·3	·371	14·1
40	2·28	32	10·9	21·1	·410	12·4
45	2·42	32	10·4	21·6	·420	11·6
50	2·56	32	9·9	22·1	·429	11·0
60	2·79	32	9·3	22·7	·441	10·1
70	2·94	32	8·9	23·1	·448	9·6
80	3·00	32	8·7	23·3	·452	9·4
90	3·02	32	8·6	23·4	·454	9·3
1	2	3	4	5	6	7

It may not be amiss to observe, that the numbers in the 6th column, in thousands of an ounce, express the resistance to the plane placed at the angle, indicated by the corresponding line in the 1st column, when the velocity in feet per second is that shown in the last column.

Motion, impulse, and resistance of air.

153. These experiments having shown that the resistance is the same, under the same circumstances, whichever side of the plane was put in the direction of the arm, the next object was to determine the law of the resistance as it depended upon the angle of inclination, with which view the following experiments were performed:

TABLE *of experiments on oblique planes.*

Angle with the horizon.	Actuating weight.	Time of one revolution.	Velocity per second.
	oz.	sec.	feet.
0°	19	1·50	18·76
	20	1·43	19·68
5°	22	1·43	19·68
	23	1·33	21·16
10°	24	1·43	19·68
	25	1·40	20·10
20°	26	1·76	16·00
	27	1·73	16·27
	28	1·67	16·85
	29	1·65	17·06
	30	1·58	17·81
	32	1·56	18·04
	34	1·52	18·51
	36	1·48	19·02
	38	1·44	19·54
	40	1·36	20·69
30°	44	1·60	17·59
	48	1·54	18·27
	52	1·43	19·68
	56	1·40	20·10
40°	64	Thread broke.	

Similar experiments continued with less velocities.

Angle with the horizon.	Actuating weight.		Time of one revolution.	Velocity per second.	Difference of weights.		Resistance to 12 feet velocity.	The same reduced to the body.	Ratio to the last.	Sines of the angles.
	oz.	dr.	sec.	feet.	oz.	dr.				
0°	10	0	2·50	11·25	0	0	0	0	0	0
5°	10	4	2·67	10·54	0	4	0·81	·016	19	87
	10	12	2·43	11·68	0	12				
10°	12	0	2·50	11·25	2	0	2·28	·044	52	173
20°	15	0	2·73	10·31	5	0	6·84	·133	158	342
	16	0	2·50	11·25	6	0				
30°	24	0	2·37	11·87	14	0	14·31	·278	331	500
	26	0	2·26	12·45	16	0				
	28	0	2·19	12·85	18	0				
40°	32	0	2·40	11·72	22	0	23·09	·448	583	643
	34	0	2·30	12·23	24	0				
	36	0	2·26	12·45	26	0				
50°	39	0	2·40	11·72	29	0	31·34	·609	724	766
	40	0	2·39	11·77	30	0				
	41	0	2·38	11·83	31	0				
60°	43	0	2·50	11·25	33	0	37·60	·730	868	866
70°	44	0	2·58	10·91	34	0	41·49	·805	957	940
	45	0	2·55	11·04	35	0				
80°	46	0	2·60	10·88	36	0	43·06	·836	994	985
	47	0	2·52	11·14	37	0				
90°	48	0	2·50	11·25	38	0	43·35	·841	1000	1000
1	2		3	4	5		6	7	8	9

154. In the above table, the numbers in the 3d column show the medium time as deduced from the whole time of the weight descending, after the motion became uniform, as explained in the preceding experi-

ments. These times correspond to the angle of inclination in the 1st, and the actuating weight in the 2d column. The 4th column shows the velocity in feet per second, found by dividing the circumference of the circle described, by the time of describing it. Then the first weight of 10 ozs. when the thin edge of the plane only was opposed to the air, being deducted from every weight in the 2d column, leaves the weights in the 5th, which are to be considered as the actual resistances applied to the circumference of the barrel.

The 6th column gives the resistances to a velocity of 12 feet per second, as deduced from those in the 4th, to which number these are nearly equal, by stating the resistances to be proportional to the 2·04 power of the velocities, agreeable to the medium above found for slow motions (art. 148).

In the 7th column the resistances in the 6th are reduced, by dividing the latter by 55·1, the ratio between the radii of the two circles: these, therefore, may be considered as the true measures of the resistances when applied to the centre of the body, and corresponding to a velocity of 12 feet per second.

The 8th column shows the ratio of all these resistances, to the last or greatest of them, in order to compare them with the sines of the same angles, placed in the 9th and last column.

Law of resistance determined.

155. From this comparison, it appears that the resistance is in nowise proportional to those sines, for, from the direct vertical position, or 90°, the sine of the angle of inclination decreases faster than the sines, but nowhere so fast as the square of the sines, and much less of their cubes, as the theory requires. In fact, the resistance varies neither as the first, second, nor third power of the sine of inclination.

156. The index of the power of the sine being, therefore, some fractional and variable number, the author, in order to determine it, proceeds as follows:

It is manifest that the exponent must be some variable quantity, which at 60° must be equal to 1, or nearly so; below 60°, it must be greater than 1, but less than 2, and above 60°, it must be between 1 and 0. Now the cosines of angles increase from 90° to 0°, and by such a law, that their doubles have the above requisite property, that at 60°, $2 \cos = 1$, above 60°, $2 \cos \angle 1$ and $7\ 0$, and below 60°, $2 \cos 7\ 1$ and $\angle 2$. He then computes the proportions of $\sin^{2 \cos}$ for all the several angles of inclination, in order to compare them with the resistances, but the agreement is not sufficiently accurate to admit this law of variation.

Instead, therefore, of $\sin^{2 \cos}$, he assumes $\sin^{n \cos}$ for the law in question, and searches the value of n, such as to produce the requisite coincidences, and thus finds

$$n = \frac{\log. 331}{\cos \times \log. \sin} = 1{\cdot}842, \text{ nearly.}$$

The law sought, therefore, is $\sin^{1{\cdot}842 \cos}$; that is, the resistance must be supposed to vary as the sine of the angle of inclination, raised to a power whose index is equal to 1·842 times the cosine of the same angle.

In order to see how nearly this law approximates to the actual results, the following table is constructed, in which the numbers in the 2d column are those computed by means of the above expression, and those in the 3d are the resistances as determined from experiment.

Angles.	Values of $\sin^{1{\cdot}842 \cos}$.	Experimental velocities.	Difference.
5°	11	18	+ 7
10	42	52	+ 10
20	156	158	+ 2
30	331	331	0
40	536	533	− 3
50	730	724	− 6
60	876	868	− 8
70	962	957	− 5
80	995	994	− 1
90	1000	1000	0

The agreement here between the actual resistances and those computed is such as seems to justify us in assuming the above expression for the law we were in search of; and the formula being multiplied by ·841, in order to bring it to ounces, we shall have $\cdot 841 \sin x^{1{\cdot}842 \cos x}$, for the resistance in ounces, on a plane whose area is $\frac{2}{9}$ths of a foot, and whose angle of inclination is x, and the velocity 12 feet per second; whence the resistance due to any other surface, and for any proposed velocity, may be readily computed by means of the laws established in the preceding articles.

It is proper to observe, that the law of resistance thus obtained, will only apply in the case of small velocities: when a body moves with a very great velocity, as in the case of a projectile from a piece of ordnance, which is frequently at the rate of 1600 or 2000 feet per second, a different law takes place. We cannot here enter into the detail of the experiments from which the law alluded to is deduced, but shall merely state the result, which is, that r being taken to denote the resistance in lbs. v the velocity in feet per second, and d the diameter of the ball in inches, shall have, in case of very great velocities,

$$r = (\cdot 0000073\ v^2 - \cdot 0015\ v)\ d^2.$$

We have been highly indebted to the third volume of Dr. Hutton's Tracts for the foregoing tables and deductions, where the subject is carried to a much greater length than is consistent with the plan of this work; those readers, therefore, who are desirous of pursuing the inquiry, and of tracing its operation in practical gunnery, are referred to the volume above quoted. See also the article GUNNERY in our alphabetical arrangement.

Smeaton's experiments.

157. We have seen from the deductions of the experiments in the preceding articles, that the general expression for the resistance of the air to an oblique surface, is of the form $A\, v^m . \sin x^{n \cos x}$, which is unfortunately very ill suited to the purposes of analytical investigation. If, for example, it were required to determine the best position for the sails of a windmill, in order, with a given quantity of cloth, to produce the greatest effect; we should have both the root and index indeterminate, and consequently our fluxional expression extremely complicated and unmanageable, supposing we even assumed the index m, appertaining to the velocity, to be equal to 2. The common solution of this problem gives for the angle of the sail 54° 44′, but this merely contemplates the action of the air upon a plane at rest, or when all the parts are

moving with the same velocity; and, therefore, will not properly apply to a windmill-sail, in which the velocity of every part, from the axis outwards, describes different circles. Maclaurin, in the solution which he has given of this problem, in his "Account of Sir Isaac Newton's Discoveries," and in his Fluxions, (art. 914,) has treated it with reference to this consideration; and the theorem which is deduced from this view of the subject is as follows: "Suppose the velocity of the wind to be represented by a, and the velocity of any given part of the sail to be denoted by c, then the effect of the wind on that part of the sail will be the greatest when the tangent of the angle in which the wind strikes it is to radius, as

$$\sqrt{\left(2 + \frac{9\,c^2}{4\,a^2}\right)} + \frac{3c}{2a} \;:\; 1''.$$

This result is doubtless much nearer the truth than that above-mentioned; but as the force is even here estimated according to the common doctrine of resistance, as it depends upon the sine of the angle of incidence, it is probably still far from the best practical mode of construction; it moreover assigns no actual angle, nor what velocity any one given part of the sail ought to have in respect to the wind; so that much is still left for practical determination; and we hope not to be suspected of any want of respect for the powers of analysis, when we give it as our opinion that in such determinations as these, much more rational and useful information may be expected from well-conducted experiments, than from the most profound mathematical researches. Being fully impressed with this idea, and well assured of the great accuracy with which Mr. Smeaton always examined all mechanical effects which fell practically under his observation, we propose to conclude this section by a detail of certain experiments performed by him on the construction and effect of windmill-sails, the results of which are published in the Phil. Trans. for 1759.

158. The machine by means of which these experiments were performed, acted on similar principles to that employed by Robins and Hutton in their experiments. Referring to (fig. 20-*a*), ABC is a pyramidical frame for supporting the moving parts. DE is an upright axis, on which is framed FG, an arm for carrying the sails at a proper distance from the centre of the upright axis. H is a barrel on the same axis, on which is wound a cord, which, being drawn by the hand, gives a circular motion to the axis and to the arm FG, and thus carries the axis of the sail forward in the circumference of a circle, whose radius is DI; causing the sails to strike the air, and thus to revolve on their own axis. At L is fixed the end of a small line, which, passing over the pulleys M, N, O, terminates on a small cylinder, or barrel, on the axis of the sails, and by winding on it, raises the scale P, with the weights placed in it for trying the power of the sails. Two parallel pillars, Q, R, stand on the arm FG, for supporting and keeping the scale P steady, which is kept from swinging by means of S, T, two small chains which hang loosely round the two pillars. W is a weight for bringing the centre of gravity of the moveable part of the machine into the centre of motion of the axis DE.

VX is a pendulum composed of two balls of lead, which are moveable on a wooden rod, and can thus be adjusted so as to vibrate in any time required. This pendulum hangs on a cylindrical wire, on which it vibrates as on a rolling axis.

The pendulum being so adjusted as to make two vibrations in the time that the arm FG is intended to make one revolution, and the pendulum being set vibrating, the experimenter pulls the cord at Z with sufficient force to make each half revolution of the arm to correspond to each vibration as accurately as possible during the time of the experiment, which is rendered easy by a little practice.

Experiment with the above machine.

159. The machine being properly adjusted, the experiments detailed in the following table were made. In No. 1, the sails were set at the angle recommended by M. Parents and many other mathematicians, the plane of the sails making nearly an angle of 55° with the axis, the complement of which, or 35°, being, therefore, the angle with reference to the direction of motion, and which is indicated in the 3d and 4th columns of the table. In the same columns, also, are shown the angles at the extremities and the greatest angles of the sails, in those cases where different inclinations were given, agreeably to Maclaurin's theorem, and according to the constructions practised by Dutch millwrights. The 5th column exhibits the number of turns made when the sails were unloaded; and the 6th, the number of turns made by the same when the effect was at a maximum; that is, when the product of the load raised, multiplied by its velocity, was a maximum. The 7th column shows this load; and the 8th, the greatest load, or that which stopped the motion. The 9th column contains the products above-mentioned; the 10th, the quantity of surface of the sails; and the 11th, 12th, and 13th, the ratios, indicated by the heads of the respective columns.

Maximum effect determined.

The method of obtaining the maximum effect, or maximum product, will be best seen by a statement of an experiment at length.

Specimen of a set of experiments.

Radius of the sails	21 inches.
Length of ditto in cloth	18 inches.
Breadth of ditto	5·6 inches.
Angle at the extremity	10 degrees.
Ditto at the greatest inclination	25 degrees.
20 turns of the sail raised the weight	11·3 inches.
Velocity of the centre of the sail, in the circumference of the great circle, per second	6 feet.
Time of the experiment	52 seconds.

No.	W^t in the scale.	No. of turns.	Product.
1	0 lbs.	108	0
2	6	85	510
3	$6\frac{1}{2}$	81	$526\frac{1}{2}$
4	7	78	564
5	$7\frac{1}{2}$	73	$547\frac{1}{2}$ *max.*
6	8	65	520
7	9	0	0

Note.—The weight of the scale and pulley was 3 ozs. and 1 oz. suspended on one of the radii, at $12\frac{1}{2}$ inches from the centre of the axis, just overcame the friction, scale, and load of $7\frac{1}{2}$ lbs; and placed at $14\frac{12}{20}$ inches, it overcame the same with 19 lbs.

No. 5 being taken for the maximum, the weight in the scale, and of the scale itself, was 7 lbs. 11 ozs. or, 123 ozs. which, added to the resistance due to the friction of the machinery, ascertained to be 16 ozs. give 139 ozs., or 8·69 lbs. for the whole resistance. Now the number of turns was 73, and multiplying 8·69 lbs. by 73, we have

$$8·69 \times 73 = 634,$$

which may be called the representation of the effect produced, and the numbers thus obtained, are given in the 9th column of the following table.

Table of results.

Experiments on windmill-sails of various constructions, positions, &c.

Description of the sails.	N°	Angles at the extremities.	Greatest angle.	N°. of turns of the sails unloaded.	N°. of ditto at the maximum.	Load at the maximum.	Greatest load.	Product.	Quantity of surface.	Ratio of the greatest velocity to the velocity at a maximum.	Ratio of the greatest load to the load at the maximum.	Ratio of the surface to the product.
		deg.	deg.			lbs.	lbs.					
Plane sails, at an angle of 55°.	1	35	35	66	42	7·56	12·59	318	404	10:7	10:6	10:7·9
Plane sails, weathered according to the common custom.	2	12	12	...	70	6·3	7·56	441	404	......	10:8·3	10:10·1
	3	15	15	105	60	6·72	8·12	464	404	10:6·6	10:8·3	10:10·15
	4	18	18	96	66	7·0	9·81	462	404	10:7	10:7·1	10:10·15
Weathered according to Maclaurin's theorem.	5	9	26⅓	...	66	7·0	...	462	404	......	......	10:11·4
	6	12	29½	...	70½	7·35	...	518	404	......	......	10:12·8
	7	15	32½	...	63½	8·3	...	527	404	......	......	10:13·
Weathered after the Dutch practice.	8	0	15	120	93	4·75	5·31	442	404	10:7·7	10:8·9	10:11·
	9	3	18	120	79	7·0	8·12	553	404	10:6·6	10:8·6	10:13·7
	10	5	20	...	78	7·5	8·12	585	404	......	10:9·2	10:14·5
	11	7½	22½	113	77	8·3	9·81	639	404	10:6·8	10:8·5	10:15·8
	12	10	25	108	73	8·69	10·37	634	404	10:6·8	10:8·4	10:15·7
	13	12	27	100	66	8·41	10·94	580	404	10:6·6	10:7·7	10:14·4
Ditto; but enlarged towards the extremities.	14	7½	22½	123	75	10·65	12·59	799	505	10:6·1	10:8·5	10:15·8
	15	10	25	117	74	11·08	13·69	820	505	10:6·3	10:8·1	10:16·2
	16	12	27	114	66	12·09	14·23	799	505	10:5·8	10:8·4	10:15·8
	17	15	30	96	63	12·09	14·78	762	505	10:6·6	10:8·2	10:15·1
8 sails, being sectors of ellipses in their best positions.	18	12	22	105	64½	16·42	27·87	1059	854	10:6·1	10:5·9	10:12·4
	19	12	22	99	64½	18·06	...	1165	1145	10:5·9	......	10:10·1
1	2	3	4	5	6	7	8	9	10	11	12	13

Practical remarks,

160. An estimate of these effects may be made immediately by an inspection of the 9th and 13th columns, and they are decidedly in favour of the Dutch construction, although those sails that are weathered according to the principle of Maclaurin, approximate nearly towards them with respect to the maximum effect.

In No. 5, the result is nearly the same as with the plane sails in their best positions; but being turned round in their sockets, so that every part of each sail stood at an angle of 3°, and afterwards of 6° greater than before; that is, their extremities being moved from 9° to 12° and 15°, the products were increased to 518 and 527 respectively; the latter appears, therefore, to be nearly the most favourable position for sails of this construction.

It is to be observed, that a sail inclined according to the theorem above referred to, presents a convex surface to the wind; whereas, the Dutch, and all our best modern millwrights, although they make the angle to diminish, yet they constantly do it in such a manner as that the surface of the sail may be concave towards the wind. In this manner the sails made use of in Nos. 8, 9, 10, &c. were constructed, and the several results show the advantage of this form, and the best position that can be given to them.

The foregoing series of experiments furnished Mr. Smeaton with the following practical maxims:

1. The velocity of windmill-sails, whether unloaded or loaded so as to produce a maximum effect, is nearly as the velocity of the wind, their shape and position being the same.

2. The load at the maximum is nearly, but somewhat less than, as the square of the velocity of the wind, the shape and position of the sails being the same.

3. The effects of the same sails at the maximum, are nearly, but somewhat less than, as the cubes of the velocity of the wind.

4. The load of the same sails at the maximum, is nearly as the squares, and their effects as the cubes of their number of turns in a given time.

5. When sails are loaded, so as to produce a maximum at a given velocity, and the velocity of the wind increases, the load continuing the same; 1st, The increase of effect, when the increase of velocity of the wind is the same, will be nearly as the squares of those velocities. 2dly, When the velocity of the wind is double, the effects will be nearly as 10 to 27½. But, 3dly, When the velocity compared is more than double of that where the given load produces a maximum, the effects increase nearly in the simple ratio of the velocity of the wind.

6. In sails where the figure and position are similar, and the velocity of the wind the same, the number of turns in a given time will be reciprocally as the radius or length of the sail.

7. The load at a maximum that sails of a similar figure and position will overcome, at a given distance

from the centre of motion, will be as the cube of the radius.

8. The effects of a sail of similar figure and position are as the square of the radius.

9. The velocities of the extremities of the sails, in all their usual positions, when unloaded, or even loaded to a maximum, are considerably quicker than the velocity of the wind.

Other practical deductions obtained from the preceding results may be found in the volume of the Phil. Trans. above referred to. See also Phil. Trans. abridged by Hutton, &c. vol. xi. p. 360.

§ IX. *On the nature and propagation of sound.*

161. The doctrine of Acoustics, according to the most extensive signification of this term, includes inquiries concerning the nature and origin of sound, its propagation and velocity, its operation and effect on the ear, the consequences of the combinations of sounds variously related, constituting the doctrine of harmonics, the construction and theory of musical instruments, &c. &c. It is not our intention, however, in this section, to consider the subject under this general point of view; we shall confine our inquiries only to the two former of these heads, reserving what relates to the latter for its proper place in our treatise on MUSIC.

Dr. Young defines sound to be a motion capable of affecting the ear with the sensation peculiar to the organ. It is not simply a vibration or undulation of the air, as it is sometimes called; for there are many sounds in which the air is not concerned, as when any sounding body is held by the teeth: nor is sound always a vibration or alternation of any kind, for every noise is a sound, and a noise, as distinguished from a continued sound, consists of a single impulse in one direction only, sometimes without any alternation; while a continued sound is a succession of such impulses, which, in the organ of hearing, at least, cannot but be alternate. If these successive impulses form a connected series, following each other too rapidly to be separately distinguished, they constitute a continued sound, like that of the voice in speaking, and if they are equal among themselves in duration, they produce a musical or equable sound, as that of a vibrating chord or string, or of the voice in singing. Thus, a string striking against a piece of wood, causes a noise, but, striking against the teeth of a wheel or a comb, a continued sound; and if the teeth of the wheel be at equal distances, and its velocity or motion constant and uniform, a musical note.

Nature and propagation of sound.

162. These definitions being understood, we may observe, that the particles of an elastic body, when they are drawn momentarily from their natural position, will return towards it again by a series of isochronal oscillations. These vibrations communicate themselves to the air, which is also a compressible and elastic body, and produce in it alternate condensations and dilations, which are, in the first instance, excited in the beds of this fluid nearest to the body put in motion, and are hence transmitted to others more distant, through the entire mass of air, in the same manner as the waves formed in tranquil water, by a stone thrown into it, are propagated in successive circles from the general centre—

> —— As the small pebble stirs the peaceful lake,
> A circle formed, another straight succeeds,
> Another still, and still another spreads.

When these dilations and contractions succeed each other with sufficient rapidity, they excite, as we have said above, in our ear, the sensation of sound; and the greater or less rapidity of their succession, constitutes all the difference between acute and grave tones, by which sounds are distinguished from each other.

It appears, therefore, that air is the principal medium for the transmission of sound; and on this account, the doctrine of Acoustics is commonly considered, and treated of as a branch of the general science of Pneumatics.

When solid bodies are struck in such a manner as to produce a distinct sound, they vibrate with great rapidity; for, if while giving out the sound, we touch them slightly with the end of the finger, or with the edge of a small metallic lamina, we may perceive very distinctly a great number of pulsations succeeding each other with extreme rapidity: a bell that is struck, or a stretched wire made to give forth a sound, will furnish abundant proofs of the truth of this assertion.

Sound is the result of rapid vibrations.

163. In order to demonstrate that sound is really the effect of these vibrations when carried on with sufficient rapidity, we need only observe what follows when we render them very slow, as by stretching a musical string by a very small weight: in this case, the vibrations may be made so slow, that they may be counted; and the consequence will be, that no sound will be produced. In order that sound may be obtained, the tension of the chord must be increased; and it will be found, that the greater the tension is, the length remaining the same, the more acute the sound will become; in the mean time, the vibrations will succeed each other so rapidly, that they cannot be followed by the eye; and we must, therefore, have recourse to calculation, in order to determine the duration of the successive vibrations, which may always be effected when we know the length and weight of the string and the degree of tension to which it is submitted. In this way it has been found, that the sound given forth by a vibrating chord, ceases to be distinctly appreciable to the most delicate ear, when it makes less than 32 vibrations per second. This will, of course, be only considered as a rough approximation; it is obvious, that it must be very different in different persons, and that it is a determination which cannot be expected to be made with great accuracy.

Sound not produced in a vacuum.

164. Having thus shown that sound is excited by the rapid vibrations of elastic bodies, we shall next endeavour to demonstrate, that the transmission is made by means of the air; at least, that it is only made when there is this fluid, or some other conducting medium, between the sonorous body and the ear. In order to prove this, it is sufficient to suspend a small bell in the moveable receiver of an air-pump. When the receiver is full of air, if we shake the machine, the clapper or tongue will strike the bell, and the sound will be heard distinctly; but, if we afterwards exhaust the air, and again shake the machine, so as to produce the same motion as before, we shall hear no sound; again, let us admit a small portion of air, and the sound again becomes sensible, and more and more distinct according as the rarefied air ap-

proaches nearer and nearer to the state it was in at first.

Air is not the only medium of transmission.

Air, however, is not the only medium through which sound may be propagated, as has been demonstrated by introducing into the exhausted receiver many other elastic fluids. Vapours, also, of water, ether, spirit of wine, &c. are capable of this transmission, as philosophers have proved in numerous instances; on which account it is necessary, in performing the experiment above referred to (viz. to show that sound cannot be transmitted through a void), to place in the receiver small pieces of some absorbing substance that shall attract to itself the aqueous vapours which may remain there; and which would otherwise transmit the sound in a perceptible manner, although very weak, in consequence of their little density.

Sound may be transmitted through fluids,

Again, elastic fluids are not the only bodies which transmit sound, for it is also propagated by means of non-elastic fluid bodies. This may be shown by striking together two stones, or other bodies under water, in which we not only hear the sound, but the noise of the blow, if the head be placed under water, is audible at considerable distances. Dr. Franklin assures us, that he has thus heard noises at the distance of half a mile.

And by solid bodies.

Lastly, sound is likewise transmitted through solid bodies. This is a fact well known to military miners, who not only hear the miners opposed to them, but are able thence to judge of their position and direction. Some very interesting experiments on the subject of the transmission of sound through solid bodies, were made by M. Biot, on the aqueducts of Paris, of which we shall give some account in a subsequent article; we shall here merely observe, that a cylindrical tube being struck with a hammer at one extremity, a double sound was very distinctly heard at the other: the one, through the solid matter, being transmitted much more rapidly than the other through the medium of the atmosphere.

Sound; how propagated.

165. These points being established, let us now examine a little more minutely, how the perturbations produced by the vibrations of sonorous bodies in the particles of air in their vicinity, can be thence transmitted to the ear. Since the continuity of the vibrations does nothing more than render the transmission continual and of greater duration, it is obvious that, in order to consider the phenomenon in its simplest form, we must first examine how a single perturbation is propagated, as for example, the explosion from a cannon or pistol.

Let us, in order to simplify this inquiry, suppose the explosion to happen in a spherical mass of air. Then, at the moment it takes place, the particles comprised in this sphere, will be strongly pressed upon those near them. But these opposing a resistance which must be overcome, it follows, that the first are compressed in the same time that they are displaced. Those which encompass them, yield in part to their effort; and are, therefore, in like manner, displaced and compressed, but in a less degree, till at length the motion and compression become insensible at a certain distance from the phonic centre, or centre of explosion. All this takes place in the first instant, when the explosion having ceased, the particles which had been compressed, dilate themselves in all directions in consequence of their elasticity, and thus again re-act upon all the particles which oppose their motion; these, in like manner, act upon all the particles in their vicinity, which had not been effected in the first instance, and they are thus also compressed in their turn. The effect then becomes the same for these, as for those in the first instant, and by these alternate condensations and dilations, the indulation is propagated successively in all directions from the general centre through the mass of air.

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166. Dr. Young illustrates this matter somewhat differently; he observes, "that in order to form a distinct idea of the manner in which sound is propagated through an elastic substance, we must first consider the motion of a single particle, which, in the case of a *noise* is pushed forwards, and then either remains stationary, or returns back to its original situation; but in the case of a *musical sound*, is continually moved backwards and forwards with a velocity always varying, and varying by different degrees, according to the nature or quality of the tone; for instance, differently in the tones of the bell and a trumpet. We may first suppose, for the sake of simplicity, a single series of particles to be placed only in the same line with the direction of the motion. It is obvious, that if these particles were absolutely incompressible, or infinitely elastic, and were only retained in contact with each other by an infinite force of cohesion or of compression, the whole series must move precisely at the same time, as well as in the same manner. But in a substance which is both compressible and extensible, or expansible, the motion must occupy a certain time in being propagated to the successive particles on either side, by means of the impulse of the first particle on those which are before it, and by means of the diminution of its pressure on those which are behind; so that when the sound consists of a series of alternations, the motion of some of the particles will be always in a less advanced state than that of others nearer to its source; while at a greater distance forwards, the particles will be in the opposite stage of the undulation, and still further on, they will again be moving in the same manner with the first particle, in consequence of the effect of a former vibration. The situation of a particle at any time may be represented by supposing it to mark its path, on a surface sliding uniformly along in a transverse direction. Thus, if we fix a small pencil in a vibrating rod, and draw a sheet of paper along, against the point of the pencil, an undulated line will be marked on the paper, and will correctly represent the progress of the vibration. Whatever the nature of the sound transmitted through any medium may be, it may be shown that the path thus described will also indicate the situation of the different particles at any one time. The simplest case of the motion of the particles is that in which they observe the same law with the vibrations of a pendulum, which is always found opposite to a point supposed to move uniformly in a circle; in this case, the path described will be the figure denominated the harmonic curve; and it may be demonstrated, that the force, impelling any particle backwards and forwards, will always be represented by the distance of the particle before or behind its natural place; the greatest condensation and the greatest direct velocity, as well as the greatest rarefaction and retrograde velocity, happening at the instant when it passes through its natural place."

167. From all that has now been stated, it appears that the motion and condensation which take place every instant in the mass of air, are in reality only the repercussions of the motion and of the condensation impressed on the first particles on which the explosion, or blow acts; and since in free air, as the undulations are more and more extended, a greater number of particles are put in motion; it follows, as a necessary consequence, that the effect on each becomes weaker; and the intensity of the sound ought, therefore, to become less and less, as we are further removed from the centre of explosion, an effect that is constantly observed to take place with respect to the propagation of sound in the open atmosphere. If we conceive the effect to be produced on a central particle, and to be thence transferred to other particles, all in the same plane, it is obvious that as the number of particles put in motion are thus greater, the intensity of the sound ought to diminish as the distance increases; but if the central particle be supposed to communicate its perturbations to a sphere of particles, then the effect must diminish, as the square of the distance from that centre, and such seems to be the law of decrease, as far as we are able to ascertain the fact from observation. If the central particle only act in a lineal direction, that is, we suppose it to communicate its effects to the particles on each side of it, and these again to the next, and so on, as in the communication of sound through a cylindrical, confined space, we can then see no reason for the sound becoming more weak at one distance than at another, unless, indeed, the friction against the sides of the tube may have such an effect.

M. Biot submitted this to the test of experiment in the tubes of the aqueducts of Paris, these being about 951 metres, or 3120 feet in length. At this distance the lowest voice was heard very distinctly; the words were clearly understood, and a conversation maintained with the greatest ease. This effect is known to many of our London tradesmen, who have workmen employed in the upper part of their houses; a tube of copper or tin is conducted from the upper rooms to the shop or warehouse, and any question is asked and the answer received with the greatest facility. M. Biot endeavoured to ascertain the lowest pitch of voice that could be distinguished at the above distance, but he could come to no determinate limit; "in order not to hear" he observes, "it was absolutely necessary not to speak at all."

168. That he might ascertain whether acute and grave tones passed with the same velocity, this author proceeded as follows. He caused a man at one extremity of the tube to play several airs on a flute, while he stationed himself at the other. Now, we know generally, that a piece of music is subject to a certain measure, which rules very accurately the intervals of the successive tones, and consequently, if there were at the distance of 3120 feet any difference in the velocity of the different notes, the music would become very confused and imperfect at the other extremity of the tube. This, however, was far from being the case, the harmony being as perfect at the above distance as in the immediate focus of the sounds.

169. It follows, then, from what has been above stated, that sound is transmitted through the medium of the air by means of the undulation of this fluid, caused by the vibrations of the sonorous body; that these vibrations must be performed with a certain velocity in order to render the sound audible; and that the limit of this velocity for the most delicate ear is 32 vibrations per second. Again, it has been shown, that sounds made in the open air ought to become more and more weak the further we are removed from the phonic centre; but, that in a confined cylindrical space, such a diminution of intensity is not perceptible even at the distance of 3120 feet; and that acute and grave tones are transmitted with equal velocities. It has also been shown, that not only air, but all other elastic fluids will also act as the medium of the transmission of sound; as will also, all vapours, fluids, and solid bodies. We shall examine in a subsequent part of this article, the greater or less facility of transmission which these several bodies possess.

Nature and propagation of sound.

Theoretical deductions.

170. The above deductions (each of which has been verified by experiment), as far as they respect common air, are the simple consequences of the physical constitution of the atmosphere, being such as we should have naturally inferred independent of any practical data; we may, therefore, with confidence use them as the basis of our inquiries respecting any other of the circumstances attending the transmission of sound; the most important of which is the velocity. Is this constant under all circumstances; and if not, what ought to be the measure of its motion under the medium state of the atmosphere? It may be shown analytically, that in an elastic fluid whose temperature is constant, the velosity of transmission is constant also; for the square of the velocity of transmission is directly as the elasticity, and reciprocally as the density: and consequently, if we suppose, as we commonly do, that the elasticity is as the density, it will follow that while the temperature remains the same, the velocity will be uniform under all degrees of pressure. It also appears that the actual velocity is that which is due to a heavy body in falling freely through half the height of a homogeneous atmosphere of equal pressure, or, as it is sometimes expressed, in falling through half the height of the modulus of elasticity.

The above result does not agree with experiment.

171. The least elastic substance that has been yet examined is carbonic acid gas, or fixed air; which is considerably denser than atmospheric air exposed to an equal pressure. The height of the air supposed to be homogeneous in ordinary circumstances at the surface of the sea, we have seen (art. 27), is computed to be about 27818 feet, and the velocity acquired by a body in falling through half this height, or 13909 feet, is found to be

$$2\sqrt{(13909 \times 16\tfrac{1}{12})} = 946 \text{ feet per second;}$$

and this, therefore, as far as our present investigation extends, ought to be the velocity with which sound is transmitted through the atmosphere.

Sound travels 1130 feet per second.

But from a comparison of the accurate experiments of Derham made in the day-time, with those of the French Academy made by night, it appears that the actual velocity of sound, at the zero of temperature of the centigrade thermometer, is about 1130 feet per second, which agrees also with other accurate experiments made by Professor Pictet. This difference between calculation and experiment for a long time engaged the attention of philosophers, but the difficulty appears to have been at length removed by the fortunate suggestion of Laplace, who attributes the discrepance in question to the elevation of temperature, which is always found to accompany the action of condensation, and to the depression produced by rarefaction. It appears, however, that this hypothesis will require a greater change of temperature than we are justified, from the experiments of Mr. Dalton, in supposing actually to take place; at the same time it has been observed, that Mr. Dalton's experiments do not perfectly accord with each other, and that the experiment by which tow is set on fire by a sudden condensation, seems to show that Dalton's results are too low, and adds greater probability to Laplace's suggestions; admitting them to be accurate, the theory may be correctly reconciled with experiments; we may estimate the modulus of the air's elasticity, which is the measure of its immediate force, from the velocity which is actually observed, and we shall thus find it to be about 39800 feet, instead of 27818 feet, commonly assumed as the height of a homogeneous atmosphere.

Velocity effected by the temperature.

172. This velocity, as we have seen above, remains unchanged by any alteration of pressure indicated by the barometer; because it varies as the elasticity directly, and as the density reciprocally; the temperature, however, will have an effect, because this increases the elasticity, while the density remains the same. The equality of motion, under different pressures, may be satisfactorily shown by means of experiments on the sounds of organ-pipes, which are intimately connected with the velocity of the transmission of sound through the atmosphere, and which are found to remain precisely the same, however the air may be rarefied or condensed. The Academicians del Cimento enclosed an organ-pipe, with bellows worked by a spring in the receiver of an air-pump, and of a condenser, and they found that as long as the sound was audible, its pitch remained unchanged. Papin screwed a whistle on the orifice which admits the air into the receiver of the air-pump, and Dr. Young has fixed an organ-pipe in the same manner, and the result agreed with the experiments of the Academicians.

But if the density of the air be changed, while its elasticity remains unaltered, as in the case of its being expanded by heat, or contracted by cold, the height of the column, and consequently the velocity, will also be altered; so that for each degree of Fahrenheit's thermometer, the velocity will vary about one part in a thousand.

Bianconi's experiments.

173. Bianconi says he has actually observed this difference of velocity according to the different heights of the thermometer, and it may be shown less directly by means of the sounds of pipes; but it has not been accurately determined whether or no the correction, on account of the effect of compression in causing heat, remains unaltered, although Bianconi's experiments agree very well with the supposition that no material change takes place in this respect. The velocity of sound must also be in some measure influenced by the quantity of moisture contained in the atmosphere; it must be a little diminished by cold fogs, which add to the density without augmenting the elasticity, and increased by warm vapours, which tend to make the air lighter; and these two opposite states are probably often produced in succession in wind-instruments blown by the mouth, the air within them being at first cold and damp, and afterwards warm and moist.

In pure hydrogen gas, the velocity of sound ought, from calculation, to be more than three times as great as in common air, but the difference does not appear to have been so great in any experiment hitherto made on the sounds of pipes in gasses of different kinds.

Steam, at the temperature of boiling water, is only one-third as heavy as common air; consequently, the velocity of sound in steam ought to be nearly three-fourths greater than in air.

174. It does not appear that any direct experiments have been made on the velocity with which an impulse is transmitted through a liquid, although it is well known that liquids are capable of conveying sound without difficulty: Professor Robison informs us, for example, that he heard the sound of a bell, transmitted by water, at the distance of 1200 feet. It is, however, easy to calculate the velocity with which sound must be propagated in any liquid of which the compressibility has been measured. Mr. Canton has ascertained, that the elasticity of water is about 22000 times as great as that of air; it is, therefore, measured by the height of a column, which is as many times 34 feet, that is, 748000 feet, and the velocity corresponding to half this height, is 4900 feet in a second. In mercury, also, it appears from Mr. Canton's experiments, that the velocity must be nearly the same as in water, and in spirit of wine a little less. These experiments were made by filling the bulb of a thermometer with water, and observing the effects of placing it in an exhausted receiver, and in condensed air; taking care to avoid the changes of temperature, and other sources of error. See (art. 2), HYDROSTATICS. A slight correction is, however, required on account of the expansion and contraction of the glass, which must have tended to make the elasticity of the fluids appear somewhat greater than it really was.

175. It is also well known that solid bodies are good conductors of sound; we have already referred to M. Biot's experiments on this subject, and we may now state them a little more minutely. They were made, as we have already observed, upon the tubes forming the aqueducts at Paris, the length of which was 3120 feet. At one extremity of these tubes was adapted a ring of metal, of the same diameter as the orifice, in the centre of which were fixed a clock-ball and hammer, which could be made to strike at pleasure, and the blow was made equally upon the ball and the ring of metal above-mentioned, the sound from the latter, therefore, ought to be transmitted through the tube, and the former through the air; and by this means, any perceptible difference in the time of transmission might be determined. And such was actually the case, for by placing the ear against the other extremity of the pipe, two sounds were very distinctly heard; the time was noted very accurately, by means of a seconds watch, and from a mean of many experiments it appears, that sound is transmitted with $10\frac{1}{2}$ times the velocity through the metal of which these tubes were composed, than through the air; that is to say, at the rate of about 11865 feet per second.

It is probable that different metals and woods possess different degrees of facility of transmission, but we have very few experiments from which any conclusions can be drawn.

176. We have stated (art. 171) that from a medium of several of the best-conducted experiments, the velocity of sound through the atmosphere, at the zero of temperature, is about 1130 feet per second; but this must be understood to appertain to a perfect calm;

for there can be little doubt, although the contrary has been stated by some writers, that the velocity of transmission must be effected by that of the wind. If the wind blows directly from the point where the sound is generated to where it is observed, its velocity must be added to the natural velocity of sound; and if directly from it, it ought to be deducted; and in any other direction the actual velocity must be determined on the same principles as we have laid down in Dynamics for the composition of motion.

177. It is probable that a want of attention to this and other circumstances, important to be noted, has been the cause of that difference which is observed in the deductions of different authors on the subject of the velocity of sound. We have stated 1130 feet per second as the result which agrees best with what are generally considered the most accurate experiments; but we ought to observe, that 1142 feet and 1150 feet per second, are numbers very commonly assumed by English writers; and, from a series of experiments conducted with much apparent accuracy by Don. J. de Espinosa and Don F. Bauza, at St. Jago, the mean result gives 1218½ feet per second for the velocity of sound. These experiments having been made under high degrees of temperature, ought, according to what is stated above (art. 172), to give a greater velocity, but the increase of $\frac{1}{1000}$th part for every degree of Fahrenheit's thermometer will not be sufficient to reconcile these results with the former: at the same time, the accuracy with which they appear to have been conducted, and their number, certainly entitle them to the notice of philosophers. The detail of them, as published in the "Annales de Chimie et de Physique," vol. vii. is as follows:

On a plane of great extent, a station was fixed upon for the position of the gun, which was a Spanish eight-pounder, and from this station, which may be called B, were measured, very accurately, four distances to four other stations, these being called C, D, E, F; the several lengths were

BC=13841; BD=50316; BE=43365; BF=29558.

The times of observing the flash of the gun and of hearing the report were observed by means of two excellent seconds watches, one at the gun, and one with the observer. The following are the results.

1. *Cannon at* B; *observer at* E.

January 13th, 1794 at sun-set; very calm weather, the wind very weak, from the south-west; sky cloudy. Barometer 25·75 inches; centigrade thermometer 22·5°. The cannon, with regard to the station E, was 69° from the north towards the west.

Distance 43365 feet.

Saw the flash at			Heard the report at		
8^{h}	0′	0″	8^{h}	0′	38″
8	10	0	8	10	38
8	20	0	8	20	38

2. *Cannon at* B; *observer at* E.

January 14th—morning; weather calm, but a little foggy. Barometer 25·75 inches; thermometer 20°.

Distance 43365 feet.

Saw the flash at			Heard the report at		
4^{h}	0′	0″	4^{h}	0′	38″
4	10	0	4	10	38
4	20	0	4	20	38

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3. *Cannon at* B; *observer at* D.

January 14th—evening; wind south-west, fresh at station B; but at the station D, it was shifting from the north; the cannon, with referrence to the station D, was 44° towards the west. Barometer 2·75, thermometer 28·8°.

Distance 50316 feet.

Saw the flash at			Heard the report at		
7^{h}	30′	0″	7^{h}	30′	43″
7	40	0	7	40	43
7	50	0	7	50	43
8	0	0	8	0	43·5
8	10	0	not heard.		
8	20	0	8	20	43

4. *Same station.*

January 15th—morning; weather calm, with slight fogs. Barometer 25·75 inches, thermometer 25°. Wind as above.

Distance 50316 feet.

Saw the flash at			Heard the report at		
4^{h}	0′	0″	4^{h}	0′	43·5″
4	10	0	4	10	43
4	20	0	4	20	43·5
4	30	0	4	30	43·5
4	40	0	—	—	—
4	50	0	4	50	43·5

5. *Cannon at* B; *observer at* F.

January 16th—evening; wind fresh at B, from the south-west, but at the station F it blew slightly from the north-east; the direction of B from F was 72° from the north towards the west. Barometer 25·75 inches; thermometer 25°.

Distance 29558 feet.

Saw the flash at			Heard the report at		
8^{h}	0′	0″	8^{h}	0′	26″
8	10	0	8	10	26
8	20	0	8	20	26
8	30	0	8	30	26
8	40	0	8	40	26

6. *Cannon at* B; *observer at* C.

January 17th—at night. Barometer 25·75 inches; thermometer 22·5°; clear weather; wind south-west, pretty fresh; the azimuth of B from C 20° to the north-west.

Distance 13841 feet.

Saw the flash at			Heard the report at		
7^{h}	30′	0″	7^{h}	30′	13″
7	35	0	7	35	12
7	40	0	7	40	12
7	45	0	7	45	12
7	50	0	7	50	12

Such are the results of this series of experiments, from which we readily deduce the following mean velocity.

Station.	Distance.	Time. seconds.	Vel. per sec. in metres.	Bar.	Ther. cent. degrees.
1	43365	38·0	370·7	25·75	21·3
2	50316	43·3	377·3	ditto	25·0
3	29558	26·0	369·3	ditto	25·0
4	13846	12·2	368·6	ditto	22·5

Mean velocity per second, 371·5 metres, or 1218½ English feet. The medium temperature being 23½° of the centigrade, or 74·3° of Fahrenheit's thermometer.

Mean results. 178. This result being reduced to the zero of temperature, or to 32° of Fahrenheit's scale, agreeably to the proportion given in (art. 172), becomes 1167·3 feet per second; which exceeds very considerably the velocity as deduced from a mean of Derham's and the Academy experiments, which we have seen give only a velocity of 1130 feet. Upon the whole, we cannot consider this point at present as well established; other experiments performed under all varieties of wind and weather would be highly interesting. It has also been suggested, that light itself has a certain influence on the nature and velocity of sound, and an interesting course of experiments, made with a view to a determination of its actual operation, performed by M. Parolette, may be seen in Nicholson's "Journal of Natural Philosophy," vol. xxv. See also an article on the same subject, by Hassenfratz, in the 2d volume of the Memoirs presented to the Institute of France by Learned Strangers; and Chladni's "Treatise on Acoustics," 2d edition, 1802.

§ X. *Pneumatic machines and experiments.*

Pneumatic machines 179. Pneumatic machines are such as are principally dependent in their operation upon the properties of elastic fluids, as the air-pump, condensers, bellows, steam-engines, &c. These are purely pneumatic machines, besides which, we have many that may be called hydro-pneumatic engines, which are nearly as much related to hydrodynamics as to pneumatics; such as the siphon, pumps, various engines for raising water, &c. We have described many of the latter kind, in the last chapter of our treatise on Hydrodynamics; a few others will be found at the conclusion of this section.

It is not, of course, our intention in this place to enter at length into a detailed description of all the machines which fall within the class properly called pneumatic, as this would far exceed the limits we propose for this chapter, and would, at the same time, be inconsistent with our plan and arrangement: generally, compound machines, or such as admit of a variety of constructions, as bellows, blowing-machines, pumps, fire-engines, &c. will be treated of, as to their construction and practical effects, under the respective heads in our alphabetical division; in the present section we shall confine ourselves chiefly to those depending on the most simple principles, and which form the fundamental parts of the more complex and compound machinery above referred to.

I. *Common syringe, rarefying syringe, &c.*

Fig. 21. Common syringe. 180. Let the barrel of the syringe AB (fig. 21) communicate with the vessel V, having a stop-cock C between them; and suppose it to communicate with the external air by another orifice D in any convenient situation, also furnished with a stop-cock: conceive this syringe to have a piston very accurately fitted to it, so as to touch the bottom when pushed down, and to have no vacancy about the sides.

If we now suppose the piston to be at the bottom, the cock C open, and the cock D shut, and that the piston is then drawn upwards, the air which filled the vessel V, will expand so as to fill both that vessel and the barrel AB, and be uniformly diffused through both; or, if instead of air the vessel be supposed to contain water, then this water will rush up into the barrel, whence it may be expelled by the contrary motion of the piston. The common water-syringe consists only of the barrel and piston, and when used, its end is dipped into water, which is drawn into the barrel as above described.

When it is employed for exhausting air out of any vessel, as above explained, it differs nothing in principle from the barrel and piston of an air-pump; and the formulæ given in (art. 41) for that machine, will equally apply in this case. We have seen there, that the air can never be wholly exhausted, but only rarefied to a very high degree; and, the same may be observed with regard to the exhausting syringe: for, when the piston has reached the bottom, there will remain a small space between it and the cock C, filled with common air. When the piston is drawn up, this small quantity of air expands, and also a similar quantity in the neck of the other cock; and no air will come out of the receiver V till the expanded air in the barrel is of less density than the air in the receiver; which circumstance evidently directs us to make the two spaces as small as possible, or, by some contrivance, to fill them up entirely; and certain constructions have been recommended for accomplishing this purpose: we shall not, however, dwell on the subject in this place, but pass to a description of the common valve piston.

The pipe of communication MN (fig. 21-*a*) has a male screw in its extremity, and over this is tied a slip of bladder or leather, M. The lower half of the piston has also a male screw on it, covered at the end with a slip of bladder Q. This is screwed into the upper half of the piston, which is pierced with a hole H coming out of the side of the rod. If, now, we suppose the syringe screwed to the conducting-pipe, and that screwed into the receiver V, and the piston at the bottom of the barrel; when the latter is drawn up, the pressure of the external air shuts the valve O, and a void is left below the piston: there is, therefore, no pressure on the upper side of the valve M to balance the elasticity of the air in the receiver, which before balanced the weight of the atmosphere. When the piston is let go, it descends, because the elasticity of the expanded air is not a balance for the pressure of the atmosphere, and therefore, presses down the piston with the difference, keeping the piston valve still shut. The valve M also shuts at the same time; for, the latter is at first opened by the prevailing elasticity of the air in the receiver, and while it is open, the two airs become of equal density and elasticity: but, the moment the piston descends, the capacity of the barrel, or of that part containing the air, is diminished, and the elasticity of the air is increased, which prevailing over that in the receiver, shuts the valve M, as above stated.

But the piston will soon have arrived at such a part of the barrel, that the air in it is of the same density with the external air, and there is no force to push it further down, which must therefore now be done by the hand. This tends to condense the air in the barrel, and therefore increases its elasticity, so that it lifts the valve O, and escapes, and the piston gets to the bottom. When drawn up, greater force is required than in the former instance, because the elasticity of the included air is less than in the preceding stroke; the piston rises further before the valve M is lifted up, and when it has reached the top of the barrel the density of the included air will be further diminished. The piston now, when let go, will descend further than it did in the former descent, before the valve opens, and therefore there will be less to effect with the hand, in order to cause the air to escape through O. Thus the rarefaction may be carried on at pleasure, by repeating the strokes of the piston. This is certainly by far the simplest form for an exhausting syringe, but it is far less powerful in its operation than one with cocks. It is evident that no air can come out of the receiver, unless its elasticity exceed that of the air in the barrel by a difference sufficient to raise up the valve M. A piece of oiled leather tied across this hole can scarcely be made tight, and certain of clasping to the hole, without some small straining, which must therefore be overcome; and it must be very gentle indeed not to require a force equal to the weight of 2 inches of water, and this is equal to about the 200th part of the whole elasticity of common air; and therefore if nothing further counteracted the operation, we could not rarefy the air more than 200 times; but the valve O cannot be raised without a similar effort, or prevalence of the elasticity of the air in the barrel above the weight of the atmosphere; consequently, uniting the two causes, it will be very difficult, if not impossible, to rarefy the air 100 times, and very few such syringes will rarefy it more than 50 times; whereas a good syringe with cocks will rarefy 1000 times; but it must be observed that these latter are very expensive, difficult to make, and, what is still worse, are very soon, by the repeated turning of the cocks, put out of order; on which account the attention of instrument-makers have been principally directed towards the improvement of the valve piston, as we shall see in the following description of the air-pump, of which the syringe must be considered as a component part.

II. *Air-pump.*

181. Of all pneumatic machines, the air-pump seems entitled to the highest consideration; since it is to this we owe much of our knowledge of the physical properties and mechanical operations of common or atmospheric air as a fluid body. The invention of this noble instrument forms so important a part of the history of the science of Pneumatics, that we conceived we should have left our first chapter imperfect, had we not given a description of one or two of the first constructions of this kind; but we have pursued the subject no further than was deemed absolutely necessary for giving a connected sketch of the progress of the science generally. Again, in our third section, we have examined and illustrated the general theoretical principles of the air-pump, and have investigated such analytical formulæ as are necessary for computing its effects; it only remains, therefore, to enter here upon the mechanical improvements that this machine has undergone since the ingenious but imperfect construction of Boyle, whose apparatus has been already described in the section above referred to.

182. *Boyle's air-pump improved by Hawkesbee.*—The description of this machine as given by Desaguliers, in vol. 2 of his Exper. Philos. is as follows:

It consists of two brass barrels *a a*, *a a* (fig. 22), 12 inches high and 2 wide. The pistons are raised and depressed by turning the winch *b b*. This is fastened to an axis passing through a strong toothed wheel, which lays hold of the teeth of the racks *c c c c*, the one being raised while the other is depressed; by which means the valves, which are made of limber bladder, fixed in the upper part of each piston, as well as in the openings into the bottom of the barrels, perform their office of discharging the air from the barrels, and admitting into them the air from the receiver to be afterwards discharged; and when the receiver becomes pretty well exhausted of its air, the pressure of the atmosphere in the descending piston is nearly so great, that the power required to raise the other is little more than is necessary for overcoming the friction of the piston; this renders the pump preferable to all others, which require more force to work them as the rarefaction of the air in the receiver advances.

The barrels are set in a brass dish about two inches deep, filled with water or oil to prevent the insinuation of air. The barrels are screwed tight down by the nuts *e e*, *e e*, and thus force the frontispiece *f f* down on them, through which the two pillars *g g*, *g g* pass.

Gage to the pump.

From between the barrels rises a slender brass pipe *h h*, communicating with each by a perforation in the transverse piece of brass on which they stand. The upper end of this pipe communicates with another perforated piece of brass, which screws on underneath the plate *i i i i*, of 10 inches diameter, and surrounded with a brass rim to prevent the shedding of water used in some experiments. This piece of brass has three branches: 1st, An horizontal one communicating with the conduit-pipe *h h*. 2d, An upright one screwed into the middle of the pump-plate, and terminating in a small pipe *k*, rising about an inch above it. 3d, A perpendicular one, pointing downwards in the continuation of the pipe *k*, and having a hollow screw in its end receiving the brass cap of the gage-pipe *l l l l*, which is of glass, 34 inches long, and immersed in a glass cistern *m m* filled with mercury. This is covered on the top with a cork float, carrying the weight of a light wooden scale divided into inches, which are numbered from the surface of the mercury in the cistern; and the scale will therefore rise and fall with the mercury in the cistern, and indicate the true elevation of that in the tube.

There is a stop-cock immediately above the insertion of the gage-pipe, by which its communication may be cut off; and another at *n*, by which a communication is opened with the external air, for allowing its re-admission; there is sometimes also another immediately within the insertion of the conduct-pipe for cutting off the communication between the receiver and the pump. This is particularly useful when the rarefaction is to be continued long, as there are by these means fewer chances of the insinuation of air by the many joints.

The receivers are made tight by simply setting them on the pump-plate with a piece of wet or oiled leather between; and the receivers, which are open at the top, have a brass cover set on them in the same manner. In these covers there are perforations and contrivances for various purposes. The one in the figure has a slip wire passing through a collar of oiled leather, having a hook or a screw in its lower end for hanging any thing on, or for producing a variety of motions.

Sometimes the receivers are set on another plate, which has a pipe screwed into its middle, furnished with a stop-cock and a screw, fitting the middle pipe *k*. When the rarefaction has been made in it, the cock is shut, and then the whole may be unscrewed from the pump, and removed to any convenient place. This is called a *transporter-plate.*

It only remains to explain the gage *llll*. In the ordinary state of the air its elasticity balances the pressure of the incumbent atmosphere. We find this from the force that is necessary to press it into less bulk in opposition to this elasticity. Therefore the elasticity of the air increases with the vicinity of its particles; consequently it is reasonable to expect, that when we allow it to occupy more room, and its particles are further asunder, its elasticity will be diminished, though not annihilated; that is, it will no longer balance the WHOLE pressure of the atmosphere, though it may still balance part of it. If, therefore, an upright pipe have its lower end immersed in a vessel of mercury, and communicate by its upper end with a vessel containing rarefied, and therefore less elastic, air, we should expect that the pressure of the air will prevail, force the mercury into the tube, and cause it to rise to such a height that the weight of the mercury, added to the elasticity of the rarefied air, acting on its upper surface, shall be exactly equal to the whole pressure of the atmosphere. The height of the mercury is the measure of that part of the whole pressure which is not balanced by the elasticity of the rarefied air, and its deficiency from the height of the mercury in the Torricellian tube, is, in like manner, the exact measure of this remaining elasticity.

It is evident, therefore, that the pipe will be a scale of the elasticity of the remaining air, and will indicate the degree of rarefaction; for there must be some analogy between the density of the air and its elasticity, and we have seen (except in extreme cases of condensation and rarefaction) that they increase and diminish together and according to the same law.

This gage must be considered as one of the most important improvements introduced into this machine by Hawkesbee; at the same time it must be acknowledged to have certain imperfections and inconveniences attending it, which are obviated in other contrivances of a like kind; some of these will be described in what follows:

This construction imperfect.

183. It is, however, not only in the gage that inconveniences are experienced, but in the pump itself, constructed on the principles above explained, and which are, indeed, common to all the machines of the old or common form. It will be immediately perceived, that air-pumps merely serve to rarefy the air to a greater or less degree, and that none of them can produce a complete exhaustion; the same will appear by consulting the analytical formula given (art. 41) for expressing the degree of rarefaction after any number of strokes, from which we perceive, that nothing short of an infinite number can produce a complete exhaustion. But, independent of this theoretical impossibility, there are mechanical defects which prevent us from carrying on the exhaustion according to the law laid down in theory; few pumps will bring the mercury, in such a gage as that above described, to within $\frac{1}{10}$th of an inch; and that of Hawkesbee's, fitted up according to his own instructions, will seldon bring it within $\frac{1}{5}$th. Pumps with cocks of the best construction, and under the most favourable circumstances, will bring it within $\frac{1}{40}$th; but this degree of rarefaction has never been attained, when valves fitted up with wet leather were employed: in such cases, $\frac{1}{5}$th of an inch is the least that the gage falls short of the Torricellian column.

Pne ma an peri

Pumps with stop-cocks, when well made and newly put together, are generally found to rarefy the air to a greater degree than those which are made with valves; but after being used for some time they become less accurate than those with valves. The valves, however, are also imperfect, because the external air, pressing upon that in the piston, prevents its rising when the elastic force of air in the receiver under exhaustion is much diminished; and attempts have therefore been made, particularly by the Abbé Nollet and Gravesande, to perfect the construction of cocks.

Gra san air-

184. In the latter author's double-barrelled-pump, the cocks at the bottom of the pistons are turned by an apparatus that is moved by the handle of the pump. The piston has no valve, and the rod is connected with it by a stirrup, as in a common pump. This rod has a cylindrical part, which passes through the stirrup and moves stiffly in it, through the space of about half an inch, between a shoulder above and a nut below. The stirrup supports a round plate, which has a short square tube that fits tight into the hole of a piece of cork, and which has also a square shank that goes into the square tube. Between the plate and the cock is put a piece of thin leather, soaked in oil, and another is placed between the cork and the plate, which forms the sole of the stirrup. When the winch is turned to raise the piston from the bottom of the barrel, the friction of the piston against it keeps it in its place, and the rod is drawn up through the stirrup. The wheel has thus liberty to turn about an inch, and this is sufficient to turn the cock, so as to cut off the communication with the external air, and to open that with the receiver. When this is done, the continued motion serves to raise up the piston to the top of the barrel. When the winch is turned in the opposite direction, the piston remains fixed till the cock is turned, so as to shut the communication with the receiver and open that with the external air. The cock has one perforation diametrically through it, and another in a perpendicular direction to this, and after reaching the centre, it passes along the axis of the cock, and communicates with the external air.

By means of this communication, when it is opened, the air rushes in and ballances the pressure on the upper side of the piston in this barrel, so that the pressure on the other must be counteracted by the person who works the pump. In order to obviate this inconvenience, Gravesande put a valve on the orifice of the cock, by tying over it a slip of wet bladder, or oiled leather; and by means of this the piston is pressed down as long as the air in the barrel is rarer than the outward air, just as if the valve were in the piston itself.

This is all that is necessary to be described in Gravesande's pump; its performance is highly extolled by him, as far exceeding his former pumps with valves; and the same preference was given to it by his successor Muschenbroeck. Hawkesbee's pump, however, maintained its pre-eminence in this country, and, indeed, pretty generally on the continent, except in France, till about the year 1750, when Smeaton, so celebrated for his mechanical skill, undertook to improve and perfect the cock-pumps, which, however, he at length, after various attempts, gave up, finding it absolutely impossible to bring them to that degree of accuracy he had in view; and he then turned his attention to the valve-pumps, and the great advantages gained by his construction it is now for us to explain.

His first object was to diminish the resistance to the entry of the air from the receiver into the barrels, which he rendered almost nothing, by enlarging the surface on which this feebly elastic air was to press. Instead of making these valves to open by the pressure of the air on a circle of $\frac{1}{20}$th of an inch in diameter, he made the valve-hole an inch in diameter, thus enlarging the surface 400 times; and to prevent this piece of thin leather from being burst by the great pressure upon it, when the piston in its descent was approaching the bottom of the barrel, he supported it by a delicate but strong grating, dividing the valve-hole like the section of a honey-comb; and that the points of contact between the bladder and grating might be as few as possible, the holes were made hexagonal, and the partitions filed almost to an edge. The breadths of these hexagons are $\frac{1}{10}$ths of an inch, and, consequently, the surface nine times larger than common; and, as the circumference is three times greater than that of the common valve, and the cohesion to be overcome is, in the first moment of the air exerting its force, proportional to the circumference of the hole, the valve over any of these holes will be raised with three times more ease. Beside, the raising of the valve over the centre hole is aided on all sides by those that are placed round it; and, as they all contribute as much to raise the bladder over the centre hole, as the air acting immediately under it, the valve will be raised with much more facility than we have already supposed, that is, with about one-sixth of the force commonly necessary.

The other defect in the common construction would still hinder the rarefaction from being carried on beyond a certain degree; for, as the piston does not fit so closely to the bottom of the barrel as totally to exclude the air, this air, as the piston rises, will expand itself, and, pressing upon the valves in proportion to its density, prevent the air within the receiver from coming out. Therefore, if a vacancy, for example, equal to 150th part of the capacity of the whole barrel had place, no air could pass out of the receiver when expanded 150 times, though the piston were constantly drawn to the top; because the air in the receiver would be in equilibrio with that in the barrel when in its highest state of rarefaction.

In order to obviate this inconvenience, Mr. Smeaton shut up the top of the barrel with a plate, having in the middle a collar of leather, through which the cylinder works that carries the piston; by which means the external air is prevented from pressing on the piston; but, for the discharge of the air that passes from below through the valve of the latter, there is another valve applied to the plate at top, which opens upwards. By this construction, when the piston is put down to the bottom of the cylinder, the air under it will evacuate itself so much the more, as the valve of the piston opens more easily when pressed by the rarefied air above it, than when pressed by the whole weight of the atmosphere; and, as the piston may be made to fit as nearly to the top of the cylinder as it can to the bottom, the air may be rarefied as much above the piston as it could before have been in the receiver; whence it follows, that the air may now be rarefied in the receiver, in the duplicate proportion to what it could be upon the common principle. By this means a pump, consisting of a common barrel, may be worked with more ease than the common pump with two barrels, because the pressure of the outward air is taken off by the upper plate; and when a considerable degree of rarefaction is required, it will produce it more speedily.

Smeaton's air-pump, by Nairne.

185. Mr. Smeaton's air-pump, as constructed by Mr. Nairne, may be described as follows:

Fig. 23.

Upon a solid base or table are set up three pillars F, H, H (fig. 23); the pillar F supports the pump-plate A; and the pillars H, H, support the front or head, containing a brass cog-wheel, which is turned by the handle B, and works in the rack C fastened to the upper end of the piston-rod. The whole is still farther steadied by two pieces of brass *c b* and *o k*, which connect the pump-plate with the front, and have perforations communicating between the hole *a* in the middle of the plate and the barrel. DE is the barrel of the pump, firmly fixed to the table by screws through its upper flanch; *e f d c* is a slender brass tube screwed to the bottom of the barrel, and to the under hole of the horizontal canal *c b*. In this canal there is a cock which opens a communication between the barrel and the receiver, when the key is in the position represented in the figure; but when the key is at right angles with this position, this communication is cut off. If that side of the key which is here drawn next to the pump-plate be turned outward, the external air is admitted into the receiver; but if turned inwards, the air is admitted into the barrel.

g h is another slender brass pipe, leading from the discharging valve at *g* to the horizontal canal *h k*, to the under side of which it is screwed fast. In this horizontal canal there is a cock *n* which opens a passage from the barrel to the receiver when the key is in the position here drawn; from the barrel to the external air when the key is turned outwards, and from the receiver to the external air when the key is turned inwards. This communication with the external air is not immediate, but through a sort of box *i*; the use of which is to receive the oil which is discharged through the top valve *g*. In order to keep the pump tight, and in working order, it is proper sometimes to pour a table-spoonful of olive oil into the hole *a* of the pump-plate, and then to work the pump. The oil goes along the conduit *b c d f e*, gets into the barrel and through the piston-valve, when the piston is pressed to the bottom of the barrel, and is then drawn up, and forced through the discharging valve *g* along the pipe *g h*, the horizontal passage *h n*, and finally into the box *i*. This box has a small hole in its side, near the top, through which the air escapes.

From the upper side of the canal *c b* there arises a slender pipe, which bends outward and then turns downwards, and is joined to a small box, which cannot be seen in this view. From the bottom of this box proceeds downwards the gage-pipe of glass, which enters the cistern of mercury G fixed below.

On the upper side of the other canal at *o* is seen a small stud, having a short pipe of glass projecting horizontally from it, close by and parallel to the front piece of the pump, and reaching to the other canal. This pipe is closed at the further end, and has a small drop of mercury or oil in it at the end *o*. This serves as a gage in condensing, indicating the degree of condensation by the place of the drop; for this drop is forced along the pipe, and condenses the air before it in the same degree that it is condensed in the barrel and the receiver.

In constructing this pump, Mr. Smeaton introduced a method of joining together the different pipes and other pieces, which has great advantages over the usual manner of screwing them together with leather between, and which is now much used in hydraulic and pneumatic engines.

Fig. 24.

The piece *h i p* in fig. 24, is the same with the little cylinder observable on the upper side of the horizontal canal *c d*, in fig. 23. The upper part *h i* is formed into an outside screw, to fit the hollow screw of the piece *d e e d*. The top of this last piece has a hole in its middle, giving an easy passage to the bent tube *c b a*, so as to slip along it with freedom. To the end *c* of this bent tube is soldered a piece of brass *c f g*, perforated in continuation of the tube, and having its end ground flat on the top of the piece *h i p*, and also covered with a slip of thin leather strained across it and pierced with a hole in the middle.

It is plain, from this form, that if the surface *f g* be applied to the top of *h i*, and the cover *d e e d* be screwed down on it, it will draw or press them together, so that no air can escape by the joint, and this without turning the whole tube *c b a* round, as is necessary in the usual way.

The conduit-pipe E *e f c* (fig. 23) is fastened to the bottom of the barrel, and the discharging pipe *g h* to its top, in the same manner.

Advantages of this construction.

186. The above description of Mr. Smeaton's pump will be sufficient for enabling the reader to understand its operation, and to appreciate its advantages over other pumps of the common construction: in order to place the latter more immediately before the reader, we may observe, that, supposing the stiffness of the valve equal to the weight of $\frac{1}{10}$th of an inch of mercury, when the barometer stands at 30 inches, and that the pump-gage stands at 29·9; then, in the ordinary pump, the valve in the piston will not rise, till the latter has got within the 300th part of the bottom of the barrel, and it will leave the valve-hole filled with air of the ordinary density. But, in this pump, the valve will rise as soon as the piston quits the top of the barrel, and when it is quite down, the valve-hole will contain only the 300th part of the air which it would have contained in a pump of the usual form. Suppose further, that the barrel is of equal capacity with the receiver, and that both pumps are so badly constructed, that the space left below the piston is the 300th part of the barrel, then, in the common pump, the piston-valve will rise no more, and the rarefication can be carried no further, however delicate the barrel-valve may be; but in this pump, the next stroke will raise the gage to 29·95, and the piston-valve will again rise as soon as the piston gets half way down the barrel.

Another advantage attending this construction, is that in drawing up the piston, we are not resisted by the whole pressure of the air; because the air is rarefied above this piston as well as below it, and the piston is in precisely the same state as if connected with another piston in a double pump. The resistance to the ascent of it, is the excess of the elasticity of the air above it, over the elasticity of the air below; this, towards the end of the rarefaction, is very small, while the piston is near the bottom of the barrel, but gradually increases as the piston rises, reduces the air above it into smaller dimensions, and becomes equal to the pressure of the atmosphere, when the air above the piston is of the common density. If we should raise the piston still higher, the air would be condensed above, but, to prevent this, Mr. Smeaton has there made an issue for the air, by a small hole in the top of the barrel, covered with a delicate valve, which allows the air to escape, and shuts again as soon as the piston begins to descend, leaving almost a perfect void behind.

A third advantage belonging to this construction is, that the pump may be in a moment converted from a rarefying to a condensing machine, by simply turning the cocks introduced for that purpose.

187. We have had occasion, in describing Hawkesbee's, or Desagulier's air-pump, to mention the gage employed by that author for estimating the degree of rarefaction produced by the operation of the machine; we have also stated that it was subject to some inconveniencies, and particularly on account of its length, which rendered its application very incommodious. In order to remove this inconvenience, a *short barometer-gage* was next introduced. This was, in fact, only the bottom part of a common barometer, about eight or nine inches in length, filled with mercury, and immersed with its aperture into a small quantity of mercury contained in a glass vessel, which forms the cistern. This may either be placed under the receiver, upon the principal plate of the pump, or under a small receiver, upon a small auxiliary plate, which is annexed to some air-pumps for this purpose. As this gage is not equal to a whole barometer, it will not show the first stages of the rarefaction; but its indications will commence when about three-fourths of the air have been removed from the receiver; that is, when the air has been rarefied till its remaining elasticity is not able to suppor that short column of mercury.

188. The *siphon-gage*, which is shown in the figu of Smeaton's air-pump, differs from the short baromete gage merely in this circumstance, that, instead of te minating in a cistern, the tube is here bent and ris upwards with its aperture, which, by means of a bra tube, is made to communicate with the inside of t pump, so that the ascending leg of the tube perfor the office of a cistern. Hence, in rarefying the air, t mercury descends from the closed end of the tube, a rises into the ascending leg; and, consequently, t altitude of it in one leg, above its altitude in the oth shows the degree of rarefaction; and this altitude, in the long and short barometer-gages, is shown by annexed scale of inches and parts.

The gages above-mentioned evidently indicate elasticity of the fluid, which remains in the receiver the pump after a certain degree of rarefaction; an is immaterial whether that elastic fluid be air or vapo

189. Mr. Smeaton invented another instrument this kind, which, from its form, is called a *pear-ga* this shows (not at the actual time, but after the admission of the air into the receiver) how much that fluid had been left in the receiver in the preced rarefaction. This gage is also shown in the figure Smeaton's pump above described. It consists of a gla vessel, which has a small projecting orifice, and at *th*

other end it is extended into a tube closed at the upper extremity; the capacity of which is 100th part of that of the whole vessel. The instrument is suspended with its aperture downwards to the lower end of a slip of wire, or a wire which passes through a collar of leather, within the glass receiver of the pump, and exactly under it, a small cup, containing quicksilver, is placed upon the plate of the pump. When the machine has been worked to the intended degree, the air in the pear-gage is evidently rarefied as much as it is in the receiver; and in that state, by lowering the slip of wire, the pear-gage is let down till its aperture has reached the bottom of the mercury. This done, the external air is admitted into the receiver; which cannot, as is obvious, enter the pear-gage, on account of the aperture being immersed in the quicksilver; but the pressure of the atmosphere, now acting on the surface of the mercury, forces that fluid into the gage, and fills it up to a certain degree, for instance to *r*; then the upper part of the gage will contain all the air or vapour which occupied the whole cavity of it during the rarefaction. To the upper part of the gage is annexed a divided scale, which shows what part of the capacity of the whole gage is filled with air, and of course measures the degree to which the rarefaction of the air had been carried.

190. When Mr. Smeaton first made use of this gage, he was much surprised to find that it did not give always the same results as the common gages above described; but it was afterwards found that it arose from the vapour which was mixed with, and formed a part of the elastic fluid in the receiver. When the receiver contains no other fluid besides air, the pear-gage and the other gages will indicate the same degree of rarefaction; but if the receiver contain the vapour of water or of other liquor, then the pear-gage will indicate a much greater degree of rarefaction than the siphon or short barometer-gage; because the vapour, which has elasticity sufficient to supply the place of air in the receiver, on the re-admission of air, is condensed into a much smaller space than the same quantity of rarefied air would be, so that the pear-gage shows the quantity of air alone which had been left in the receiver, whereas the other gages show the quantity of elastic fluid which is actually remaining in it.

191. This cause, however, of the difference in these two kinds of gages was not in the least suspected when the circumstance was first observed. Mr. Nairne, who constructed many instruments and gages after the plan of Mr. Smeaton, was much struck with the disagreement above alluded to, and made a great variety of experiments, in order, if possible, to discover the reason of it; some of which being repeated before the Honourable Mr. Cavendish, the latter accurate philosopher immediately accounted for the amazing difference which the two gages indicated on the principles above stated, by referring to some experiments of his father, Lord Charles Cavendish, from which it appeared that water, whenever the pressure of the atmosphere on it is diminished to a certain degree, is immediately turned into vapour, and reduced as suddenly to water again on restoring the pressure. The degree of pressure varies according to the temperature of the water; for when the heat is 72° of Fahrenheit's scale, it is converted into vapour as soon as the pressure is reduced to three-quarters of an inch of quicksilver, or about one-fortieth of the usual pressure of the atmosphere; but when the heat is only 41° of Fahrenheit's scale, the pressure must be reduced to that of a quarter of an inch of quicksilver, or to a $\frac{1}{120}$th part of the usual pressure, before the water turns into vapour. According to this hypothesis, whenever the air in the receiver is exhausted to the above-mentioned degree, the moisture adhering to the different parts of the machine will be converted into an elastic vapour, and supply the place of the air which is drawn away by the working of the pump, and the fluid left in the receiver and pear-gage will be chiefly this vapour. When the air is let into the receiver again, the vapour within the pear-gage will be reduced to water, and only the real air will remain uncondensed; consequently, this gage shows only how much real air is left in the receiver, and not how much the pressure or spring of the included air is diminished; whereas either of the other gages shows how much the included elastic fluid is diminished, and that equally, whether it consist of air or vapour.

Pneumatic machines and experiments.

The above hypothesis confirmed by Mr. Nairne.

192. In order to ascertain the truth of this plausible theory, Mr. Nairne proceeded to free every part of his apparatus as much as possible from any adhering moisture, concluding that by this means he should be able to bring the two gages to an agreement. Instead, therefore, of placing the receiver on leather as before, he put it on the pump-plate, made as clean and dry as possible, and applied a cement round its edge to exclude the outward air. When the pump, in this state, was worked for ten minutes, the barometer-gage indicated a degree of exhaustion nearly equal to 600; and on letting the air into the receiver, the pear-gage showed also an exhaustion but little exceeding the former. In another experiment he put a piece of the oiled leather above-mentioned in the receiver, and found, on working the pump, that the barometer-gage indicated a degree of exhaustion of nearly 300, while, on the re-admission of the air, the pear-gage gave an exhaustion of little less than 4000, and thus confirmed the hypothesis advanced by Mr. Cavendish. He performed, likewise, numerous other experiments relating to this subject, which may be seen in vol. lxvii. Phil. Trans.

Imperfection of Smeaton's pump.

193 Let us, after this necessary digression, return to the subject more particularly in question, which we shall do by observing, that there still remained one imperfection which Mr. Smeaton had not attempted to remove. The discharging-valve was still opened against the pressure of the atmosphere; and although the ingenious contrivances of this celebrated mechanist had greatly diminished, it had not annihilated the obstructions to the passage of the air from the receiver into the barrel. His success, however, encouraged other attempts, but they are too numerous for us to enter upon in detail in this article; we shall, therefore, pass over some of minor importance, and proceed at once to describe that of Mr. Prince, a native of America, who conceived the idea of dispensing with the valves entirely.

Prince's air-pump.

194. In order to effect this, he removed the lower valve, and opened the bottom of the barrel into a cistern on which it was placed, and which had a free communication with the receiver; for the valve on the upper plate on the top of the barrel, constructed like Mr. Smeaton's, made it unnecessary that there should be any at the bottom, in order to rarefy the air in the receiver. The cistern was made deep enough to admit the piston descending into it below the bottom of the

barrel. If the piston be solid, that is, without a valve, when it enters the barrel and rises to the top plate, which is made air-tight with a collar of leather like Smeaton's, it forces out all the air above it; and as the air cannot return into the barrel on account of the valve in the top plate, when the piston descends there will be a vacuum between it and the plate, every thing being supposed perfect. But, in working the pump, the piston is not allowed to descend entirely into the cistern, so far as to leave the barrel open; it descends only below a hole in the side of the barrel near the bottom, which opens a free communication between the barrel, cistern, and receiver. Through this hole the air rushes from the cistern into the exhausted barrel, when the piston has dropped below it; and by the next ascent, this air is forced out as the other was before. If the capacity of the receiver, cistern, pipes, &c. below the bottom of the barrel taken together, be equal to the capacity of the barrel, half the remaining air will be expelled at every stroke. But in order to save the labour that would attend the working of this pump with a solid piston, the constructor pierced it with three holes, at equal distances from each other, and by a circular piece of bladder tied over the top of the piston, formed a sort of valve over the holes, which opened with sufficient ease to save any labour in working the pump, by allowing the air to pass through the piston in its descent. The escape of the air, however, does not depend upon a passage through the piston into the barrel; for when the air, weakened by rarefaction, cannot open this valve, it will still get into the barrel when the communication is opened with the hole at the bottom. This piston will, therefore, descend as easily as any other, nor will the rarefaction be in any respect impeded by the valves. By this construction the valves, made to open with more ease by Mr. Smeaton, are rendered useless for rarefying the air, and that at the bottom of the barrel is entirely removed; the valve on the top plate being the only one necessary in the rarefaction.

The next object of this mechanist was to expel the air more perfectly out of the barrel than was effected in the machine last described, by making a more complete vacuum between the piston and the top plate, so that more of the air might be allowed to expand itself into the barrel from the receiver. Mr. Prince also contrived to connect the valves on the top plate with the receiver, occasionally, by means of a pipe and cock, by the turning of which the machine might be made to exhaust or condense at pleasure. It still remained to remove the pressure of the atmosphere from the valve on the top plate, so that this might open as easily as the piston-valve; with which view he connected with the duct on the bottom piece, which conveys the air from the valves to the cock, a small pump of the same construction as the large one, having the barrel opening into the cistern; the piston-rod, which is solid, moving through a collar of leathers, and a valve at the top, through which the air is forced into the atmosphere. This is called the valve-pump; its principal use being to rarefy the air above the valves, or to remove the weight of the atmosphere from them. When this valve-pump is used, the passage through the cock is shut up; and, therefore, instead of placing three ducts at equal distances round the cock, after the manner of Mr. Smeaton, the whole is divided into five equal parts, leaving the distance of one-fifth part between th
leading from the cistern and the valves to th
and two-fifths between each of these and the o
ing from the cock to the receiver; therefore, w
communication is open between the receiver
valves for condensation, the other hole thro
cock opens the cistern to the atmosphere; b
the communication is made between the ciste
the receiver, for exhaustion, a solid part of
comes against the duct leading to the valve,
up, and the air which is forced out of the barrel
through the atmosphere into the valve-pump.

195. A great advantage in this construction
the pump with two barrels may be made like the
pump, which cannot be conveniently done w
lower valve is retained. In this pump the pis
not move the whole length of the barrel, a ho
section being made in them a little more than h
from the bottom, where the top plates are inser
which account it is more convenient and simple
head of it is brought down from the top of the
in the same manner as in the common air-pum
barrels also stand upon the receiver-plate,
raised high enough to admit the long barometer
32 or 33 inches to stand under it, without incon
in working the pump; as the winch moves th
smaller portion of an arc at each stroke, than i
do if the piston moved through the whole lengt
barrels.

196. When the condensation is to be mea
gage is placed between the barrels, having a fr
munication with the valves, cock, &c. which is
so constructed that it will serve to measure t
faction above the valves, when the air is worke
the valve-pump. This consists of a pedestal, t
which is made of glass (which forms a ciste
mercury), a hollow brass pillar, and a glass t
metically sealed at one end, which moves up a
in the pillar through a collar of leathers. W
pump is used as a condenser, the degree of cond
is shown by a scale marked on one edge of th
and when as an exhauster, the rarefaction is m
the other edge. This gage will also show
valves have done playing, either with the weig
atmosphere on them, or taken off. For the pur
great condensation, the inventor has fitted a c
of a smaller bore than the barrel of the great
the cistern of a valve-pump, to be screwed on o
ally. Or even without this condenser, the val
may be adapted to the purpose, by being mad
larger, and by having a plate made to screw
bottom of the cylinder, with a valve on it ope
the cistern; a hole must be made to open on
occasion near the top of the cylinder, to let air
the piston, when this is drawn up above it. T
mon gage, which is generally placed under the
plate, is here situated in the front of the pump
it may be seen by the person working the mach
that the plate itself may be left free for other
plate is also so fixed to the pipe leading to t
that it may be taken off at pleasure, and u
transferrer, or for other purposes.

197. We may likewise observe, that the
made whole, except a small piece on the bac
the wheel is let in; the latter being freed from th
rods by pushing it into the back part of t

where it is kept in its place by a button, screwed into the socket in the axis behind. The piston-rods are thus dislodged from the wheel, and let down into the cistern when the pump is not in use, in which cistern they may have the advantage of being covered with oil; the same may also be done with all the principal joints which are sunk into the sockets, that the leathers which close them may be covered with oil, to prevent their leaking after a long disuse.

A perspective view of Prince's air-pump, as constructed by Mr. Jones, is shown in (fig. 25), which will serve to render the above account more intelligible, and save the necessity of further description.

198. *Cuthbertson's air-pump.* The last improvement in the construction of air-pumps which it will be necessary for us to mention in this place, is that by Mr. Cuthbertson; this machine is so excellent in its structure, and so powerful in its effects, as to claim particular notice and description.

A perspective view of it is given in (fig. 26), with its two principal gages screwed into their places, which need not, however, be used together except in cases where the utmost exactness is required. In common experiments one of them is removed, and a stop-cock put in its place. When the pear-gage is used, a small, round plate, on which the receiver may stand, must be first screwed into the hole at A; but this hole is stopped on other occasions by a screw. When all the three gages are used, and the receiver is exhausted, the stop-screw B, at the bottom of the pump, must be unscrewed, to admit the air into the receiver; but when they are not all used, any of the stop-screws will answer the purpose. Fig. 27 represents a cross bar for preventing the barrel from being shaken, either by working the pump, or by accident; its place in fig. 26 is represented by the dotted lines; where it is confined and kept down close to the barrels by two slips of wood NN, which must be drawn out, as well as the screws OO, when the pump is taken asunder. The figures 28, 29, &c. exhibit a section of all the working parts of the pump, except the wheel and rack, in which there is nothing uncommon. Fig. 28 is a section of one of the barrels, with all its internal parts; and figs. 29, 30, 31, and 32 are different parts of the piston, proportioned to the size of the barrel.

In fig. 28, CD represents the barrel, F the collar of leathers, G a hollow cylindrical vessel to contain oil, R is also an oil-vessel to receive the oil which is drawn along with the air through the hole *a a*, when the piston is drawn upwards; and when this is full, the oil is carried over with the air, along the tube T, into the oil vessel G; *cc* is a wire, which is driven upwards from the hole *a a* by the passage of the air; and as soon as this has escaped, it falls down again by its own weight, shuts up the hole, and prevents all return of air again into the barrel.

At *d d* are fixed two pieces of brass to keep the wire *c c* in a vertical position, in order that it may accurately shut the hole; and H is a cylindrical wire or rod, which carries the piston I, being made hollow to receive a long wire *g g*, which opens and shuts the hole L. On the other end of the wire O, is screwed a nut, which, by stopping in the narrowest part of the hole, prevents the wire from being driven up too far; these are seen more distinctly in figs. 29 and 33; they slide in a collar of leather *r r*, figs. 29 and 32, in the middle piece of the piston. Figs 31 and 32 are the two mean parts which compose the piston, and when the pieces fig. 30, and 33, are added to it, the whole is represented by fig. 29; and fig. 32 is a piece of brass of conical form, having a shoulder at bottom, in which is cut a long hollow screw, about two-thirds of its length, and the remaining part of the hole, in which there is no screw, is of about the same diameter as the screwed part, except a thin plate at the end, which is of a width exactly equal to the thickness of *g g*.

That part of the inside of the conical brass in which no screw is cut, is filled with oiled leathers, having holes, through which *g g* can slide stiffly. An external screw, with a hole in it, is also fitted to *g g*, and serves to compress the leathers *r r*. In fig. 31, *a a a a* is the outside of the piston, the inside of which is turned so as exactly to fit the outside of fig. 32; *b b* are round leathers, about sixty in number; and *c c* is a circular piece of brass of the size of the leathers, while *d d* represents the screw serving to compress them. The screw at the end of fig. 30 is made to fit the screw fig. 32, and if fig. 33 be pushed into fig. 32, this into fig. 31, and fig. 30 be screwed into the end of fig. 32, these will compose the whole piston, as shown in fig. 28. In fig. 28 and 29, H represents the same part, being that to which the rack is fixed; if, therefore, this be drawn upwards, it will cause fig. 32 to shut close into fig. 31, and drive out the air above it; and on the contrary, when it is pushed downwards, it will open as far as the shoulder *a a* will permit, and suffer the air to pass through. AA (fig. 34) is the receiver-plate, BB is a long square piece of brass, screwed into the under side of the plate, through which a hole is drilled, corresponding to that in the centre of the receiver-plates, and with three internal screws, *b*, *b*, *c*.

Fig. 34.

Process of rarefaction.

199. Having thus described at large the several parts of this machine, let us now endeavour to explain the process of rarefaction, which is carried on as follows. Conceive the piston to be at the bottom of the barrel, the inside of which, from *a* (fig. 28), contains common air. Now when the rod is drawn up, the upper part of the piston sticks fast in the barrel, till the conical part connected with the rod shuts the conical hole, and its shoulder applies close to its bottom. The piston being now shut, the whole is drawn up by the rack-work, driving the air before it through the hole *a c* into the oil-vessel at R, and out into the atmosphere by the tube T. The piston will then be at the top of the barrel at *a*, and the wire *g g* will stand nearly as shown in the figure, just raised from the hole L, where it is prevented from rising higher by the nut O. During this motion, the air will expand in the receiver, and come along the bent tube *m*, into the barrel. By this means the barrel will be again filled with air, which, as the piston rises, will be rarefied in the proportion of the capacity of the receiver, pipes, and barrel, together, to that of the latter alone.

When the piston is moved down again by the rack-work, it will force the conical part (fig. 32) out of the hollow part (fig. 31) as far as the shoulders *a a*; (fig. 29) will rest on *a a* (fig. 31), which will then be so far open as to permit the air to pass freely through it, while at the same time the end of *g g* is forced against he top of the hole, and by shutting prevents any air from returning into the receiver; thus the piston, moving downwards, suffers the air to pass out between the sides of figs. 31 and 32; and when it is at the bottom of the barrel, it will have the column of air above it, and, consequently, when drawn up, it will shut, and drive out this air, and, by opening the hole at L, will, at the same time, give a free passage to more air from the receiver. This process being continued, the air of the receiver will be rarefied as far as its expansive power will permit; for, in this machine, there are no valves to be forced open by the elasticity of the air in the receiver, which at last it is unable to effect; there is, therefore, nothing to prevent the air from expanding to its utmost degree, which is the peculiar excellency of this construction.

Condensation.

Fig. 35.

200. When this machine is used for condensation, at the same time that it rarefies, or separately, the piece containing the bent tube T must be removed, and fig. 35 put in its place, and fixed by its screws. This figure, as drawn in the plate, is intended for a double-barrelled-pump; but for a single-barrel, only one piece is used, represented by *b a a*, the double piece being cut off at the dotted line *a a*. In this piece is an internal screw, to receive the end of a long brass tube, to which a bladder (if sufficient for the experiment of condensation), or a glass, properly secured for this purpose, must be screwed: then the air, which is abstracted from the receiver on the pump-plate, will be forced into the bladder, or glass; but, if the pump be double, the apparatus fig. 35 is used, and the long brass tube screwed on at *c*.

Figs. 36 and 37.

Figs. 36 and 37 represent two gages, the former of which is screwed into *c b*, or into the screw at the other end of *c* (fig. 34), and fig. 37 into the screw *a b* (fig. 34). When the machine is used as a single pump, either to rarefy or condense, the screw K, which fastens the rack to the piston-rod H, must be taken out; then turning the winch till H is depressed as low as possible, the machine will be fitted to exhaust as a single pump, and if it be required to condense, the directions above given must be observed with regard to the tube T and fig. 35.

Mr. Cuthbertson, by a variety of experiments with this air-pump, has shown its great power of exhaustion with the double siphon-gage, as also with the long gage, compared with an attached barometer, in which the mercury had been repeatedly boiled; and the difference between the heights of the mercurial columns was found not to exceed $\frac{1}{40}$th of an inch, the barometer standing at 30 inches, which gives a rarefaction of 1200. On some occasions, when the air was in a very dry state, the difference has been observed as low as $\frac{1}{100}$th part of an inch, a rarefaction more than double the former. See a Description of an improved Air-Pump by John Cuthbertson; and Nicholson's Phil. Journal, vol. i.

Prince's and Cuthbertson's pumps compared.

201. The latter author, in examining the respective merits of Prince's and Cuthbertson's air-pumps, offers, in that comparison, the following judicious observations:

"There is no provision to open the upper fixed valve of Prince's greater barrel, except the difference between the pressures of the elastic fluid on each side of the stripe of bladder: and this may reasonably be inferred to limit the power of his small pump. In Cuthbertson's pump, the same valve is exposed to the action of the atmosphere, together with that of a column of oil in the oil-vessel. The mischief in either instrument is probably trifling; but in both the valve might have been opened mechanically. If this were done, the small pump of Prince might, perhaps, be unnecessary in most states of the atmosphere. With regard to the lower valves, Cuthbertson, by an admirable display of talents as a workman, has insured their action; Prince, on the other hand, has, by the process of reasoning, so far improved the instrument, that no valves are wanted. In this respect he has the advantage of simplicity and cheapness with equal effect. The mechanical combination of Cuthbertson's pump, reduces the operation to one simple act of the handle; but Prince's engine requires some manipulation with regard to the play of the small pump; though this might have been remedied by a more skilful disposition of the first mover.

"The most perfect scheme for an air-pump, taking advantage of the labours of these judicious operators, seems to be that in which two pistons, of the construction of Prince, should work in one barrel, one piston being fixed at the lower end of the rod, and the other at the middle. The lower piston must come clear out of the barrel when down, and work air-tight through a diaphragm at an equal distance from the effective ends of the barrel. In the diaphragm must be a metallic valve of the form of Cuthbertson's lower valve, but with a short tail beneath, that it may be mechanically opened when the piston comes up. Above the diaphragm must work the open piston, similar to the first; but as it cannot quit the barrel when down, a small portion of the barrel must be enlarged just above the diaphragm, so that the leathers may be clear in that position. Lastly, the top of the barrel must be closed, and fitted with a valve and oil-vessel, according to the excellent contrivance of Cuthbertson."

If we suppose the workmanship of such a pump to leave the space between the diaphragm and lower piston, when up, equal to one thousandth part of the space passed through by the stroke of that piston, the rarefaction produced by this part of the engine will, in theory, bear the same proportion to that of the external air; and the same supposition applied to the upper piston, would increase the effect one thousandth part more: whence the rarefaction would be 1000000. How far the practical effect might fall short of this, from the unavoidable imperfections of the workmanship, or the nature of the air, which, in very high rarefactions may not diffuse itself equally through the containing spaces, or from other yet unobserved circumstances, cannot be deduced from mere reasoning, independent of experiment.

202. We have given above, as far as was consistent with the plan of this work, such a view of the successive improvements in the construction of the air-pump as will furnish, in connection with what has been given in our historical chapter, a connected sketch of the particular history of this important pneumatic machine; at the same time, it must be obvious, that many minute particulars, and a variety of constructions, have necesssarily been omitted; an entire volume, perhaps, being barely sufficient to enter upon this subject at full length; but if we have introduced all the more important particulars, and described one or two of the best modern machines, we trust it is all that is to be expected in a treatise of this kind. We shall conclude our sketch with the description of a table, or portable air-pump, which is, from its size and figure, very

convenient and sufficiently powerful for most common experiments.

203. *Table, or portable air-pump.* This has two brass barrels, which are firmly retained in a perpendicular situation to the square wooden table on which they rest, by a transverse beam, which is pressed upon them by screws at the top of the two pillars. From the hole in the centre of the pump-plate there is a perforation, or canal, in a brass piece to the fore part of the frame of the pump; and from this canal there is a perforation, at right angles to the former, passing to the centre of the base of each barrel. At each of these centres a valve is placed, opening upwards, to admit the air into the barrels; and to each barrel a piston is so fitted that the air cannot pass between it and the sides of the barrel. Both pistons have a valve opening upwards, that the air in the lower part of the barrel may escape through them into the room or atmosphere. They are also connected by a rack, and raised or depressed by a handle, the lower part of which is fixed to the axis of a cog-wheel, whose teeth lay hold of the rack. One piston is raised, and the other is depressed by the same turn of the handle, and the operation of exhaustion is the same as in the common pump first described. The two barrels are advantageous, because they perform the work more speedily, and also because the weight of the atmosphere, pressing upon the rising piston, is counterbalanced by an equal weight pressing upon the other piston descending. Behind the large receiver on the pump-plate, there is sometimes a small plate for sustaining a small receiver; and from the hole, at the centre of this plate, there is a canal communicating with that, which passes from the large receiver to the barrels. Under the receiver is a small bottle containing mercury, and a small tube filled with mercury, freed from air, inverted with the open end in the mercury; which is, in fact, the short barometer-gage described (art. 187). This pump, without the gage, is shown by fig. 38.

Experiments with the air-pump.

204. In the introductory part of this treatise we have endeavoured to demonstrate experimentally all the most important mechanical properties and effects of common air; in which, however, we have avoided, as much as possible, such as required the assistance of the air-pump; for, since the operation of this machine depends upon these same properties, we did not consider it satisfactory to employ it as the ground-work of our theory; but having now established the principles of its operation, and illustrated its construction under a variety of forms, it may be interesting to many of our readers to see our preceding deductions confirmed and established by experiments on this important pneumatic apparatus.

III. *Experiments for showing the weight and pressure of atmospheric air.*

205. 1. One of the most incontestible experiments of this kind is to take a large copper ball, as light as is consistent with the purpose it is intended to answer, having a small neck to it, furnished with a stop-cock; this neck is to be screwed on the receiver-plate, so that it may be exhausted of air after the same manner as the common receiver; this being done, turn the stop-cock, which will prevent any air from entering when the ball is unscrewed. Let now the weight of the ball be taken by a very accurate balance, and when this is well ascertained, turn again the stop-cock, and the air will immediately rush in, and the ball will in consequence preponderate. The weight necessary to bring the balance to an equilibrium will be equal to the weight of the air re-admitted. In this manner, the specific gravity of air has been ascertained to be ·00128, that of water being 1.

2. Place a receiver on the plate of the air-pump, having a hole an inch or an inch and a-half in diameter at the top; cover this hole with the palm of the hand, while another person works the machine so as to produce the exhaustion; and the pressure which, after two or three strokes of the piston, will be experienced on the hand from the external atmosphere, will be such as to convince the most sceptical of the truth of the proposition. If the exhaustion be carried to any considerable degree, it will be difficult to remove the hand without violence, or without re-admitting the air.

3. Place a small receiver O (fig. 39) over the hole of the pump-plate, and, upon exhausting the air, the receiver will be fixed down to the plate by the pressure on its outside; then, by turning the cock of the pump, and re-admitting the air, the receiver will become loose. In order to prove that the receiver O is held down by the pressure of the air, suspend it on the hook of the wire PP passing through the collar of leathers at the top of the receiver M, by which it is covered, and thus let it down on the plate of the pump; and when the air is exhausted from both receivers, the large receiver M will be fixed to the plate by the pressure of the external air; but the small one will be loose and easily removed; then again, in letting in the air, the lesser one will be fixed to the plate, and the other will be released. Fig. 39.

4. The same may be otherwise shown by the glass employed in experiment 2, viz. by tying over the hole at top a piece of bladder; then, as the air is exhausted, the bladder will be pressed into the receiver in a concave hemispherical form, and, by carrying the exhaustion to a sufficient extent, the bladder will at length burst with a loud report; or, instead of the bladder, a piece of common window-glass may be employed, resting on a rim of leather; when it will be found that the pressure of the external air will break the glass as soon as the air is exhausted.

5. Join together the two hollow brass hemipheres A and B (fig. 40), the edges of which must be made very perfect, or otherwise they must have a circle of wet leather placed between them; screw the end D into the plate of the pump, and open the stop-cock E of the pipe CD communicating with the hemispheres; and having exhausted the air, turn the cock again so as to stop the pipe; remove the ball from the plate, and screw on at the end D the handle FH. Two strong men now, one taking hold of each handle, and pulling straight against each other, will find considerable difficulty, or will be, perhaps, wholly unable to separate the two hemispheres. The degree of strength that will be requisite depends upon the diameter of the sphere and the nature of the exhaustion. If the latter be nearly complete, and the diameter be 4 inches, the area of section will be Fig. 40.

$$4^2 \times {\cdot}7854 = 12{\cdot}5664 \text{ inches};$$

and, assuming the pressure on every square inch to be about 15 lbs. we shall have

$$12{\cdot}5664 \times 15 = 188 \text{ lbs.}$$

for the force or strength requisite to produce the separation.

Pressure and weight of the atmosphere.

6. The pressure of the air may be also shown as follows: Set a square phial of thin glass upon the pump-plate, and, to prevent accidents, cover it with a wire cage, and place both under a close receiver. The phial must be supplied at top with a small valve opening upwards, so as to be exhausted with the receiver, but which shuts and prevents the air from afterwards entering. Let the air be exhausted from the receiver and phial, and it will be found that, upon re-admitting the air into the former, the latter will be crushed into a number of small pieces. Quicksilver may also be forced into the pores of wood, and made to pass through it by the pressure of the atmosphere.

Fig. 41.

7. Immerse the neck *c d* of the hollow glass ball *e b* (fig. 41) in the water of the vessel *a a*; place it on the plate of the pump, and cover it and the hole of the plate by the receiver A; exhaust this receiver, and the air will escape by its spring, from the ball *e b*, through the neck *c d*, rise in bubbles through the water, and pass off into the external air. When the water has done bubbling, turn the cock of the pump, and the air that is admitted will, by its pressure on the surface of the water, force it up in a jet into the ball *e b*, and nearly fill it; the small quantity of remaining air which occupied the whole ball, and which is now reduced to a small space by condensation, is all that prevents the water from filling the whole cavity of the ball.

Fig. 42.

This experiment may be varied by screwing the end A of the brass pipe ABF (fig. 42) into the hole of the pump-plate, and placing it, by means of a circle of leather, upon the plate *c d* a tall receiver GH, close at top, exhausting the receiver of air, and stopping the pipe by the cock *e*: when this is done, remove the apparatus from the pump, set its end A in a bason of water, and open the pipe by turning the cock *e*, when it will be found that the pressure of the air on the water will force it up through the pipe, so that it will ascend in a jet to the top of the receiver.

Fig. 43.

8. Place the tall open receiver AB (fig. 43) on the pump-plate over the jar D, containing quicksilver; the latter being placed near the hole or pipe communicating with the barrels. Into the plate C, placed upon the upper end of this receiver, introduce the open glass tube *g f*, immersed at its lower extremity in the quicksilver of the jar D, and screwed by a brass top annexed to it at *h*, to the syringe H, which is itself screwed to the plate C. By means of the ring I, draw up the piston of the syringe, and thus exhaust the tube of its air; and the quicksilver in the bason, pressed by the undilated air of the receiver AB, will ascend in the tube. That this ascent is owing to the pressure of the air, and not to what is vulgarly called suction, may be shown by exhausting the receiver of its air, which will cause the quicksilver to descend into the jar, and, by re-admitting the air, it will rise again in the tube, although the piston of the syringe be not moved.

If the tube be 32 or 33 inches in length, the quicksilver will rise nearly as high in the tube as it stands at that time in the barometer; and if the syringe have a small hole at *m*, and the piston be drawn up above that hole, the air will pass through it into the syringe and tube, and the quicksilver will immediately fall down into the jar.

Pn ma ex per Fig.

9. The jar A (fig. 44) being filled with quicksilver, and placed on the pump-plate, cover it with the receiver B, and push the open end of the glass tube *d e* through the collar of leather in the brass neck C, almost down to the quicksilver in the jar; then exhaust the receiver B of its air, and the tube *d e*, which is close to the top *f*, will at the same time be exhausted. When the exhaustion has been carried on to a sufficient extent, push the open end of the tube into the quicksilver of the jar, and it will be found, that although the tube is exhausted, the mercury will not rise in it, because there is no pressure on the surface of that in the jar; but upon admitting the air into the receiver, the quicksilver will immediately rise, and stand as high as it did, in consequence of the action of the syringe in the preceding experiment.

These latter experiments not only exhibit the weight and pressure of the atmosphere, but they also show that they are increased or diminished in proportion to the increase or decrease of the density of the air.

IV. *Elasticity of the air.*

Ela the

206. 10. This property of air may be shown in the most satisfactory manner, by placing under the receiver of an air-pump a flaccid bladder, well tied and secured at its neck, so as to prevent the air within it from escaping. Exhaust the air out of the receiver, and the bladder, as the process proceeds, will continue to expand, till at length it will appear like one full blown, and even burst if the exhaustion be carried on to a sufficient degree; but upon re-admitting the air, it will quickly return to its original flaccid state.

11. In experiment 7, we supposed the glass ball (fig. 41) to be filled with water, with the exception of a small bubble of air; if, in this state, it be placed with its neck downwards into the empty jar *a a*, and covered with a close receiver, and the air of the latter be exhausted, the air-bubble will expand itself, and by its elastic force protrude the water out of the globe into the jar.

The same may be shown by screwing the pipe AB (fig. 42) into the pump-plate, and by placing the tall receiver GH upon the plate *c d*; for now, exhausting the receiver, remove the apparatus, and screw it into the copper vessel CC (fig. 45), half filled with water; then turn the cock *e* (fig. 42), and the air confined in this vessel will, by its spring, force the water through the pipe AB, and cause it to form a jet into the exhausted receiver equal to that which was produced by the pressure of the air in the former experiment.

12. The little amusing experiment called the *Cartesian devil*, depends upon principles nearly the same as the above. The figure of a man, made of glass, or enamel is so constructed as to have the same specific gravity as water, and will therefore remain suspended in [illegible] mass of that fluid. A bubble, similar to that in the last experiment, communicating with the water, [illegible] placed in some part of the figure, sometimes in a small globe, as shown at *m* (fig. 46). At the bottom B of the vessel is a diaphragm of bladder, which can be pressed upwards by applying the finger to the extremity *e* of a lever *e o*, moving about a centre *o*. The pressure applied to *e* is communicated through the water to the bubble of air which is thus compressed. The specific gravity of the figure is thereby increased, and it consequently sinks to the bottom; but by removing the pressure, the figure again rises, so that it may be made

to oscillate or dance in the vessel without any visible cause. Fishes made of glass are sometimes substituted for the human shape, and when a common jar is used for the experiment, the pressure is applied to the upper surface of it at A, which is, in this case, a piece of bladder, instead of being placed at the bottom, as shown in the figure.

The same effects will be produced by placing the jar under the receiver of an air-pump, and varying the pressure by rarefaction; but, in this case, the specific gravity of the figure ought a little to exceed that of the water in the jar.

13. Pour mercury into the bottle A (fig. 47), and screw the brass collar *c* of the tube BC, into the brass neck *b* of the bottle, and let the lower end of the tube be immersed into the quicksilver, so that the air above it may be confined there. Cover this tube, which is open at the top, with the receiver G and large tube EF, fixed by collars to the receiver and close at the top. Exhaust the receiver and its tube, and the air will be thus exhausted out of the inner tube BC through its open top C; and then the air confined in the bottle A will, by its elasticity, force the mercury into the inner tube, in the same manner as the pressure of the atmosphere caused it to rise in experiment 6; whence we learn, that the elasticity of the air is equivalent to to its weight.

14. We have had occasion, in a preceding article, when describing the air-pump, to speak of a *transferrer;* but the nature of this apparatus will be better understood by attending to the following experiment: Referring to (fig. 48), screw the end C of the pipe CD into the hole of the pump-plate, and open the communication between the three pipes E, F, and DC, and the hollow trunk AB, by turning the three cocks *d*, G, H. Cover the plates *g* and *h* with wet leather, having holes in their middle, so as to communicate with the pipes; place the close receiver I upon the gage-plate *g*, shut the pipe F by turning the cock H, and exhaust the air out of the receiver. Shut out the air by turning the cock *d*; remove the machine from the pump; screw it to the wooden stand L, and put the receiver K upon the plate *h*, in which it will be loose while it is full of air; but upon turning the cock H, and opening the communication between the pipes E and F through the trunk AB, the air in K will, by its spring, pass from K to I till it become of equal density in both receivers; and they will then be held down with equal force upon their respective plates by the pressure of the atmosphere, and the force with which K was held down will be divided between K and I; whence it appears that a force equal to half the elastic force of common air, will act within the receivers against the whole pressure of the common air on their outsides.

15. We shall only detail one other experiment as a proof of the expansible property of atmospheric air. Every one must have observed a cavity at the broad end of an egg, between the shell and the white. The white and yoke are contained in a thin membrane, or bladder, which adheres loosely to the shell, but is detached from it at that part; and this cavity increases by keeping the egg in a dry place. One may form a judgment of its size, and therefore of the freshness of the egg, by touching it with the tongue; for the shell, where it is not in contact with its contents, will presently feel warm, being quickly heated by the tongue, while the rest of the egg feels quite cold.

Pneumatic machines and experiments.

If a hole be made in the opposite side of the egg, and it be set on a little tripod, and put under a receiver, the expansion of the air in the cavity of the egg will force the contents through the hole till the egg be quite emptied: or, if nearly one half of the egg be taken away at the other end, the white and yoke be taken out, the shell put under a receiver, and the air abstracted, the air in the cavity of the egg will expand, gradually detaching the membrane from the shell, till it causes it to swell out, and gives the whole the appearance of a whole egg. In like manner, shrivelled apples and other fruit will swell in vacuo by the expansion of the air confined in their cavities. And if a square phial, such as that employed in experiment 6, be placed in like manner, having its open end well corked and cemented, it will be burst into pieces by the expansive force of the included air. In this last case it is best to place the phial under a wire cage, to prevent the pieces of glass from damaging the principal receiver.

207. *Experiments to shew the resistance of the air.*

Experiments showing the resistance of the air.

16. Bodies cannot move about in the atmosphere without displacing it, which requires force, and therefore the resistance of the air always diminishes the velocity of moving bodies; but this diminution will be greater or less, according to the density of the falling or moving mass; for which reason, those bodies which we call light, fall much slower than more dense or heavy substances. To show that this is actually the case, we need only let fall a feather and a piece of metal out of our hands at the same instant, and it will be found that the metal has reached the floor while the feather is yet falling; but place them both under the receiver of an air-pump and exhaust the air, and then let them fall, and they will both be found to reach the bottom in the same time: an apparatus for performing this experiment is shown in (fig. 49), the two bodies being laid together on the brass flap *d* or *e*, which may be at any time let down, by simply turning the wire *f*, which passes through a collar of leathers *g*, placed in the head of the receiver AB.

Fig. 49.

Fig. 50.

17. Figure 50 represents another apparatus for showing the same thing. It consists of two sets of brass vanes put on separate axles, in the manner of windmill-sails. One set has the edges placed in the direction of their whirling motion, that is, in a plane to which the axis is perpendicular. The planes of the other set pass through the axis, and they are, therefore, trimmed so as exactly to front the air through which they move. Two springs act upon pins projecting from the axis, and their strengths or tensions are so adjusted, that when they are disengaged in a vacuum, the two sets continue in motion equally long; but if they are disengaged in air, those vanes which oppose the air by their planes will stop long before those that cut it edgewise.

We might enumerate a great variety of experiments on the air-pump, which are certainly highly interesting, but we conceive that many of them belong more properly to a course of experimental philosophy, than to a regular treatise of Pneumatics; we shall, therefore, proceed no further with our detail in this section, but simply refer the reader to the works of Franklin, Cotes, Helsham, Ferguson, Adams, &c. where he will find ample information on this subject.

V. *Of the condenser, condensing syringe, &c.*

Condenser, and condensing syringe.

208. We have seen in the preceding account of the air-pump, that the most modern of these machines may be made use of either for the purpose of rarefaction or condensation, but in many cases the latter operation is required to be performed when no pump is at hand, and moreover, it does not always require the same delicacy of process as that of rarefaction, and more simple apparatus are, therefore, more frequently employed.

Fig. 51.

Take, for example, a prismatic tube AB (fig. 51), shut at one end, and fit it with a piston, or plug C, so that no air can pass by its sides, which, in a cylindrical tube, is best done with a turned stopper, covered with oiled leather, and fitted with a long handle CD. When this is thrust down, the air, which formerly occupied the whole capacity of the tube, is condensed into less room, and the force necessary to produce any degree of condensation, may be concluded from the weight necessary for pushing down the plug to any depth; but it is obvious that the instrument in this form is not sufficiently accurate for us to deduce, from experiments upon it, any very accurate conclusions; the following is, therefore, more commonly the form of the condenser.

Condensing syringe.

The end of the tube, instead of being closed as described above, is perforated with a very small hole *e f*, and being externally turned to a small cylinder, a narrow slip of bladder, or of thin leather, soaked in a mixture of oil and tallow, must be tied over the hole. Now, let us suppose that the piston is forced down to the bottom of the barrel, to which it applies close; then, when it is drawn up to the top, it leaves a void behind, and the weight of the external air presses on the slip of bladder, which, therefore, claps close to the brass, and thus performs the part of a valve, and keeps it close so that no air can enter. But, the piston having reached the top of the barrel, a hole F in the side of it is just below the piston, and the air rushes through this hole and fills the barrel. Push the piston down again, it immediately passes the hole F, and no air escapes through it, therefore it forces open the valve at *f*, and escapes while the piston moves to the bottom.

Now, let any vessel E, as, for instance, a glass bottle, or brass ball, have its mouth furnished with a brass cap firmly cemented to it, having a hollow screw, which fits a solid screw *p q*, turned on the cylindric nozzle of the syringe, and this vessel may be considered as a receiver. Screw the syringe into this cap, and it is evident that the air forced out of the syringe will be accumulated in this vessel; for upon drawing up the piston, the valve *f* always shuts by the elasticity or expansive force of the air in E; and on pushing it down again, the valve will open as soon as the piston has got so far down that the air in the lower part of the barrel is more powerful than the air already in the vessel. Thus, at every stroke, an additional barrel-full of air will be forced into the vessel E; and it will be found that, after every stroke, the piston must be pushed down further before the valve will open; for, as stated above, it cannot open till the air, condensed in the barrel by the downward motion of the piston, exceeds that in the receiver, and as the latter is increased at every stroke, so also must the stroke be longer every time before the valve will be opened.

Pne ma ar per

A construction somewhat different to the above is now very commonly adopted: this consists in supplying the piston itself with a valve; that is, instead of being, as above described, one solid piece, it is made of two pieces perforated. The upper part, *i k m n*, is connected with the rod or handle, and has its lower part turned down to a small cylinder, which is screwed into the lower part *k l o n*; and has a perforation *g h* going up the axis, and terminating in a hole *h* in one side of the rod, and a piece of oiled leather is stretched across the hole *g*. When the piston is drawn up, and a void left below it, the weight of the external air forces it through the hole *h g*, opens the valve *g*, and fills the barrel. Then, on pushing down the piston, the air being compressed into less space, presses on the valve *g*, shuts it, and none escaping through the piston, it is gradually condensed, as the piston descends, till it opens the valve *f*, and is added to that already accumulated in the vessel E.

And con tion

We may thus force any quantity of air consistent with the strength of the vessel into it, and it will be ready for any experiments that we may be desirous of making with it. Instead of the glass bottle, or simple hollow globe supposed above, we may use the latter furnished with a neck and stop-cock; in this case, the cock may be turned, the ball unscrewed from the syringe, and be thus in a state to be transferred and applied to any other purpose required.

The condensation thus produced may be measured by a gage fitted to the instrument as follows: a glass tube GH (fig. 51-*a*), of cylindrical bore, and close at the end, is screwed into the side of the cap on the mouth of the vessel E; a small drop of water or mercury is taken into this tube by warming it a little in the hand, which expands the contained air, so that when the open end is dipped into water, and the whole allowed to cool, the water advances a little into the tube. The tube is furnished with a scale, divided into small equal parts, numbered from its close end. Now, since this tube communicates with the vessel, it is evident that the condensation will force the water along the tube, acting like a piston on the air beyond it, and the air in the tube and vessel will always be of one density. Suppose, for example, the number at which the drop stands, before the condensation is made, to be *c*, and that it stand at *d*, when the condensation has attained the required degree; the density of the air at the remote end of the gage, and consequently in the vessel, will be $\frac{c}{d}$. It was by such means that the laws of the condensation and elasticity of common air were deduced, a statement of which has been already given in the introductory section of this treatise.

VI. *Of the air-gun.*

210. One of the most interesting applications of condensed air is that where it is made to project bullets, as in the air-gun.

Pri the

The first account we meet with of an air-gun is in the "Elemens d'Artillerie" of David Rivaut, who was preceptor to Louis XIII. of France: he ascribes the invention to Marin, of Lisieux, who presented one to Henry IV. We have seen, however (art. 2), that instruments of this kind were not wholly unknown to the ancients.

In order to construct a machine of this kind, it is only necessary to use a condenser such as that above described; and then attaching it to a barrel containing a bullet; when, upon turning a cock, and opening a communication between the condensed air and the bullet, the latter will be projected forward with a greater or less velocity, according to the state of condensation and the weight of the projected body. The effect, therefore, will be similar to a gun charged with powder, and hence we may readily form a comparison between the effects of the two fluids; for inflamed gunpowder is nothing more than very condensed air, so that the two forces are exactly similar. The elasticity of condensed air has been estimated, by Mr. Robins, as equal to about 1000 times that of common air: admitting, therefore, this to be correct (although there seems to be great reason to suppose it to be much under-rated), it would be necessary that air should be condensed 1000 times more than in its natural state, to produce the same effect as gunpowder. There is, however, this important consideration to be attended to, viz. that the velocities with which equal balls are impelled are directly proportional to the square roots of the forces; so that if the air in an air-gun be condensed only 10 times, the velocity with which it will project a ball will be $\frac{1}{10}$th of that arising from gunpowder; and if the air were condensed 20 times, it would communicate a velocity of $\frac{1}{4}$th of that of gunpowder, and so on.

Air-guns, however, project their balls with a much greater proportion of velocity than that stated above, and for this reason, that as the reservoir, or magazine of condensed air is commonly very large in proportion to the tube which contains the ball, its density is very little altered by expanding through that narrow tube, and, consequently, the ball is urged all the way by nearly the same uniform force as at the first instant; whereas the elastic fluid arising from inflamed gunpowder is but very small in proportion to the tube or barrel of the gun, occupying, indeed, but a very small portion of it next the butt-end; and, therefore, by dilating into a comparatively large space as it urges the ball along the barrel, its elastic force is proportionally weakened, and it acts always less and less on the ball in the tube. Whence it happens that air condensed into a pretty large machine only 10 times will project its ball with a velocity but little inferior to that given by gunpowder; and, if the valve of communication be suddenly shut again by a spring, after opening it to let some air escape, then the same charge may serve to impel several balls in succession. In all cases where a considerable force is required, and, consequently, a great condensation of air, it will be requisite to have the condensing syringe of a small bore, perhaps not more than half an inch in diameter; otherwise the force requisite to produce the compression will become so great, that the operator cannot work the machine; for, as the pressure against every square inch is about 15 lbs., and against every circular area of an inch diameter, 12 lbs.; if the syringe be an inch in diameter, it will require a force of as many times 12 lbs. as the density of the air in the receiver exceeds that of the common atmosphere; so that when the condensation is 10 times, the force required will be 120 lbs.; whereas, with a half-inch bore, it will only amount to 30 lbs.

Construction of the air-gun. Fig. 52.

211. There are air-guns of various constructions: an easy and portable one is represented in (fig. 52), which is a section lengthwise through the axis, to show the inside. It is made of brass, and has two barrels; the inner barrel KA, of a small bore, from which the bullets are shot; and a larger barrel ESCDR, on the outside of it. In the stock of the gun there is a syringe NPS, whose rod M draws out to take in air; and, by pushing it in again, the piston SN drives the air before it, through the valve PE, into the cavity between the two barrels. The ball K is put down into its place in the small barrel with the rammer, as in another gun. There is another valve, at SL, which, being opened by the trigger O, permits the air to come behind the ball, so as to drive it out with great force. If this valve be opened and shut suddenly, one charge of condensed air may make several discharges of bullets, because only part of the injected air will then go out at a time, and another bullet may be put into the place K; but, if the whole air be discharged on a single bullet, it will impel it more forcibly. This discharge is effected by means of a lock (fig. 53), when fixed to its place as usual in other guns; for, the trigger being pulled, the cock will go down and drive a lever which opens the valve.

Fig. 53.

Dr. Macbride (Exper. Ess. p. 81) mentions an improvement of the air-gun, made by Dr. Ellis, in which the chamber containing the condensed air is not in the stock, but in five or six hollow copper balls, about 3 inches in diameter. These are fitted to a screw on the lock of the gun, and are so furnished with valves as to retain at pleasure the air forced into their cavities, whereby they may be carried about from place to place, ready charged with condensed air, and the gun itself is rendered as light and portable as the smallest fowling-piece.

Martin's air-gun. Figs. 54 and 55.

212. But the best construction is, perhaps, that of Martin, which is shown in (fig. 54). It consists of a lock, stock, barrel, ramrod, &c. of about the size and weight of a common fowling-piece. Under the lock, at *b*, is screwed on a hollow copper ball *c*, perfectly air-tight. This ball is fully charged with condensed air, by means of the syringe B (fig. 55), previous to its being applied to the tube *b* of (fig. 54). Being charged and screwed on as above stated, it is evident that if a bullet be rammed down in the barrel, and the trigger *a* be pulled, that the pin in *b* will, by the spring-work within the lock, forcibly strike out into the copper ball, and there, by pushing in suddenly a valve within it, let out a portion of the condensed air, which air will rush up through the aperture of the lock and forcibly act against the bullet, driving it to the distance of 60 or 70 yards, or further, if the air be strongly condensed. At every discharge only a portion of the air escapes from the ball; therefore, by re-cocking the piece, another discharge may be made, and the same be repeated fifteen or sixteen times. Sometimes an additional barrel is made, and applied for the discharge of shot, instead of the one above described.

Method of charging.

213. The air in the copper ball is condensed by means of the syringe B, in the following manner. The ball is screwed quite close on the top of the syringe; at the end of the steel-pointed rod *a* is a stout ring, through which passes the rod *k*; upon this rod the feet should be firmly set, then the hands are to be applied

to the two handles *i i*, fixed on the side of the barrel of the syringe; when, by moving the barrel B steadily up and down on the rod *a*, the ball *c* will become charged with condensed air; and it may be easily known when the ball is as full as possible, by the irresistible action that the air makes against the piston when you are working the syringe. At the end of the rod *k* is usually a square hole, which, with the rod, serves as a key to fix the ball *c* fast to the screw *b* of the gun and syringe, close to the orifice in the ball *c*. In the inside is fixed a valve and spring, which gives way for the admission of air, but, upon its emission, comes close up to the orifice, shutting up the external air. The piston-rod works air-tight by a collar of leather on it, in the barrel B; it is therefore obvious, that when the barrel is drawn up the air will rush in at the hole *h*. When the barrel is pushed down, the contained air will have no other way to pass from the pressure of the piston but into the ball *c* at the top. The barrel being drawn up, the operation is repeated, until the condensation is so strong as to resist the action of the piston.

Various other constructions have been devised for guns of this description, which it would be inconsistent with the plan of this work to detail. The most important of these is the magazine air-gun of Colbe, by which ten balls may be discharged in succession, and so quick as to answer the purpose of as many different guns. See "Adams's Lectures, by Jones," vol. i.; and "Martin's Phil. Brit." vol. ii.

VII. *Air-cane.*

Air-cane.
Fig. 56.

214. This is the same in principle as the air-gun, the only difference being that the syringe is applied to the end of the barrel C (fig. 56), and the lock and trigger shut up in a brass case, and the trigger pulled, or the discharge made by pulling the chain *b*. In this form of the machine, there is a round chamber for the condensed air at the end of the string at *e*, and it has a valve acting in a similar manner to that of the copper-ball. When this instrument is not in use, the brass case *d* is made to slide off, and it then becomes a walking-stick; from which circumstance, and the barrel being made of cane, or wood, it derives its name of air-cane; generally, however, the barrel is made of brass, or other metal. The head of the cane unscrews or takes off at *a*, where the extremity of the piston-rod in the barrel is shown. An iron-rod is placed in a ring at the end of this, and the air is condensed in the barrel in a manner similar to that of the gun above described, but its force and action are not near so great as in that machine.

Air-lamp.

Air-lamp.

215. The air-lamp, although depending upon principles different from those which have been illustrated in the preceding sections, is generally classed with Pneumatic machines, we, therefore, make no apology for introducing a description of it in this place. The construction of it is as follows:

Fig. 57.

A (fig. 57) is a glass jar, containing inflammable air, B an open glass urn holding water, by the pressure of which the air is forced out of the jar A, through the brass pipe *a*; C is the stop-cock, so perforated that the water may descend from B into A, and the air pass out through the pipe *a*. By turning the bar of the stop-cock into a horizontal position, the communication between the two vessels is cut off, and the passage of the air obstructed; but by placing it in a vertical position, the communication is again opened. The lower jar A is supplied with inflammable air by means of the bladder (fig. 58), and two bladders of this kind are attached to each lamp. The method of using the machine is as follows:—Take off the cover D from the lamp, and turn the stop-cock upwards, then pour as much clean water into it as will fill the vessel A up to the pipe *a*, unscrew this pipe, and put in its place the small brass piece W, and to this screw one of the stop-cocks and bladder (fig. 58).

With the bladder under one arm, and with one hand applied to the cock at C, while the other is directed to the bladder, press the latter, and, at the same time, open the apertures; by which means the air will be forced upon the water in A, and driven up the glass pipe through the tube into B. When the vessel A is thus charged with air, the stop-cocks are to be turned, so as to cut off the communication with the external air; and care must be taken that the common atmospheric air does not mix with the inflammable; for, if this take place, the explosion would be great, and might be attended with unpleasant circumstances.

For lighting the lamp the electric fluid is employed, the apparatus being as follows: The wooden base EF (fig. 59), is a sort of box, about 12 inches square and 5 inches deep; and in this is placed an electrophorus, consisting of a resinous cake *c*, and metallic plate *d*, which, by a hinge at its back, admits of being pulled upwards and downwards by the silken string *b*, connected both with it and with the stop-cock C. When this cake is once excited, its electric effect upon the metallic plate will be continued a long time. A metallic chain G, communicates with a wire and ball, passing through a brass tube below, in the box over the plate; and above, with a fine wire passing through a glass tube. This upper wire is bent to about $\frac{1}{8}$ of an inch distance from the flame-pipe. It is evident that when the electrophorus in the box is previously excited, and the stop-cock C turned, the silken string *b* will raise the metallic plate; and this will give an electric spark to the ball and wire above, which will convey it instantly to the flame-pipe, and inflame the air issuing out of the pipe, in consequence of the pressure of the water in its descent into the vessel A. The cock C being turned back, the flame ceases; and turned again, appears; and will serve to light a candle, match, &c. whenever it may be thought proper.

The number of times in which light may be produced will be very great, depending on the quantity of inflammable air in the vessel A. If the cock be not turned back, the flame will continue till the whole of the inflammable air is consumed. The light thus produced will be sufficient for reading a large print in the night, or seeing the hour by a watch. When the electrophorus is to be excited, the silken string *b* is unhooked from the plate, and the apparatus taken out of the box; and the metallic plate is lifted up, whilst with a silken string, or dry cat-skin rubber, you may briskly rub the surface of the resinous cake. About 20 revolutions in rubbing will be sufficient, so that the plate will give a spark to the knuckle about the distance of an inch; and by the strength of the spark the degree of excitation is to be estimated. The silken string and small glass tubes, through which the wire G passes, should always be very dry, that the passage of the electrical spark may be quite perfect. The whole length of this apparatus is about 22 inches; but it may be made of any dimensions. Dr. Ingenhouz used a small apparatus, constructed upon a similar principle, in obtaining light for domestic purposes, both when at home and on his travels. "Adams's Lectures, by Jones," vol. ii. p. 99, &c.

Principle of the steam-engine.

216. The steam-engine is, without doubt, one of the most noble and most important machines ever produced by the genius of man; but its constructions are so various, and its purposes so numerous, that it would be wholly inconsistent with our arrangement to enter into a minute detail of its history and successive improvements in this place; at the same time, the general principle of its operation is so intimately connected with the properties of air and elastic fluids, treated of in the preceding pages, that we must not pass it over without endeavouring concisely to explain the nature of its operation.

Let a cylinder ABCD (fig. 60), placed vertically, have a piston working in it, the rod of which is fixed to the end of a beam GH, turning on the axis O, and loaded at its other extremity by a weight W; and when the machine is at rest, let the beam incline towards the side where the weight is. Then, if a vacuum be any way made in the part of the cylinder under the piston, the whole pressure of the atmosphere will tend to depress the piston, and will raise the weight W, at least, if its momentum be not greater than that of a column of water of 33 or 34 feet, having for its base the section of the cylinder, and acting by the lever GH.

Thus far it is unimportant how the vacuum is obtained, the principle of operation being still the same, but in the steam-engine, the steam of boiling water is the agent employed for this purpose, and hence the name of the machine. Let us. therefore, suppose the steam from a boiler to be introduced through a valve into the cylinder, and to expel the air; when this is done, the valve to be shut, and another valve opened, by which a jet of cold water is injected into the cylinder; by this means the steam will be suddenly condensed, a vacuum will be produced, and the piston will be forced down by the pressure of the atmosphere. The introduction of the steam at the bottom of the cylinder will again elevate the piston, and the reproduction of the vacuum will cause the depression of it; so that the end of the lever to which W is attached, may be employed to raise any weight, to pump up water, &c. &c.

217. This is the simplest form of the steam-engine; its operation depends upon the pressure of the atmosphere, and its effects may be computed as follows: Let b denote the length of the stroke of the piston, r the radius of the cylinder, h the height of a column of water which balances the pressure of the atmosphere, π the semi-circumference of a circle whose radius is 1, or the area of a circle whose diameter is 2; then $\pi r^2 h$ will be the moving power, and $\pi r^2 h b$ its momentum. If the radius of the piston of the pump worked by the beam be r', d the depth of the water, and h' the height to which it is raised at one stroke, then, $\pi r'^2 d h'$ will be the amount or momentum of the resistance. In order, therefore, that the machine may be effective, we must have

$$r^2 b h > r'^2 d h'.$$

This will, undoubtedly, be understood as merely indicating the nature of the calculation, and not the actual process by which its power is computed: for, in order to form a complete theory of the steam-engine, much more is necessary than the knowledge that $r^2 b h$ must be greater than $r'^2 d h'$; for the rate of working depends on that excess, and must be determined on other principles; the weight of the pump-rods must also be included; the effects of the friction duly estimated, &c. &c. before we arrive at a conclusion which is required by the constructor. We have likewise above supposed, that a perfect vacuum is produced in the cylinder by the jet of cold water, which is not the fact; for by the alternations of the heat and cold, to which the cylinder is exposed, it can neither acquire the heat necessary to the full elasticity of the steam, nor the cold requisite for its complete condensation; and on this account the effect of the machine falls much under the computed value.

Improved steam-engine.

218. We have observed above, that the form we have described is the simplest possible construction of this machine; and the inconvenience to which we last alluded, is the greatest. To obviate this in some measure, it may be remarked, that when the cylinder is full of steam, if a valve be opened, by which the steam is allowed to escape into a another vessel, where a jet of cold water is introduced, the condensation is much more complete, and the heat of the cylinder being preserved, the steam will possess its full elasticity. This improvement was made by Mr. Watt, who thus completely changed the character of the steam-engine. In the old form, that is, in the one described above, the power was reduced nearly to half its real value, so that the moving force, instead of amounting to 14 lbs. on every square inch of the piston, was reduced to little more than 7 lbs. By the last construction, the moving force is very nearly 12 lbs. per square inch.

Further improvement.

219. Both the above forms are now distinguished by the name of atmospheric steam-engines, because in both the action depends upon the pressure of the atmosphere; but a further improvement was afterwards made by injecting the steam into the cylinder, alternately above and below the piston, so that the whole motion is produced by the elasticity of steam, and has no dependence on the weight of the atmosphere. This improvement we owe also to Mr. Watt. For a further account of this important machine, the reader is referred to the article STEAM-ENGINE, in our alphabetical arrangement.

VIII. *Principle of the barometer.*

Principle of the barometer.

Fig. 61.

220. We have already described the principle of this instrument, when speaking of the Torricellian tube, and the experiment of the Puy de Domme; it is, in fact, no more than this, ABD (fig. 61) represents a glass tube about three feet long, recurved at D, and terminating at A in an open bulb or cistern, the other end being hermetically sealed, or rendered perfectly impervious to the air. Conceive now quicksilver to be poured in at A, and the tube inclined, and finally reversed, so that the entire capacity is filled with this fluid, and all air-bubbles wholly expelled. Then, when the tube is again reversed, so as to bring it into the

position shown in the figure, the column of mercury of 36 inches (the length we have supposed) being more than equal to the atmospheric pressure, it will descend by its superior gravity, leaving a vacuum behind it, till the equilibrium between the pressure of the air and that of the mercurial column is established; and the quicksilver, supposing the whole bore to have been quite full, will run over at A during this descent, after which it will remain at rest, as we have already stated in describing the Torricellian tube; or, if the bore be not quite full, the equilibrium will ultimately obtain when the surface of the mercury in the bulb has risen to a certain height, suppose to *ef*, the largest section of the bulb. In this state, the distance between the surface *ef* of the fluid in the bulb, and the upper surface of the column in the tube, will be the height due to the atmospheric pressure; in order to estimate which, a scale of inches and parts, with a noninus, or vernier scale, is fixed towards the upper part, extending to a greater or less distance, according to the intended purpose of the instrument; when it is only used for the house, to indicate the changes in the state of the atmosphere, the scale need not exceed 3 or 4 inches, 3 inches being the greatest variation ever experienced; but when the instrument is designed for the barometrical measurement of heights, the scale must, of course, descend lower. Instead of the recurved tube and bulb, as described in the figure, a leather bag of mercury is more commonly fixed to the end of a straight tube, which obviously answers the same purpose, and is much more convenient for instruments designed to be carried from place to place, as in those for the purposes of measurements. Such is the most simple construction and general principle of this instrument, but there are many particulars relating to its form, the preparation of the mercury, &c. to be observed beyond the simple description above given, which will be found under the article BAROMETER, in our Alphabetical Division.

IX. *Of the thermometer.*

Principle of the thermometer.

221. We have already referred to the thermometer in our sketch of the history of Pneumatics, but what is there stated relates merely to the invention of this instrument, little being said respecting the principles of its operation and construction. Experiment teaches us, that all bodies expand by heat, and are contracted by cold, and hence the degree of expansion and contraction becomes the measure of the degree of heat. Fluids are more convenient for this purpose than solids, and of the former, quicksilver is now commonly preferred to any other. Let AB (fig. 61-*a*) represent a very fine glass tube, having a pretty large hollow ball at the bottom, and conceive the ball and about half the tube to be filled with quicksilver, at any medium state of temperature. Then conceive the instrument in this state to be heated to a very great degree, which will expand the quicksilver till it flows out of the open end of the tube. While exposed to this heat, and while the the quicksilver is still flowing out at the end, let the tube be hermetically sealed, so as perfectly to exclude all communication with the outward air. Then, in cooling, the fluid will contract, and, consequently, its surface will descend in the tube till it comes to a certain point, corresponding to the temperature or heat of air, and it will afterwards ascend and descend as the heat of the air or the medium in which it is immersed is increased or diminished.

Fig. 61-a.

After various attempts, philosophers have agreed to fix, for the two standard points of the thermometer, those which correspond to the freezing and boiling of water, which, under the same barometrical pressure, are found to be the same in all parts of the world. The instrument being, therefore, made as above described, it is first immersed in ice just thawing, or water just freezing, and the point to which the mercury descends is marked. This is called the freezing point. Then immersing it in water just boiling, the mercury rises; and this point, being marked, is called the boiling point: and on a supposition that equal quantities of expansion take place with equal degrees of temperature, that is, that the expansion is proportional to the difference of temperature, the distance between those two points is divided into a certain number of equal parts, each one of which answers to one degree of temperature.

222. With respect to the extent of those degrees, or the number of degrees between these extremes, there is still considerable difference. In the *centigrade* thermometer this distance is divided into 100 equal parts, and the scale is continued considerably above and below the two standard points; the lowest point, or that answering to freezing, is marked 0, or zero, and the degrees are marked off from it both ways, those below being considered as negative; so that when the mercury descends below the freezing point any number of degrees, as, for example, 9°, we say the temperature is — 9°, or 9° *below zero*; and when it ascends above, they are called positive. It is not, however, commonly so expressed, but is always understood. This is by far the most simple form of the thermometer; but in England, and indeed in many parts of the continent, *Fahrenheit's* scale is employed; here the freezing point is marked 32°, and the boiling point 212°, and the distance between them is divided into 180 equal parts, or degrees, numbered from 32° upwards; and the scale extends downwards to 0; which, at the time this division was projected, was considered to be the maximum degree of cold. Another scale of very common use is *Reaumur's*: in this the freezing point is zero, as in the centigrade; the boiling point is 80, and the scale is continued both ways, the same as in the latter.

These are the scales most commonly employed; but various others have been proposed, and a great variety of constructions have been given which we shall not here attempt to describe; they will be found under the article THERMOMETER, in our Miscellaneous Division.

It follows, from what has been stated above, that the number of degrees between any two points, according to these several scales, are as 100, 180, and 80, or as 5, 9, and 4; whence they may be readily converted from one scale to another.

The same proportions will also convert the registered degree of the one thermometer to that of another, observing only, that 32° must always be added to the result, when the conversion is to Fahrenheit's, because the zero on each of the others is marked 32° on his. This conversion will, however, be more simply effected by the following theorems, in which C, F, and R, denote the degrees on each instrument respectively.

when the two plates are fixed together by screws passing through them, they close the tow round the tube, both to steady it and to prevent the escape of steam.

Scale.
Fig. 65.

"Fig. 65 is the scale, 5 inches long, 0·9 broad; between the two standards NO, a length of 4·15 inches is divided into 100 parts, and by the vernier into 1000, giving 241 parts to the inch; this was accidental, being occasioned originally by the thread, of my adjusting screw, which assisted me in making the divisions. The scale is fixed down to the upper plate (fig. 64), within the collar, by screws passing through the flanch P at the back of the scale. It would be well that a piece of thick leather, or soft wood, should be screwed between these two, if it can be done with sufficient firmness, for the purpose of preventing the scale getting inconveniently hot.

"The adjusting screw, which carries the vernier, is raised by the standards above the scale, and is placed opposite the centres of the plates (fig. 63 and 64), by which means the milled head Q goes better into the case hereafter described. The tube of the thermometer, when passed through the central holes in the two plates, turns by its bend to the left hand, and up the side of the scale, being slightly fastened to it at the top only with a small piece of cork under it, to keep it clear from the scale. The vernier has fixed to it, by means of a screw head, two pieces of thick paper laid upon each other, the one black and the other white; half of the outer paper being cut away straight, makes a line between the black and white better than any that can be drawn for adjustment to the top of the mercurial thread."

Boiler.

In boiling, the bulb should be exposed to steam only, as being steadier in its heat than water. Mr. Wollaston's boiler is a tin cylinder, 5·5 inches deep, and 1·2 inch in diameter, with an external cylinder of 1·4 inch diameter, to prevent the transmission of heat; the bottom only is single. The interior cylinder has a brass collar soldered into it, having an internal screw which fits to either of the external screws on the plate (fig. 64), so that what is the boiler when fixed below the plate, becomes a case, to protect the scale when screwed to the upper side of that plate. The top of the external cylinder being closed into the same brass collar, becomes slightly conical, and is to be soldered into it. An opening of 0·2 inch diameter, is made through both cylinders, immediately under the collar, as a vent for the steam from within, but is prevented from communicating with the annular space between the vessels, lest inconvenience should arise from water accidentally getting between them. Another tin cylinder 1·2 inch diameter, and 2·1 deep, with a similar screw collar at top, forms a case for protecting the bulb when screwed to the under side of plate (fig. 64), and is also a measure for the quantity of water put into the boiler, which should not touch the bulb; it is here 1·25 inch below it.

Apparatus for packing.

For the purpose of rendering every thing requisite for use portable with the thermometer, a stand is made, which will be immediately understood from the following description: Round the outside of the boiler, and just below the conical closing of it, is soldered a ring of brass wire ST (fig. 66). Fitting to the conical top of the boiler is made another short cone of thick tin, which may be fastened down by screwing the thermometer into the boiler; or, which is better, by a separate collar U for the purpose, to screw into the boiler, having the same internal screw above to receive the thermometer. This conical cap has a wire soldered round it at VWX; and on this wire turns, by eyes at their ends, seven wires, 9 inches long, and of sufficient strength. They are placed at six equal distances round the cap, two of them being placed close together. These wires bearing on the ring ST, are spread outward; being connected with gores of thin linen, sewed between them from top to bottom, they are prevented from spreading beyond a certain point; they form a very steady base for the support of the whole instrument, and, at the same time, a bell tent, to protect the lamp and boiler from the wind. The two wire legs which are placed together, are not connected otherwise than by a hook at bottom, and will allow the tent to be opened at that part, for examining and adjusting the lamp, while the instrument stands firm on the remaining legs. The lamp (fig. 67) is a cylinder 1·8 inch in diameter and 0·9 deep, having a tube in the centre to carry a wick and a cover, with six holes round it of 0·2 inch diameter each, and an opening in the middle 0·8 inch in diameter. A copper pipe 0·85 inch in diameter and 1·1 inch long, turns over the opening by means of a hinge, and forms a chimney, to prevent smoke, on the principle of Argand's lamp. Mr. Wollaston burns oil, mixed with tallow, to make [illegible] congeal. The lamp has a rod of strong wire fixed to its circumference within, and sliding in a tube YZ on the outside of the boiler. The instrument, with its stand, fit for use, is shown in (fig. 68). For some further particulars respecting this instrument, the reader is referred to Phil. Trans. Part ii. p. 183, for 1817.

Fig. 66.

XI. *Of the siphon.*

224. In consequence of the close connection between certain pneumatic and hydraulic machines, or rather on account of many machines uniting in themselves the principles both of Pneumatics and Hydraulics, it is difficult to draw an exact line of separation between them, on which account we have already, in our treatise on the latter subject, been obliged to introduce a slight description of the siphon; but we could not in that place so well explain the principle of its operation; we therefore again recur to the subject in this place, because what has now been delivered will enable us to illustrate its uses more particularly than in the chapter above alluded to.

225. The common siphon is represented in (fig. 69), where we suppose the shorter leg AB to be immersed in a fluid contained in the vessel MNOP. Now, by applying the mouth to the orifice D, and sucking out the air, the water will ascend till it rises to the upper arch; after which it will descend by its own gravity, and continue to be discharged at D till the vessel is completely emptied.

In order to illustrate this, let us suppose that the siphon had branches of equal lengths, such as AB, BC, and that the water were drawn up by suction, or rather pressed up by the weight of the atmosphere upon the surface of the water; then it is obvious, that as this pressure is equal to the pressure of the air at the extremity C, and as the altitudes of the columns of the fluid AB, BC, are equal, there is no force which will determine the water to discharge itself at C. [illegible] when the leg BC is lengthened, so as to excee as to D, then the water will be discharged at the latter orifice by the pressure of the additional column CD, and the velocity of discharge will be equal to that due to the difference in the lengths of the two branches.

226. The above is the common form of this instrument: an improved construction is as follows. D (fig. 70) is a stop-cock fixed at the extremity of the longer branch AB; a small bent tube ED, lying along the outside of the same branch, communicates with the cavity of the leg AB above the stop-cock. When the aperture C is placed in the fluid to be drawn off, the mouth of the stop-cock D is closed, and the air is drawn out of the longer branch by suction at E; or, instead of a stop-cock, the finger may be applied till the air is sucked out at E.

227. By means of the principles above illustrated, we may explain the phenomena of reciprocating springs, that is to say, those springs which ebb and flow alternately, and those which have a periodical swell, or which discharge a greater quantity of water at one time than at another, the changes taking place at equal intervals.

The first of these kind of springs is very easily accounted for, by supposing the channel which carries the water off from a cavern to have the form of a siphon. In this case the water will only flow when it rises in the cavern to a height equal to that of the upper curve of the siphon-formed canal, and it will fail again when it descends below it.

The following explanation of the second kind of intermitting springs was suggested about a century back by Dr. Atwell, of Oxford, by attending to the phenomena of Laywell spring, at Brixam, near Torbay, in Devonshire. Let AA (fig. 71) be a large cavern near the top of a hill, which derives its supply of water from the rains or melted snow filtering through the crevices of the mountains; and let CC represent the small channel which conveys the waters of the cavern to the opening G in the hill, where they are discharged in the form of a small spring. From the cavern AA let there be a small channel D, which carries water into another cavern B, and conceive the water in the second cavern to be carried off by a bent channel E *e* F, wider than D, and joining the first channel CC at *f*, before it issues from the mountain, the point of junction *f* being below the level of the bottoms of both the caverns. Then, as the cavern B fills with water, the fluid will ascend to the same height in the channel E *e* F, but it will not be discharged by it till the surface in B is on a level with *e*, the highest part of the channel. The water will then be carried off by the natural siphon E *e* F G, till the whole is discharged, and consequently there will be a great swell in the spring at G. This spring will now cease, because the channel D does not convey the water into B so fast as the siphon E *e* F carries it off; and it will again commence as soon as the water in B rises to a level with the summit *e*. A machine for illustrating these phenomena is described by Ferguson in his Lectures, vol. ii. p. 106. This subject will be treated of at greater length under the article SPRING.

XII. *Experiments with the siphon.*

Pneumatic machines and experiments.

Fig. 72.

228. *Construction of a vessel which discharges its contents when the fluid attains a certain height.* This is familiarly known under the name of Tantalus' cup, and consists of either a glass or metal vessel ABCD (fig. 72), divided into two compartments by the partition EF. A glass tube H *h*, open at both ends, is inserted in the opening H, in the partition EF; the lower end being allowed to reach a little below EF, the tube H *h* must then be covered by a small glass receiver *a b c*, or a wide tube hermetically sealed above, a small aperture being left at the bottom of this tube to admit the water. This mechanism is generally covered by the figure of a man representing Tantalus, as shown in the drawing. If the water be now poured into the vessel, it will get admittance into the receiver or wide tube, and will always stand at the same height in this tube that it does in the vessel. The water will therefore be retained in the vessel as long as it does not enter the tube H *h*, but as soon as the water rises in the vessel to the same level as the point *h*, it will flow down the tube H *h*, according to the principles of the siphon, and will discharge the whole fluid in the vessel. If water be poured slowly in with the intention of making it rise to the lips of Tantalus it will never reach them, provided the siphon carries off the water faster than it is poured in. In the lower compartment of the vessel there ought to be a small air-hole near the top, to allow the air to escape when the water takes its place.

Second experiment.

Fig. 73.

229. *To construct a vessel which retains water when upright, but discharges it when inclined.* Let ABCD (fig. 73) be the vessel, divided as above into two compartments by the partition EF. Into this partition insert the longer branch *b c* of a siphon *a b c*, whose shorter branch *b a* reaches nearly to the bottom of the vessel. If water be now poured into the vessel till it stand a little below the lower side of the bent part of the siphon, it is obvious that no water will descend through the siphon, as it has not risen high enough in the shorter branch to enable it to pass through the bent portion. If the vessel, however, be inclined on one side, as it is in the act of drinking, the water will rise higher in the short branch *a b*, pass over the bent part of the siphon, descend in the longer branch, and carry off all the water into the lower compartment of the vessel. In order that this experiment may succeed, the sides of the vessel ought not to be symmetrical round the point *b*, at the summit of the siphon; for in this case no inclination of the vessel, however great, will cause the water to flow over the point *a*. The siphon should therefore be placed towards one side of the vessel, and the vessel inclined to the same side.

Fig. 73-*a*.

A similar effect may be produced much more elegantly by using the double cup, shown in (fig. 73-*a*), where *a b c* represents the siphon. The person who tries to drink must apply his lips to the side *b* of the siphon, otherwise the experiment will not succeed.

Fig. 74.

230. The following is another very pretty machine, whose operation depends upon principles nearly the same as the above. ABCD (fig. 74) is a vessel divided into three compartments by the partitions EF and GH. In the partition EF, are inserted two tubes, one of which, LM, forms a communication between the bottom of the compartment BG and the bottom of FC, while the other tube IK forms a communication between the upper part of FC and the upper part of HF. A third tube, NO, is fixed in the cover AB, extending from near the bottom of HF, and rising from the tapering bore to the point O, through the middle of the vessel SR, intended to receive the water which falls

from the pipe NO. The figure of a bird, with its bill immersed in the water in the basin SR is placed on one side, and through its body passes a bent siphon QP, the lower branch of which goes into the compartment BG.

When the two upper compartments are nearly filled up to a little below K with water, through two apertures for that purpose, and when these apertures are shut, it is obvious that the cock of the pipe LM being opened, the water will descend through LM, and occupying the compartment EC, will drive the air up through the pipe IK, and compress the air contained in the cavity HF. This condensed air pressing on the surface HK of the water, will raise it in the tube NC, and cause it to be projected upwards in a jet d'eau. The water from the jet, after being carried to a height due to the pressure which is experienced, will fall down into the vessel SR; but as the water escapes from the compartment BG, the air in that compartment will be rarefied, and will therefore not be sufficient to balance the pressure of the exterior air upon the surface of the water SR. This unbalanced pressure will therefore force the water up the siphon QP, through which all the water in the vessel SR will be conveyed into the cavity BG, as if it had been drank by the bird.

XIII. *Of the siphon-fountain.*

The siphon-fountain. Fig. 75.

231. Let AB (fig. 75) represent a tall receiver, standing in a wide basin DE, which is supported on the pedestal H by the hollow tube FG. In the centre of the receiver is a jet-pipe C, and in the top a ground-stopper A; near the base of the pillar is a cock N, and in the pedestal is another, O.

Fill the basin DE with water within half an inch of the brim; pour in water at the top of the receiver (the cock N being shut) till it is about half full, and then put in the stopper. A little water will run out into the vessel DE; but before it runs over, open the cock N, and the water will run into the cistern H; and by the time that the pipe C appears above the water, a jet will rise from it, and continue as long as water is supplied from the basin DE. The passage into the base cistern may be so tempered by the cock N that the water within the receiver shall keep at the same height, and what runs into the bass may be received from the cock O into another vessel, and returned into DE, to keep up the stream.

Pneuma machi and ex perime

Fig. 7

This pretty philosophical toy may be constructed in the following manner: BB (fig. 76) is the ferrule, or cap, into which the receiver is cemented. From its centre descends the jet-pipe C *a*, sloping outwards, to give room for the discharging-pipe *b d*, of larger diameter, whose lower extremity *d* fits tightly into the top of the hollow pillar FG.

The operation of the toy is easily understood. Suppose the distance from C to H (fig. 75) 3 feet, which is about $\frac{1}{11}$th of the height at which the atmosphere would support a column of water. The water poured into AB would descend through FG (the hole A being shut) till the air has expanded $\frac{1}{10}$th, and then it would stop. If the pipe C *a* (fig. 76) be now opened, the pressure of the air on the surface of the water in the cistern DE will cause it to spout through C to the height of 3 feet nearly, and the water will continue to descend through the pipe FG. By tempering the cock N, so as to allow the water to pass through it as fast as it is supplied by the jet, the amusement may be continued a long time. It will stop at last, however; because, as the jet is made into rarefied air, a little air will be extricated from the water, which will gradually accumulate in the receiver, and diminish its rarefaction, which is the moving cause of the jet. This, indeed, is an inconvenience felt in every employment of siphons, and so much the more remarkably, as their top is higher than the surface of the water in the cistern of supply.

Cases of this employment of a siphon are not unfrequent. When water collected at A (fig. 77) is to be conducted in a pipe to C, situated in a lower part of the country, it sometimes happens, as between Lochend and Leith, that the intervening ground is higher than the fountain-head, as at B. A forcing-pump is erected at A, and the water forced along the pipe. When once it runs out at C, the pump may be removed, and the water will continue to run on the siphon principle, provided BD do not exceed 33 feet. But the water in that part of the conduit which is above the horizontal plane AD, is in the same state as in a receiver of rarefied air, and gives out some of the air which is chemically united with it. This gradually accumulates in the elevated part of the conduit, and at last choaks it entirely. When this happens, the forcing-pump must again be worked. Although the elevation in the Leith conduit is only about 8 or 10 feet, it will seldom run for twelve hours.

TABLE

OF THE PRINCIPAL MATTERS IN THE

TREATISE ON MECHANICS.

*** The connection and importance of the three Sciences already completed; may render a Table of the Principal Matters contained in these Treatises acceptable to the reader.

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construction of
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Velocity of the wind
of sound through the air
through solids
Whirling machine
Wind-gages
Windmill-sails
Wind, velocity of
effect of, on the propagation of sound
Wollaston's thermometrical barometer

OPTICS.

Optics (ὀπτική, from οπτομαι, *to see,)* is a physico-mathematical science, by which we investigate the properties of light and vision.

In this science, as it is now taught, light is regarded as proceeding from the object to the eye in three different ways: 1st, *directly,* or without any change in its course; 2dly, in a course which is bent or *refracted;* 3dly, in a path which, meeting with some other object, becomes *reflected.* We denominate that part of the science which relates to direct vision, *optics properly speaking, or simple optics* (to distinguish it from the general term *optics,* which includes all the varieties). The part which relates to vision, occasioned by refracted light, is called *dioptrics:* and that which relates to the vision, which results from reflected light, is called *catoptrics.*

§ 1. *History of Optics.*

It would be more easy to become the encomiast of this science than to trace its history; for there is no department of philosophy more deserving of our study, whether we consider its beauty or the multiplicity of its phenomena. Air, which serves as the medium of speech, and the vehicle of sound, enables us to carry on an intellectual intercourse with our fellow-creatures; but how considerably is that intercourse improved and facilitated by light, which brings before us their image—their image which tells us so much of their character and of their thoughts! The eye, so susceptible of multifarious impressions, conveys to the mind ideas of the forms by which bodies are limited, the colours by which they are adorned, their relative positions, and their motions. By a single look this admirable organ enables us to seize the indefinite modifications of the numerous objects that diversify our richest landscapes; and when it becomes aided by the instruments furnished by our applications of reflected and refracted light, it contemplates the two kinds of infinity that would otherwise have remained unknown—that of animalculæ and of small inanimate objects, imperceptible by reason of their minuteness—and that of the celestial bodies, invisible by reason of their remoteness; thus opening to natural history a new field, to astronomy a new heaven, and inviting us successfully to contemplate the universe of the poet,

> ———————— "Without bound,
> Without dimension, where length, breadth, and height,
> And time and place are lost."

But our present employment must not be that of eulogy.

The ancients for several centuries seem, as was naturally to be expected, to have had no knowledge of the theory of Optics, and to have made no advances of consequence in the construction of optical instruments. The observance of a straight rod partially immersed in water, would suggest to them the idea of refraction; and the sight of their own image reflected from the smooth surface of a quiescent liquid, would naturally lead them to attempt the construction of artificial mirrors. Accordingly, we find mention not merely of mirrors, but of *metallic mirrors,* in the earliest writings now extant, those of Moses. In Exodus xxxviii. 8, though Luther, and some few after him, translate the passage "He made the hand-bason of brass, and its stand also of brass, *in the presence of the women* who served before the door of the tabernacle;" yet they have been censured for this, since the Septuagint, the Vulgate, the English and the Dutch Bibles, agree in translating *Beramoth* "of the mirrors," made, say many of the commentators, of polished brass. In the book of Job, too, now generally assigned by biblical critics to Moses as the author, we have (xxxvii. 18) in the address of Elihu to his afflicted friend, the inquiry,

History.

Hebrew mirrors.

> "Hast thou with him (God) spread out the heavens,
> "Polished as *a molten-mirror?*"

Pliny assures us* that the pagan women, when attending the worship of their deities, were ornamented with metallic mirrors; and it seems extremely probable, as Cyril, of Alexandria, has affirmed,† that the Israelitish women borrowed this custom from the Egyptians, and attempted to introduce it into their own worship. These early mirrors were flat, and so they appear to have been, generally, down to the time of Prasitelis, who lived in the reign of Pompey the Great.

Prasitelis's mirrors. B. C. 60.

His mirrors chiefly consisted of hammered plates of pure silver, as we learn from the words of Pliny:—*Lamina duci et specula fieri non nisi ex optimo (argento) posse creditum fuerat.* But the silver was sometimes mixed with other metals:—*Id quaque jam fraude corrumpitur.* Pliny further informs us, that "Specula quoque ex eo (stanno) laudatissima, Brundusii temperabuntur, donec argenteis uti cæpere et ancillæ." Highly-praised mirrors were manufactured at Brundusium, *till the very maid-servants began to use silver ones.* The monster, Nero, who it seems was short-sighted, employed as a mirror an *emerald,* reduced to a polished surface, on which he viewed by reflection, the combats of the gladiators. Here, however, is no optical science.

Aristotle. B. C. 350.

Aristotle is the earliest author whose writings on the subject of Optics have reached our times; but, unfortunately, he has not been more successful in this branch of research than he was in reference to Mechanics. His speculations on the nature of the rainbow, on the manner in which we perceive objects, and on different optical phenomena, are not merely crude, but generally erroneous; and in this treatise, *περι χρωματων, de coloribus,* every thing is so vague and foreign from

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* Lib. xxiii. cap. 9. Lib. xxxiv. cap. 17.

† Lib. ii. De adoratione in spiritu.

Optics.

correctness of explication, that we should not hold ourselves justified in presenting any detail.

Euclid. B. C. 300.

Soon after Aristotle, the celebrated geometer Euclid composed a book on this subject. It appears under the title Ὀπτικα (neuter plural), and has been sometimes ascribed to another author bearing the same name. We are of opinion, however, that it fairly belongs to the geometrician, and that it is the *introduction* only which was written by another hand. As the deductions of Euclid, though founded upon a wrong hypothesis, are curious, considering the state of mixed mathematics at the epoch in which they appeared, we shall present a synopsis of them in this place.

Propositions.

Light propagates itself in right lines, as is shown by the shadows of bodies, and by the passage of light through a door or window.

If the luminous object be equal to the object illuminated, the sections of the shadow are equal to the object, because the extreme rays are parallel. If the illuminated body be less than the luminous body, the shadow will successively diminish: on the contrary, if the illuminated body be largest, the shadow will become successively larger and larger.

Hypothesis. Visual rays issue from the eyes in diverging right lines, so as to form a pyramid, or cone, whose vertex is in the eye, and whose base encircles the object which we contemplate. Objects to which these rays are directed are seen by us; but we cannot see those towards which these visual lines do not point.

Objects appear larger, smaller, or equal, according as the angles under which we see them, are greater, less, or equal. The object is always seen in the direction of the visual ray; and those which are seen by the greatest number of rays are most distinct.

We never see the whole of an object. Of two equal objects, the nearest will be seen most distinctly. Every visible object becomes invisible at a certain distance.

Of equal parts of a right line, those which are most remote are seen under the smallest angle, and appear smallest.

Equal magnitudes, seen at unequal distances, appear unequal; that which is nearest will appear greatest.

Parallel lines, viewed from a distance, appear to converge.

If a horizontal surface be lower than the eye, the part which is most remote will appear to be elevated: if the horizontal be higher than the eye, the most remote portion will appear depressed.

A circle, viewed in the direction of its own plane, will appear as a right line.

When we look at a sphere with one eye, we never see so much as its half.

Viewed from a distance, a sphere appears as a circle.

When we look at a sphere with both eyes, if its diameter be equal to the distance between the two pupils, we see its half: if the interval between the pupils be greater, we see more than half; if the said interval be less, less than the sphere half will be seen.

If we look at a cylinder with one eye, we shall not see its half; as we approach nearer to it, we see less and less.

If the eye be in a line that passes through the centre of a circle perpendicularly to its plane, all the radii of the circle appear equal.

A circle, seen obliquely, appears flattened or contracted.

If several objects are in motion, and only one quiescent, that one will seem to move in a contrary direction.

If several bodies move with unequal velocities, and the eye is carried along in the same direction, those objects which have the same velocity as the eye will appear stationary; those which have greater velocities will appear to advance, while those which have less velocities will seem to recede.

If several objects have equal velocities, those which are most remote will appear to move most slowly.

If the eye advance, distant objects will appear to be left behind.

If an object appear to augment, we judge that it is approaching towards the eye.

Objects unequally distant, which are not in a right line, may sometimes give the idea of a concave surface, and at others, the idea of a convex surface.

These propositions relate to direct vision: there are a few which relate to reflection. Among these we find the problem, to find the height of an object by its shadow, or, in the absence of the sun, *by means of a mirror*, on the principle of the equality of the angles of incidence and reflection.

Euclid also attempts to determine the burning point of a concave spherical mirror, but errs in his conclusion; for he supposes that this point was the centre of the concavity, or the centre of the sphere.

How long this error in theoretical deduction remained uncorrected, it is not easy to say: it is well known that the ancients employed concave mirrors to re-kindle the vestal fires. Plutarch, in his Life of Numa, gives a description, not very distinct, however, of the σκαφεῖα, or dishes, which they thus employed. They seem to have been concave polished hemispheres, or segments nearly hemispherical; and Plutarch tells us, that the combustible matter was placed in their *centre*. This could not be; for the focus is at the distance of *half* the radius.

Imperfect, however, as the theory appears to have been, there can be no question that the practice of setting fire to substances by placing them in the foci of catoptric and dioptric instruments, was known some centuries before the Christian æra. In addition to what is already adduced, we may cite a passage from the Clouds of Aristophanes, in which he introduces Socrates as giving lessons to Strepsiades. The object of the dramatist is to ridicule the philosopher. Strepsiades proposes an expedient by means of which he intends to pay his debts.

"*Strep.*—You have seen at the druggists that fine transparent stone with which fires are kindled.

"*Soc.*—You mean *glass*, do you not?

"*Strep.*—Just so.

"*Soc.*—Well, what will you do with that?

"*Strep.*—When a summons is sent to me I will take this stone, and, placing myself in the sun, I will, though at a distance, melt all the writing of the summons."

Writing, in those times, was traced on *wax* spread upon a more solid substance. Hence we see why Strepsiades should propose to *melt* the writing.

From this use of burning glasses, the transition to the mirrors, said to have been employed by Archimedes, is not either so extraordinary, or so difficult, as has been usually imagined. It has been repeatedly affirmed on the authority of Hero, Diodorus Siculus, Lucian and Pappus, that Archimedes, by means of burning mirrors, set fire to the Roman fleet that was drawn up to besiege Syracuse. This, however, has been often denied. Descartes, and many after him, have regarded the thing as impossible. To the discussion of this question we cannot devote much space; it will be expected, however, that we do not pass it over in total silence.

Father Kircher, although he was among the incredulous, in reference to the Archimedean mirrors, concluded from an actual survey of the site of the town and harbour of Syracuse, that the distance to which the philosopher had to project the solar rays, was not more than *thirty* paces. And whatever may have been the doubts formerly entertained on this subject, it is now well known, that the solar rays may, after reflection, be thrown to an effective focus at a much greater distance than this. Our deduction will not be speculative, but historical.

Zonaras affirms, from the authorities above specified, that Archimedes set fire to the Roman fleet by means of the solar rays collected and reflected by a polished mirror. He then adds, that Proclus, copying his example, burnt with mirrors of brass the fleet of Vitalian, who besieged Constantinople, under the emperor Anastusius, in the year 514.

Tzetzes, who also quotes the same authorities, presents a particular explication of the mechanism of Archimedes' burning mirrors. "When Marcellus, (says he) had removed his fleet out of the reach of the darts, Archimedes brought into play a hexagonal mirror, composed of several other smaller mirrors, each of which had twenty-four angles, and which could be moved by means of their hinges, and of certain plates of metal. He placed this mirror in such a position that it was in the midst of the meridional solar rays both in summer and in winter; so that those rays, being received on the mirror, were reflected by it, and kindled such a fire as reduced the Roman vessels to ashes." This is much such a description as might naturally be expected from a person not skilled in either Optics or Mechanics; and such a person was Tzetzes.

A very obscure hint, however, is sufficient to bring real genius into action; and it is highly probable, that the celebrated Napier, putting a happy construction upon the words of Tzetzes, recovered the admirable invention of Archimedes. In a paper of Napier's, bearing date June 2, 1596, and containing hints of secret inventions, we meet with the following:—

"The invention, proof, and perfect demonstration, geometrical and algebraical, of a burning mirror, which receiving of dispersed beams of the sun, doth reflex the same beams altogether united, and concurring precisely in one mathematical point, in the which point most necessarily it engendereth fire; with an evident demonstration of their error, who affirm this to be made a parabolic section. The use of this invention serveth for the burning of the enemy's ships at whatsoever appointed distance.

2dly. "The invention and sure demonstration of another mirror which, receiving the dispersed beams of any material fire, or flame, yieldeth also the former effect, and serveth for the like use."

Long after this, viz. in 1726, M. Du Fay found that, at 200, 300, and even as far as 600 French feet (about 640 English), the rays of the sun received on a plane mirror, one foot square, and thence reflected to a concave one, 17 inches in diameter, consumed combustible bodies in the focus of the latter."

Buffon.
A. D.
1747.

The success of this interesting experiment, doubtless, stimulated Buffon to attempt the production of fire at a distance, after the manner of Archimedes, by one reflection only. In the year 1747, after various trials with combinations of plane mirrors (in number sometimes amounting to 400), placed in a square frame, and brought to bear upon the object by means of screws, he succeeded in melting lead and tin at the distance of about 50 English yards; and in burning lighter substances, at the distance of 75 yards. This was affected in March and April. With summer heat, and a better apparatus, he expresses a certainty of producing combustion at more than 140 of our yards; a distance, probably, double that at which Archimedes produced his conflagration. Since the publication of Buffon's results, the scepticism which prevailed in reference to the burning mirror of Archimedes, has been rapidly wearing away. The philosopher had not to *invent* the apparatus for the purpose, but simply to apply what he had previously invented. The vessels in the harbour might have been left dry by the receding tide; and, of consequence, there would not be sufficient water at hand to check the progress of combustion. With regard to the probable construction of Archimedes' appartus, since this would not be the proper place to enter into detail, we refer to the speculations in Peyrard's edition of Archimedes, tom. ii. pp. 464—508. We must now return to the point at which we commenced this inquiry.

Peyrard.
A. D.
1808.

Ptolemy
Euergetes.
B. C. 224.

In the same century with Archimedes, lived Ptolemy Euergetes, celebrated by historians for causing to be placed on the tower of the Pharos at Alexandria, a mirror, which represented accurately all that was done on water or land within its scope; and by means of which, as some authors relate, an enemy's fleet was seen at the distance of 600,000 paces. We do not hold ourselves responsible for the truth of this. Father Abat, whose *Amusemens Philosophiques* were published in 1763, has an ingenious attempt to prove the probable existence of such a mirror at the time specified; and a copious abridgment of his arguments, by a very able writer, was given in the nineteenth volume of Tilloch's Philosophical Magazine.

Ptolemy the
astronomer.
A. D. 140.

Among the writings which still remain of the celebrated Claudius Ptolemy, the Alexandrian astronomer, is one on Optics. It is comprised in five discourses, or books, of which the first is lost; most part of the remaining four are preserved, and have been carefully examined in the MSS., both by Delambre and by Venturi.

Although, as we have just remarked, the first book is wanting, we are not entirely ignorant of its contents, because each book commences with a recapitulation of what had been taught in the former. Thus we learn, that the first book treated of the relations between light and vision, of their resemblance and of their difference. It was probably a philosophical dissertation after the manner of Aristotle. Ptolemy supposes that vision is effected by means of a pyramid of visual rays, of which the vertex is at the eye, and the base at the object seen. This agrees with the notions of Euclid: some writers, earlier than either of these, taught, as the moderns do, that the rays of light proceed from the visible object; but this notion had few partizans, while the notions of Euclid and Ptolemy prevailed extensively. Vision by the axis of the pyramid is, according to Ptolemy, more correct and perfect than that by oblique rays. Vision makes bodies known, reveals their magnitude, colour, figure, rest, and motion; but all this requires *light*.

Shadows are not *seen:* we know them only by privation.

We can see better with two eyes than with one: with one only we do not see the object precisely at the same place as with two. We see the object simple, if the two axes of the pyramids are directed in the same manner upon the object; we see the same object double, if the axes are not directed naturally, and if the distance is a little less than that between the two eyes.

Colour makes part of bodies, it is the exterior crust. The eye perceives the direction of the visual ray which it sends towards the body; it perceives, in like manner, the length; it judges of the magnitude of the object, from the length of the pyramid combined with the magnitude of its base. If the humidity of the visual ray be promptly dissipated, bodies are seen better when near; if it be slowly dissipated, they are seen better at a distance.

That which causes certain persons to see better than others, is the abundance of the *visual virtue;* which, like all other faculties, fails in old men.

The moon has a colour which is peculiar to it, and which is only perceived in eclipses.

When we have long contemplated an object highly coloured, and then direct our eyes to another object, we attribute to that the colour of the former.

Things which we see by reflection, partake of the colour of the mirror; as those which we see through a diaphanous or transparent body, assume its colour.

When we observe a fire or a light at the horizon, beyond a pool of water, we perceive a long luminous train which follows our motion.

A sail seen from far, appears more curved than it is in fact; because the middle, which is seen directly, is perceived better than the edges, which appear to fly. Thus painters, when they would excite the idea of any thing concave, give a less vivid tint to the middle than to the edges; and the contrary, if they would give the idea of convexity.

Similar to these are the remaining speculations in the second book.

In the third book Ptolemy proceeds to the subject of mirrors. The principal propositions are these:—In the plane mirror, the object is seen in the continuation of the perpendicular let fall from the object itself upon the plane, and as far behind the plane as the object is before it. This had been previously taught by Euclid; as had been the equality of the angles of incidence and reflection.

Ptolemy then recurs to the consideration of objects which appear in different places at the same time, though simple; and to those which, though more than two, appear in one and the same place. We cannot detail his speculations on these points, but must limit ourselves to one only, and that relating to astronomy. "It results (says he) from the preceding, that of things which are in the sky and subtend equal angles, those which are nearest the zenith ought to appear less, and those which are near the horizon appear greater; because we see the latter in a manner to which we are more accustomed. Elevated objects are seen in a way with which we are less familiar, and with difficulty of action." Thus, according to Ptolemy, the moon in the zenith, appears smaller; because he who looks towards the zenith, is in a less natural position than when he looks at any object posited horizontally.

The author next returns to plane mirrors, and shows that in them objects are not disfigured; but that the right becomes the left, and *vice versa.*

In concave mirrors, objects appear concave; in convex mirrors, they appear convex. In convex mirrors objects seem diminished. Thus, also, taught Euclid.

In a concave mirror, a curve line may, according to circumstances, appear either convex, concave, or rectilineal. In a convex mirror, objects appear on the side on which they really are; yet, by reason of our habit of judging, the right will seem to be on the left, and the left on the right.

Ptolemy's fourth book relates principally to concave mirrors.

He treats of the place of the image, and shows when it may be upon the surface of the mirror; when before that surface, when behind the eye, when behind the mirror.

When the image is behind the mirror, the distance of the object from the mirror is less than that of the image.

When the image is between the eye and the mirror, the distance of the object from the eye, will be greater, equal, or less, than the distance of the image from the mirror, according to circumstances.

When the object is between the eye and the mirror, it appears in a different place from that in which it really is; and when we move it in one direction, it will appear to move in the contrary one.

Ptolemy next passes to the consideration of mirrors compounded of a plane and a concave, or of a convex and a concave; and explains the cases in which the image is direct or inverted, augmented or diminished; after which he traces the peculiarities of pyramidal mirrors with circular or pyramidal bases, having the eye placed in the axis of the pyramid. In all this Ptolemy never determines the precise point of reflection, when the place of the eye and that of the image are known: he satisfies himself with showing, generally, that the object is before or behind the mirror, or the eye, or nearer to the mirror, or more remote than the eye; the relations being not measured, but indicated vaguely.

The fifth book of Ptolemy's Optics contains his researches into the nature of refraction.

He explains the experiment of the piece of money so placed in a vessel that its edges render it invisible, until water is poured in, when the money is brought into sight, while it has remained quiescent. After this, he proceeds to a curious set of experiments, which we cannot here detail, in order to determine the relation between the positions of the incident and refracted ray, the media being air and water, for all degrees of incidence, varying by tens, up to 80°. The medium ratio of the sine of incidence to that of refraction, when the ray passes from air into water, is 4 to 3·06936: according to the experiments of Newton, the ratio of these sines is 4 to 2·99432. When the ray passes from air into glass, the result of Ptolemy's experiments is, that the sines of incidence and refraction are as 3 to 2·02158. Newton gives for the ratio of these sines

3 to 1·93048. The correspondence between these respective ratios is greater than might reasonably have been expected, considering that the instruments employed by Ptolemy would not enable him to measure angles to nearer than half a degree. Newton employed *rain* water: Ptolemy has simply informed us that his water was always of the same density. Newton, again, employed *common* glass: Ptolemy calls his the purest glass: what that was we cannot say, because we know nothing of the glass-manufactory among the Egyptians in the time of Ptolemy.

In the explication of astronomical refraction, Ptolemy proceeded in several respects as Cassini did in the last century. He, also, taught expressly, that the more a star is elevated, the less will be the difference between the true and the apparent place, and that this difference is nothing when the star is in the zenith, because the vertical ray does not undergo any flexure. This Ptolemy demonstrates by means of a figure; from which it appears, that in all cases the refraction carries the star towards the zenith.

Ptolemy afterwards describes different experiments connected with the subject of refraction; but his deductions from them are, in general, erroneous. Altogether, however, this fifth book of his Optics is highly curious and interesting; and, indeed, the whole work is methodical and instructive; on which account, we have entered more fully into description of it than has been usual among the historians of Optics.

It is an interesting inquiry, but by no means of easy determination, how far the ancients attempted to assist sight by dioptrical instruments. Roger Bacon, in his piece "On the Secret Works of Nature and Art, and on the nullity of Magic," says, "transparent bodies may be so figured, that things at the greatest distance may appear to be the nearest, and the contrary; so that, from an incredible distance, we may read the smallest letters, and number things, however minute: thus it is thought that Julius Cæsar, on the coast of Gaul, discerned, by or through very large glasses, the disposition and situation of the camps and (coast) cities of Britannia Major." We here render *per ingentia specula*, by or through very large *glasses*, because the author is speaking of, *perspicua*, transparent things. On what evidence he grounds his assertion we know not.

The ancients are well known to have used dioptric as well as catoptric burning glasses; and it is not probable that they would employ the former thus, and yet remain ignorant of their magnifying power. The contrary, indeed, is plainly affirmed by Seneca: Literæ quamvis minutæ et obscuræ, per vitream pilam aquâ plenam, majores clarioresque cernuntur. "Letters, though minute and obscure, appear larger and clearer through a glass bubble filled with water."* Such a phenomenon, often observed, would naturally lead inquisitive men a few steps farther. But they could make no important advance (says Dr. Hook) without the art of grinding glass. This they had; so at least says Pliny. "Some glass is fashioned by blowing; some is ground upon a wheel, or in a turning lathe; and some is engraved like silver. Sidon was celebrated for its glass works, having also invented *specula*. Such was the ancient art of glass."* What were here meant by *specula?* The phrase "siquidem etiam specula *excogitaverat*," points evidently to some notable invention.

That glass was ground by the ancients, is also fairly deducible from the language of Seneca. He tells us that prisms were in use among the virtuosi of Rome, in the days of Nero; and how could a glass prism be made by blowing? "A rod, or bar of glass (says he) is made with several angles; and if the rays of the sun pass through it, such colours are made as we see in the rainbow."† Seneca also speaks of multiplying-glasses, the several faces of which must have been cut upon a wheel.

Pisidas.
A. D. 670.

All this, however, brings us not to any such invention as that of telescopes. Nor are we aware of any thing in antiquity that can indubitably be so interpreted. We have seen adduced, for this purpose, a passage from *Pisidas*, a Christian writer, who flourished at Constantinople in the seventh century:—*Τα μελλοντα ως δια διοπτρȣ συ βλεπεις*; "you see things future as by a *dioptrum*." They who contend for the early invention of telescopes, say that this dioptrum was a prospective glass. But, if it were the same as the dioptrum employed by Hipparchus, and afterwards by Ptolemy, it was no other than a straight ruler of about four feet long, on which were fixed three equidistant "sights," as they are technically called. A figure of it is given in M. Halma's edition of the Almagest.

From what has been stated, then, it will be seen that the ancients had but little that deserved the name of optical science; that what they possessed was confined almost entirely to catoptrics, and was even in reference to that department, extremely imperfect; that their instruments were chiefly catoptrical, some of which, however, they carried to very high perfection; and that we have no proof that their dioptric instruments went beyond single lenses, prisms, and multiplying glasses. In reference to refraction, the few researches which they had instituted, seem to have been ingenious and partially successful.

Alhazen.
A. D. 1100.

After an interval of some centuries, we arrive at the time of Alhazen, an Arabian mathematician; in whose work, translated from the Arabic, and published in 1572, under the title of *Thesaurus Opticæ*, we probably have a summary of the optical knowledge of Persia and Arabia, at the period in which he lived, and which knowledge he carried into Spain. It does not appear from an inspection of this Thesaurus, that Alhazen was at all acquainted with the optical writings of Ptolemy. But we find, first, that Alhazen has refuted the system of vision which prevailed among the Greeks. 2dly. That he has attempted the anatomical description of the eye, and has assigned the respective functions of the several parts in vision. 3dly. That he succeeds better than Ptolemy in accounting for the single vision with both eyes. 4thly. That he has investigated the nature of *seven* kinds of mirrors, while Ptolemy had contemplated only *three*. 5thly. That in speculating upon astronomical refraction, he was led to some inquiries respecting *twilight*, in which he concludes that the morning twilight commences when the sun is 18° below the horizon. 6thly. That he attempts the explication of the phenomenon of the *horizontal moon*. This attempt,

Alhazen: Twilight; and horizontal moon.

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* Seneca, Nat. Quest. lib. i. cap. 7.

* Plin. Nat. Hist. lib. xxxvi. cap. 26.

† Senec. Nat. Quest. lib. i. cap. 7.

however, is extremely unsuccessful, and need not to be detailed in this place.

R. Bacon, born A. D. 1214. Died A. D. 1294.

Roger Bacon, who flourished in the thirteenth century, has many claims to the notice and gratitude of posterity; and among these, on account of his discoveries in Optics. He has treated very ingeniously on the subjects of astronomical refraction, the apparent magnitudes of objects, the extraordinary size of the sun and moon near the horizon, and the place of spherical foci. That his theory and practice of optical glasses were much farther advanced than many moderns have been willing to admit, will be evident from the following quotation from the *Opus Magnus:* "Greater things than these may be performed by refracted vision. For it is easy to understand, by the canons above-mentioned, that the greatest things may appear exceedingly small, and contrarily. For we can give such figures to transparent bodies, and disperse them in such order, with respect to the eye and the objects, that the rays shall be refracted and bent toward any place we please, so that we shall see the object near at hand, or at a distance, under any angle we please; and thus, from an incredible distance, we may read the smallest letter, and may number the smallest particles of dust and sand, by reason of the greatness of the angle under which we may see them; and, on the contrary, we may not be able to see the greatest bodies just by us, by reason of the smallness of the angle under which they may appear. For distance does not affect this kind of vision, excepting by accident, but the quantity of the angle does. And thus a boy may appear to be a giant, and a man as big as a mountain; forasmuch as we may see the man under as great an angle as the mountain, and as near as we please. And thus a small army may appear a very great one, and though very far off, yet very near us; and contrarily. Thus, also, the sun, moon, and stars may be made to descend hither in appearance, and to appear over the heads of our enemies; and many things of the like sort, which would astonish unskilful persons."

It would not be right to infer from this passage, that Roger Bacon had precisely such instruments as the common telescopes and microscopes of the nineteenth century; yet this we think is fairly admissible, that he could, and did so combine and fix transparent bodies, as, by means of them, to make distant objects appear as though they were near, or contiguous objects as though they were remote, or minute objects as though they were large; that is, that he produced effects in practical optics precisely analagous to those which we now accomplish by means of telescopes and microscopes. But further, Bacon, in his work, addressed to Clement IV. after proposing some expedients for promoting Optics and Astronomy, adds, "But what is much more (important or requisite) than these things; men ought to be had who understand

Optical instruments used in astronomy.

perspective and its instruments well; because the instruments of astronomy only proceed by vision, according to the laws of that science." From this, bearing in mind that in the time of R. Bacon, the term *perspective* was employed to signify *optics* in general, it is not unnatural to conclude, that he made use of, and recommended optical instruments in astronomical observations. Bacon himself tells us, that the first burning glass which he made cost him 20 *l.* equivalent to about 187 *l.* of our money; and that he expended in chemistry, "*experimental* perspective, and *practical* astronomy," in twenty years, "no less than 2,000 *l.*" about 18,733 *l.* as at present valued. Much that he knew and accomplished is, doubtless, lost; but surely there can be no reason to conclude, as many have done, that the most valuable application of his optical knowledge did not equal in utility a common pair of spectacles.

Contemporary with Roger Bacon was Vitellio, or Vitello, a Polish philosopher. His treatise on Optics was published with Alhazen's, in the *Thesaurus Opticæ*, 1572. He observed, that light was always lost by refraction, but does not attempt to estimate the quantity lost. He remarked, also, that refraction is necessary as well as reflection, to produce the rainbow. He ascribed the twinkling of the stars to the motion of the air in which the light is refracted; and to illustrate his hypothesis, remarked, that they twinkle still more when viewed in water that is in gentle motion.

John Baptista Porta, a Neapolitan, published his *Magia Naturalis* towards the end of the fifteenth century. In the sixth book of that work he treats of optical refractions, and teaches, that, if there be made a small orifice in the window-shutter of a room, from which the light is every where else completely excluded, the external objects will depict themselves on a white wall, or on a piece of paper, in their natural colours; and he adds, that, by placing a small convex lens at the aperture, the objects will appear so distinct as to be instantaneously recognised. This is no other than the *camera obscura;* from the invention of which, to the true explication of the mechanism of vision, there was but one step; that step, however, though indicated by Porta, was not by him taken.

Robert Recorde, an early English writer on arithmetic, algebra, and geometry, must not be omitted in a history of Optics. In the preface to his "Path-way to Knowledge," a book on the elements of geometry, printed in 1551, and dedicated to our King Edward VI. (whose physician Recorde was), we meet with this remarkable passage: "But to retourne againe to Archimides, he did also by art prospectiue (whiche is a part of geometrie)* deuise suche glasses within the towne of Syracusa, that did burne their enemies shippes a greate waie from the towne, which was a meruailous politike thynge. And if I should repeate the varietie of such straunge inuentions, as Archimedes and others haue wrought by geometrie, I should not onely excede the order of a preface, but I should also speake of suche things as can not well bee understoode in talke without some knowledge in the principles of geometrie. But this will I promise, that if I maie perceiue my paines to be thankfully taken, I will not onely write of suche pleasaunte inuentions, declaryng what they were, but also *will teache how a great nomber of them were wrought, that they maie be practised in this tyme also.* Whereby shall be plainly perceiued that many thynges seeme impossible to bee done, which by arte maie verie well bee wrought. And when they bee wrought, and the reason thereof not understoode, then saie the vulgare people, that those thynges are doun by negromancie, And hereof came it that frier Bacon was accompted so greate a negromancier, whiche neuer used that art (by any coniecture that I can finde), but was in geometrie and other mathematicall sciences so experte that he could doe by them suche thynges as appear wonderfull in the sight of moste people. Great talke there is of a glasse that he made in Oxforde, in whiche men might see thinges that weare doen in other places, and that was iudged to bee doen by power of euill spirites. *But I knowe the reason of it to bee good and naturall, and to be wrought by geometric (sith perspective is a parte of it), and to stande as well with reason, as to see your face in a common glasse.* But this conclusion, and other diuers of like sort, are more meete for princes, for sundry causes, then for other men, and ought not to be taught commonly."

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* Recorde had been *professor* of geometry at Oxford.

Maurolicus, a teacher of mathematics at Messina, about 1575, published two treatises, entitled *Theoremata Lucis et Umbræ*, and *Diaphanorum Partes*, &c. These treatises exhibit some curious speculations on the measure and comparison of the effects of light; on the different degrees of brilliancy which is communicated to an opaque body from a luminous body, according to their different distances, &c. Maurolicus demonstrates that the chrystaline humour of the eye is a lens which collects the rays of light that issue from an object, and throws them upon the retina, where is the focus of every pencil. From this he explained why some persons were short-sighted, and others long-sighted; and why the former are relieved by concave, the latter by convex glasses. He also correctly explained a phenomenon which had exceedingly embarrassed Aristotle and others among the ancients, viz. that the rays of the sun when they pass through a small hole of any shape, (as triangular, square, &c.) always form a luminous *circle*, when received on a paper perpendicular to the course of the rays, and at a suitable distance from the orifice.

In the year 1570 was first published, the first English edition of Euclid's Elements, by Sir Henry Billingsley, a highly interesting and curious work. It is here mentioned, on account of an erudite mathematical preface to it by the celebrated John Dee, "specifying the chiefe mathematicall sciences, what they are, and whereunto commodious: where, also, are diclosed, certain new secrets mathematicall and mechanicall, untill these our daies greatly missed." In this preface, Dee defines *perspective* to be "an art mathematicall, which demonstrateth the nature and properties of all radiations, direct, broken, and reflected:" and "Glasse," he says, "is a generall name in *catoptrike*, for any thing from which a beam reboundeth." He describes a great variety of the most curious phenomena of refraction and reflection, the latter from mirrors of different kinds. In this connection, he says, "I speake nothing of *Naturall Philosophie*, which without *perspective* cannot be fully understanded, nor perfectly atteined unto. Nor of *astronomie*, which without *perspective* cannot well be grounded." He means, we conceive, that the former requires instruments analogous to our microscopes, the latter instruments analogous to our telescopes. He certainly speaks of them as his own, and gives the initials, S. W. P., of a gentleman who possessed "a marueillous glasse." In the same preface we meet with this extraordinary passage:—"No small skill ought he to have, that should make true report, or neer the truth of the numbers and summes, of footmen or horsemen, in the enemies ordering. Afarre off, to make an estimate between neer terms of More and Lesse, is not a thing very rife, among those that gladly would do it."—"The Herald, Pursuivant, Serjeant Royall, Captain, or whosoever is carefull to come neer the truth herein, besides the Judgment of his expert eye, his skill of Ordering *Tacticall*, the help of his Geometricall instrument: Ring or Staffe Astronomicall: commodiously framed for carriage and use. *He may wonderfully help himself by perspective Glasses.* In which (I trust) our posterity will prove more skilfull and expert, and to greater purposes, than in these dayes, can (almost) be credited to be possible." This passage we regard as furnishing decisive evidence that "perspective glasses," and some of their most useful effects, were known in England at least forty years earlier than their supposed invention in Holland. Another quotation or two from writers of the same age, will tend to render this important fact still more unquestionable.

L. Digges.
A. D.
1571.

In the Pantometria of Leonard Digges, first published in 1571, and afterwards in 1591, with improvements, by his son, Thomas Digges, there are some curious investigations relative to Optics. Both father and son were educated at Oxford, and both were respectable mathematicians. In the preface to the edition of 1591, Thomas Digges, after speaking of Archimedes' burning mirrors, and similar contrivances known to the ancients, adds, "But to leave these celestiall causes, and things doone of antiquitie long agoe, my father, by his continuall painfull practises, assisted with demonstrations mathematicall, was able and sundrie times hath, by proportionall glasses, duely situate in convenient angles, not onely discouered things farre off, read letters, numbered peeces of money with the verye coyne and superscription thereof, cast by some of his freends of purpose, upon downes in open fields, but also seuen myles off declared what hath been doone at that instant in priuate places. He hath also sundrie times, by the sunne beames, fired powder and dischargde ordinance half a mile and more distante; which things I am the boulder to report, for that there are yet liuing diuerse, of these his dooings *oculati testes* [eye witnesses], and many other matters farre more strange and rare, which I omit as impertinent to this place."

Thomas Digges further informs us, that his father's great knowledge and dexterity in Optics "partly grew by the aid he had of one old *written* book of the same Bacon [of Oxford]'s experiments, that by strange adventure, or rather destinie, came to his hands; though chiefly by conjoyning continuall laborious practise with his mathematicall studies;" and in the twenty-first chapter of the first book of this curious treatise, we meet with the following interesting paragraph:

"Thus much I thought good to open concerning the effects of a plaine glasse,* very pleasant to practise, yea most exactly seruing for the description of a plaine champion country. But marueillous are the conclusions that may be performed by glasses concaue and conuex of circulaire and parabolicall formes, using for multiplication of beames sometime the aide of glasses transparent, which by fraction should unite or dissipate the images or figures presented by the reflection of other. By these kinde of glasses, or rather frames of them, placed in due angles, yee may not onely set out the proportion of an whole region, yea represent before your eye the liuely image of euery towne, uillage, &c. and that in as little or great space or place as ye will prescribe, but also augment and dilate any parcell thereof, so that whereas at the first appearance, an whole towne shall present itselfe so small and compact together that yee shall not discerne anye difference of streates, yee may by application of glasses in due proportion, cause any peculiare house or roome thereof dilate and shew itself in as ample forme as the whole towne first appeared, so that yee shall discerne any trifle, or reade any letter lying there open, especially if the sunne beames may come unto it, as plainely as if you were corporally present, although it be distante from you as farre as eye can discrie: but of these conclusions I minde not here more to intreate, hauing at large in a uolume by itself opened the miraculous effects of perspectiue-glasses."

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* By *glasse* the author here means any plane reflecting surface; for, in the chapter from which the above is quoted, he says, "The best kinde of *glasse* for this purpose is of *steele* finely polished, neither convex nor concave, but flat."

Quotations from books, so scarce and so little known as those which we have here cited, are not only curious, but important in the history of this branch of science. Considered jointly, they present, in our opinion, decisive evidence that optical instruments were employed in astronomy, &c. long before the time to which it has been customary to refer the invention of the telescope. We know that Harriot, our celebrated analyst, observed the spots in the sun, between 1610 and 1613, by means of telescopes "which had magnifying powers of 10, 20, and 30 times;" and though it has been generally supposed that Harriot procured his telescopes from Holland, yet, since he was born and educated at Oxford, where Bacon and the Diggeses preceded him, and where, if any where, their discoveries and inventions would be known, we think it highly probable that the instruments which he employed were of English, not of foreign construction. The Diggeses, Dee, and Bacon, evidently possessed instruments which served for the purposes to which we now devote telescopes; yet neither of them claimed the invention for himself. Thomas Digges tells us, that his father profited by a manuscript of Roger Bacon; and Bacon tells us, that Julius Cæsar had an instrument which answered a similar purpose. From this induction of particulars, which we have extended to a length commensurate with its importance, we regard it as neither rash nor inconsistent with the undoubted pretensions of modern inventors, to infer that the knowledge of some such instrument as the telescope has been concealed among the learned from very remote ages; and that, on the revival of letters, learned and scientific men in Britain, so applied the theoretical knowledge they undoubtedly possessed, to the hints which they drew from the writings of antiquity, as to re-invent instruments for like purposes *several years* before they were heard or thought of in any other part of Europe.

Harriot. A. D. 1610.

Telescopes known and employed earlier in England than in any other part of Europe.

Antonio de Dominis. A. D. 1611.

In the year 1611, Antonio de Dominis, archbishop of Spalatro, published his treatise *De Radiis Visus et Lucis*, in which he seems a little to have improved Vitello's theory of the rainbow: he also attempted to account for the inferior bow, but without success.

Kepler, born A. D. 1571. Died A. D. 1630.

Kepler, so justly celebrated for his indefatigable application as an astronomer, made some researches in Optics, on which he published two treatises. He succeeded better than any of his predecessors in explaining the nature of vision: yet he failed (as many have done since his time) in accounting for the circumstance of an inverted image on the retina, exciting in the mind the idea of an erect object.

It has often been remarked, in the history of science, that what reiterated investigation has failed to accomplish, has been effected by *apparent* accident, without labour: and thus has it been with respect to the telescope. Bacon, Dee, and Digges, as we have just seen, invented instruments by which minute objects were made to appear large, and remote objects to appear near; yet the happy results of their investigations never became generally known, even in England, much less on the continent. Towards the end of the sixteenth century, however, a fortunate concurrence led to the invention of dioptric telescopes in Holland, and, speedily, to the active and successful employment of this valuable invention in the region of astronomical discovery. Borelli informs us, in a treatise composed not so long after the circumstance as to render his authority doubtful,* that the children of Zachariah Jansen, a spectacle-maker of Middleburgh, amusing themselves in their father's shop, placed by chance a convex and a concave lens in such a manner, that on looking through them at the weathercock of the church, it seemed to them nearer and much larger than usual. The father was called to witness this, and immediately fixing the glasses upon a board, that their relative position might be rendered permanent, he presented either these, or others similarly arranged, to his patron, Prince Maurice. This was about the year 1590. No sooner was this interesting discovery known, as it speedily was, over the principal part of the continent of Europe, than Galileo, Kepler, Descartes, and other philosophers, bent the whole force of their genius to the improvement and employment of so useful an apparatus; and since that epoch, theoretical and practical men have vied with each other in improving its construction and extending its powers.

Since the microscope is an instrument whose construction flows from the same principles as that of the telescope, it is natural to look for the invention of both at nearly the same time. Conformably with this idea, we find that microscopes were in frequent use within thirty years of the lucky incident so well improved by Jansen. We cannot speak positively respecting the real inventor of the microscope, but it is generally imagined to be Cornelius Drebbel of Alkmaer. He had obtained telescopes of Jansen about 1600, and, doubtless, constructed and employed microscopes between the years 1618 and 1620.

We are indebted to Willebrord Snell for the knowledge of the laws to which the refraction of light conforms. After a variety of delicate experiments, he found that, by prolonging the incident and refracted rays on each side of the point where the refraction [illegible] made, and drawing any vertical line, the parts of the two rays comprised between the above point and the vertical line, always preserve a constant ratio to each other, whatever the obliquity may be. This is precisely the same thing as saying, that when a ray of light passes out of one medium into another, the sines of the angles of incidence and refraction, always preserve a constant ratio for every variety of incidence in the same media. Snell, however, did not perceive that the two propositions were the same; but Descartes did, and in 1637 published the discovery *as his own*, in his Treatise on Dioptrics. Huygens, who knew

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* Borelli, De vero Telescopii inventore, Hagæ Com. 1655.

" I then proceeded to examine more critically, what might be effected by the difference of the incidence of rays coming from divers parts of the sun; and to that end, measured the several lines and angles belonging to the image. Its distance from the hole, or prism, was 22 feet; its utmost length $13\frac{1}{4}$ inches; its breadth $2\frac{5}{8}$; the diameter of the hole $\frac{1}{4}$ of an inch; the angle, which the rays, tending towards the middle of the image, made with those lines in which they would have proceeded without refraction, was 44° 56′. And the vertical angle of the prism, 63° 12′. Also the refractions on both sides the prism, that is, of the incident and emergent rays, were, as near as I could make them, equal, and consequently about 54° 4′. And the rays fell perpendicularly upon the wall. Now subducting the diameter of the hole from the length and breadth of the image, there remains 13 inches the length, and $2\frac{3}{8}$ the breadth, comprehended by those rays, which passed through the centre of the said hole, and consequently the angle of the hole, which that breadth subtended, was about 31′, answerable to the sun's diameter; but the angle which its length subtended, was more than five such diameters, namely 2° 49′.

" Having made these observations, I first computed from them the refractive power of that glass, and found it measured, by the ratio of the sines, 20 to 31. And then, by that ratio, I computed the refractions of two rays flowing from opposite parts of the sun's discus, so as to differ 31′ in their obliquity of incidence, and found that the emergent rays should have comprehended an angle of about 31′, as they did before they were incident. But because this computation was founded on the hypothesis of the proportionality of the sines of incidence and refraction, which though, by my own experience, I could not imagine to be so erroneous as to make that angle but 31′, which in reality was 2° 49′; yet my curiosity caused me again to take my prism. And having placed it at my window, as before, I observed, that by turning it a little about its axis to and fro, so as to vary its obliquity to the light, more than an angle of four or five degrees, the colours were not thereby sensibly translated from their place on the wall, and consequently by that variation of incidence, the quantity of refraction was not sensibly varied. By this experiment, therefore, as well as by the former computation, it was evident, that the difference of the incidence of rays, flowing from divers parts of the sun, could not make them, after a decussation, diverge at a sensibly greater angle, than that at which they before converged; which being at most but about 31 or 32 minutes, there still remained some other cause to be found out, from whence it could be 2° 49′.

" Then I began to suspect whether the rays, after their trajection through the prism, did not move in curve lines, and according to their more or less curvity tend to divers parts of the wall. And it increased my suspicion, when I remembered that I had often seen a tennis ball, struck with an oblique racket, describe such a curve line. For, a circular as well as a progressive motion being communicated to it by that stroke, its parts on that side, where the motions conspire, must press and beat the contiguous air more violently than on the other, and there excite a reluctancy and reaction of the air proportionably greater. And for the same reason, if the rays of light should possibly be gl bodies, and by their oblique passage out of on dium into another acquire a circulating motion, ought to feel the greater resistance from the an æther, on that side where the motions conspire thence be continually bowed to the other. But, withstanding this plausible ground of suspicion, I came to examine it, I could observe no such c in them. And besides (which was enough for m pose) I observed, that the difference between the l of the image and diameter of the hole, through the light was transmitted, was proportionable to distance.

" The gradual removal of these suspicions, at l led me to the experimentum crusis, which was t took two boards, and placed one of them close b the prism at the window, so that the light migh through a small hole, made in it for the purpos fall on the other board, which I placed at about l distance, having first made a small hole in it als some of that incident light to pass through. T placed another prism behind this second board, s the light trajected through both the boards, migh throught that also, and be again refracted before rived at the wall. This done, I took the first pri my hand, and turned it to and fro slowly about its so much as to make the several parts of the imag on the second board, successively pass through th in it, that I might observe to what places on th the second prism would refract them. And I sa the variation of those places, that the light tend that end of the image, towards which the refract the first prism was made, did in the second prism a refraction considerably greater than the light tend the other end. And so the true cause of the length image was detected to be no other, than that ligh sists of rays differently refrangible, which, witho respect to a difference in their incidence, were, acc to their degrees of refrangibility, transmitted to divers parts of the wall.

" When I understood this, I left off my afo glass works; for I saw, that the perfection of teles was hitherto limited, not so much for want of g truly figured according to the prestriptions of authors (which all men have hitherto imagined because that light itself is a heterogeneous mixt differently refrangible rays. So that, were a gl exactly figured, as to collect any one sort of ray one point, it could not collect those also into the point, which, having the same incidence upon the medium, are apt to suffer a different refraction. I wondered, that seeing the difference of refrang was so great, as I found it, telescopes should arr that perfection they are now at. For measurin refractions in one of my prisms, I found, that supp the common sine of incidence upon one of its was 44 parts, the sine of refraction of the utmos on the red end of the colours, made out of the into the air, would be 68 parts, and the sine of refr of the utmost rays on the other end 69 parts: so the difference is about a 24th or 25th part of the refraction; and, consequently, the object-glass o telescope cannot collect all the rays which come one point of an object, so as to make them conv its focus in less room than in a circular space, diameter is the 50th part of the diameter of its aperture; which is an irregularity, some hundreds of times greater than a circularly figured lens, of so small a section as the object-glasses of long telescopes are, would cause by the unfitness of its figure, were light uniform.

"This made me take reflections into consideration, and finding them regular, so that the angle of reflection of all sorts of rays was equal to their angle of incidence; I understood that by their mediation, optic instruments might be brought to any degree of perfection imaginable, provided a reflecting substance could be found, which would polish as finely as glass, and reflect as much light as glass transmits, and the art of communicating to it a parabolic figure be also attained. But there seemed very great difficulties, and I have almost thought them insuperable, when I further considered, that every irregularity in a reflecting superficies makes the rays stray 5 or 6 times more out of their due course, than the like irregularities in a refracting one: so that a much greater curiosity would be here requisite, than in figuring glasses for refraction.

"Amidst these thoughts, I was forced from Cambridge by the intervening plague, and it was more than two years before I proceeded further. But, then, having thought on a tender way of polishing, proper for metal, whereby, as I imagined, the figure also would be corrected to the last, I began to try what might be effected in this kind, and by degrees so far perfected an instrument (in the essential parts of it like that I sent to London), by which I could discern Jupiter's four concomitants, and showed them divers times to two others of my acquaintance. I could also discern the moon-like phase of Venus, but not very distinctly, nor without some niceness in disposing the instrument.

"From that time I was interrupted till this last autumn, when I made the other. And as that was sensibly better then the first (especially for day objects), so I doubt not, but they will be still brought to a much greater perfection by their endeavours, who, as you inform me, are taking care about it at London.

"I have sometimes thought to make a microscope, which in like manner should have, instead of an object glass, a reflecting piece of metal. And this I hope they will also take into consideration. For those instruments seem as capable of improvement as telescopes, and perhaps more, because but one reflective piece of metal is requisite in them.

"But to return from this digression, I told you, that light is not similar, or homogeneal, but consists of difform rays, some of which are more refrangible than others: so that of those, which are alike incident on the same medium, some shall be more refracted than others, and that not by any virtue of the glass, or other external cause, but from a predisposition, which every particular ray has to suffer a particular degree of refraction.

"I shall now proceed to acquaint you with another more notable difformity in its rays, wherein the origin of colours is unfolded: concerning which, I shall lay down the doctrine first, and then, for its examination, give you an instance or two of the experiments, as a specimen of the rest.—The doctrine you will find comprehended and illustrated in the following propositions:—

"1. As the rays of light differ in degrees of refrangibility, so they also differ in their disposition to exhibit this or that particular colour. Colours are not qualifications of light, derived from refractions, or reflections of natural bodies (as it is generally believed), but original and connate properties, which in divers rays are diverse. Some rays are disposed to exhibit a red colour, and no other; some a yellow, and no other; some a green and no other; and so of the rest. Nor are there only rays proper and particular to the more eminent colours, but even to all their intermediate gradations.

"2. To the same degree of refrangibility ever belongs the same colour, and to the same colour ever belongs the same degree of refrangibility. The least refrangible rays are all disposed to exhibit a red colour, and contrarily, those rays which are disposed to exhibit a red colour, are all the least refrangible: so the most refrangible rays are all disposed to exhibit a deep violet-colour, and contrarily, those which are apt to exhibit such a violet-colour, are all the most refrangible. And so to all the intermediate colours, in a continued series, belong intermediate degrees of refrangibility. And this analogy betwixt colours, and refrangibility, is very precise and strict: the rays always either exactly agreeing in both, or proportionally disagreeing in both.

"3. The species of colour, and degree of refrangibility proper to any particular sort of rays, is not mutable by refraction, nor by reflection from natural bodies, nor by any other cause, that I could yet observe. When any one sort of rays has been well parted from those of other kinds, it has afterwards obstinately retained its colour, notwithstanding my utmost endeavours to change it. I have refracted it with prisms, and reflected it with bodies, which in day-light were of other colours; I have intercepted it with the coloured film of air interceding two compressed plates of glass; transmitted it through coloured mediums, and through mediums irradiated with other sorts of rays, and diversely terminated it; and yet could never produce any new colour out of it. It would, by contracting or dilating, become more brisk, or faint, and by the loss of many rays, in some cases very obscure and dark; but I could never see it change in specie.

"4. Yet seeming transmutations of colours may be made, where there is any mixture of divers sorts of rays. For, in such mixtures, the component colours appear not; but, by their mutual allaying each other, constitute a middling colour. And, therefore, if by refraction, or any other of the aforesaid causes, the difform rays, latent in such a mixture, be separated, there shall emerge colours different from the colour of the composition; which colours are not new generated, but only made apparent by being parted: for if they be again entirely mixed and blended together, they will again compose that colour which they did before separation. And for the same reason, transmutations made by the convening of divers colours are not real; for when the difform rays are again severed, they will exhibit the very same colours which they did before they entered the composition; as you see, blue and yellow powders, when finely mixed, appear to the naked eye green, and yet the colours of the component corpuscles are not thereby really transmuted, but only blended. For, when viewed with a good microscope, they still appear blue and yellow interspersedly.

"5. There are, therefore, two sorts of colours. The one original and simple, the other compounded of these. The original, or primary colours are red, yellow, green, blue, and a violet-purple; together with orange, indigo, and an indefinite variety of intermediate gradations.

“ 6. The same colours in specie with these primary ones may be also produced by composition: for a mixture of yellow and blue makes green; of red and yellow makes orange; of orange and yellowish green makes yellow. And in general, if any two colours be mixed, which in the series of those, generated by the prism, are not too far distant one from another, they by their mutual alloy compound that colour, which in the said series appears in the midway between them. But those which are situated at too great a distance, do not so. Orange and indigo produce not the intermediate green, nor scarlet and green the intermediate yellow.

“ 7. But the most surprising and wonderful composition was that of whiteness. There is no one sort of rays which alone can exhibit this. It is ever compounded, and to its composition are requisite all the aforesaid primary colours, mixed in a due proportion. I have often with admiration beheld, that all the colours of the prism being made to converge, and thereby to be again mixed as they were in the light before it was incident upon the prism, reproduced light, intirely and perfectly white, and not at all sensibly differing from a direct light of the sun, unless when the glasses I used, were not sufficiently clear; for then they would a little incline it to their colour.

“ 8. Hence, therefore, it comes to pass, that whiteness is the usual colour of light; for light is a confused aggregate of rays indued with all sorts of colours, as they are promiscuously darted from the various parts of luminous bodies. And of such a confused aggregate, as I said, is generated whiteness, if there be a due proportion of the ingredients; but if any one predominate, the light must incline to that colour; as it happens in the blue flame of brimstone; the yellow flame of a candle; and the various colours of the fixed stars.

“ 9. These things considered, the manner how colours are produced by the prism, is evident. For, of the rays constituting the incident light, since those which differ in colour, proportionably differ in refrangibility, they, by their unequal refractions, must be severed and dispersed into an oblong form in an orderly succession, from the least refracted scarlet, to the most refracted violet. And for the same reason it is, that objects, when looked upon through a prism, appear coloured. For the difform rays, by their unequal refractions, are made to diverge towards several parts of the retina, and there express the images of things coloured, as in the former case they did the sun's image upon a wall. And by this inequality of refractions, they become not only coloured, but also very confused and indistinct.

“ 10. Why the colours of the rainbow appear in falling drops of rain, is also from hence, evident. For, those drops which refract the rays disposed to appear purple, in the greatest quantity to the spectator's eye, refract the rays of other sorts so much less, as to make them pass beside it; and such are the drops on the inside of the primary bow, and on the outside of the secondary or exterior one. So those drops, which refract in greatest plenty the rays apt to appear red, toward's the spectator's eye, refract those of other sorts so much more, as to make them pass beside it; and such are the drops on the exterior part of the primary, and interior part of the secondary bow.

“ 11. The odd phænomena of an infusion of lignum nephriticum, leaf gold, fragments of coloured glass, and some other transparently coloured bodies, appearing in one position of one colour, and of another in another, are on these grounds no longer riddles. For, those are substances apt to reflect one sort of light, and transmit another; as may be seen in a dark room, by illuminating them with similar or uncompounded light. For, then they appear of that colour only, with which they are illuminated, but yet in one position more vivid and luminous than in another, accordingly as they are disposed, more or less, to reflect or transmit the incident colour.

“ 12. From hence, also, is manifest the reason of an unexpected experiment, which Mr. Hook, somewhere in his micrography, relates to have made with two wedge-like transparent vessels, filled the one with red, the other with a blue liquor: namely, that though they were severally transparent enough, yet both together became opaque; for, if one transmitted only red, and the other only blue, no rays could pass through both.

“ 13. I might add more instances of this nature; but I shall conclude with this general one, that the colours of all natural bodies have no other origin than this, that they are variously qualified to reflect one sort of light in greater plenty than another. And this I have experimented in a dark room, by illuminating those bodies with uncompounded light of divers colours. For, by that means, any body may be made to appear of any colour. They have there no appropriate colour, but ever appear of the colour of the light cast upon them, but yet with this difference, that they are most brisk and vivid in the light of their own day-light colour. Minium appears there of any colour indifferently, with which it is illustrated, but yet most luminous in red; and so bise appears indifferently of any colour with which it is illustrated, but yet most luminous in blue. And, therefore, minium reflects rays of any colour, but most copiously those indued with red; and, consequently, when illustrated with day-light, that is, with all sorts of rays promiscuously blended, those qualified with red shall abound most in the reflected light, and by their prevalence cause it to appear of that colour. And for the same reason bise, reflecting blue most copiously, shall appear blue by the excess of those rays in its reflected light; and the like of other bodies. And that this is the entire and adequate cause of their colours, is manifest, because they have no power to change or alter the colours of any sort of rays, incident apart, but put on all colours indifferently, with which they are enlightened.

“ These things being so, it can be no longer disputed whether there be colours in the dark, nor whether they be the qualities of the objects we see; no, nor, perhaps, whether light be a body. For, since colours are the qualities of light, having its rays for their entire and immediate subject, how can we think those rays qualities also, unless one quality may be the subject of and sustain another; which, in effect, is to call it substance. We should not know bodies for substances, were it not for their sensible qualities; and the principal of those being now found due to something else, we have as good reason to believe that to be a substance also.

“ Besides, whoever thought any quality to be a heterogeneous aggregate, such as light is discovered to be.

commended one which is much more simple. In his reflecting telescope, the rays from the object first strike on the whole of the concave and well-polished bottom of the tube, whence they are reflected to a plane mirror, which convey them to the eye of the observer, by means of a lens adapted to an aperture made in the side of the tube. In this disposition of the mirrors, however, objects are not seen directly; a lateral telescope, therefore, has been added, to direct the instrument towards the object which the observer wishes to contemplate.

Cassegrain's telescope, Gregory's modified.

In 1672, M. Cassegrain published in the French memoirs, a description of a reflecting telescope, which he pretended was invented prior to that of Newton. That, however, is not correct: nor, indeed, is Cassegrain's telescope any thing else than James Gregory's a little disguised, by making the small mirror convex, instead of concave, as Gregory proposed. This change, as Newton showed (Phil. Trans. May 4, 1672), would be productive of disadvantages, where its author expected advantages.

Newton and Gregory had a friendly correspondence in reference to their respective inventions. Each of these was a man of genius, and each knew and highly estimated the talents of the other. Their letters on this subject (preserved in the supplement to Dr. David Gregory's "Catoptrics and Dioptrics") are fine specimens of the candour, urbanity, and mildness, with which scientific discussions ought always to be conducted.

Hadley. A. D. 1723. Short. 1734.

The constructions of both the Gregorian and the Newtonian telescopes, were much improved by Mr. Hadley, the inventor of the reflecting quadrant. And about the year 1734, Mr. James Short, of Edinburgh, brought the Gregorian telescope to very great perfection. But these are points to which we can but just allude.

Alhazen's problem, solved by Huygens, Slusius, and Robins. A. D. 1673. A. D. 1761.

The invention of reflecting telescopes, caused the minds of investigators to revert to a problem proposed long ago by Alhazen, which was this:—"Having given a concave or a convex speculum, as also the place of the eye and the visual point, to find the point of reflexion." In this problem, if reduced to geometrical terms, it is required in a given circle whose centre is A, and radius AP, to find some point P, such that drawing the lines PE, PB, to the given points E, B, unequally distant from the centre A, the line AP produced, may bisect the angle EPB." This admits of various cases, most of which have been investigated and elegantly solved, with practical applications, by Huygens and Slusius, in the Philosophical Transactions for 1673. The general construction, by means of the hyperbola, is also most neatly and perspicuously effected by Robins, in the Remarks on Dr. Smith's Optics, in the second volume of his Mathematical Tracts.

Halley, on the rainbow. A. D. 1700.

The first who exhibited a correct theory of the rainbow, was Descartes. He showed that the first rainbow, or that which is most usually seen in the heavens, is formed by one reflection and two refractions of the sun's rays falling upon drops of rain; that the second rainbow is formed by the rays falling upon drops of rain, and emerging after two refractions and two reflections; and that so we might go on to the third, fourth, or any other iris, which are made successively when the rays emerge out of the drops after three, four, or more reflections. Such was the general theory. In 1700, Dr. Halley proposed to determine the colours and diameter of the rainbow from the given ratio of refraction; and the contrary. He demonstrates that in all the bows there is this general rule, that the excess of four, or five, or more refracted angles (the number of reflections increased by unit) above one angle of incidence, must be a maximum. That maximum being doubled, is always the distance of the iris from the point opposite to the sun, when the number of reflections is odd. But, if that number be even, the double of the said maximum angle is the distance of the iris from the sun itself. Hence, by elegant constructions and theorems, he shows how to determine the dimensions of the successive rainbows in any liquid whose refraction is known; and, conversely, knowing the angles under which colours, by refraction from liquid drops, are exhibited, he shows how to determine the refraction of the liquid employed in the experiment. Admitting the ratio of Descartes, Halley shows that the primary rainbow is distant from the point that is opposite to the sun 41° 30′, the secondary rainbow 55° 55′; and that the third and the fourth (if they can be seen at all) will appear at 40° 20′ and 45° 30′ respectively, from the sun itself. More compact and perspicuous elucidations of the nature and cause of the rainbow, have been given by different writers since the time of Halley; but we recollect none which contain more curious or useful results.

Among the different topics of optical research, one of the most difficult, as well as most interesting, relates to the nature of vision itself. Ever since the time of Baptista Porta, the eye has been justly considered as a natural optical instrument, or *camera obscura*, in which pictures of the external objects are pourtrayed upon the retina, by rays introduced through the aperture of the pupil. This is beautifully demonstrated by the famous discovery of Scheiner. If we take the eye of an ox, recently killed, and strip the sclerotic coat with the choroides from its posterior portion, carefully preserving the retina as it lies upon the vitreous humour, and place the eye in a suitable aperture in the window shutter of a darkened chamber, with the corner outwards, a transparent miniature painting of the external landscape, in all its variety of figures and colours, will be exhibited upon the retina. Such was Scheiner's experiment; in which, however, the whole scene appears completely inverted.

This inverted appearance of the picture on the retina, though evidently agreeable to the general laws of Optics, involves an important question: How comes it to pass, since they appear *invertedly* upon the retina of another eye, that every person perceives the objects of vision that are in his own eye, in a different, or *erect* position? This question, and others, in reference to the apparent place of the object, the limits of distinct and indistinct vision, &c. have much exercised the talents of mathematicians, physiologists, and metaphysicians. But after all the attention which has been paid to them by Kepler, Descartes, Barrow, Berkeley, Picard, Le Cat, Mariotte, Pecquet, Reid, Porterfield, Jurin, Robins, Young, Hosack, Horn, and others, there still remain doubts as to the most usual explications. This, however, is not the place to detail them: the theories and their defects will be shown in our treatise.

Newton, in attempting to explain the reflection of light, remarks, that many of the rays which impinge

Here the first reflection under the angle 3° 35′, was nearly as vivid as from mercury itself. It is thus with all plane bodies whatever. They are all very good reflectors, when the incident rays make very small angles with their surface; but their reflecting force becomes rapidly weakened as the incidence of the rays becomes elevated, and approaches the perpendicular. In this, they evidently differ from bodies which have a strong reflecting force; for, in them, the intensity of the reflected light experiences only slight variations under different incidences. On mercury, for example, and on the reflectors of telescopes, the total extent of this variation between 0° and 90° of incidence, does not exceed $\frac{1}{5}$ or $\frac{1}{6}$. For the incidence of 21° counted from the surface, mercury will reflect about 637 rays out of 1,000: in the reflection of all other angles, the reflected rays vary between 700 and 600. This metal, therefore, although it be probably the best reflector of all bodies, extinguishes in its substance, according to Bouguer, more than a quarter of the light which strikes it; and the corresponding absorption is much greater with regard to those bodies, which are not so good reflectors as mercury.

Bouguer also made researches into the comparative intensities of light reflected by the moon and planets, with respect to each other, or with regard to its own light at different elevations above the horizon. His estimates in this respect have been much corrected by Professor Leslie, and other philosphers.

Count Rumford. A. D. 1794.

Many years after the time of Bouguer, Count Rumford devoted much time and ingenuity to the subject of the comparative intensities of light, and the best means of ascertaining them experimentally. His principal apparatus is described in the eighty-fourth volume of the Philosophical Transactions, under the name of *Photometer*. The first object, in his experiments, was to determine the certainty of the assumed law of variation of intensity; namely, the inverse ratio of the squares of the distances. This he found to obtain. Then he ascertained that the air did not resist light: after which he investigated the loss of light in its passage through plates or panes of different glass; the loss of light in its reflection from the surface of a plane glass mirror; the relative quantities of bees-wax, tallow, olive oil, rape oil, and linseed oil, consumed in the production of light; the relative quantities of oil consumed, and of light emitted, by lamps of different kinds; the fluctuations of the light emitted by candles, and the transparency of flame. Several of his results are very curious; but they are not susceptible of the kind of abridgment which will render them admissible into this history.

Photometer.

Achromatic telescopes invented by Mr. C. M. Hall.

One of the principal defects of refracting telescopes has been long known to be that which arises from the aberration of refrangibility; in consequence of which the images of objects, as seen through these instruments, appear tinged with colours which do not appertain to the objects themselves. This defect has been in a great measure removed, by the invention of achromatic telescopes; an invention, the honour of which has been ascribed to different individuals of different nations. The truth, we are persuaded, is, that this kind of telescopes was invented by Mr. Chester More Hall (a private gentleman of Essex), in the year 1729. After maturely deliberating upon the nature of the different humours of the eye, he concluded that they were endowed with such properties, and so placed as to correct the different refrangibility of light. He then conceived, that if he could find substances which had such properties as he supposed those humours to possess, he should be enabled to construct an object-glass that would exhibit objects colourless. After many experiments, he had the good fortune to find those properties in two different kinds of glass, which, by dispersing the rays of light in different directions, effected the achromatism. About 1733, he completed several achromatic object-glasses, which bore an aperture of $2\frac{1}{2}$ inches, though the focal length did not exceed 20. Some of these object-glasses are still in existence; and achromatic telescopes, of Mr. Hall's construction, are well known to have been on sale in the year 1754.

A. D. 1729.

In the year 1747, the celebrated Leonard Euler, by whose labours many departments of mathematics and natural philosophy have been considerably enriched, published, in the Berlin Memoirs, a short paper, in which he deduced, from a fanciful optical principle which he had devised, the composition of a lens formed according to certain proportions with glass and water, which should entirely remove all extraneous colours, whether occasioned by the unequal refraction of the several rays, or by spherical aberration. John Dollond, who did not see Euler's paper till the beginning of 1752, called in question the truth of Euler's theorem, and assigned his reasons in a letter addressed to Mr. James Short, and published in the 48th volume of the Philosophical Transactions. Euler immediately replied to the animadversions of the English artist; and the dispute soon excited the attention of other philosophers. Klingenstierna, a Swedish geometer of some eminence, sent to Dollond, in 1754, some investigations, tending to show that the Newtonian principle to which Dollond had appealed, was, in some extreme cases, incompatible with the phenomena, and therefore ought not to be received as an undoubted law of nature. This stimulated Dollond to recur to the test of experiment; and, after a series of researches of this kind, conducted under several interruptions, he at length discovered as well the cause of the mistake which had been imputed to Newton, as the means of correcting in a very high degree the defect in telescopes of the dissipation of the rays that arises from their unequal refrangibility. He combined together, so as to appear but one, two lenses, one of flint glass, the other of crown glass, which have different refractive and dispersive powers, and which, by means of the combination, are made to correct each other's effects and he found that, by rightly proportioning the conspiring and counteracting curvatures, it was possible to remove almost entirely the errors that arise from spherical aberration. "Notwithstanding (says he, in concluding his paper to which the Royal Society voted the Copleian medal) so many difficulties as I have enumerated, I have, after numerous trials, and a resolute perseverance, brought the matter at last to such an issue, that I can construct refracting telescopes with such apertures and magnifying powers, under limited lengths, as, in the opinion of the best and undeniable judges, who have experienced them, far exceed any thing that has been produced, as representing objects with great distinctness, and in their true colours."

Many investigations of mathematicians contributed to the perfection of this invention. Euler, Bevis, Clair-

of the substance in which the property is destroyed, Dr. Brewster called the *depolarizing* axis; and the axis in which it is not altered, the *neutral* axis.

8. Mica and topaz, while they possess, in common with other bodies, the neutral and depolarizing axes, have also axes of a different kind. Each depolarizing axis of the mica is accompanied with an *oblique neutral axis*, while the neutral axis between the two common depolarizing axes, has an *oblique depolarizing* axis.

9. When the images of a luminous object are depolarized by the mica, they exhibit, by a gentle inclination of the plate, very singular alternations of the prismatic colours. The same colours were observed in the topaz; and in a more perfect manner in a rhomboid of iceland spar, which exhibited some new phenomena.

10. Light suffers a peculiar modification when reflected from the oxidated surface of polished steel, from which it was inferred that the oxide is a thin transparent film.

11. Light is partially polarized when it is reflected from polished metallic surfaces.

12. The light that is reflected from the clouds, the blue light of the sky, and the light which forms the rainbow, are all polarized.

13. From a variety of experiments, it appears, that bodies exert a different action upon the different coloured rays; oil of cassia, for example, having the least, and sulphuric acid the greatest, action upon *green* light.

14. The existence of a third, or of a *tertiary spectrum*, has been established by numerous experiments; and Dr. Brewster has shown how to employ this spectrum as a measure of the action, which different bodies exercise upon the different coloured rays.

These discoveries of Dr. Brewster, were made during a time of suspended intercourse between Great Britain and the continent. Soon afterwards, however, he learnt from the abridged account of the labours of the French Institute, for 1812, and from some other notices, that M. Arago had discovered the depolarization of light by transparent bodies; that Malus had ascertained before his death, that light was polarized by reflection from metallic surfaces; and that M. Biot had discovered the law which generalizes many of these and other facts.

Brewster, Biot, and Dr. Seebeck, have with much skill and assiduity continued their researches, independently of each other, in this new and inviting region of inquiry. The results at which they have arrived, approximate more and more to a systematic shape; and we hope that, by the time we arrive at the physical doctrine of light and colours in this general arrangement, we shall be able to present an accurate view of the curious facts thus developed, and of the theory which will best serve to connect them together.

Having thus brought down our history to the present period, it remains for us to remark, that, interesting as is this branch of science, there is scarcely a single treatise which the reader can consult, from which he will obtain a perspicuous and comprehensive view of Treatises. the present state of Optics. Among the works which may always be consulted with advantage, as exhibiting perspicuous representations of the science at the age in which they were written, are Barrow, Lectiones Opticæ, London, 1669; Jac. Gregory, Optica Promota, London, 1663; Grimaldi, Physico-Mathesis de Lumine, Coloribus, et Iride, Bononia, 1665; and, still more beneficially, Newton's Optics and Lectiones Opticæ, 1704. The performances of David Gregory, Emerson, and Martin, cannot now be regarded as of much importance; and a much more recent work, that of Dr. Wood, though neat and perspicuous as far as it goes, is too concise and incomplete to be intrinsically so valuable as some other treatises by that author. The two quarto volumes published at Cambridge, by Dr. Smith, convey much useful information, but not in a manner that can be commended for its perspicuity and order. Both that work, and its abridgment by Dr. Hipling, are now, we believe, out of print. Bouguer's Treatise on the Gradation of Light, already mentioned in this history, contains many curious and valuable experiments and deductions. But, probably, the best view of the modern state of Optics, though defective in arrangement, is that given in the third and fourth volumes of M. Biot's Traité de Physique.

We shall now proceed to exhibit, as succinctly as is consistent with perspecuity and utility, the mathematical principles of this science, referring the physical facts and several deductions from them, to our subsequent disquisitions on LIGHT and COLOURS, and simply introducing in this treatise so many of those facts as serve for the basis of the correct and received theory of Optics.

PART I.

SIMPLE OPTICS.

§ I. *Fundamental facts.*

1. *Preliminary definitions.*—A space which light traverses, is denominated a *medium*. 2. If that space be either absolutely void, or filled with matter of such a nature as opposes no obstacle to the motion of light, it is called a *free medium*. 3. If the space be occupied by some matter which, though it suffer light to pass, interposes some principle which modifies its motion, and lets it proceed with a facility varied by that principle, we call it a *diaphanous*, or *transparent medium*. 4. If the interposed matter be throughout the same, it is called a *homogeneous medium*: if the parts of the interposed matter differ in their nature, the medium is then called *heterogeneous*.

Here we assume the motion of light: it will not lon[illegible] remain a subject of mere assumption.

5. The principles which serve for the foundation o[illegible] optical science, must be drawn from experience. The[illegible] are facts which every one may realize: we may deduc[illegible] them all from an examination of the circumstances [illegible] the following experiment:

Let a chamber, or other apartment, be closed on a[illegible] sides, so that the light shall nowhere enter, except [illegible] one very small aperture. Then, if the day be clea[illegible], there will appear on the opposite wall of this apartme[illegible] (if it be smooth and white), images or pictures of a[illegible] the objects that are conveniently posited without, eac[illegible] depicted in its proper colour; that colour, however, le[illegible] brilliant in the picture than in the object. The pictur[illegible] of fixed or stationary objects, as of houses, trees, &[illegible] will appear unmoveable; while those of objects i[illegible] motion as of men, horses, oxen, &c. will appear to b[illegible] in motion. The whole will appear inverted; and th[illegible] for a reason which will be explained when we treat [illegible] Dioptrics. If the sun shine through the aperture, w[illegible]

Fig. 2. Thus, supposing rays of light to emanate from a radiant point P (fig. 2. Pl. I. OPTICS), and we for simplicity confine our attention to the cone of rays PR″r″ lying in one general direction. The same number of rays which at the distance PC fall upon the circle R r, will, at the distanee PC′ be diffused over the larger circle R′ r′, and at the distance PC″ over the still larger circle R″ r″. The intensities of light, therefore, as intercepted at those respective circles, will be inversely as the areas of those circles, that is, by the principles of geometry, inversely as the squares of the radii CR, C′R′, C″R″; that is, further, by similar triangles (PCR, PC′R′, PC″R″), inversely as the squares of the respective distances PC, PC′, PC″. The same reasoning may, evidently, be extended to pyramids of rays of any bases whatever.

13. This law of the intensity of light varying inversely as the squares of the distances from the luminous body, is that which was employed by Bouguer in the experiments (of which we made brief mention in the preceding history) for determining the intensities of different lights.

Clairaut, and many others, have also employed this law in solving the problem which requires *the determination, in a right line which joins two lights of different intensities, of the point of equal intensity.* A more general problem, of easy solution, upon the same principles, is the following:

Problem. PROBLEM.—Having given the position of two lights of known intensities, to determine the nature and equation of the surface, of which every point shall be equally illuminated by the two lights.

Fig. 3. Let A and B (fig. 3) be the two points at which the lights are placed. Assume for the axis of the ordinates denoted by x the line AB which joins those points, and one of those points for the origin. Let $a = AB$, be the absciss of the other point, and let m and n be the intensities of the two lights at any assumed unit of distance. For a point in space of which the rectangular co-ordinates are x, y, and z, the intensity of the light proceeding from the first luminous point will be expressed by $\frac{m}{x^2 + y^2 + z^2}$, while that of the light from the second luminous point will be expressed by $\frac{n}{(a - x)^2 + y^2 + z^2}$. Consequently, for a point of the surface sought, we shall have the equation

$$\frac{m}{x^2 + y^2 + z^2} = \frac{n}{(a - x)^2 + y^2 + z^2}.$$

Reducing this equation to its simplest expression, and making $x = x' - \frac{m\,a}{n - m}$, it becomes

$$x'^2 + y^2 + z^2 = \frac{a\,n\,m}{n - m}.$$

The surface required is, therefore, a sphere of which the radius $= \frac{a}{n - m} \sqrt{m\,n}$, and whose centre has for an absciss the quantity $\frac{m\,a}{m - n}$.

If $m = 2\,n$, then the centre has for its absciss $2\,a$, and the radius $= a \sqrt{2}$: describing from B, as a centre, the circumference ADE, and from E, as a centre, a sphere passing through D (BD being perpendicular to AE) this sphere will be the surface sought.

In the case where $m = n$ the radius is infinite, and so, likewise, is the abscissa from the centre: thus is indicated, instead of a sphere, a plane at a finite distance. Indeed, if we make $m = n$ in the first equation above, we shall have $x = \frac{1}{2}\,a$. Consequently, the required surface in this particular case, becomes a plane perpendicular to the middle of the right line AB; as is, otherwise, sufficiently evident.

14. *Remark.*—Though the *force* of light decreases as rapidly as we have seen, in passing from its origin, yet the brightness of the same luminous body, seen at any distance whatever, through a perfectly free medium, and with the same opening of the pupil, is constant.

For the brightness is the same, when the same quantity of light falls upon the same portion of the retina of the eye (as will be seen when we treat of vision); that is, when the density of light upon the retina is the same. But the density of light falling on the pupil, is reciprocally as the square of the distance; and the apparent magnitude of the object is (from the principles of geometry) reciprocally as the square of the distance. Therefore, the quantity of light on the pupil is as the apparent magnitude of the object; that is, as the space taken up on the retina. Consequently, the quantity of light being as the space taken up, the *brightness* is given, or constant.

15. It is true, however, in our actual experience, that the same objects appear so much the less bright as they are more remote *(cæteris paribus)*, and that at length they cease to be visible. But this obscurity arises from the circumstance of their passing through the air, which is a medium sufficiently dense, especially near the surface of the earth, and which, of consequence, dissipates a prodigious quantity of the rays, in the interval between a remote object and our eyes.

16. PROP. III.—The effect of a uniformly dense diaphanous medium, is to produce a diminution in a geometrical progression of the intensity of light propagated by any rays whatever.

Imagine that the uniform density of a medium, for example, a piece of glass, consists in this, that each of the numerous small solid portions of glass which stops the light in its passage, is the $\frac{1}{n}$th part of the volume of the glass. Imagine also, that the thickness of this glass is divided into laminæ, each equal in thickness to the diameter of one of these solid parts, which we suppose to be respectively equal. It is evident, that if a bundle of rays of light, however disposed, and regarded as unity, fall upon this glass, the $\frac{1}{n}$th part of these rays will be stopped in passing the first lamina; so that there will issue only $1 - \frac{1}{n}$, or $\frac{n - 1}{n}$. And, since the second lamina is homogeneous, and equal to the former, it will stop in like manner the $\frac{1}{n}$th part of the rays which arrive at it; that is, $\frac{1}{n}$ of $\frac{n - 1}{n}$, or $\frac{n - 1}{n\,n}$. There will, therefore, issue from the second lamina, the portion $\frac{n - 1}{n} - \frac{n - 1}{n\,n} = \frac{n^2 - 2\,n + 1}{n^2} = \frac{(n - 1)^2}{n^2}$. In like manner it may be shown, that there will issue from the third lamina, the portion $\frac{(n-1)^3}{n^3}$; from the fourth, the portion $\frac{(n-1)^4}{n^4}$, and so on. This is evidently a geometrical progression.

Remark.—This law of the density of light, in a diaphanous medium, is the origin of the *lucigrade* of father François-Marie, an instrument composed of plates of glass.

He observes the light through these plates; and, adding plate after plate till the light can no longer be distinguished, determine the ratio of the intensities from the number of equally thick plates.

Now, let m be the number of plates of glass necessary to intercept the whole of that light, whose intensity is denoted by I, and n the number of such plates necessary totally to intercept the other light of intensity K. Let $\frac{1}{p}$ be the portion of light intercepted by each plate. The portion of light which will arrive at the eye, after having traversed m plates, will be $\mathrm{I}\left(\frac{p-1}{p}\right)^m$, and that of the light K which will arrive at the eye, after passing through n plates, will be $\mathrm{K}\left(\frac{p-1}{p}\right)^n$. Making $\frac{p-1}{p} = k$, the light which will arrive at the eye, after having passed through m and n plates respectively, will be $\mathrm{I}\,k^m$ and $\mathrm{K}\,k^n$. But as, in each case, the intensity thus reduced is the limit of visible light, we may suppose it equal to a constant quantity k. Thus we shall have $\mathrm{I}\,k^m = \mathrm{K}\,k^n = a$. Consequently,

$$\mathrm{I} = \frac{a}{k^m}, \text{ and } \mathrm{K} = \frac{a}{k^n};$$

and the ratio of the intensities $\mathrm{I} : \mathrm{K} :: k^n : k^m$.

17. Prop. IV.—In a diaphanous medium of uniform density, the intensity of light which diverges from a luminous point taken in that medium, will decrease according to the series $\frac{n-1}{n}$, $\frac{(n-1)^2}{4n^2}$, $\frac{(n-1)^3}{9n^3}$, $\frac{(n-1)^4}{16n^4}$, $\frac{(n-1)^5}{25n^5}$, &c. in which n expresses the portion of rays of light stopped by the medium at each interval of distance from the luminous point.

For, at the end of each successive interval of distance, by reason of the divergence (art. 13), the intensity is as the corresponding fraction in the series 1, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$, $\frac{1}{25}$, &c. and, by reason of the uniform density of the medium, it is (art. 16) as the corresponding terms of the series $\frac{n-1}{n}$, $\frac{(n-1)^2}{n^2}$, $\frac{(n-1)^3}{n^3}$, $\frac{(n-1)^4}{n^4}$, &c. The fractions in these respective series blended, term by term, give those in the proposition.

18. If, for example, it is gathered from experiment that, at 403 yards distance, light loses $\frac{1}{100}$ of its rays, at the mean density of the air near the earth's surface, it will follow that the intensity of the light by which we see an object at a quarter of a mile distance, is to that by which we see it at the distance of a mile, as $\frac{96059601}{100000000}$ to $\frac{90}{100}$, or, very nearly, as 33 to 2.

Light lost in traversing the atmosphere.

19. *Remark* I. As the light which comes to us from the heavenly bodies passes through the atmosphere which surrounds the earth on all sides, the quantity of light stopped and absorbed by it will vary with the portion, greater or less, which it has to pass through. Let ABC (fig. 4) be a portion of the circumference of the earth, abc a concentric arc to represent the limit (as to sensible effects) of the atmosphere. Let DB be a ray of light which comes from the zenith, vertically, to an observer at B, EB a ray which comes obliquely, and FB another which comes horizontally. It is evident that the vertical ray DB has only the shortest distance, bB, of the atmosphere to pass through, while the oblique ray, EB, has to pass over a larger portion, GB, and the horizontal ray FB, a still larger portion, HB. From which it is manifest, without entering prematurely into the consideration of the effect of air of different densities, or of the refraction of the rays in passing through the atmosphere, that light in the direction EB will be more weakened in its progress through the atmosphere, than light in the direction DB; and light in the direction FB still more so.

Fig. 4.

Bouguer's experiments.

20. Bouguer, to whose researches we have more than once adverted, made several experiments in reference to this very point. Of 10,000 rays, which he supposes to come from a star, he finds, that when the star is vertical, 8,123 rays will reach the eye; at an elevation of 80 degrees, 8,098 rays; and at elevations of 70, 60, 50, 40, 30, 20, 10, 5, 3, and 1 degrees, the respective numbers of rays that reach the eye, are 8,016, 7,866, 7,624, 7,237, 6,613, 5,474, 3,149, 1,201, 454, and 7. This happens when the air is clear and free from dew or fog.

Hence we see why the sun and moon appear with greater faintness when near the horizon, than when at higher elevations. They also appear *larger;* but the reason of that phenomenon is not yet to be assigned.

21. *Remark* II. Though the light of luminous bodies which shine in the dark, as *lamps, torches,* &c. suffers a continual diminution by the extinguishing operation of the medium the farther it goes; yet, since they emit (art. 10) considerably more particles than are requisite for distinct and continuous vision, they are, notwithstanding the extinction of many rays, seen with comparative brightness at very great distances.

Velocity of light 200,000 miles per second.

22. *Remark* III. A property of light to which we have already alluded, viz. its *rapid propagation,* is made known to us by the observations of astronomers. It was by observations on the satellites of Jupiter that the actual velocity of light was appreciated. Roemer first noticed this, and Pound confirmed it. It has since been found by repeated experiments, that when the earth is exactly between Jupiter and the sun, the satellites of that planet appear eclipsed about 8¼ minutes *sooner* than they should, according to astronomical tables; and, on the contrary, that when the earth is in the opposite point of its orbit, those eclipses occur about 8¼ minutes *later* than the tables alone would have led us to expect them. Thus it appears, that light occupies about 16½ minutes in passing over a space equal to the diameter of the earth's orbit, that is, in passing over about 190 millons of miles. This gives us the astonishing velocity

of 200,000 miles per second! How exceedingly minute must the corpuscles be, that their impression upon the eye, and even upon substances generally, should not be, in all cases, injurious, and even dangerous!

Light emanating from a point.

23. Prop. V.—If rays of light, emanating from a *point*, pass through an orifice into a darkened apartment, and are received on a plane, parallel to that of the orifice, they will form on that plane a figure similar to that of the orifice, and so much the greater as it is more remote from the orifice.

For, in the case of this proposition, the luminous point is the vertex of a pyramid of light, the faces of which are determined by those rays which pass by the margin and the sides of the aperture: and the plane on which the rays are received, being, by supposition, parallel to that of the orifice, the luminous image, and the section of the said orifice, are analogous to parallel sections of a pyramid, which, from the principles of geometry, are known to be similar; and that most distant from the vertex of proportionally larger dimensions.

24. If the plane on which the rays are received, be posited obliquely to that of the orifice, the luminous figure upon it will still have as many sides as the orifice, but the figures will not be similar. If the orifice be circular, the figure received upon the oblique plane will be that of a conic section.

25. It is, further, evident that the luminous figure is nothing else than a compact group of images of the luminous point.

Light emanating in parallel rays.

26. Prop. VI.—When the light that comes from the sun, or from the full moon, passes through a small orifice of any figure whatever; if it be received on a plane parallel to that of the aperture and very near to it, there will be depicted upon it a luminous figure similar to that of the aperture: but if the light be received on a plane at a sufficient distance, the luminous figure will appear circular.

For the surface of the aperture may be regarded as composed of an infinitude of points which are like so many contiguous small holes, through each of which the rays of light pass that come from all points of the disc of the luminary. Each point of the surface of the aperture, therefore, is the vertex of a luminous cone whose base is the disc of the luminary: the axis of the cone is the ray which passes from the centre of the disc to that point, and the angle at the vertex of the cone is the angle which the luminary subtends [about 32 minutes for the sun, and about 31 for the moon; both, however, variable]: the rays of light passing beyond the orifice cross each other there, and form another luminous cone inverted with respect to the former, having the same vertex, the same axis, and the same vertical angle; but extended indefinitely beyond the orifice, opposite to the luminary. And since the breadth of the orifice is indefinitely small, with regard to its distance from the luminary, the axes of all the cones may be considered as parallel. Also, because of the smallness of the angle at the vertex of each cone, the slant sides of such cone will not, for a small distance, be distinguishable from the axes. Consequently, a plane, posited near the orifice, will receive the light from the luminary, as though it came by the axes only; and these, being parallel as to sense, are arranged in the same order as all the points of the surface of the orifice; and, therefore, the luminous figure on this plane, will be similar to the figure of the orifice itself.

But, 2dly, as the plane is removed further from the orifice, the slant sides of the luminous cones deviate more and more perceptibly from the axes, the cones themselves continually widen, so that at a considerable distance from the orifice the luminous figure is composed of all the bases of these cones, which are circles. The centres of these circles, determined by the intersection of the axes of the cones with the intercepted plane, are, in fact, arranged upon that plane similarly to the corresponding points of the surface of the orifice; but their circumferences are mutually blended one with another, in such manner as to constitute a figure that differs insensibly from a circle. Thus the seven circles, whose centres are A, B, C, D, E, F, G, (fig. 5) form a curvilinear heptagon, which, at a moderate distance, would be scarcely distinguishable from a circle: and an increased number of equal circles, drawn from different centres included within the same general space, would, evidently, still further diminish the irregularities of contour. Whence both parts of the proposition are manifest.

27. *Remark* I. It is, however, meant in the above, that the dimensions of the orifice are not much distorted, but approximate to regularity, though they are not circular. If the figure of the orifice be oblong, that of a parallelogram, for example, the luminous figure would be a parallelogram terminated at the two opposite ends by a semicircle. In general, all luminous figures thus caused by the sun, or by the moon, have their angles rounded off to a certain distance.

28. If the plane, spoken of in the proposition, be not parallel to that of the orifice, the luminous figure will be oval, because the bases of all the cones of lines will then be ellipses, and their mutual blending will constitute an oval outline.

29. If we stop up a part of the aperture, thus changing its figure, that of the luminous image will still remain the same; it will simply become less vivid, or, in some cases, a little smaller.

30. Hence we perceive how it happens, that when we are in a walk overshaded by a thick arch of trees, we sometimes see light passing from the sun through small openings between the leaves producing luminous *circles* on the ground.

31. If there be various orifices near to each other, *three*, for example, through which the solar light passes into a darkened apartment, we shall observe at first, at a certain distance, three distinct luminous circles: in proportion as the plane which intercepts the light is removed further from the apertures, the circumference of the luminous circles approach each other then they touch; afterwards become blended; and at length appear as though they were all united to form one circle.

§ III. *Properties of shadows.*

32. Prop. I.—An opaque body, partly illuminated casts a shadow that is bounded by right lines, and precisely opposite to the light.

For, since light is always propagated in right lines (6), the rays of light which pass contiguously to the limits of any body, pass on rectilinearly, and constitute the boundaries to the shadow behind the body that intercepts the light.

33. Prop. II.—The shadow of a body, partly illuminated, produces an obscurity so much the more evi-

dent as the light which illuminates the opposite part is the stronger.

For then the contrast of the light which passes in the vicinity of the shadow, is the more sensible and striking.

34. When the same body is exposed to several different lights, situated nearly on the same side, they cast on the opposite side as many different shadows, which become partially confounded near the base of the body: we may learn from this proposition, why the darkness of these shadows is so much the greater as there are a greater number confounded together.

35. Prop. III.—The shadow occasioned by the interposition of an opaque body in an illuminated medium, and received on a plane, is always terminated by a penumbra, increasing in extent, by reason of the magnitude of the body from which the light emanates, the distance of the opaque body from the plane which receives its shadow, and the obliquity of the shadow to that plane: and the intensity of the penumbra diminishes as it is more remote from the simple shadow.

Let AB represent the sun, or other body from which the light issues (fig. 6), ED an object placed upon the earth DI. Having drawn the rays BF, CG, AH, it is manifest that an eye which shall advance from I towards H, will perceive the whole of the luminous body. Arrived at H, it will begin to lose sight of the lower part of the body about A, and to lose the benefit of a portion of the illumination. Continuing to move on towards D, he will see less and less of the body AB; when arrived at G, for example, only the upper half of the disc will be visible; proceeding onwards, less and less will be seen; till, when the eye has arrived at F, it will have lost sight entirely of the body, and will enter the pure shadow which occupies the space FDE. Hence it follows, 1st, That we perceive the object less and less clearly as we approach the true shadow, the space HF being covered by a penumbra which gradually deepens from H to F, where the true shadow commences; 2dly, That in the triangle FEH, the side FH, which is a measure of the penumbra, is so much the greater, jointly as the opposite angle FEH (which measures the apparent diameter AB of the luminous object) is greater, as the distance ED of the extremity E of the body which casts the shadow, is greater from the plane DI which receives it, and as the right lines EH, EF, are more oblique. This reasoning may evidently be generalized to apply to all analogous cases.

36. Hence we see why the limit of the shadow of bodies illuminated by the sun is always confused, especially when the shadow is raised on a plane which is far from the body that causes it. And since the diameter of the sun appears at a medium under an angle of 32 minutes [the limits are 32′ 35″ in January, and 31′ 31″ in July], it is evident from the principles of trigonometry, that the magnitude FH of the penumbra of an object, is to the distance, EF, of the extremity E of the object from the commencement F of its pure shadow, as the sine of 32′ is to the sine of the angle EHD, that is, of the apparent elevation of the sun's lower limb above the plane DI which receives the shadow.

What is here said of the sun may also be extended to the moon, and, generally, to all luminous bodies which have a sensible diameter, and by which a penumbra is formed. Those which have simply a luminous point form no penumbra.

Lengths of shadows.

37. Prop. IV.—The lengths of the true shadows of bodies that present obstacles to the light which comes from the sun or moon, are directly as the heights of the bodies, and directly as the co-tangents of the elevations of the upper limb of the respective luminary above the plane which receives the shadow.

This is evident from fig. 6, in which DF, the dimension of the pure shadow, obviously increases with DE, the altitude of the opposing object, as it does also with the tangent of the angle DEF or the co-tangent of EFD to the radius DE.

38. Hence, having given two of these three things, viz. the angle of elevation of the upper limb of the luminary, the perpendicular altitude of an object above the plane from which we estimate the elevation of the luminary, and the length of the true shadow of that object (measured from the foot of the pendicular let fall from the object on the plane), we may determine the third by the resolution of a right-angled triangle.

Shadows received by globular bodies.

39. Prop. V.—If light emanating from a luminous globe, be projected upon an opaque globe that is larger than the former, the part which causes the illumination will be greater than, and the part which receives the illumination less than, a hemisphere. If the globe from which the light emanates be greater than the other, the contrary will take place. And, if both globes be equal, the half of the radiant globe will illuminate half of the opaque globe.

Fig. 7.

Suppose that light emitted from the whole surface of the globe B (fig. 7), illuminate the face which is towards it of the larger globe C; let the tangent planes which in the figure are projected into the lines LQ, KV, be drawn to touch both spheres. Then it is evident, that every part of the surface of the smaller sphere which lies between L and K towards C, communicates its portion of light towards the illumination of the larger globe; and, on the other hand, that no part of the surface of that larger globe, which is beyond the points P and O, can receive direct light from the smaller globe. If, now, perpendicularly to the right line BC, which joins the centres of the two globes, there be drawn the two diameters IH, NM, being projections of the great circles which divide each into its respective hemispheres, it will be manifest that the portion LRK of the smaller sphere, which illuminates the larger, exceeds a hemisphere; while the portion OSP, of the larger sphere, which receives illumination, is less than a hemisphere. The contrary will obviously be the case, if C be the luminous, and B the opaque globe. And, if C and B were equal in size, LP and KO would both be parallel to BC; BI and BK would coincide with BH and BI; CP and CO would coincide with CM and CN: and, consequently, the portion of one globe which tended to illuminate the other, and the portion of the second, which received the illumination, would both be hemispheres.

Deductions.

Remarks.—It is evident, from the similarity of the right-angled triangles LBH, PMC, KBI, OCN, that the axes IH, PM, KI, ON, being the measures of equal angles, contain an equal number of degrees; and consequently, the arc of the luminous globe, which measures its effective portion in illuminating the other, is the supplement to 360° of the arc which measures the illuminated portion of that second globe.

41. Hence, also, the portion of the opaque globe which remains dark, is *similar* to the portion of the luminous globe which communicates illumination to the other globe, and *vice versa.*

42. And because of the similar triangles ABL, BLH, ABK, BKI, we have $LBH = LAB$, $KBI = KAB$, $LAK = LBH + KBI = PCM + OCN$: consequently, the difference between the illuminated part and that which remains dark, is measured by the angle of inclination LAK of the tangent rays.

43. A luminous globe illuminates the half of an equal globe, at whatever distance they may be the one from the other; but a globe which throws light upon a smaller one, illuminates so much the greater portion as it is nearer, and reciprocally.

45. The shadow of a globe that is illuminated by an equal globe, is cylindrical and indefinitely long. The shadow of a less globe, illuminated by a greater (as of the earth, or of the moon, illuminated by the sun), is a cone of finite length, and computable when the diameters and distances of the globes are known. And lastly, the shadow of a globe, illuminated by one that is smaller, extends itself indefinitely in a truncated cone, perpetually enlarging.

46. These properties of shadows, and others, which readily flow from them, will find their application partly in the doctrine of eclipses in Astronomy, and partly in perspective.

§ IV. *Of the nature and properties of light, with regard to vision and to colours.*

Light, with reference to vision and colour.

47. It is not our intention to attempt the full explication of vision (because it is not possible), till we have exhibited the principles of Catoptrics and Dioptrics: we may then describe the eye with a certain degree of minuteness, as an optical instrument, and speak of common vision distinct, and indistinct; as well as of vision when aided by lenses, mirrors, &c. At present, we shall only consider vision in its obvious analogy to the phenomena of the *camera obscura;* and of these general properties of colours, which connect them with the mathematical part of Optics; leaving the physical properties of light to be traced in a more suitable portion of our work.

48. In the general terms of analogy, we may say, that, to us, the eye answers a similar purpose with a darkened apartment, or *camera obscura.* The pupil is a small orifice through which the rays of light pass, and from whence, after mutually crossing each other, they go on to the bottom of the eye; where, on a suitable membrane, they depict inverted images of all the objects which are exposed to our view in such manner, that the diameters of the images thus formed, are nearly proportional to the angles formed at the pupil of the eye, by the two rays which issue from the two extremities of the object, when those angles are small; or, which amounts to the same, the diameters of the images of the same object, as the distance of the object is less; and *vice versa.* Though the images of direct objects appear inverted, they, notwithstanding, excite the idea of *directness* of position, for reasons which we shall attempt to explain when we treat of vision.

49. Although we cannot give a complete explication of the manner in which light forms the images of
objects in the eye, till we have established th
ciples of Dioptrics, yet we may here explain
what experience teaches us, as to the manner in
light acts on the organ of sight; and the ideas
are excited by that action.

50. Light is constituted of a prodigious qua
moleculæ or corpuscles, distinguished one fro
ther, of an indefinite minuteness, extremely e
moving with an astonishing velocity: when arr
the organ of sight, they strike it with forc
jointly proportional to their density, mass, and v
and excite different impressions; which, by re
the intimate union that subsists between our
and our soul, occasion in our mind different, and
distinguishable ideas, respecting the objects from
the luminous moleculæ emanate.

51. Either the luminous moleculæ are of di
kinds, or they have particular properties whic
tinue invariable with regard to each class, and
pendent of the different modifications which lig
undergo in its route. It is not all necessary to o
sent purpose, to inquire which of these is th
probable.

52. We may now define *a ray of light* more p
larly, as a stream of luminous atoms, conti
homogeneous, and of the same kind. Mathem
speaking, single particles of light succeeding
other in a right line, constitute a ray of light
physically speaking, a ray is the least part o
on which alone experiments can be made. In re
to colours, there are as many kinds of luminou
as there are kinds of luminours moleculæ, or as
are classes of invariable properties. These di
kinds, are distinguished by the different sen
which the organ experiences; and these differe
sations, excite in the mind ideas which we deno
those of different *colours.*

53. Though we are not able to make an
classification of all the rays of coloured light,
may be produced in various ways, yet we may
distinguish seven kinds, forming so many di
colours, which, since the time of Newton, hav
usually and correctly called *primitive.* In the o
which they occur in the rainbow, they are name
orange, yellow, green, blue, indigo, violet. In sp
of the rays of light, we sometimes, for the sake
venience, speak of the *red rays,* the *blue rays,*
denote streams of luminous atoms which, striki
eyes, excite in us peculiar sensations, as, for ex
those of a blue, or of a red colour, &c.

54. An object may be visible, 1st, because i
particles of light directly to our eye; and in th
we call it, as we have done all along, a *luminous*
such as the sun, a torch, a lighted candle, &c.
from an object of this kind is called *direct ligh*
object may become visible, 2dly, because whe
struck by rays that proceed from a luminous ob
reflects and sends them back towards our ey
exciting in us an idea of its presence. In thi
the object is said to be illuminated, and the ligh
prised between the eye and it, is called *reflected*

As the sun is, with respect to us, the most lu
of all objects, we shall here explain, generally,
causes us to see objects. The same explicati
plies, *mutatis mutandis,* to other luminous object

55. The sun throws out from every point of

face, and in all directions, to an immense distance, a prodigious quantity of rays of all kinds, so blended together that no kind predominates perceptibly, and that throughout the known regions of the universe, there is not any sensible point that does not posses its light; or, in other words, there is no place where solid matter is found, in which we do not also find the true shadow of some body impenetrable to light, which decidedly indicates the existence of light itself.

56. The rays which our eye receives directly from all points of the sun's surface exposed to our view, form a cone of which that surface is the base, and the point of admission into the iris the vertex. The prolongation of those rays beyond the pupil (neglecting in the present account the effects of the different humours of the eye), forms within the organ another cone which is terminated at the bottom of the eye, and which there makes an impression on a circular space (of greater or less dimensions according to circumstances), which, in its turn, through the intervention of appropriate nerves, excites the idea of a round and luminous body, and this we call *the sun*.

57. In what follows, we shall generally distinguish by the term *images of objects in the eye*, the spaces at the bottom of the eye in which the rays of light are stopped; and where, of consequence, the impressions of which we have been speaking, are made. We thus denominate them, because, when, according to the experiment of Scheiner, we present an eye, divested of its exterior coats, to a strongly luminous object, we actually see an image of that object depicted in its proper colours at the bottom of the eye. When rays from the sun do not come to us till after reflection, or, more generally, when those rays meet with a body, there may happen four cases:

58. *Case* I.—When the solid parts of that body (which we here suppose impenetrable to light) are situated so regularly with regard to each other, at the part of the surface on which the light falls, that they send back all those rays in the same order as that in which they arrived at it,[*] it is evident that an eye that shall be in the path of those reflected rays, will receive an impression of precisely the same kind as though those rays came directly from the sun. The eye, therefore, has precisely the same kind of sensations as it would receive from the sun itself; an image of the sun is formed at the bottom of the organ, and the body which has thus reflected the rays is, in fact, a true *mirror*, though it may be invisible to the eye. But, since the reflected rays have changed their direction, the place in which the sun will appear will not be the same as if it were seen directly; because, we suppose that objects are situated in the right line which coincides with the direction of the rays at the moment when they reach the eye.

59. We know from experience, that the more highly the surface of a body exposed to the sun is polished, the more completely it answers the purpose of a mirror; that is, it becomes so much the less visible in itself, while it more vividly reflects the image of a luminous object: and, since the surfaces of bodies can never receive a perfect polish, that is, can never be entirely free from protuberances differently inclined, &c. however minute, we shall, through the remainder of this section, suppose that no bodies can become *perfect* mirrors, in the sense just described.

Imperfect mirrors.

60. *Case* II.—If the solid parts of a body are so situated at that portion of the surface where the rays fall, that they reflect nearly all the rays of the sun, or, at least, that they do not absorb perceptibly more of one kind of rays than of another, in such manner that these rays are reflected confusedly, some on one side, some on another, according to the position of the surface of the small solid part which received them, the eye which finds itself in the stream of the light thus confusedly reflected, will receive rays that come from all parts of the reflecting surface. Those rays will form a pyramid, of which that surface will be the base, and the pupil of the eye the vertex: their prolongation will, as in the cases previously described, form within the eye another pyramid, terminated at the bottom of that organ, by a base similar (very nearly) to that of the exterior pyramid. Now, each solid particle of the reflecting surface is a small mirror, which can only refer to the eye a very small part of the image of the sun; the irregular situation of all these solid particles, render them, in effect, so many minute mirrors differently posited; which causes so many different positions in the appearance of each portion of the sun's image. Whence it follows, that the aggregate impression made in the whole extent of the base of the pyramid within the eye, will excite the idea of an assemblage of luminous parts, terminated by a figure similar to that of this base.

61. This may be better comprehended by means of the following example. The diameter of the sun appears to us to subtend in the sky an arc of about 32 minutes. If, therefore, we reflect, by means of a plane mirror exposed to the sun, an image of that luminary towards an eye, that image will seem to occupy a moderately considerable portion of the mirror. Suppose that we covered nearly the whole of that portion, leaving only a small part uncovered: it is evident, 1st, That we should now be able to see only a small part of the image of the sun, which we had before seen entirely. 2dly, That the part seen would appear of the same figure as the uncovered part of the mirror. And, it would obviously be the same in effect, if, instead of a large mirror almost entirely covered, we had simply employed a small mirror equal and similar to the part which we have been supposing to be uncovered. This granted, imagine that there be taken several pieces of looking-glass, each too small to exhibit an entire image of the sun; and that we dispose each of those pieces in any figure whatever, as a hexagon, for example, in such manner, too, that each shall reflect to the same eye that part of the image of the sun which it is capable of reflecting (the manner of effecting which will be shown in the sequel); it is clear that in this case the eye will see as many portions of images of the sun as there are mirrors, and that all those portions of images will form a luminous figure similar to that which results

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[*] There is still a question agitated among philosophers, whether the reflection of light be made similarly to that of elastic bodies by a simple decomposition of the motion on meeting the solid parts of bodies; or, on meeting with an elastic matter which is diffused over the surfaces of bodies; or, whether the reflection of light be the effect of a repulsion occasioned by an active power which operates on light when it approaches bodies. This interesting question must be considered, when we treat of the physical properties of light; in the meanwhile, we shall speak of the reflection of light as though it were made from the solid parts of bodies.

from the assemblage of mirrors (a hexagon, for example); so that, as we add or take away a piece of glass, there will appear, or disappear, a corresponding portion of the sun's image, which will change the luminous figure in the same manner as the assemblage of portions of glass changes its figure.

62. We may further perceive, 1st, That we may so dispose these pieces of glass, that there shall be no perceptible interval between the portions of the solar images which they reflect; and that thus the luminous figure shall appear continuous and without interruption. 2dly, That, according to the clearness and polish, will be the brilliancy of the part of the luminous figure which it forms. 3dly, That the luminous figure will make the same impression in the eye, as if the rays which reach that organ came directly from the sun, and that consequently, it will be of the same colour as the sun, namely, white.

Distinguishing property of bodies which appear white.

63. If, therefore, we regard the surface of a body which reflects a very great quantity of solar rays of all kinds, without absorbing more of one kind than of another, as terminated by solid particles, which are polyedra separated one from another, so that their faces constitute so many small mirrors placed irregularly, and in different planes; we may conceive that this body ought to appear *white*, and terminated by a figure similar to that of its image in the eye: and the parts of the surface of this body will appear of a white, more or less bright, according to the greater or less compactness of the tissue of these small polyedra, which leave, of consequence, more or fewer dark intervals, according to their polish, and to the position of their faces with regard to the eye and to the sun.

64. On this hypothesis, therefore, it appears, that *white* bodies are those which reflect towards our eyes rays of all kinds mixed together, in the same proportion, or nearly, as the blended rays of the sun.

Opaque bodies.

65. *Case* III.—If the solid parts of bodies are so situated with regard to the eye, and to the sun; or, if they are of such a nature that they send back very few rays, almost the whole being absorbed by reason of some peculiarity in the constitution of such bodies; then the eye will receive such excessively minute portions of images of the sun, that they will not make any sensible impression, or will merely make such an impression as will enable us to perceive that there are before the eye some bodies, or parts of bodies, which reflect a little light. Such bodies will either be not visible, or scarcely visible, and we shall have no idea of their presence and figure, except as other neighbouring objects are brighter, and make a strong contrast with it. These we call *opaque* bodies.

66. On this hypothesis, therefore, *opaque* bodies are those which reflect only very few rays of light.

Production of colours.

67. *Case* IV.—If the solid parts which terminate the surface of a body, are of such a nature that they absorb almost all the rays of light, except those of one kind, nearly all of which they reflect, an eye which is in the direction of the rays thus reflected, will receive so many small portions of images of the sun as there are solid particles that have reflected the rays: but those minute portions of images being, in this case, all produced by one kind of rays, their assemblage will excite the idea of a body of a certain colour (namely, that which belongs to the class of rays which are reflected), and of a figure determined by that of this assemblage.

68. From the preceding it follows, I. That bodies of a certain colour, as red, green, &c. are those which absorb all the different kinds of rays except one; and that kind which is reflected, is the appropriate kind of rays to excite the idea of the specific colour, red, green, &c.

II. That the shades of colours depend on the combination of the different kinds of rays reflected.

Diffe colou

III. That, in order a body may have a certain colour or tint, in some cases its interior parts, in others those at the surface, must have a certain specific and appropriate arrangement; or, a peculiar foreign matter must be made to enter its pores; or its surface must be covered by a coat or varnish; or, by some other suitable means, the body must become disposed to reflect rays all of one species; or, at least, that that class of rays shall make an impression upon the optic organs of observers, so as decidedly to predominate over the rays that are fitted to excite the ideas of other colours.

69. It follows also, from the preceding general explication of vision and colours, *that a particle of light possesses the power of producing the image of the luminous point whence it comes.* If a yellow ray proceeding from the sun meets with a red body, or one so tinged or coloured as to appear red, that ray will not be reflected: it may either penetrate the body, or it may be stifled and absorbed within it; or, after having made many changes in its course, it may issue again, but not in such a way as to reach the eye. But, if the said yellow ray meet a yellow body, it will be reflected without penetrating; and when it reaches the eye, will excite the idea of its proper colour.

New ideas colou

70. Such is the popular representation which is usually given of the nature of the colours of bodies. Newton, who never examined any subject superficially, has given a much more recondite view of the cause of colours; to which, however, we must here advert only briefly. He has irrefragably proved, that every ray of light, in its passage through any refracting surface, is put into a certain transient constitution or state, which, in the progress of the ray returns at equal intervals; and disposes the ray *at* every return to be transmitted, and *between* the returns to be reflected by it. Hence, he assigns as the reason why the surface of bodies reflect part of the light that falls on them, and transmits the rest; that some rays at their incidence are in fits of easy reflection, and others in fits of easy transmission; the returns of these dispositions of the rays being denominated *fits* of easy reflection and transmission respectively. Hence, also, he teaches, 1st, That those surfaces which refract the rays most strongly in fits of easy refraction, reflect them most strongly in fits of easy reflection. 2dly, That when rays of a deep yellow pass perpendicularly out of any medium into air, the interval of the fits of easy reflection are the 89,000th part of an inch; and that the fits of easy transmission have the same interval. 3dly, That when a plate is thinner than the 178,000th part of an inch, the rays are all transmitted. 4thly, That the intervals of the fits are shortest in the most refrangible rays. 5thly, That all bodies appear transparent, whose particles and pores are too small to cause any reflections. 6thly, That the opacity of bodies arises from the multiplicity of reflections within them. 7thly, That the

parts of bodies on which their colours depend, are denser than the medium included in the pores. 8thly, That bodies whose colours appear most thin, have their constituent parts greatest. 9thly, That a body is rendered transparent by filling its pores with a fluid of equal density. These are natural consequences of Newton's exquisite theory of light and colours. They have been frequently questioned by hasty sciolists, who have not either sufficient patience, or sufficient skill to repeat his experiments. But the objections have been proved nugatory, and the theory of Newton has received an unexpected confirmation in the recent discoveries respecting polarization, of which we hope to present a satisfactory view in its proper place.

§ V. *Of the ideas which light occasions in the mind.*

71. Most of our sensations are caused (we need not here inquire in what manner) by impressions from external objects; and particular sensations never fail to excite in us specific ideas; those ideas, or something essential at least in their composition, we undoubtedly acquire by experience. How else could a particular sound produce the compound idea of a bell, a drum, a man, a friend? And language is a striking instance of the great strength and quickness of our retentive faculties, in associating together a vast variety of ideas, which before were collected and treasured up, as the proper significations of such particular words or sounds. In like manner, different smells and tastes are now accompanied with ideas which before had been treasured up in the memory. Our sensations themselves are probably coeval with us, and perhaps differ not very much all our lives, except in point of vividness and perfection. But the ideas that accompany them are acquired; yet, for the most part are got so early, that we may forget our having learned, as it may with propriety be called, to see, to hear, &c. Still, that seeing and hearing are arts to be learned, is evident from the case of persons who, having been blind or deaf from their infancy, at adult age acquire the use of their eyes or ears; for it is always found that they are not able to derive any practical advantage from their newly restored sense, till they have time and opportunities to ascertain the use of them by means of repeated experiments, observations, trials, and comparisons. The theory of seeing, little as we know of it, is better understood (and that because it is more entirely susceptible of reference to mathematical principles), than that of any of our senses. It is very manifest from what has gone before, that the same object placed in a different light, in a different position as to high or low, right or left, &c. or at a different distance from the eye, will occasion a different image upon the bottom of the eye: still, extreme variety of these circumstances will cause no difference in the actual idea; for, having learned from experience that the difference in the sensations arises from appreciable difference of circumstances, we instantly and imperceptibly form the correct notion; that is, if the usual sources of accurate judgment are within our reach.

There seems but little room to doubt that like sensations always excite like ideas (unless we are enabled to correct them, however unconsciously to ourselves, by means of the judgment), although the causes of these similar sensations may be different. The lightning-like flashes, sometimes arising from a sudden blow upon the head, or a blow or partial squeeze upon the eye, are probably owing to this, that the same effect is produced upon the muscles and nerves of the eye, as by the impressions of real lightning. After the same manner, we may make some probable conjectures as to the causes of other phenomena of the like sort. Thus, the circle of colours, like those in the feather of a peacock's tail, that appear upon pressing in the dark one corner of the eye with the finger, are probably owing to the like motions being excited at the bottom of the eye, by the pressure of the finger, as at other times are excited there by light for causing vision. And these phenomena are supposed by some to corroborate the hypothesis, of vision being caused by some motion excited in the optic nerve, by the impulses of light upon the bottom of the eye.

Impression of light on the eye.

72. The ideas we have from the impressions of light do not instantly vanish, but remain for some small time. This must undoubtedly be true, with respect to all our sensations; for all motions must be performed in time, though imperceptible to us. The phenomenon of a burning coal appearing like a ring of fire, when whirled rapidly about in the circumference of a circle, seems to prove that visible sensations have a sensible duration. For the impressions of light from the coal in every part of the circle probably remain entire on the eye, till they are renewed by the arrival of the coal at the same part again; otherwise, instead of a ring of fire, we should see the coal distinctly in the successive places; as the case is when it is moved but slowly. In like manner, on going suddenly out of a strong light into the dark, the impressions of the light are perceptible for some time; and the uneasiness we feel at such times, shows that a sudden change, even from light to darkness, is hurtful to the eyes: the same kind of uneasiness is likewise felt in going suddenly out of a dark room into a strong light. These phenomena also show that the eye requires some time to adapt itself to different degrees of light.

Thus far, then, it is evident, that we judge of the presence and of the figure of objects by means of the impressions made in the eye itself, at the bottom of which, images of those objects are depicted; that we infer their magnitude, position, motion, and distance, in part by those impressions, and in part by certain judgments to which we are accustomed; which, however, are often erroneous, and require to be corrected by reasoning, or by the aid of other senses besides that of seeing.

Of the magnitude of images.

73. There is a certain ordinary limit of our sight, or a certain distance within which the objects are found with which we are most conversant in the commerce of life. When objects are seen within this limit, although the dimensions of their images in our eye change prodigiously in consequence of our approaching to them, or receding from them, but a little, yet the objects do not appear to change their magnitude. Beyond that limit, however, objects appear to diminish in proportion as we recede from them, and reciprocally. If, for example, I place my eye, successively, at 2, 4, and 6 feet distance from the same man, it is evident that the lineal dimensions of his image will be successively as 1, $\frac{1}{2}$, $\frac{1}{3}$; and, of consequence, that, so far as the *magnitude* of the image directs our judgment, he ought to appear less and less in the ratio of those numbers: yet, notwithstanding this, we know that, in

fact, no such diminution is perceived. To be convinced that this arises from habit, we have only to consider that if we see a man before us, at the distance of 40 yards, he will not appear by any means so small as if we saw him (ourselves being near the bottom) at the top of a tower of 40 yards high. We are in the *habit* of seeing men on the same horizontal plane, but not much above us or below us; so that, in the latter case, we derive no assistance from experience, and infer the magnitude of objects principally by that of their images in our eye.

74. When, therefore, an object is within the ordinary limit of our sight, it appears that we judge of its magnitude and distance by the knowledge which we have acquired, through long and familiar use, of the dimensions of all which is seen between our eye and that object; and that this judgment does not depend upon the dimensions of its different images in our eye. Also, that beyond that limit, or when some obstacle hides from us the intermediate objects, as when we look at some object through a telescope, or a microscope, or merely through a small orifice pierced in an opaque substance, the magnitude and the distance of that object appear to us to depend principally on the different dimensions of its image in our eye, in such manner, that if that image be enlarged or diminished by some optical artifice, the object, though fixed, will appear to us to change its magnitude and distance, that is, either to have approached and become larger, or to have receded and become smaller.

Ideas of blind persons restored to sight.

75. Dr. Smith, in his Optics, and many subsequent writers, relate in connection with this subject, the case of the boy of thirteen years old, on whom Mr. Cheselden performed the operation for the cataract, and who, when he first began to see, thought all visible objects touched his eyes. In a paper of Sir Everard Home's, published in the Philosophical Transactions for 1807, he remarks, that in all cases "where the eye, before the cataract is removed, has only been capable of discerning light without being able to distinguish colours, objects, after removal, will seem to *touch* the eye, and there will be no knowledge of their outline." But, that "where the eye has previously distinguished colours, there must also be an imperfect knowledge of distances, but not of outline; which, however, will soon be acquired."

In one boy of seven years of age, on whom the late Mr. Ware performed the operation successfully, in October, 1807, the progress in the faculty of seeing and discerning is thus described. He had before been able to distinguish bright colours with tolerable accuracy; ten minutes after the operation, a round piece of card, of a yellow colour, was placed before his eye; he said immediately that it was yellow; and on being asked its shape, said, "let me touch it, and I will tell you." Being told that he must not touch it, after looking for some time, he said it was round. A square blue card, nearly of the same size, being put before him, he said it was blue and round. A triangular piece he also called round. He directed his eyes to objects at different distances, but seemed to see best at six or seven inches. His focal distance has been since determined to be seven inches. Being asked whether the object seemed to touch his eye, he said "no," but could not form any estimate of its distance. He was highly delighted with the pleasure of seeing, and said it was "so pretty," even when no particular object was before him. About two hours after the operation, as nothing could induce the child to abstain from the employment of his new sense, the experiments were repeated. At first, he again called all the cards round, but being asked if he could find any corners in the square card, he was very desirons of touching it; this being refused, he examined it for some time, and said at last that he had found a corner, and then counted the four corners of the square, and afterwards the corners of a triangle; but in doing so he directed his eye along the edge, naming each corner when he arrived at it. The next day a guinea being held at the distance of fifteen inches from his eye, he said it was a seven shilling piece; but when placed about five inches from his eye, he knew it to be a guinea; and he made the same mistake as often as the experiment was repeated. Three days afterwards, he was allowed to go to a window, and immediately called out, "what is that moving?" being asked what he thought it was, he said, "a dog drawing a wheelbarrow; there is one, two, three dogs drawing another; how very pretty!" It may easily be imagined that these were carts and horses, which he saw from a two-pair-of-stairs window. All this is very consistent with our preceding remarks; and will receive still farther explication when we come to treat of the optical nature of vision with both eyes.

VI. *On the different appearances of objects whether quiescent or in motion, when seen at a distance.*

76. Def.—*Visual angle*, is that which is formed in the pupil of the eye, by the two rays which proceed from the two extremities of an object; or from certain points of two separte objects.

When we thus speak, we do not mean to have it understood that the eye measures that angle as with an instrument; but simply that, as the magnitude of an image upon the bottom of the eye, is always at equal distances proportional to that of the visual angle of the object, so the difference of sensations that accompany different degrees of the one, enable us, *cæteris paribus*, to judge of different degrees of the other. Thus, lines and angles in Optics, are not merely useful for the mutual communication of ideas, but really have their foundation in nature.

77. Prop. I.—Objects, whether equal or unequal, seen under the same angle, appear equal; unless there be some particular cause which changes the appearance. This is evident from the preceding observations.

78. Prop. II.—Objects presented in the same manner to our view, appear to diminish in magnitude in proportion as they are further from the eye.

This, in all cases where other circumstance do not modify the judgment, is a necessary consequence of geometrical principles. For the lineal dimensions of those objects are constant bases of a triangle whose sides are the distances of their two extremities from the eye; these sides increase in proportion as the object or eye retires; their opposite angles, consequently, increase; and, therefore, the third angle, or that opposite to the constant side, must diminish, and form in the eye images smaller in proportion.

79. *Corollary* I.—The apparent magnitudes, or the visual angles of objects, are inversely as their distances from the eye, when those angles are small.

80. *Cor.* II.—Equal parts of a large object, beyond the

ordinary limit of our view, do not appear equal. For the parts that are most remote from the eye, subtend smaller angles, and, of course, seem smaller; and *vice versa.*

81. *Cor.* III.—The smallest of two parts of an object may appear to be greatest; if the whole be so posited that the smallest part appears under the greatest angle.

82. Prop. III.—Parallel lines, being prolonged to a great distance, appear to concur and form an angle at their extremities.

For the lines which measure the distance between those parallel lines, being always equal, subtend visual angles which incessantly diminish, and at length become insensible, when they are seen at an indefinitely large distance: whence the proposition is manifest.

83. *Remarks.*—Different degrees of apparent brightness in objects placed along these parallel lines, may so modify the result, in some cases, as to make the lines instead of converging right lines, appear converging curves. But, without here regarding this effect, we may learn from the proposition, 1st, Why a high tower, or other upright building, seems to lean forward. For, if the tower be vertical, the spectator compares it with the vertical line which passes through his eye. These two vertical lines are parallel, which from the proposition appear to approach each other. Therefore, the vertical line, or face of the tower, will seem as though, at its upper part, it approached towards the vertical line which passes through the eye, so that the wall will seem to hang over, or reclined upon the vertical.*

2dly, Why the sea appears to be more elevated as it is further from the shore, and as we observe it from a more elevated spot. Here, for a similar reason, we compare the surface of the sea, which is horizontal, with the line of level which passes through the eye: these two levels being parallel, seem to approach as they recede from the eye. The apparent approximation is the greater, as we see a greater extent of the sea; and that, obviously is, as we are more raised above its surface: consequently, on both these accounts, the surface appears to approach the ideal level which passes through our eye.

3dly, Why in a long gallery the floor appears to rise, and the cieling to be depressed, as they are further from us. This is, because we refer both of them to the line of level which passes through the eye.

4thly, Why, when we are walking through a long avenue, or by a long wall, the parts which are to the right, appear to tend more and more towards the left; or, if we are between two rows of trees, they appear to separate from each other as we approach them.

85. Prop. IV.—The apparent figure of an object is determined by the situation of the points of that object which can send rays to the eye. This is evident.

86. *Cor.* I.—A right line so posited that its prolongation will pass through the centre of the pupil, perpendicularly to the surface of the eye, will appear simply as a point.

87. *Cor.* II.—A plane so posited that its prolongation passes through the centre of the pupil, will appear merely as a line.

88. *Cor.* III.—A solid which presents towards the eye only one of its faces, will appear simply as a surface.

89. Prop. V.—To an eye which is in the plane of any considerable line whatever, very remote, whether regular or irregular, such line appears as an arc of a circle whose centre is the eye itself.

For, since the points G, F, A, B, C, D, O (fig. 8), Fig. 8 of the irregular curve GFAO, are in the plane which passes through the eye E, and are, by hypothesis, much beyond the ordinary limit of view, the eye cannot judge which are the nearest, or which the most remote points. It cannot, for example, distinguish the difference between EP and ED, because it is very small with regard to either of those lines; and, therefore, being unable to judge of the difference between those radii, it supposes them equal. The same is the case with respect to EB and EL, EI, and EF, &c. It, therefore, conceives itself at the centre of a circle in whose circumference those points all lie.

90. *Remark.*—If the differences are extremely unequal, we may sometimes ascertain which are the nearest parts by the greater vividness of the light, or by their greater apparent magnitude. And, if the eye were very far from the plane of that curve, it might also perceive those inequalities, provided the lines DP, BL, IF, be not indefinitely small with respect to ED, EB, EF; for, being exposed more directly to veiw than when the eye is in their plane, they would become sensible.

91. *Corol.*—A *small* irregular line, as ABCDO, seen from a distance, will appear like a right line; that is as an arc of a circle so small as not to be distinguishable from its chord.

92. Hence it is, 1st, That when we are in an extensive plane terminated irregularly, we seem always to be in the centre of a large circle, the elevated distant objects appearing as though they were arranged in the circumference. 2dly, That in such a plane we do not seem to advance, although we be always moving, because we are still in the centre. 3dly, That the sky appears like a hollow sphere, somewhere on one of the diameters of which our eye seems to be pointed. 4thly, That a remote sphere, as the sun or the moon, appears only as a flat circle. 5thly, That a regular polyhedron viewed from a moderate distance, appears like a globe; and, from a considerable distance, like a circle. 6thly, That a tower on a polygonal base, appears either round or flat, according to the distance. 7thly, That we cannot perceive a revolving globe, at a moderate distance, to turn on its axis, unless there be one or more spots upon it, and the globe turns slowly.

93. Prop. VI.—To an eye placed in a line elevated perpendicular to the plane and passing through the centre of a regular polygon, that polygon will appear regular; but if the eye is out of that line, the polygon will appear irregular.

For the lines drawn from the several angles of the polygon to the eye in the line perpendicular to its centre, will form a right pyramid with a regular base, in which all the angles composing the solid angle of the vertex of the pyramid will be respectively equal, and the lines which bound those plane angles, also equal; therefore, the sides of the polygon will appear to the eye under equal angles, and symmetrically posited. But, if the eye be placed out of that perpendicular line, the equality of the plane angles and their respective sides will no longer obtain, the sides of the polygon will be unequally distant, and appear under unequal angles, and consequently the polygon will seem irregular.

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* It is proper, however, to remark, that this phenomenon may be partly owing to another cause. In looking upwards, after inclining our heads all we can, we are apt to imagine that we are looking directly towards the zenith; yet this, in truth, is not the case, but our eyes look forwards some degrees from the zenith: the natural consequence is, that whatever is actually in the zenith seems to be as many degrees behind us. This, also, evidently accounts for the mistakes which persons usually make concerning the altitudes of stars that have high elevations.

94. *Corol.*—A regular polygon viewed obliquely, will appear irregular, and a circle will appear as an oval; or, indeed, as an ellipse.

95. *Remark.*—The object of PERSPECTIVE is to represent geometrically, the appearances explained in the preceding propositions, and other such appearances to which no allusion has here been made.

96. PROP. VII.—Objects situated in a tract of country open to our view, appear so much the more dark and confused as they are more remote: and, contrarily, with colours so much the more vivid, and with so much the more distinctness, as they are nearer.

This is matter of fact, confirmed by experience: and the principal reason is, that distinctness of appearance and vividness of colouring depend on the intensity of the light, which, as we have already shown (art. 15), decreases in proportion to the distance of objects, in consequence of the interposition of the air comprized between the object and the eye.

Apparent magnitude and distance.

97. Hence it is, 1st, That objects elevated above the usual surface of the earth, such as those that are on the tops of mountains, are seen much more distinctly than those that are below, because the air is rarer and more free from vapours on such elevated spots than at the earth's surface; 2dly, That by means of light and shade skilfully managed, painters make some objects appear to stand forward and seem prominent, others to retire, as if into the back ground.

98. PROP. VIII.—Objects which appear dark and indistinct, are, in consequence, referred by the mind to a greater distance.

The reason is, that being accustomed to see only remote objects appear indistinct, we are inclined to refer all indistinctness to the same cause, namely, remoteness.

99. *Remark.*—If it happen that, from any cause whatever, an object beyond the ordinary limit of our view, but of a magnitude to which our eye is accustomed, or possessing certain characteristics with which we are familiar, become more dark and indistinct, we immediately judge that it is more remote: yet, as it remains at the same distance, and, of consequence, forms in our eye an image which is not diminished, we conclude that the object is larger than it is in fact.

100. Hence we may at once perceive, 1st, Why objects appear larger when seen in a mist; 2dly, Why objects seen in the night appear larger and more remote than they are in reality; 3dly, Why the sky appears to us as a depressed concavity, and not a concave hemisphere. For the light of the stars being less vivid (art. 20) as they are nearer the horizon, those stars which are most elevated appear less remote, and *vice versa:* thus it is found by experiment, that the apparent distance from our eye to the horizon is nearly thrice the apparent distance from the zenith. Such is this apparent depression of the upper parts of the sky, that if we attempt to estimate by sight the middle between the zenith and the horizon, we shall assume some point about 23° or 24° in elevation; instead of which, if the sky appeared perfectly hemispherical, this point would be at 45°. 4thly, Thus, also, we may learn why the sun and moon appear so much larger when near the horizon, and seem to diminish as they rise above it, while accurate measures of their diameters with astronomical instruments prove that the contrary actually obtains. Let AO (fig. 9) be the horizon, O the place of the observer; the luminary at different elevations in the sky, at BC, DH, FG; AKMR the apparent depressed concavity of a portion of the sky. It is evident, that in whatever place the luminary may be in the circle ADHFG, of which O is the centre, its diameter will appear nearly under the equal angles BOC, DOH, FOG. But, on account of the depressed apparent concavity of the sky, the luminary will be thought at KI when it is really at BC; and, in like manner, at PN, and TS, when it is at DH and FG; and the apparent disc of the luminary, which is bounded by the tangents drawn from the real disc (as in the diagram), will be less at the elevation AM than at AL, and less at the elevation AR than at AM; although its apparent diameter in the three cases is measured by the angles IOK, PON, TOS, equal respectively to the true angles BOC, DOH, FOG.

Hori moon sun. Fig.

Other causes may contribute a little to modify this result, so as to make the horizontal sun or moon to appear larger at some times than at others; but the principal occasion of the phenomenon is as we have here explained.

101. PROP. IX.—Objects appear more remote and of greater magnitude, in proportion as we see a greater number of objects, and a greater extent of ground between the eye and those objects; and, reciprocally, they appear nearer and smaller, as we see fewer objects and less extent of ground between them and the eye.

For a great quantity of intermediate objects and ground give the idea of a great distance, and, consequently, of a magnitude so much the more considerable, and reciprocally.

Simil illusi

102. Hence it is, 1st, That the horizon appears contiguous to the sky, because we see nothing between the one and the other. 2dly, That when we do not perceive a valley which is posited in the midst of a comparative plane, the objects which are beyond that valley appear very near to us; but recede to their actual remoteness when we approach the edge of the valley. 3dly, That at evening, objects which are a little elevated and full in view, appear far off and large, because night preventing us from judging of their distance, by the quantity of ground or the number of objects which intervene, we refer the objects seen, to the horizon; and, of consequence, regard them as proportionally remote and large.

103. PROP. X.—If two objects unequally distant from the eye, pass over parallel and equal spaces in the same time, the most remote body will appear to move most slowly, and the nearest to move most rapidly.

This is evident, because the space described by the most distant object, in any given time, will subtend the smallest angle at the eye.

Corol.—The apparent velocities of bodies moving in parallel lines, at different distances from the eye, are directly as the real velocities, and reciprocally as the distances.

104. *Remark.*—If the directions of the motions are not parallel, it may happen that the nearest body shall appear to move most slowly, although it moves in reality

Optics.

most swiftly; because the space that it describes may be so oblique to the visual rays, that they shall form smaller angles at the eye, than the smaller spaces described by the more distant body in the same time. Thus, in (fig. 10), although the nearer body should move uniformly from C to D, in the same time that the more remote body moved uniformly over the shorter space AB; yet, since the angle AEB at the eye of the observer, exceeds the angle CED, which would measure the simultaneous motion of the nearer body, it would seem to have moved more slowly, though it have, in truth, moved more rapidly.

10.

blem pecting parent ions.

105. Problem.—Having given the position of the place S, where the spectator thinks himself immovable, and a certain number of points A, B, C, of the real path of a body in any plane whatever, with the points a, b, c, in which the eye of the spectator really is, at those moments, to determine the apparent path of the moving body.

11.

Having drawn the right lines A a, B b, C c (fig. 11), draw through the point S lines S α, S β, S γ, respectively equal and parallel to the former; and the points α, β, γ, will be those through which the apparent path of the body will pass. For the right line, S α for example, being equal and parallel to A a, the point α is situated in the same manner with respect to S, and at the same distance from it, as the point A with respect to the point a. Therefore the spectator, who supposes his eye to be in S, will of consequence suppose the object to be in α, when it is really in A; and he will conclude the same with regard to β and γ, when the object is at B and C. And so of other points.

113. *Cor.* I.—The true and imaginary place of the eye, and the true and imaginary place of the object, always occupy the angles of a parallelogram. The true place of the object, and the imaginary place of the eye are always opposite angles; the apparent place of the object and the true place of the eye, always at the other two opposite angles; so that the object appears always in a situation opposite to that of the true place of the spectator's eye.

114. *Cor.* II.—If the object be immovable in A, its apparent path $\alpha \beta \gamma$ (fig. 12) is a line equal to the real path of the eye, and situated in a parallel plane.

For, because of the parallelogram $a \alpha$, $b \beta$, $c \gamma$, of which SA is a common diagonal, and at the same time a common intersection of their planes, and of which the bases S a, S b, S c, are situated in the same plane, which is that of the path of the eye; their parallels and equals A α, A β, A γ, must be also in the same plane parallel to the plane of the path of the spectator's eye, and form angles α A β, β A γ, equal respectively to the angles a S b, b S c. Therefore the points α, β, γ, must be in a line equal to the line $a\,b\,c$, and in a parallel plane; it is evident also, that $a\,b\,c$ and $\alpha \beta \gamma$ will be inverted with regard to each other. So that, if the object be placed in the plane which contains the path of the eye, the apparent path of the object will be also in the same plane.

115. *Cor.* III.—If the object be immovable, and situated in the place where the observer imagines his eye to be, the object will appear at the extremity of an equal line, and in the same direction as the line drawn from the true place of the eye to its imagniary place. Thus, if the eye move in a circle of which the object occupies the centre, and where the observer imagines himself to be, the object will appear to describe the same circle; but to be all along in the points diametrically opposite to those in which the observer's eye is successively found; and consequently the object will *appear* to move with a velocity equal to that of the eye.

Simple Optics.

Certain celestial phenomena.

116. *Remark.*—Terrestrial objects which surround us on all sides, and which are fixed with respect to us, or to one another, although they are really carried with us about the sun, lead us to conclude that we are at rest in the centre of the world, and that the sun revolves about us (or about the earth) though it in truth be fixed; also, that the planets, which in fact revolve about the sun, seem to describe irregular curves, in which they proceed sometimes from east to west, at others from west to east. Now, by the preceeding problem, we may describe all these apparent motions on a plane; if we draw two concentric circles, one to represent the orbit of the earth, the other to represent that of a planet. Thus, taking Jupiter, for example, the radii of the two circles must be in the ratio of the distances of the sun from the earth, and from the planet, viz. nearly as 1 to 5; these two circles must be divided in the ratio of the velocities of the earth and planet, which is nearly that of 11 to 1; that is, we may divide the circle which represents the earth's orbit into portions of 11° each, the other circle into degrees; then making the consecutive divisions of the earth's orbit by a, b, c, d, &c., and those of Jupiter's orbit by A, B, C, D, &c. (commencing by simultaneous points assumed at pleasure), putting S for the common centre of the two circles; it will be easy to find all the points of the apparent curve described by Jupiter, and consequently to elucidate and explain the peculiarities of the apparent motion of that planet.

The same may be done for the other planets. Astronomers have other methods of representing the epicycloidal curves thus formed; but, as they evidently appertain to another department of science, we need not dwell upon the subject here.

General scholia relative to apparent motion, the minimum visible, &c.

Minimum visible, &c.

117. *Schol.* I.—We have seen that, after the same manner, as objects in motion will have different apparent velocities, according to their different distances from a spectator, so to a spectator in motion, objects at rest will have different apparent velocities. And hence it is, that a passenger in a coach observes the trees in the next hedge to move swiftly backwards, in appearance, whilst those in the field beyond move slower: those further off, slower still, and so on, till those that are very remote seem scarcely to move, if at all. And if a spectator in motion keep his eye fixed upon an object at some distance, stationary objects which are tolerably near to that will appear at rest, as in this case their images upon the bottom of the eye will keep a constant distance; whilst nearer objects will seem to go backwards, and more remote ones progressively forwards, the same way with the spectator.

Illusions experienced by landsmen when at sea.

These, and other illusions of a similar class, are most effectual in children. Men correct them partially by the deductions of the judgment from long experience; but whenever they are thrown into a new classification of moving scenery, they remark the same effects. So it is with landsmen who take a short voyage, and observe ships at anchor, or moving in different directions, as the vessel in which they are sails along.

Objects at rest may seem in motion, and the converse when the spectator is at rest.

118. *Schol.* II.—If two or more objects that have the same apparent velocity, move all the same way, an object at rest, by which they pass, may appear to move the contrary way, whilst the objects in motion may seem at rest. For, since their images within the eye keep the same distances from each other, no motion among them is perceptible; and if the spectator insensibly move his eye, so as to keep these images in the same place, the image of the object at rest will pass successively over them, after the same manner as it would if that object had been in motion the contrary way. The same phenomenon may happen if the single object be in motion, either the same or the contrary way; only its apparent motion will be quicker or slower, direct or retrograde, according to the different circumstances. Thus, when the clouds move successively over the moon, she seems to move with their velocity the contrary way, whether that be eastward or westward.

119. *Schol.* III.—From the preceding propositions, we may perceive how difficult it is to form a just estimation of the real velocities of objects, from the apparent ones; since, to make a correct inference, we ought to know both the directions and distances of the moving objects, neither of which, in numerous cases, can be guessed at with any tolerable accuracy: and even, though we had them both given, the computation is too perplexed to be made *extempore*.

Situations of objects frequently mistaken.

120. *Schol.* IV.—By mistaking the nearest extreme, or the nearest of a series of objects, to be the most remote, which may frequently happen from a variety of causes, we also mistake the situation with respect to right and left, of the side nearest to us; and the more is the real obliquity, the more liable we are to commit these mistakes, and dependent ones in reference to motion.

Fig. 13.

Let A, B, C, D, (fig. 13) be a row of lamps, trees, columns, the arches of a bridge, &c. Now, if a spectator at O, by reason of some partial obscurity in the nearest object A (or from any other cause), be led to imagine it the remotest in the series, as at *a*; he will of consequence, imagine all the rest to be, as at *b*, *c*, *d*, in a line extended on his right hand, contrary to the situation of the line AD, which here is to his left. And thus, while he is under this mistake, if he see a body moving from A towards D, it will appear to him to take quite a different course, as from *a* towards *d*. The position of the line *ad* will be variable, according to the imaginary proportion of O*a* to O*d*. For the visual angles AOB, BOC, COD, will be the same, wherever in the lines OA, OB, OC, OD, the points *a*, *b*, *c*, *d*, be taken.

The direction of rotatory motion (as of mill-sails) often mistaken.

121. When the object is but of small extent, in proportion to its distance from the spectator, it is so much the more difficult to determine its true position, or motion. And hence, beyond a certain distance, we are very liable to mistake the position of a weather-cock, the fly or sails of a wind-mill, &c., and consequently to mistake the course of their circular motion. Thus, if a spectator at O (fig. 14) situated nearly in the plane of the sails, imagine the nearest end A, of the sail AE, to be the most remote, and the real motion of the sails be according to the order of the letters ABCDE, then when A has arrived at B it will seem in D, as if it had moved from E to D; and so the sail will seem to go round according to the order of the letters EDCBA, contrary to the true course of its motion. This illusion is so strong, that it is frequently an even wager, that one mistakes the true course of such circular motion. We are also liable to the same mistakes, and that for a similar cause, with regard to the circular course of a branch or hoop of lighted candles turning round at some distance.

Fig. 14.

122. *Schol.* V.—Apparent motion, or the motion of the images of a moving object upon the nervous expansion at the bottom of the eye, must have a certain limited degree of velocity, both as to quickness and slowness, to become perceptible: that is, the space described in a given time by that image must be neither less than some determined space, nor greater than some other determined space.

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123. If the motion of the sun's shadow, at the distance of 5 feet from the gnomon that casts it, be observed at about a foot distance from the shadow, when the sun changes his azimuth 10 degrees in an hour, the apparent motion of the shadow will be then as *slow* as can well be perceived,* and the space described by it upon the floor will be about 10½ inches in an hour; and the motion of the image of the shadow, on the bottom of an eye which observes it at the distance of 11 inches, will be at the rate of about $\frac{6}{10}$ths of an inch in an hour, a hundredth of an inch in a minute, or a six thousandth part of an inch in a second.

124. It is obvious, from the above experiment, that the space described in an hour by an object that moves with the slowest perceptible motion, and the distance of the eye from the object are nearly equal: and therefore, since radius is equal to the chord of 60°, the space described in a minute of time, by an object that has such a rate of motion, subtends an angle at the eye of about *one degree:* or the space described in a minute by an object that moves in a line parallel to the face of the spectator, and with such a velocity as to be just distinguishable from quiescence, is about $\frac{1}{60}$ of the distance between the object and the spectator.

Hence the reason why the stars are not seen to move is manifest; the spaces described, even by those of them that are in the equator, in a minute, not subtending an angle of more than a quarter of a degree. The apparent motion of the moon never exceeds $15' 40''$ in a minute of time, or $15\frac{2}{3}''$ in a second; and this also is imperceptible by the naked eye.

125. *Schol.* VI.—It is more difficult to assign the angular space which is described in a given time by an object which has the greatest perceptible motion; and the difference may be very great betwixt that degree of velocity with which something may be perceived to cross the field of view, and that with which the same object may be distinctly seen, as it moves. Of this latter kind, let that of a falling body, at the distance of 20 feet from the spectator, and in that part of its fall when its velocity is 19 feet per second, be the quickest perceptible motion (as it will be to most eyes,

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* This motion of the shadow, however, would not be perceptible, if we did not attend to fixed marks upon the floor; and when we cannot compare the situations of moving bodies with others at rest, a motion much swifter than the above observed motion of the shadow will be imperceptible.

and, excepting in few cases, an object cannot be seen that is less than $\frac{1}{430}$ of an inch in diameter; an object of that breadth subtending an angle of 1 minute at the distance of 8 inches from the eye. But, further, an object placed on a field that differs insensibly from it in colour, is not perceptible under a less angle than about 4 minutes; and, in such circumstance, the smallest visible object is scarcely less than $\frac{1}{100}$ of an inch in diameter. At a medium, the breadth of the least globular object that is discernible by the naked eye, is probably about the $\frac{1}{300}$ of an inch.

Greatest apparent distance, or visible extent of horizon.

132. *Schol.* VIII.—Before we terminate these scholia, it may not be amiss to make an inquiry concerning the limit or extent of the greatest apparent distance, when that is not regulated by the apparent magnitude of distant objects, as trees, churches, &c. whose real magnitude we nearly know. It is very certain, that no objects appear further from us than the visible horizon; and that the horizon itself appears most remote when the whole intermediate tract of land is visible. But, as that visible extent will be different according to the height of the spectator, the evenness or roughness of the ground, according also, as the ground is horizontal or lies either ascending or descending from the place of the spectator, it will be well to have recourse to some general estimate, always bearing in mind, however, that a pre-conceived notion of a greater distance will, inevitably, in some measure modify our idea of apparent distance.

Suppose, now, that the moon, for instance, when near the horizon, appears to be a yard in diameter, or equal to a round table just at hand, whose diameter is a yard. The visual angle subtended by the moon is (art. 26) about half a degree, and, therefore, in this case, the apparent distance of the moon is the radius of a circle whose circumference is twice 360 yards: for it is evident that twice 360 apparent moons (each a yard in diameter), would go round the periphery of a circle of the above-stated magnitude. Thus, upon this supposition, the apparent distance of the moon will be about 115 yards. In like manner, knowing the visual angle, and the apparent magnitude of any other object, we may ascertain its visible or apparent distance. But, perhaps, few objects furnish so good a scale of comparison as the horizontal moon; with regard to which, if the apparent magnitude, as above specified, be too large, the resulting maximum apparent distance will be proportionally too great; 100 yards will, probably, be a medium measure. But in some cases, as when we stand on an eminence, viewing an extensive vale, or a gradual ascent, on which are a great number of objects whose different distances are very perceptible, the greatest apparent distance may reach to a quarter, or even half a mile.

This will, doubtless, appear extraordinary to most persons; but it is an accurate deduction notwithstanding.

A principle purpose of vision being to discover, as nearly as circumstances will permit, the *true* places of objects, we do not rest satisfied with mere apparent distances; but from thence, and from such other helps as occur, we always make some judgment of the real distances of terrestrial objects. And hence it is, that most persons are apt to think the simple apparent distance of a remote object to be much more extensive than it really is. It is unquestionable, that the apparent distances of two steeples, one a furlong off, the other 10 miles, are much the same, if no intervening objects are perceived; and yet we are seldom without helps, by which we can readily distinguish that one is more remote than the other. Upon these, however, we cannot now dwell; but must proceed to other topics.

PART II.

CATOPTRICS AND DIOPTRICS.

§ I. *General notions respecting Catoptrics and Dioptrics.*

I. *On images, foci, &c.*

133. Catoptrics is that part of Optics which relates to the *reflection* of light, and the various phenomena which depend upon it.

Reflexibility is the disposition of the rays to be turned back into the medium whence they came; and the change of motion which the ray in this case undergoes, is called *reflection.*

The angles which the incident and reflected rays contain with the perpendicular to the reflecting surface at the point of incidence, are called the *angles of incidence* and of *reflection.*

Any smooth surface that reflects light, is called a *speculum*, or *mirror*, and is evidently either plane, convex, or concave.

A line drawn through the centres of the sphere and of the lesser circle which terminates a spherical speculum, is called the *axis.*

134. Dioptrics is that part of Optics which relates to the *refraction* of light, and the various phenomena which result from it.

When light moves through a homogeneous medium, it preserves a continued rectilineal course; but in passing from one medium to another of different density, it deviates from its former course; the change of direction is called *refraction.*

The lines which a ray describes before and after it enters the refracting medium, are called the *incident* and *refracted rays:* the angle contained between the incident ray, and a perpendicular to the surface drawn from the point on which the ray falls, is called the *angle of incidence:* the angle contained between the refracted ray and the said perpendicular, is called the *angle of refraction.*

The difference of the angles of incidence and refraction, is the angle by which the ray deviates from its original direction, and is called the *refracted angle*, or the *angle of deviation.*

135. A slender portion of rays separated from the rest, is called a *pencil of rays.*

Pencils of rays are either conical or cylindrical; the axis of the pencil is the same with the axis of the cone or cylinder.

136. *Parallel* rays are such as proceed equally distant from each other through all their course. Rays which proceed from any point, and, as they proceed, recede more from each other, are called *diverging;* and such as tend towards a certain point, at which they would at last unite if not prevented, are called *converging* rays.

137. By the aid of glasses and mirrors, and the reflexibility and refrangibility of light, we may collect

at the same sensible point, a very great number of the rays that emanate from the same point of an object: and since each ray carries with it the property of exciting the image (art. 50) of the point whence it has issued, the combination of many such rays cannot do otherwise than form an image, which will be so much the more vivid as there are more rays united, and so much the more distinct, as they preserve in their combination the order in which they were transmitted from the luminous object. If a smooth white plane be placed at the point of convergence of these rays, the image in its natural shape and colours will there be depicted.

138. The *focus* is that point from which rays diverge, or towards which they converge. The point from which rays *seem* to diverge, when in reality they diverge from another, is called a *virtual* or *imaginary* focus. The point to which parallel rays are reflected or refracted, is called the *principal* focus. The focus before reflection is called the *focus of incident rays;* and the focus after reflection, the *focus of reflected rays;* both together are called *conjugate foci:* and they are so related, that if either of them be the radiant point, or the focus of incident rays, the other will be the focus of reflected rays.

139. From this fact, that each ray carries with it the image of the object whence it issues, it follows, that if rays, after mutually intersecting and forming an image at their intersection, can again be brought together by any refraction or reflection, they will form a new image; and so on, so long as their order is not confounded. We may, therefore, form as many images of the same object, as we can effect new re-unions of the rays (which had been separated) without confounding them.

140. It follows, that so long as the question relates only to the progress of the luminous rays, we may regard the image as the object, and the object as the image; and even a second image, as if the first image had been the object which produced it; and so on.

II. *Laws or principles deduced from experiment, on which the demonstration of Dioptrics and Catoptrics are founded.*

141. 1st Law.—Every luminous ray which passes through a medium, on meeting with another medium of a different density, or a different nature, changes its direction. If it do not penetrate that medium, it will be reflected at its surface; if it enter the medium, it will be bent or refracted at its entrance; except it meet the medium perpendicularly to its surface at the point of entrance.

This is matter of universal experience. With regard to *reflection,* the proof is furnished by the surfaces of mirrors and other smooth bodies. A simple instance of the effect of refraction, is that of a straight staff plunged obliquely into a vessel of quiescent water. The staff appears as if it were broken at the point of immersion; and the part immersed, although it appears rectilinear, seems to approach more to the horizontal position than the part which is out of the water. The same thing is observed in all inclinations of the staff.

Let AC (fig. 15) be a ray which, passing through the air falls upon the surface PQ of a solid piece of glass PS. At the point C (where the surface of the new medium meets the ray), that is, *the point of incidence,* draw the perpendicular MD, sometimes called the cathetus of incidence. If the incident ray AC finds some obstacle which prevents its penetrating the glass, it will change its direction by reflection, and will take its course along CI; so that, while ACM is *the angle of incidence,* MCI is *the angle of reflection.* But if the incident ray AC can enter the glass, instead of pursuing its first direction CB, it will be refracted and take its course in some other direction CT. In that case the angle DCT is called *the angle of refraction,* TCB the *refracted angle,* BE the sine of the angle of incidence, or the *sine of incidence,* and TH the sine of the angle of refraction, or *the sine of refraction,* to the assumed radius CT.

142. 2d Law.—A luminous point which, on meeting different surfaces or media, had experienced all the reflections, refractions, &c. which they were calculated to produce, would, if it met with an obstacle which compelled it to return in a precisely opposite direction, re-issue from the last surface upon the same line as that in which it entered.

Thus, a ray AC (fig. 15), which being refracted at C described CT, would, if it were repelled from T in the direction TC, re-issue at C in the direction CA.

143. 3d Law.—The angle of reflection or of refraction, is in the same plane as the angle of incidence, and this plane is perpendicular to the surface of the medium.

For its position is determined by the cathetus of incidence, which is perpendicular to that surface.

144. 4th Law.—The sine of the angle of reflection and the sine of the angle of refraction of a ray, are in constant ratios to the sine of the angle of incidence; with regard to the same media.

Some experiments on which this law is founded, are described. We shall here state the results, and make a deduction or two.

In *reflection,* the ratio of the sines is that of equality; or, in truth, the *angles* of incidence and of reflection are equal.

Laws of refraction.

The ratio of the sine of refraction to the sine of incidence, is, when the luminous point passes from air into rain-water, nearly as 3 to 4, or more exactly as 3 to 4·0076;—from air into glass, as 2 to 3, or more accurately, as 20 to 31;—from glass into water, as 8 to 9. And reciprocally, the sine of refraction is to that of incidence, in the passage from water into air, as 4 to 3;—from glass into air, as 3 to 2; &c.

145. *Cor.* 1.—Hence we see that it is consistent with the general law, that when an incident ray is perpendicular to the surface of the medium which it meets, it is either reflected upon itself, or it is transmitted through the medium without changing its course. For then the sine of incidence being zero, the sine of reflection and that of refraction are zero also; or, which amounts to the same, the ray continues confounded with the cathetus of incidence.

146. *Cor.* II.—Under whatever angle of incidence a ray meets a medium impenetrable to light, it may always be reflected; but when the sine of the angle of incidence must, by the nature of the medium, be smaller than the sine of the angle of refraction, the ray cannot always penetrate that medium by refraction: or, which amounts to the same, there are always certain limits in the angles of incidence, beyond which a ray cannot be refracted, or pass from the medium in which it is, into that which it meets. For, if a luminous particle, passing through the air, meet a surface of

water, under an angle of incidence of nearly 90°, the angle of refraction will be about 48° 36′ or 48°$\frac{1}{2}$; because the sine of 48° 36′ is $\frac{3}{4}$ of the sine of 90°. If, therefore, on the contrary, a ray were to pass from water into air, under an incidence of 48°$\frac{1}{2}$, it would issue from the water in an angle of about 90°, and, in a manner, glide along the surface: and, if the ray had to pass under an angle of incidence of more than 48°$\frac{1}{2}$, the sine of its corresponding angle of refraction would exceed radius; which is impossible. It is, therefore, impossible that a ray with such an incidence can pass from water into air; and, we learn from experiment, that a ray so circumstanced, is reflected at the common surface of the water and the air, and remains within the former.

The same reasoning applies to other mediæ; and by means of it we may determine the limits of the possible refractions for any given ratios of the sines of incidence and refraction.

III. *Experimental and theoretical investigations of the preceding laws, and of the measure of refraction.*

Laws of refraction; how demonstrated.

147. The proof from experiment of the laws of *reflection* is very simple. Instead of employing the light which emanates immediately from the sun, we may make use of the flame of a candle or lamp, or even, of the light naturally flowing from some point of a terrestrial object, which we receive, after reflection, from the first surface of a polished plate. Let such a plate (fig. 16) HL be placed in a perfectly horizontal position; and let S be the point of an object, from which a ray, as SI, reflected from the plate at I into the direction IC, is that which we would examine. Let C be the centre of a graduated circle, furnished with sights, and otherwise fitted for measuring angles, be placed somewhere on IC: if, after having posited this circle vertically in the right direction, we give a continued motion to the index which carries the sights, we shall find that it successively points to the image, the point of incidence, and the object; which proves, that *the incident and reflected rays are both contained in one plane perpendicular to the surface of incidence* (art. 143).

Fig. 16.

Now (conceiving, through the point C, a horizontal line CH′ to be drawn) we may, by means of the circular instrument, ascertain the angle SCH′, the apparent altitude of the point S above the horizon, as well as the angle SCI between the direct ray SC and the reflected ray IC. Thus, taking the most simple case, in which the luminous point is so remote that the distance CI from the point of incidence to the centre of the circle may be insensible with respect to it. Then, the angles being measured, we shall find that SCI is double of SCH′. But, since the luminous point is supposed indefinitely remote from the circle, SC (fig. 17) will be, as to sense, parallel to SI, at the same time that LH is parallel to CH′: consequently SIH = SCH′. The angles H′CI, CIL, being alternate angles, are equal: therefore, we have, for the total value of SCI,

Fig. 17.

$$SCI = SIH + CIL.$$

But it is found, by observation, that

$$SCI = 2\,SCH' = 2\,SIH.$$

Therefore, by equating the two values of SCI, we have

$$2\,SIH = SIH + CIL, \text{ or } SIH = CIL.$$

Hence it appears, *that the incident and reflected rays form equal angles with the reflecting surface* (art. 144).

148. Strictly speaking, it is not possible that the
rays SI, SC, be exactly parallel; but the error
become so small as to be altogether imperceptible:
besides, whenever it is necessary, we can have re
to the angle which they make between them.
(returning to fig. 16), let there be drawn, throug
centre C of the circle, the line CS′ parallel to IS.
this case, if the angle CSI be sensible, we find, b
servation,

$$SCI = 2\,SCH' + CSI.$$

Adding CSI to both sides of the equation, we ha

$$SCI + CSI = 2\,(SCH' + CSI).$$

But, since CS′ is parallel to IS, it is evident
CSI = S′CS, and, consequently,

$$SCI + CSI = 2\,(SCH' + SCS') = 2\,S'CH' =$$

$$\text{and } SCH' + CSI = S'CH'.$$

Substituting these values in the equation given b
servation, it becomes

$$S'CI = 2\,S'CH':$$

which is obviously the same in effect as the prece
and thence again we obtain, as before, SIH =
an equation that establishes this law of reflection.

149. The fundamental law of *refraction*, namely
the sine of incidence of homogeneous rays, is t
sine of refraction, in a given ratio, most eleg
demonstrated by Newton, after a synthetic pr
from this supposition that bodies refract the ra
light by acting upon them in lines perpendicul
their surfaces; without stopping to decide what
is, or by what kind of force it is refracted. We
here exhibit the demonstration precisely as Newto
done.*

150. Prop.—If two similar mediums are sepa
from each other by a space terminated on each
by parallel planes; and a body in its transit thr
this space, is attracted or impelled perpendicular
wards either medium, and is not agitated or hin
by any other force; and the attraction is every
the same at equal distances from either plane,
towards the same side of the plane; then will the
of incidence on either plane be to the sine of emer
from the other plane in a given ratio.

Case I.—Let A *a*, B *b*, be two parallel planes.
the body fall upon the first plane in the direction
line GH, and in its whole transit through the i
mediate space let it be attracted or impelled tow
the medium of incidence, and by that action conce
to describe the curve line HI (fig. 18); and let it en
in the direction of the line IK. Let IM be erected
pendicular to the plane of emergence B *b*; me
both the line of incidence GH produced in M, an
plane of incidence A *a* in R: and let the line of
gence, KI, produced, meet HM in L. With the c
L at the interval LI let a circle be described, cu
both HM in P and Q, and MI produced in N:
first, if the attraction or impulsion be supposed uni
the curve HI, from what Galileo has demonst
(see also Mechanics, p. 105. art. 44.), will
parabola, of which this is a property, that the rect
under its given *latus rectum* and the line IM, is
to the square of HM. But, further, the line HM
be bisected in L; whence, if LO be let fall perp

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* Princip. Math. lib. i. Prop. xciv. Theor. 48.

cularly to MI, MO and OR will be equal; and the equal lines ON, OI, being added, MN, IR, will become equal. Wherefore, since IR is given, MN is likewise given, and NM·MI is to the rectangle under the *latus rectum* and IM, that is, to HM^2, in a given ratio. But $NM \cdot MI = PM \cdot MQ$, that is $= ML^2 - PL^2 = ML^2 - LI^2$; and HM^2 has a given ratio to its fourth ML^2: therefore, the ratio of $ML^2 - LI^2$ to ML^2, is given; and, by conversion, the ratio of LI^2 to ML^2, and its subduplicate, the ratio of LI to ML. But, in every triangle LMI, the sines of the angles are proportional to the opposite sides. Therefore the ratio of the sine of the angle of incidence, LMR, to the sine of the angle of emergence, LIR, is given; which was to be demonstrated.

Case II.—Let the body now pass successively through several spaces, terminated with parallel planes A *a b* B, B *b c* C, &c. (fig. 19); and let it be agitated by a force which is uniform in each of them separately, but different in different spaces. By what has been already demonstrated, the sine of incidence on the first plane A *a*, will be to the sine of emergence from the second plane B *b*, is a given ratio; and this sine, which is the sine of incidence on the second plane B *b*, will be to the sine of emergence from the third plane C *c* in a given ratio; and this sine to the sine of emergence from the fourth plane D *d*, in a given ratio; and so on, continually: and *ex æquo*, the sine of incidence on the first plane, to the sine of emergence from the last plane, in a given ratio. Let the intervals of the planes be now diminished, and their number increased continually, so that the action of attraction or impulse, exerted according to any assigned law, may become continual; and the ratio of the sine of incidence on the first plane, to the sine of emergence from the last plane, being continually given, will be still given. Which was to be demonstrated.

151. Newton goes on to demonstrate that, the same things being supposed, the velocity before incidence, is to its velocity after emergence, as the sine of emergence to the sine of incidence. This he effects by the following process:

Let AH, I *d*, be taken equal; and let the perpendiculars AG, *d* K, be erected, meeting the lines of incidence and emergence GH, IK, in G and K. In GH let TH be taken equal to IK; and let fall T*v* perpendicular to the plane A *a*. And (Mechanics § VIII.) let the motion of the body be resolved into two, one perpendicular to the planes A *a*, B *b*, C *c*, &c. the other parallel to them. The force of attraction, or of impulse, acting in perpendicular directions, does not at all change the motion in parallel directions; and therefore the body, with this motion, will describe in equal times, those equal parallel intervals which are between the line AG and the point H, and between the point I and the line *d* K; that is, it will describe the lines GH, IK, in equal times. Therefore, the velocity before incidence is the velocity after emergence, as GH to IK, or TH; that is, as AH or I *d* to *v* H; that is (with respect to TH or IK as radius), as the sine of emergence to the sine of incidence.

152. As all which relates to the law of refraction is too important to be passed over slightly, we shall add an analytical demonstration of the constant proportionality of the sines of incidence and refraction.

Supposing, with Newton, that the change in the direction of the refracted ray, is due to a *constant* accelerating force, the curve described by the inflected ray will be a parabola (art. 150).

Now, let $y = -ax + b$, be the equation of the incident ray, and let v denote the velocity of impulsion in a rectilinear direction, by the augmentation of the accelerating force in a unit of time. By virtue of the force of impulsion, in a time t, the ray would have described the space H n by an uniform motion, and we should have $B^2 n = v t$, or, since $B^2 n^2 = B^2 r^2 + r^2 n^2 = a^2 x^2 + x^2 = x^2 (1 + a^2)$, and $(1 + a^2) x^2 = v^2 t^2$.

But, having regard to the attraction, and putting s for the vertical space, described by reason of that force, we shall have (Mechanics, § III. p. 85) $s = g t^2$; which compared with the last equation, gives

$$s = \frac{g x^2 (1 + a^2)}{v^2},$$

for the vertical space, that the attraction will cause the particle to describe, to retain it on the curve. The ordinate of the curve will therefore be

$$y = -ax + b - \frac{2 g x^2 (1 + a^2)}{2 v^2},$$

which, making $\frac{2 g (1 + a^2)}{2 v^2} = c$, will become

$$y = -ax - cx^2 + b,$$

an equation to a parabola.

The sine of refraction, at the point I, will be expressed by $\frac{1}{\sqrt{(1 + v^2 + 4bc)}}$: for the tangent of the angle, at the point c, is found by taking the value of $\frac{\dot{y}}{\dot{x}}$ for that of x, which corresponds to $y = 0$; which gives $\frac{\dot{y}}{\dot{x}} = -\sqrt{(a^2 + 4bc)}$. The sine of incidence at the point H is $\frac{1}{\sqrt{(1 + a^2)}}$. The ratio of these sines is constant, and is equal to the ratio of $\sqrt{\left(1 + \frac{g}{v^2}\right)}$ to 1.

153. For the purpose of determining exactly the angles of incidence and refraction, and thence the ratio of their sines, different processes have been recommended and employed by different philosophers. The method exhibited by M. Biot, with respect to transparent solids, is, on the whole, as elegant as any which we have seen. A brief view of it may here be presented.

Investigation of formulæ for refraction through prisms.

Fig. 20.

Let ABC (fig. 20) be a section made in a triangular prism (constituted of the diaphanous substance which we would subject to the experiment), by a plane perpendicular to its rectangular faces. In this plane, conceive a luminous ray SI to fall on the surface of the prism at the point I, and be refracted in the direction II′. From the first law of refraction, the two lines SI, II′, will continue in one plane perpendicular to the refringent surface; they will, therefore, remain in the plane of the section ABC, which fulfils those conditions. The refracted ray, after having passed through the substance of the prism, will meet the second surface in I′, and there, on passing into the air, will again be refracted into a new direction I′O; still, however, in the same plane, for the same reason as before. A spectator, therefore, who shall place his eye on a convenient point O of the direction I′O, will receive simultaneously on the organ of vision the refracted ray I′O and the direct ray OS that proceeds from the luminous object. Now, it is evident that the angle of deviation SOI′ will, in all cases, depend upon the law of refraction; and, therefore, conversely, that a careful estimate of that angle may lead to the determination of the refraction itself.

Let ϕ be the angle SIA which the incident ray forms with the surface of the prism *towards the base*, and ϕ_1 the opposite angle AII′ formed by the refracted ray with the same surface: ϕ_2 the angle, II′C, which the refracted ray makes with the second surface towards the base, and ϕ_3 the opposite angle made by the emergent ray after its second refraction. The same kind of notation may evidently be continued to any extent.

154. Now, to verify the general law of refraction, suppose the sines of incidence and refraction, reckoned from the cathetus of incidence at each surface, to be in the ratio of n to 1. In reference to the first surface, these angles will be $90° - \phi$ and $90° - \phi_1$, supposing ϕ to be less than a right angle: so that we shall have $\frac{\cos\phi}{\cos\phi_1}$, for the ratio of their sines. In the case where ϕ is greater than 90°, the angle of incidence, reckoned from the said cathetus, will be $\phi - 90°$ and $\phi_1 - 90°$. So that we shall still have $\frac{\cos\phi}{\cos\phi_1}$ for the ratio of their sines. The like will evidently hold at the second surface, where, of course, the ratio of the sines of incidence and refraction will be $\frac{\cos\phi_3}{\cos\phi_2}$; placing always in the numerator of the ratio, that of the two angles which obtains in the air. But since these ratios are constant with regard to the same refracting substance, we shall have

$$\frac{\cos\phi}{\cos\phi_1} = n, \text{ and } \frac{\cos\phi_3}{\cos\phi_2} = n.$$

155. There must be further introduced into consideration, the refringent angle BAC (suppose $= a$) of the prism; which, evidently, has its influence on the deviation of the ray, by the inclination which it gives to the second surface, with regard to the refracted ray, &c. For, relatively to the triangle IAI′, the angle ϕ_2 is exterior; so that we have

$$\phi_2 = a + \phi_1:$$

this condition we blend with the others, thus:

$$\cos\phi = n\cos\phi_1 \ldots\ldots \phi_2 = a + \phi_1 \ldots\ldots \cos\phi_3 = n\cos\phi_2. \qquad (1)$$

Of these, the first makes ϕ_1 known, when the angle of incidence ϕ and the ratio n are given: ϕ_1 being known, as the angle a of the prism is given, the second equation will give ϕ_2; and, by means of this and n, the last equation will give the emergent angle ϕ_3.

This angle must now be connected with the deviation SOI′, in order that this, when it can be observed, may serve for the determination of n. Here the mode of connection presents itself at once; for the figure SII′O being a quadrilateral, the sum of its interior angles will be equal to four right angles. Let, then, Δ denote the deviation SOI′, and δ the angle ISO. This last, being formed by the direct ray SO and the incident ray SI, is given from the positions of the object, the observer, and of the point of incidence. The two other angles of the quadrilateral are $\phi + 180° - \phi_1$, and $\phi_2 + 180° - \phi_3$. We, therefore, have

$$\delta + \phi + 180° - \phi_1 + \phi_2 + 180° - \phi_3 + \Delta = 360°.$$

Striking 360° from both members of the equation, and putting for $\phi_2 - \phi_1$, its value a, we shall have

$$\delta + \phi + a + \Delta - \phi_3 = 0,$$

which gives $\qquad \phi_3 = \phi + \Delta + \delta + a. \qquad (2)$

By means of this equation, ϕ_3 will be entirely expressed in terms of the data of the problem, and of the observed deviation Δ. It will remain, therefore, to know ϕ_2, in order to obtain n. To effect this, we may proceed thus:

Substituting in the first of the equations (1) for ϕ_1, its value deduced from the second, and combining it with the third, we shall have

$$\cos\phi = n\cos(\phi_2 - a),$$

and $\qquad \cos\phi_3 = n\cos\phi_2.$

The difference and the sum of these equations will give, respectively,

$$\cos\phi_3 - \cos\phi = n[\cos\phi_2 - \cos(\phi_2 - a)],$$
$$\cos\phi_3 + \cos\phi = n[\cos\phi_2 + \cos(\phi_2 - a)].$$

On dividing the first of these equations by the second, member for member, n will disappear, and we shall have

$$\frac{\cos\phi_3 - \cos\phi}{\cos\phi_3 + \cos\phi} = \frac{\cos\phi_2 - \cos(\phi_2 - a)}{\cos\phi_2 + \cos(\phi_2 - a)}.$$

But, it is a general property of any two angles p and q demonstrated in our treatise on Trigonometry, that

$$\frac{\cos p - \cos q}{\cos p + \cos q} = -\tan\tfrac{1}{2}(p+q)\tan\tfrac{1}{2}(p-q).$$

The respective terms of the above equation for the cosines of ϕ, ϕ_1, ϕ_2, &c. being transformed agreeably to their corresponding values in the tangents, we have

$$\tan\tfrac{1}{2}(\phi_3 + \phi)\tan\tfrac{1}{2}(\phi_3 - \phi) = \tan(\phi_2 - \tfrac{1}{2}a)\tan\tfrac{1}{2}a.$$

Of this equation, the first member contains only known quantities. The second contains no other unknown quantity than $\tan\phi_2$. This quantity, therefore, will be determined. But it may be previously remarked that, from the value of ϕ_3 given by equation (2) we have

$$\tfrac{1}{2}(\phi_3 + \phi) = \phi + \tfrac{1}{2}(\Delta + \delta + a);$$
$$\tfrac{1}{2}(\phi_3 - \phi) = \tfrac{1}{2}(\Delta + \delta + a.)$$

Substituting these values in the equation which gives ϕ_2, all the results of the problem will be comprized in the formulæ

$$\left.\begin{array}{l} \tan(\phi_2 - \tfrac{1}{2}a) = \\ \dfrac{\tan[\phi + \tfrac{1}{2}(\Delta + \delta + a)]\tan\tfrac{1}{2}(\Delta + \delta + a)}{\tan\tfrac{1}{2}a}; \\ \phi_3 = \phi + \Delta + \delta + a; \quad n = \dfrac{\cos\phi_3}{\cos\phi_2}. \end{array}\right\} \quad (3)$$

The first of these will enable us to determine $\phi_2 - \frac{1}{2}a$, from the data furnished by observation; and to this, adding $\frac{1}{2}a$, we have ϕ_2. The second will enable us to find ϕ_3: and from this and ϕ_2, by means of the third equation, we obtain n. These formulæ are evidently fitted for logarithmic computation; and require simply the usual precautions with regard to the signs of the trigonometrical quantities.

156. Instead of eliminating ϕ_1 from the primitive formulæ, we might have eliminated ϕ_2. In that case, after operating in a similar manner, we should find

$$\left.\begin{array}{c} \tan(\phi_1 + \tfrac{1}{2}a) = \\ \dfrac{\tan[\phi + \frac{1}{2}(\Delta + \delta + a)] \tan\frac{1}{2}(\Delta + \delta + a}{\tan\frac{1}{2}a}; \\ n = \dfrac{\cos\phi}{\cos\phi_1} \end{array}\right\} \quad (4)$$

These formulæ are, in certain cases, not merely useful, but necessary; for when the angle of emergence ϕ_3 is a right angle, the formulæ (3) cease to be applicable. In such circumstances ϕ_2 also is a right angle, and n is presented under the form $n = \frac{0}{0}$. Then recourse must be had to the formulæ (4). These, for like reasons, cease to be applicable when ϕ is a right angle: but, whenever that is the case, we may employ the equations (3).

157. Let us now inquire, from these formulæ, what will happen when the angle a vanishes. This is interesting to know, because it is the case of refraction through a transparent plate whose surfaces are parallel. Taking the equations (1) and (2), and supposing $a = 0$, we have $\phi_2 = \phi_1$: so that the two equations in ϕ and ϕ_3 become

$$\cos\phi = n\cos\phi_1 \; \ldots \; \cos\phi_3 = n\cos\phi_1:$$

whence we have either

$$\phi_3 = \phi, \text{ or } \phi_3 = 360^\circ - \phi.$$

Of these values of ϕ_3 the second is inadmissible, because it would make the emergent ray enter within the plate, instead of issuing from it. Adopting, therefore, the former value, and introducing it into the equation $\phi_3 = \phi + \Delta + \delta + a$, where, also, $a = 0$; it will become $0 = \Delta + \delta$; whence $\Delta = -\delta$.

The interpretation of these results will be very natural on the contemplation of fig. 21. For, in the first place, since $\phi_3 = \phi$, the emergent ray I′O, and the incident ray SI make equal angles with the faces of incidence and emergence; consequently those rays are parallel. Hence their inclination to the line OS is equal, but opposite; that line passing between the two rays. This is indicated by the opposite signs, in the formula $\Delta = -\delta$. Generally, the deviation Δ is negative whenever the emergent ray I′O passes below the direct ray OS; because, in the establishment of the formulæ, we have supposed Δ to be positive in the contrary case.

The angle δ vanishes when the object is so remote that the rays which it transmits to the observer, and to the point of incidence, may be regarded as parallel. Then, also Δ vanishes; that is to say, the emergent ray I′O coincides, as to sense, with the direct ray SO.

158. It further results from this investigation, that when we behold a luminous point through a piece of glass whose two surfaces are parallel, the image of that point deviates a little in consequence of refraction, and the deviation is equal to the angle δ, formed by the two rays from the object to the observer, and to the point of incidence. If the luminous point or object is very remote, this angle vanishes, and the deviation becomes insensible.

Numerical example into a prism of flint glass.

159. To render the application of the preceeding formulæ easy and evident, we shall subjoin a numerical example. M. M. Biot and Cauchoix, observed the refraction of a prism of French flint glass, manufactured by M. d'Artigues. The refringent angle of this prism, measured by the reflection of the light, was 24° 25′ 30″. The angle of the incident ray, with the first face of the prism, reckoned from its base, was 90° 17′ 10″. The luminous point was sufficiently remote, for the rays from that point to the observer, and to the prism, to be regarded as parallel. The deviation was observed to be 17° 45′ 9″, for the green rays.

From these data we have

$$\phi = 96^\circ 17' 10'', a = 24^\circ 25' 30'', \delta = o, \Delta = 17^\circ 45' 9''$$

The calculation will, therefore, be as below:

$$\begin{array}{rr} \frac{1}{2}(\Delta + a) = 21^\circ 5' 20'' \text{ log. tan} = & 9{\cdot}5861878 \\ \phi + \frac{1}{2}(\Delta + a) = 117^\circ 22' 30''. \text{ add} & \\ \ldots\ldots\ldots\ldots \text{ log. tan} = & 10{\cdot}2858402 \\ \hline & 19{\cdot}8720280 \\ \frac{1}{2}a = 12^\circ 12' 45'' \text{ sub. log. tan} = & 9{\cdot}3353295 \\ \hline \text{Remains, log. tan. } (\phi_2 - \frac{1}{2}a) = & 10{\cdot}5366985 \end{array}$$

This tangent being negative, falls in the second quadrant, and the corresponding angle is $(\phi_2 - \frac{1}{2}a) = 180^\circ - 73^\circ 47' 45''$.

$$\begin{array}{rrr} \Delta + a = 42^\circ\ 10'\ 39'' & \quad & \phi_2 - \frac{1}{2}a = 106^\circ\ 12'\ 15'' \\ \phi = 96\ \ 17\ \ 10 & & \frac{1}{2}a = 12\ \ 12\ \ 45 \\ \hline \phi^3 = 138\ \ 27\ \ 49 & & \phi_2 = 118\ \ 25\ \ 0 \end{array}$$

$$\begin{array}{l} \text{From log. } \cos\phi_3 = 9{\cdot}8742138 \\ \text{Sub. log. } \cos\phi_2 = 9{\cdot}6774975 \\ \hline \qquad \text{Log } n = 0{\cdot}1967163; \end{array}$$

whence $n = 1{\cdot}57296$.

For the sake of verification, we may compute n by means of the angles of incidence and refraction on the first face; where

$$n = \frac{\cos\phi}{\cos\phi_1}.$$

Here we have $\phi_1 = \phi_2 - a$, and, from the preceding,

$$\phi_2 = 118^\circ 25', a = 24^\circ 25' 30''.$$

$$\begin{array}{lr} \text{Hence, } \phi_1 = 93^\circ 59' 30'' \ldots \text{ sub. log. cos} = & 8{\cdot}8426803 \\ \phi = 96\ \ 17\ \ 10 \ldots \text{ from log. cos} = & 9{\cdot}0393878 \\ \hline n = 1{\cdot}57292 \qquad \text{log. } n = & 0{\cdot}1967075 \end{array}$$

The accordance of these two results is quite as great as can naturally be expected.

160. If the experiment be repeated by varying the incidence ϕ of the ray on the anterior surface of the prism, the computation will give values of n that correspond within the limits of variation, that may naturally be assigned to errors of observation. This correspondence is a practical confirmation of the truth of the law expressed by the sines.

Process of M.M. Levy and Rodriguez.

161. The value of $n^2 - 1$, and thence of n, may be readily obtained, by another formula, first given by M.M. Levy and Rodriguez. Resume the two fundamental equations

$$\cos\phi = n\cos(\phi_2 - a), \; \ldots \; \cos\phi_3 = n\cos\phi_2.$$

Developing the first of these, and exterminating $\cos\phi_2$ and $\sin\phi_2$, by means of the second, it becomes

$$\cos\phi = \cos a \cos\phi_3 + n \sin a \sqrt{\left(1 - \frac{\cos^2\phi_3}{n^2}\right)}.$$

This equation contains one unknown quantity, that is n. Causing the radical to disappear, we have

$$n^2 \sin^2 a = \cos^2\phi - 2\cos a \cos\phi \cos\phi_3 + \cos^2\phi_3.$$

Substituting, in the second member, instead of the powers of the cosines, their values in products of the different arcs, the result will assume this form:

$$(n^2 - 1)\sin^2 a = \cos^2 a + \cos(\phi + \phi_3)\cos(\phi - \phi_3)$$
$$- \cos a\,[\cos(\phi + \phi_3) + \cos(\phi - \phi_3)]:$$

which evidently becomes

$$(n^2 - 1)\sin^2 a = [\cos a - \cos(\phi + \phi_3)]\,[\cos a - \cos(\phi - \phi_3)].$$

Each of the factors of the second member may be transformed into a product of two sines; and this transformation gives

$$(n^2 - 1)\sin^2 a = 4 \sin\tfrac{1}{2}(a + \phi + \phi_3)\sin\tfrac{1}{2}(a - \phi - \phi_3),$$
$$\sin\tfrac{1}{2}(a + \phi - \phi_3)\sin\tfrac{1}{2}(a - \phi + \phi_3).$$

Here the second member is composed entirely of known quantities. For the angle of incidence ϕ is given, as well as the refringent angle a of the prism; and with these data, and the observed deviation, we have ϕ_3 by the formula (2)

$$\phi_3 = \phi + \Delta + \delta + a.$$

If, for the sake of simplicity, we put $\frac{1}{2}(\Delta + \delta) = \mathrm{H}$, we shall have

$$\tfrac{1}{2}(a+\phi+\phi_3) = a+\phi+\mathrm{H};\ \tfrac{1}{2}(a-\phi-\phi_3) = -(\phi+\mathrm{H});$$
$$\tfrac{1}{2}(a+\phi-\phi_3) = -\mathrm{H};\ \tfrac{1}{2}(a-\phi+\phi_3) = a+\mathrm{H}.$$

Substituting these in the preceding formula, and dividing by $\sin^2 a$, there results

$$n^2 - 1 = \frac{4\sin\mathrm{H}\sin(a+\mathrm{H})\sin(\phi+\mathrm{H})\sin(a+\phi+\mathrm{H})}{\sin^2 a}.$$

Thus $n^2 - 1$, and consequently n becomes known. In practice, there will not be more facility in the use of this formula than of those before given: yet it deserves a place, on account of its analytical elegance.

162. Upon the same general principles, and with appropriate apparatus, the ratio of the sines of incidence and refraction is determined, with respect to liquids and acriform fluids; but as the practical details would occupy more space than is consistent with their importance, in a general treatise like the present, we must now turn to other subjects.

The result obtained from our investigation in the foregoing part of this section gives, for the ratio of the sine of incidence to the sine of refraction (when a ray passes out of air into glass), 1·57292 : 1, which is however usually stated as 3 to 2, or 31 : 20; and consequently the sine of incidence to the sine of refraction, when the ray passes from glass into air, as 20 to 31. In like manner, when a ray passes out of air into water, we call the ratio that of 3 : 4, and from water into air, 4 to 3; see (art. 145). The latter ratios will be determined with greater accuracy in a subsequent investigation; till then we shall suppose them to be as above stated, being near enough for our present purpose.

§ II. CATOPTRICS.

I. *Images and foci by reflection.*

Catoptrics.

163. *Def.*—When several luminous rays (as those from the sun) arrive at a concave surface, the series of intersections formed by reflected rays indefinitely near to each other, form luminous curves (fig. 22) which are called *caustic by reflection*. In surfaces of revolution, all the generating curves have equal caustics, which touch mutually in one and the same point of the axis called the *focus*. Thus it appears, that in this sense the focus is not a mathematical, but a physical point, which is the assemblage of several rays. When the surface is convex, the caustics are imaginary. These curves are rectifiable when they are derived from algebraic curves. The properties of caustics are usually investigated in our treatises on Fluxions; and M. Haüy and some other authors apply them to the determination of foci: but the method more generally employed has the preference in point of simplicity, and is, therefore, here adopted.

164. PROBLEM.—Having given a point, or an object O, posited on the axis AO of a spherical mirror BAM, whether concave, as figs. 23, 24, or convex, as fig. 25, and an incident ray OM indefinitely near to the axis AO; to find the point F of the axis through which the ray reflected from the point M will pass.

Draw from the centre C of the spherical surface to the point M, the right line MC; this being, by the principles of geometry, perpendicular to the reflecting surface at the point M, is the cathetus of incidence. The angles OMG, or CME, is, therefore, the angle of incidence: whence, making CMF = OMG, the reflected ray is MD, which intersects the axis OA in F, the point sought.

To find an analytical expression of AF, equal to MF, when M is indefinitely near to A; let OA, or OM, the distance of the object from the mirror be put $= \pm d$, $+ d$ when the mirror is concave, $- d$ when it is convex; these signs being thus determined by the position of the incident ray OM with regard to the semidiameter AC of the mirror. Let $\mathrm{AC} = r$, and FA, or $\mathrm{FM} = f$. Then we shall have $\mathrm{FC} = r - f$, figs. 23, 25, and $\mathrm{FC} = f - r$, fig. 24; also $\mathrm{CO} = -r + d$, fig. 23, $\mathrm{CO} = r - d$, fig. 24, and $\mathrm{CO} = r + d$, fig. 25. Now, by the principles of geometry, since FMC = CMO,

$$\mathrm{CO} : \mathrm{CF} :: \mathrm{MO} : \mathrm{MF}.$$

This, in figs. 23, 24, is $\mp r \pm d : \pm r \mp f :: d : f$: whence we have $f = \frac{dr}{2d - r}$

And, in fig. 25, we have CO : MO :: sin CMO or FMC : sin MCO or MCF in the triangle FMC, sin FMC : sin MCF :: CF : FM; consequently,

$$\mathrm{CO} : \mathrm{CF} :: \mathrm{MO} : \mathrm{MF},\ \text{or,}\ r + d : r - f :: d : f.$$

Whence we have, for convex mirrors, $f = \frac{dr}{2d \mp r}$.

So that the general formula for spherical mirrors both *concave* and *convex*, is

$$f = \frac{dr}{2d \mp r}.$$

II. *On the magnitude of the images.*

165. In order to ascertain the magnitude of the images, let AA′ (fig. 26), a convex spherical mirror, and of a radius equal to AC; DD′, an object whose extremities are supposed equidistant from the centre C;

169. Since all the light which emanates from the object O, and falls on the mirror at a small distance from the point A, will, in *concave* mirrors, pass either through the point F, or very near to it, there must be formed in that point a sensible image of the object O. In *convex* mirrors, the reflection disperses the rays which issue from the point O, and gives them such a direction that they appear to concur at the point F; and the reflected light gives an image of the object as at that point.

170. Since the image of the object O is placed upon the axis of the sphere which passes through that point O, it follows that, if there be any obstacle which would prevent the drawing of an uninterrupted line from the object to the centre of the sphere, there cannot be formed an image of the object.

171. If the object O emits rays of light in sufficient quantity and of such a kind as to excite a perceptible heat, without being condensed, such as are rays from the sun, from a torch, &c. it is evident that, if these rays are combined and brought nearer together by means of a concave mirror, they will produce a degree of heat proportional to their density, and to the actual heat of the incident rays. Hence it is, that the name of *focus*, or burning point, has been given to the point where the reflected rays meet.

II. *On the place, situation, and motion of images by reflection.*

Place, motion, &c. of images by reflection.

172. We have already seen how certain modifications of the general theorem, indicate the diversities in the apparent *magnitude* of the image. We may now, by making different suppositions as to the distances to which an object exposed to the reflecting surface of a spherical mirror may be removed, find the place and the motion of its image. Taking our hypotheses of distance in order:—

173. When the distance from the mirror is infinitely small, the image is infinitely near, and behind the mirror. For, when $d = \frac{1}{\infty}$, the general expression $f = \frac{d\,r}{2\,d \mp r}$ (art. 164), becomes $f = \frac{1}{\mp\,\infty}$. The sign $-$ indicates, that in the concave mirror, the image is on the opposite side to the radius of the concavity, which we suppose $= +\,r$; consequently, the image is behind the mirror, The sign $+$ shows, that with regard to the convex mirror, the image is on the side of the centre of convexity, of which the radius has been supposed $+\,r$. And, as it is evident that, whatever value we assign to d in the formula $f = \frac{d\,r}{2\,d + r}$, the value of f cannot become negative, it follows, that, in the convex mirror, the image is necessarily on the side of the centre of convexity, at whatever distance we conceive the object. This must be kept in mind throughout.

173. As the distance of the object from the mirror increases from 0 to a quantity equal to half the radius of sphericity, the image will continue to recede behind the mirror. In the *concave* mirror it will recede from apparent contact to an infinite distance; and in the *convex* from contact to the distance of $\frac{1}{4}\,r$. For, 1st, conceiving d to be less than $\frac{1}{2}\,r$, or $2\,d$ less than r, then will $2\,d - r$ be negative: therefore, in the concave mirror, f also is negative, and the image behind the mirror. But, 2dly, if we conceive $d = \frac{1}{2}\,r$, the formula for the concave mirror becomes $f = \frac{\frac{1}{2}\,r\,r}{0} = \infty$; which shows that if the object be placed at a distance equal to half the radius, the image will then be infinitely remote; or, which amounts to the same, the rays of light that pass from the object to the mirror, will be reflected in directions respectively parallel. 3dly, In the case of the convex mirror, when $-\,d = \frac{1}{2}\,r$, we have $f = \frac{-\frac{1}{2}\,r^2}{-\,2\,r} = \frac{1}{4}\,r$.

174. When the distance of the object from the mirror increases from half radius to equality with the radius, the image in the *concave* mirror is on the same side, and approaches the mirror from infinity till it arrives at the centre of concavity: in the *convex* mirror, the same motion of the object causes the image which is behind the mirror to remove from the distance of $\frac{1}{4}\,r$ to $\frac{1}{3}\,r$.

For, it is evident that so long as d is greater than $\frac{1}{2}\,r$, or $2\,d$ greater than r, the theorem for the concave mirror can never become negative; hence, the image will be always on the side of the concavity. And, if we convert the theorem into a proportion

$$d : 2\,d - r :: f : r,$$

because of d being greater than $\frac{1}{2}\,r$, and less than r, the antecedent d is greater than the consequent $2\,d - r$; consequently f exceeds r, and the image is beyond the centre. If $d = r$, then $2\,d - r = d$, and $f = r$. Therefore, in the concave mirror, the object being at the centre, the image is likewise there. But, in the convex mirror, making $-\,d = r$, we have $f = \frac{1}{3}\,r$.

175. Hence we see why a spectator, on placing his eye before a concave mirror, between the distance of $\frac{1}{2}\,r$ and the centre, cannot perceive the image. It is behind the eye. It is infinitely distant when the eye is at half the radius of concavity, and it moves from infinity to the centre, where it becomes confounded with the eye, while the eye recedes from half the radius to the centre.

176. The distance of the object from the mirror increasing from the centre of sphericity to infinity, the image will advance *towards the concave mirror*, from the centre to the distance of half the radius; and will retire *behind the convex mirror*, from the distance of $\frac{1}{3}$ to that of $\frac{1}{2}$ the radius.

For, r being then smaller than d, in the proportion $d : 2\,d - r :: f : r$, the antecedent d is less than the consequent $2\,d - r$; therefore f is less than r. And if we put $d = \infty$, the general theorem becomes $f = \frac{1}{2}\,r$.

177. Theorem.—If an arc of a circle OPQ (figs. 28, 29), having the same centre as a spherical mirror, serves as an object presented to that mirror: 1st, the image $o\,p\,q$ is a like arc of a concentric circle. 2dly, The radius of that image is greater or less, and, consequently, the image itself is greater or less, according as the said image is nearer to, or further from the centre C of the mirror. 3dly, That image will be direct, viz. its position will correspond with that of the object, so long as the image and the object are on the same side of the centre C; while, on the contrary, the image will be inverted when the centre falls between the image and the object.

For, since OPQ is concentric with BAD, the right lines OB, PA, QD, which pass through the centre C, and on which are situated the images *o*, *p*, *q*, of the points O, P, Q, are equal respectively: therefore *d*, which expresses their value in the general formula, is a constant quality, as well as *r*: consequently, *f*, which depends upon *d* and *r*, is a constant quantity; or, in other words, *o*B, *p*A, *q*D, are equal, and *opq*, OPQ, BAD, concentric arcs, since the same reasoning will apply to any number of points.

178. This premised, it is evident, 1st, that when the image and the object are on the same side with respect to the centre, as in fig. 29, the image is posited similarly with the object, since every point of the image is on a radius which passes through the corresponding point in the object. But that if the image is beyond the centre with regard to the object (fig. 28), the right lines, on which are the images of the several parts of the object, pass necessarily through the centre of sphericity of the mirror; and, therefore, those which proceed from points that are above the axis that passes through the middle of the object, will be found below, after having passed through the centre, and *vice versa.* If, therefore, one of those right lines which are above that axis, issue from the superior part of the object, which is, of consequence, above the axis, the image of that part must be below, because it cannot be formed upon that line, until after it has passed the centre: consequently, the whole image is inverted with respect to the object.

It is evident, also, 2dly, that the whole image of an object being comprised between the right lines (as CB, CD, or their continuations) which concur at the centre, it must be so much the smaller as it is nearer the centre of sphericity; and the contrary.

179. In the *convex* mirror, the image of an object that forms an arc concentric with the mirror, is always *direct,* because it is always on the same side of the centre, with regard to the observer, as the object; and it decreases as the object is more remote, because it approaches more and more to the centre. In the *concave* mirror, the image is *erect*, and continues to augment in proportion as the object recedes from the surface of the mirror to half the radius; it decreases and is *inverted* as the object passes from half the radius to the centre; it increases again, but is still *inverted* as the object proceeds from the centre till it is infinitely distant. It must be remarked, principally in this latter case, that if the object does not augment, but if it take merely a concentric figure, in proportion as it recedes its image will diminish.

180. The smaller the radius of sphericity of the mirror, the smaller, *cœteris paribus,* will be the images.

181. *Remark.*—The preceding theorem cannot be strictly applied to all sorts of objects placed before spherical mirrors: yet, supposing them to be so small that their length may be regarded as an arc concentric with the mirror, we may, by means of what has been explained above, comprehend, 1st, Why the images of objects placed before a spherical mirror are sometimes larger, at others smaller, than the objects; 2dly, Why they are sometimes direct, at others inverted; 3dly, Why they seem to recede from the object when the object is removed further from the concave mirror, &c.

182. We see, also, that the images of objects, whose surfaces are not spherico-concentric, must be so much the more disfigured, or the less similar to the objects, as their surface is the greater, and as the radius of sphericity of the mirror is the less. Thus, for example, a right line placed before a spherical mirror must have a curvilinear image, because the points of that right line being at unequal distances from the mirror, the images also of those points are at unequal distances; and the inequalities are not in a constant ratio. The images, then, are the more disfigured, as the objects are the nearer the middle of the radius, in the *concave* mirror; for in that position the images are very large, and a small difference of distance from the mirror in the several parts of the object, will cause great differences of distance and magnitude, in the corresponding portions of the image.

III. *Application of the preceding theory to plane mirrors.*

Application to plane mirrors.

183. Regarding plane mirrors as spherical ones in which the radius of sphericity is infinite, we may easily deduce their principal properties from the general theorem, by making $r = \infty$. In that case, $f = \frac{d\,r}{2\,d - r}$, becomes $f = -\,d$. From this we learn, that the images seen in plane mirrors are always as much behind as the objects are before, and that they are always direct. And since, in spherical mirrors the image of each point of an object is in the right line which passes through that point and the centre, and which is, consequently, perpendicular to the surface of the mirror, the image of each point of an object placed before a plane mirror, is in the prolongation of the perpendicular drawn from that point to the surface of the mirror. Lastly, because perpendiculars let fall from any extremities of the object upon the mirror, are respectively parallel, the images comprized between those right lines are equal to the object in all their dimensions.

Image and object equal and at equal distances from the plane mirror.

Fig. 30.

184. Independently of the spherical mirror, let DD′ (fig. 30), be an object, and AB a plane mirror, O the point at which the observer's eye is placed: the angle of incidence being equal to the angle of reflection, the angles DMA and FMA formed by the incident and the reflected rays with the reflecting plane, are perfectly equal; therefore, since the angles at A are right angles, AF = AD, that is to say, the rays reflected to the eye, or their prolongations meet at a distance from the mirror equal to that of the object; and the same will obtain with regard to rays issuing from the point D′, and all intermediate points. The image of the object, therefore, is situated behind the mirror, at a distance equal to that of the object before it; and it is direct, because the rays are reflected, without crossing in the same order as that in which they arrived at the surface. And further, since the lines DF, and D′F′ are equal and parallel, it is evident that DD′ = FF′; that is, that the image is equal to the object. If the object DD′ were inclined with respect to the mirror, the equality of image and object would still obtain; since we should have on the different sides of the mirror, equal trapezia instead of the equal rectangles ADD′N and AFF′N.

185. If in any plane speculum a person sees his own image, or that of an object that is behind him, if the image appear erect, it will appear reversed as to right

and left; because, the object and the image face each other, and, as it were, look contrary ways. The case is not unlike, when the object is between us and the mirror; though we are apt to make it different for want of considering that with respect to the image seen in the mirror, it is the back of the object that is now towards us; and so we now call that the right side of the object, which we should call the left, if we were on the other side of it.

Space occupied by an image.

186. Theorem I.—The image of an object that is placed parallel to the surface of a plane mirror, seems to occupy no more in the mirror than a space equal in each dimension to *half* of that which the object occupies.

Fig. 31.

Let AB (fig. 31) be one of the lineal dimensions of an object parallel to a mirror IG, and let AB be the image of AB behind the mirror. From any point P taken on AB, draw P*a*, P*b*, to the extremities of the image, those lines intersecting IG in the points I, E: then, it is evident, that IE is the part of the mirror occupied by the image *a b*: and because IG is midway between the object and image, it follows from the principles of geometry, that $\text{IE} = \frac{1}{2}\, ab = \frac{1}{2}\,\text{AB}$.

187. *Corol.*—Hence, that a person may see himself from head to foot, in a plane mirror, it is requisite that the mirror be at least half the length and half the breadth of the said person.

188. Theorem II.—If a plane mirror turn upon an axis, the angular motion of the images is double that of the mirror.

Relative motion of a mirror, and an image of a fixed object.
Fig. 32.

Let AB (fig. 32) be the first position of the mirror, OE an incident ray, EF the reflected ray. Suppose the mirror to turn upon an axis, which passes through the point E, and to take a new position CD; then will the incident ray OE have another ray, as EG, for that of reflection. Then, the angle FEG, which expresses the angular motion, or the quantity by which the reflected ray EG deviates from its first position, is double AEC the angular motion of the mirror. For the cathetus of incidence is always midway between the incident and the reflected rays; and as it is always perpendicular to the surface of the mirror, it has the same angular motion as the mirror itself. If, therefore, the angular motion of the mirror carry it towards the incident ray, it will cause the cathetus of incidence to remove so much further from it; and, in the same time, the reflected ray will remove from the cathetus by the same quantity, that the cathetus may, as before, be midway between the two rays. Consequently, the reflected ray is now separated from the incident ray, by a quantity double the angular motion of the lever.

189. If we give a motion of a quarter of a circle to the mirror, the reflected ray will describe a semicircle. And hence it is, that we can give so rapid a motion to the images of the sun reflected by the mirror. Thus also it is, that images of the sun reflected from water, frequently appear very agitated, especially when they are received on a plane a little remote from the point of incidence.

Two reflecting planes: angle formed by a reflected ray.

190. Theorem III.—If a ray of light be reflected once by each of two reflecting surfaces, and in a plane which is perpendicular to their common intersection, the angle contained between the first and last directions of the ray, is double the angle at which the reflectors are inclined to each other.

Let AB, CD, be two plane reflectors (fig. 33), inclined at the angle AGD; and let SB, BD, DH, be the course of a ray reflected by them. Produce HD to O, and SB, till it meets DH in H.

Then, because $\text{HBG} = \text{ABS} = \text{DBG}$;

$$\text{DBH} = \text{HBG} + \text{DBG} = 2\,\text{DBG};$$

In like manner, $\text{BDO} = 2\,\text{BDC}$:

And since,

$$\text{BGD} = \text{BDC} - \text{DBG};$$

we have,

$$2\,\text{BGD} = 2\,\text{BDC} - 2\,\text{DBG} = \text{BDO} - \text{DBH} = \text{BHD}.$$

Remark.—This theorem includes the fundamental property in the construction of Hadley's sextant and octant; and Theor. II. includes in like manner, the fundamental property in the construction of Bickford's reflecting quadrant.

191. Theorem IV. If two reflecting planes, BC, DC (fig. 34), be placed at right angles to each other, a body A moving along the line AB, perpendicular to the one speculum, and consequently parallel to the other, will have its images *a*, *b*, move the contrary ways; that is, while the body A is in motion, the image *b* goes on in the same direction, whilst *a* moves directly towards A: and the motions of A, *b*, and *a*, are all exactly at the same rate.

It is manifest, that the lines *a* B, *b c*, described by the images *a*, *b*, are the images of the line AB in the respective mirrors BC, DC; therefore *a* B = AB, and *a b* is perpendicular to BC: also *b* D = AD, and *b c* is parellel to DC. Whence the proposition is manifest.

192. If the line AB be inclined to the mirrors BC, CD (fig. 35), the lines *a* B, *b c*, described by the images *a*, *b*, may be found by drawing the perpendicular A *a*, to BC, and the perpendiculars A *b*, *h c*, to DC: then taking *d a* = *d* A, D *b* = DA, and C *c* = C *h*. Or, the lines *a* B, *b c*, may be found by making the angles *a* B *d* and AB *d* equal, and the angles *b e* D, A *e* D.

193. Theorem V.—If a plane mirror RS (fig. 36) be inclined to the horizon in an angle of 45 degrees, with its face downwards, an upright object CD will have its image *c d* in a horizontal position; and the image *a b* of an object AB, lying horizontally, will be vertical.

For, each object and its image forming equal angles with the mirror, and those which the objects form being each half a right angle, the angles CR *c*, BS *b*, are right angles. Whence the theorem is evident.

194. Hence, if RS be a plane mirror, with its face upwards, AB will be the image of an object *a b*; and the image of an object *c d* will be CD. So that a person who stands upright before such a mirror, will see his image extended horizontally, as it were, on the floor, with his face upwards.

195. A person looking in a plane mirror inclined to the horizon in an angle of 45°, with its face upwards, will see himself, as it were, in a flying posture; for he will seem suspended horizontally in the air with his face downwards; and if the mirror be sufficiently long, he will seem to fly downwards or upwards, as he walks towards or from the mirror.

196. If the plane mirror SR be inclined to the horizon in any other angle than 45°, the images *c*, *d* *ba*, of the vertical and horizontal objects CD, BA,

will be correspondently inclined from the position which they now have in the figure. And particularly, the objects CD, AB, retaining their vertical and horizontal position; if the angle RSB be greater than 45°, the image *c d* will recline upwards, the point *d* being higher than *c*; and the image *a b* will recline backwards from the speculum. The contrary will happen, when the angle RSB is less than 45°.

197. Upon a flat and smooth board FI (fig. 37), let there be made a groove or channel *a b c d e*, by means of thin ledges A, B, C, D, E, glued to the board FI; and let the edges of the boards A, B, C, &c. that are next the channel, form the same angles with the sides of the rectangular board FI.* The board being thus prepared, and ledges fixed to the sides of it, and another at *a e*; if the end FG be a little elevated, a marble or ivory ball placed at *a*, will move with tolerably uniform velocity along the channel till it gets to the bottom at *e*. Now, if the board be placed at the bottom of a box, in which is a looking-glass so inclined, that the board FI appears vertically, the end HI appearing uppermost, the ball will *seem* to roll upwards, till it arrives at the top at *e*.

This little deceit is surprizing enough till the cause is known; and to prevent its being immediately detected, the box should be placed a little above the eye, and the aperture in the front; the size of the glass being so limited that, from a certain station, nothing shall be visible but the image of the board FI. In the side *a e* of the box, and below the end *e* of the channel, may be a narrow board, whose end *e* may be lifted up for the purpose of throwing the ball back again to *a*; and then it will re-commence its journey, as before.

198. Theorem VI.—In any number of plane mirrors all lie in the same plane, there can be seen from the same place only one image of the same object.

For, in this case, there can be but one image; and the object is seen by rays proceeding directly to the eye, as from the image. Whence the theorem is evident.

199. Hence, if two plane mirrors be inclined backwards, so that perpendiculars to each meet somewhere beyond them, it is manifest that rays diverging from any object cannot be reflected by both mirrors to an eye having any given position; and therefore they can exhibit only one image of the same object. Vide fig. 35, where *a* may be supposed the object, *e* the eye, BC, CD, the two mirrors.

200. Theorem VII.—If two plane mirrors, AB, BC, be joined together so as to form any angle BAC, there will be a certain distance AF (fig. 38), within which a spectator will see his image in each mirror at the same time: and if to these be joined two other mirrors BD, CE, forming angles ABD, ACE, a spectator placed any where within a certain distance HG, will see his image in each of the four mirrors. And universally, however many plane mirrors are joined together in an arch of a circle, not too wide to be comprehended in one view, a small object placed near the centre of that circle, will be multiplied as many times as there are reflecting planes.

Upon AB, AC, raise the perpendiculars BF, CF. Now an object placed any where between F and A, will have its images in lines parallel to FB, FC, till it arrives at A, where both images will be united. Again, upon BD and CE raise the perpendiculars DG, BH, and EG, CH; and, from the same reasoning it will be seen, that a spectator placed any where between G and H, will see his image or part of it; both in BD and CE. And universally, if O be the centre of an arch ED, any number of mirrors joined together within this arch, will, each of them, exhibit the image of an eye placed at or near O. Whence the truth of the theorem is evident.

201. Hence it follows, that as a spectator walks from F towards A, his images in the two mirrors AB, AC, continually approach nearer together; and at a certain station he will see two heads and one body.

If the place of an object and of the eye be given, the phenomena may be easily deduced. Thus, if H be the place of an object, its image will be visible in the mirrors AB, AC, at all distances; because the angle of incidence HBF or HCF, is not less than the angle HFB or HFC. And in the mirrors BD, CE, the image will be also visible to a considerable distance beyond H; because the angle of incidence HDG is not much less than the angle HGD. For it is manifest that, when the angle of incidence is equal to the angle formed by the cathetus of incidence and the line AO, the reflected ray is parallel to AO.

Images formed by mirrors forming a given angle.

Fig. 39.

202. Theorem VIII.—The images of any sector SRT (fig. 39), contained between two reflecting planes RS, RT, inclined to one another in the given angle SRT, will all be concentric; and the number of these images will be such as will exactly complete the whole circle.

The images of the sector SRT, formed by the mirrors RS, RT, are respectively the sectors SRG, TRM. But it is manifest that the rays which form the image SCG fall on the speculum RT, after the same manner as if SRG were a real object; and so the said rays form a second image, or sector MRL. In like manner, the rays forming the image TRM, by means of the reflection of RS, form the image GRH; and the images MRL, GRH, form the sectors HRI, LRK, by the reflections of RS and RT respectively. Again, by the reflection of the rays forming either of the two last images at the respective speculums, the sector IRK will be exhibited, and so on till the whole circle is completed. But it is obvious that none of the rays forming the image IRK, or which exhibit the space IRK contained between the backs of the reflecting planes SR, TR produced, can be again reflected, or form another image.

203. Hence then it appears, that the images of any sector, contained between two reflecting planes inclined to each other in any given angle will be all concentric; and that the number of them will be such as will exactly complete the whole circle. But when the sector SRT is not an aliquot part of a circle, the last image as IRK, will be less than SRT, if the two last perfect images HRI, LRK, do not intersect each other. When they do so intersect, they will then frequently appear as one image, greater than SRT; but this image can never be twice as great as SRT: because it is manifest that each speculum forms the same number of images of the sector SRT; and, therefore, the remaining sector IRK may be of any magnitude not so great as the double of the original sector SRT.

204. The images A, B, C, D; *a*, *b*, *c*, *d*, of an object Q placed between two speculums RS, RT inclined

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* The breadth of the channel *a*, *b*, *c*, *d*, *e*, may be less than half the diameter of the ball; and the ball will run upon a less declivity, if the edges of the boards against which it bears be chamfered or sloped considerably.

Principle of the kaleidoscope.

to each other in any angle SRT, will be all in the periphery of the circle whose centre is R, and radius QR. There will be one image and no more in each figure of the sector SRT, and the arcs contained between the images will be alternately equal to 2 QS and 2 QT. This is sufficiently manifest from the preceding proposition; but it will not be amiss, in consequence of the very singular and popular application lately made of this principle in the construction of the *kaleidoscope*, to enter a little further at length into an investigation of this property.

Upon the reflecting planes RS, RT, let fall alternately the perpendiculars $Q m A$, $Q n a$; $A o B$, $a p b$; $B q C$, $b r c$; $C s d$, and $c t d$. Take

$$m A = m Q;\ n a = n Q;\ o B = o A, p b = p a$$
$$q C = q B;\ r c = r b;\ s D = s C, t d = t c$$

Now A, B, C, D, *a*, *b*, *c*, *d* are the images formed by the rays diverging from Q upon the speculum RS, RT respectively, after one, two, three, &c. reflections. For A, *a*, are first images of the object Q; and the rays proceeding from these images fall upon the opposite speculums so as to form the second images B, *b*; for the same reason, these again form the third images C, *c*; and lastly, the images D, *d*. But in this example, because the angle SRT, is an aliquot and also an even part of a circle, the two images D, *d*, coincide and form but one image; whence it is sufficiently manifest that, 1st, All the images are in the periphery of the circle whose centre is R, and radius QR: 2dly, There is one image and no more in every reflection of the space SRT; 3dly, The arcs.

$$A b = B c = 2 Q T;\ b C = a B = 2 Q S$$
$$C d = Q a = 2 Q T;\ D c = Q A = 2 Q S.$$

When the sector SRT is $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{10}, \frac{1}{20}$, &c. of a circle, the number of images of the said sector, and of an object Q, placed between the two reflecting planes, will be accordingly 2, 3, 4, 5, 9, 19, &c. But when the angle of the planes becomes too acute, there is a danger from the loss of light in the successive reflections of one or more of the last images becoming invisible, or very indistinct.

Parallel mirrors.

Fig. 40.

205. If the two plane speculums SR, TR (fig. 40), be placed parallel to one another, there will be a series of images of the floor, or space between them, indefinitely extended both ways; there being no other limitation of their number, than what is caused on account of the decrease of light by the continual reflections. Also an object Q, placed between the said speculums, will have a like series of images all situated in the indefinite line $B Q b$, perpendicular to the reflecting planes; and the distance between the said images will be equal to 2 Q T and 2 Q S, alternately, as is evident from what is stated above. A person standing between two such speculums will see, in that he faces, the images of both his fore and back part, repeated several times in a long row, but continually becoming fainter as the distance increases.

To determine the progress of a ray.

206. Problem. I.—Having given the angle of inclination of two plane speculums, and the situation of an object, and of the eye placed between them, it is required to trace the progress of any ray diverging from that object, through its several reflections to the eye.

Let RS, RT be the speculums, SRT the given angle, Q the object, and O the place of the eye (fig. 41).

Fig. 41.

It is obvious here, that the rays diverging from Q, and falling in the same manner on each of the two speculums RS, RT, will be reflected in a manner fectly analogous to each other; and therefore, th will be sufficient if we consider only the progress o of those rays, as for example, that which diverges Q upon RS. First, by drawing the perpendic QA, AB, BC, CD, upon the respective speculums by taking

$$m A = m Q, o B = o A, q C = q B, s D = s C,$$

we have the places of the images A, B, C, D.

Draw a line from O to any given image (A, B, &c.); and from the point where this line intersect speculum, draw a line to the next preceding im and from the point where this line meets the spect draw a line to the next preceding image, and so o the object itself Q; and this ray traced backwards Q, will be that which exhibits the given image t eye at O. Thus, to find the ray which paints the i D, draw the line $O k D$; and from the point of i section k, to the next preceding image C, the line k draw also the lines $i b B$, $h g A$, and, lastly, $g Q$; will $Q g$, $g h$, $h i$, $i k$, $k Q$, be the ray that exhibit image D to the eye at O.

For the angle $O k T = R k D = R k C$, and, t fore, the ray $i k$ is reflected from k to O. Again angle $k i S = C i k = K i B$; and the angle $i h$ $B h T = T h A$; also the angle $h g R = A g$ $S g Q$. Therefore, the ray diverging from Q to reflected from g to h, from h to i, from i to k, and k to O. In like manner, the image C is seen by m of the ray $Q d$, $d e$, $c f$, $f O$; the image B by th $Q b$, $b c$, $c O$; and the image A by the ray $Q a$, And, exactly, in the same way, we may trace the diverging from Q upon the speculum R T, till reach the eye at O.

207. It may be proper to observe here, tha pencil of rays which exhibits any image to the ey reflected orderly, as from each of the preceding im beginning with the first image, and from no other of the speculums: and consequently, the whole nu of reflections is always equal to the whole numb images; as the images A, B, C, or D, are see spectively by rays which have undergone one, three, or four reflections.

208. Again, the distance OD of any image D the eye at O, is equal to the whole ray proce from Q, till, after all its reflections, it arrives that is,

$$OD = Q g + g h + h i + i k + k O.$$

For it is manifest that

$$Q g + g h = h A = h B,$$

and $\quad i h + h B = i C$; also $i C + i k = k D$.

In like manner, we have

$$OC = Q d + d e + c f + f O,$$
$$OB = Q b + b c + c O,$$
$$OA = Q a + a O.$$

209. The last image, as D, of a small obje will be invisible, whenever the eye O is so situate the straight line OD passing through the centre so as not to touch either of the speculums when meet not in R; in that case, the next preceding i C is to be reckoned the last image. Consequent these cases, if the arc ST be $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{10}$, or $\frac{1}{20}$ of a circle; the number of visible images of a object Q, will be accordingly one, two, three, eight or eighteen, &c.

210. In like manner, as in (art. 206), we may trace the rays exhibiting any image of an object Q, to the eye at O, placed between two parallel speculums SR, RT. For the image A is seen by the rays Q *r*, *r* O, the image B, by the rays Q *s*, *st*, *to*; the image *a*, by Q *v*, *v* O, and *b* by rays Q *x*, *xy*, *y* O. But the image on one side will be visible to a much greater distance, if the eye be placed behind one of the speculums, looking through a hole in it, or over its edge at the other speculum.

211. By placing two plane speculums, one at each end of a box, parallel to one another, and looking over or by the edge of one of them, the images of the bottom of the box will appear continued to a considerable distance. Or, if the two speculums be placed upright in a box, but inclined to one another, the trapezium thus formed by the bottom of the two glasses, and the lines joining them, will be repeated on each side till the images meet; and thus form two concentric polygons, whose centre is the point where the two speculums would meet if produced. By joining four speculums together, such as those we have been describing, viz. two parallel to each other, and the other two joining those parallel planes, and fixing them thus posited in a box, the bottom of the latter will be multiplied to a very astonishing extent, there being no other limitation to the number of images but what is owing to the continued loss of light from reflection. The top of the box may be covered with thin canvas, which will let sufficient light enter to render the exhibition very distinct. This experiment may be made very entertaining, by placing on the bottom of the box some toy, as two persons playing cards, soldiers on sentry, &c.; and if these be put in motion, by wires attached to their feet, or otherwise, and passing through the bottom or side of the box, it will afford a spectacle still more entertaining. Or three plane speculums joined together will have a like effect; for the polygon formed by two of the reflecting planes, will be repeated by the other. The bottom of the box may be covered with artificial moss, shining pebbles, flowers, &c.; only in all cases it is necessary to observe, that the upright figures between the speculums should be slender, and not too many in number, as they would otherwise too much obstruct the reflected rays in their passage to the eye.

212. Another application of this principle is as follows: If the sides of a small room, whose floor or base is a polygon, be all lined or covered with looking-glasses, the room being enlightened either through the top, or by a chandelier suspended in the middle, the different objects in it will be multiplied to an almost indefinite extent, the reason and principle of which multiplication will be obvious from what is stated above. Exhibitions of this experiment are sometimes made under the denomination of *luminous amphitheatres*. The room should not be very large; for the smaller it is, the oftener will the images be repeated before they become too faint to be perceived.

213. On principles precisely similar to the above, depends the operation of an instrument described by Bradley, in his Art of Gardening, published in the early part of the last century, which has lately attracted considerable attention on account of its supposed resemblance to the kaleidoscope of Dr. Brewster. We shall examine this question more at length in a subsequent chapter.

IV. *On glass mirrors, considered with reference to their reflecting and refracting surface.*

Distinction between metallic and glass specula.

214. We have hitherto only considered the phenomena and properties of plane mirrors, under the supposition of their reflecting all the rays impinging on them at their front surface, as would obviously be the case with all the reflected rays from a polished metal plate; but when the mirror consists of a plate of glass silvered at the back, the principal reflection is from the surface of the silver; and a ray of light thus passing through the glass will be subjected both to reflection and refraction, and consequently a slight modification of the preceding deductions becomes necessary to accommodate them to this case. Without immediately pursuing this inquiry, we may observe, that as some of the rays, falling on such a medium as we are supposing, are reflected at the first surface, without ever entering the glass, while the greater number pass through, and are afterwards reflected, there will in such cases be two images of the same object, the one near and faint, the other vivid and more distant from the eye of the observer, and that the weak image will become so much the more perceptible as we view it more obliquely; because when we observe it directly, that is, perpendicularly to the surface of the mirror, it falls upon, and is confounded with the vivid image formed by the reflection from the second or silvered face of the glass. We propose here to examine the effect of this refraction as it regards glass reflecting mirrors.

The focus of refracted rays. Fig. 42.

215. Theorem IX.—If Q (fig. 42) be taken to denote the focus of the incident rays *f*, upon a refracting plane BAC, and a point *q* be taken in a perpendicular QA (produced if necessary) to that plane, so that A *q* shall be to AQ as the sine of incidence to the sine of refraction, the point *q* will be the focus of those rays that are nearest to the perpendicular Q *a* after refraction; that is, if the points of incidence B, C, of the rays QB, QC, are not remote from A, the refracted rays B *e*, C *f* produced backwards would meet in *q* nearly.

Let QC be any incident ray, and C *f* the reflected ray, which produced backward, meets the perpendicular QA somewhere as in *q*. Through the point of incidence C, draw EH parallel to QA; then will QCE = AQC be the angle of incidence, and HC *f* = A *q* C the angle of refraction. And in the triangle Q *q* C, it will be as the sine of < CQ *q* or AQC to the sine of < C *q* Q or A *q* C, so is C *q* to CQ.

But when the angle of incidence AQC is very small, it will be C *q* : CQ :: A *q* : AQ *nearly*.

And the nearer any incident ray, as Q *b* or Q *a*, is to the perpendicular QA, the nearer will the sides *q b*, Q *b*, or *q a*, Q *a* of the triangles Q *q b*, Q *q a*, approach to a ratio of equality with *q* A, QA, until at length they will coincide, and therefore the point *q*, as determined by the preceding theorem, is without sensible error the focus of the refracted rays.

Cor. I.—If *q* be the focus of converging rays (*e* B, *f* C, &c.), then Q will be their focus after refraction; and because sin CQ *q* : sin C *q* Q :: A *q* : AQ, we shall have denoting the ratio of the sines by *m* to *n*

$$\mathrm{A}q \backsim \mathrm{AQ} : \mathrm{A}q :: m - n : m \text{ or } n,$$

according as the incidence is into a rarer or a denser medium. And if the point of incidence of a ray QC be so near to A, that the angles and their sines may be reckoned proportional, it will be m, or $n : m - n ::$ $\angle$ of *incidence* AQC : $\angle$ of *refraction* Q c q.

Explanation of certain optical phenomena.

2. The image q of a small object Q, placed under water, is, obviously, from what is stated above, one-fourth nearer to the surface than the object itself; and hence, the reason why the bottom of a river or pond always appear nearer than it actually is, the water being always one-third deeper than it appears to the spectator above. Persons who are not expert swimmers should be aware of this, or they may plunge into water which appears fleet, when it is actually beyond their depth; and many melancholy accidents may probably be attributed to this optical deception. On the same principle, depends the well-known phenomenon of a shilling or other object in a basin of water, appearing more elevated than it does when the basin or vessel is empty; and that the object becomes visible to an eye in a certain position, with the vessel filled, although the view of it was intercepted by the edge of the vessel when the latter was empty.

3. If the refracting medium be glass, and the angle of incidence and refraction be so small as to be considered without sensible error, proportional to their sines, the refracted angle, or that angle contained between the incident ray produced and the refracted ray, will be $\frac{1}{3}$d of the greatest of those angles, and $\frac{1}{2}$ of the least; and hence, if the angle of incidence be increased or diminished, the refracted angle will be increased or diminished by $\frac{1}{3}$d of that augmented or diminished part, if the incidence be from air into glass; and by one-half if the incidence be from glass into air.

Aberration of the actual and geometrical focus.

Fig. 43.

216. When an incident ray, as QC (fig. 43), is oblique to the plane-refracting surface AC, the proper focus x of the refracted ray CD, will be further from Q than the geometrical focus q, that is, Q x will always be greater than $q\,x$; and the more oblique the incidence the greater will be the aberration $q\,x$. In each division of the figure, conceive a right line drawn from q to C; then, if q be supposed to represent the focus of the refracted ray CD, we ought to have the proportion

$$\text{AQ} : \text{A}q :: \text{CQ} : \text{C}q :: m : n.$$

But it is manifest that these ratios are not equal; for, if QA and Q a be made radii, CQ and C q will be the secants of the unequal angles AQC, A q C; and since we must have

$$\text{AQ} : \text{A}q :: \text{CQ} : \text{C}x,$$

it is manifest that in the first part of the figure, C x is greater than C q, and the contrary in the second; but, in both cases, Q x is greater than Q q. It is also further evident, that the greater the angle AQC, the greater will $q\,x$ become.

Determination of the focus of diverging rays, after reflection and refraction.

Fig. 44.

217. PROBLEM II.—Having the focus of incidence, of rays diverging upon a plane looking glass, whose thickness is given, to find the focus of the emergent rays after this second refraction at the foreside of the glass.

Let CF (fig. 44), represent a plane glass mirror, whose back, reflecting, or quicksilvered side is EF, the thickness of the plate being AB; and let Q be the focus of incident rays. Through the point Q, draw the indefinite line NP perpendicular to the sides of the glass; in this line take the point G, such that

$$\text{AQ} : \text{AG} : n :: m$$

($n : m$ denoting ratio of the sines of incidence and refraction); then from what is shown above (art. 215), G will be the focus of the rays, after their refraction at the first surface of the glass CD. Again, take BH =
BG, and H will be the focus of the rays after t
reflection at the surface EF; also find the poin
by means of the proportion

$$m : n :: \text{AH} : \text{AR}$$

and R will be the focus required. That is, an inci
ray, as Q a, will be first refracted into $a\,b$, so a
belong to the focus G; then at b, it will be refle
into $b\,c$, proceeding from the focus H; and lastly
c it will be refracted into $c\,k$, so as to belong to
focus R.

If A r = A Q, that is, if r be the focus of
reflected by the surface CD, before they enter
glass, we shall have $\text{R}r = \frac{2n}{m}\text{AB}$; that is, in a lo
ing glass R r = $\frac{4}{3}$ of its thickness.

For $\quad \text{AH} = \text{AG} + 2\,\text{AB}$; A r, or $\text{AQ} = \frac{n}{m}\text{AG}$

and $\quad \text{AR} = \frac{n}{m}\text{AH} = \frac{n}{m}\text{AG} + \frac{2n}{m}\text{AB},$

whence $\quad \text{R}r = \frac{2n}{m}\text{AB} = \frac{4}{3}\text{AB}.$

218. *Schol.*—Some of the rays diverging upon a lo
ing-glass, as ED (fig. 44), may be several times refle
between its surfaces, before they totally emerge, w
some other of the pencil emerge each time at the
face CD. These emergent rays will have diffe
focii, s, t, v, &c. all situated orderly one behind
other in the perpendicular BP; and the distances
$s\,t$, $t\,v$, &c. between the several focii s, t, v, &c, wil

each $= \text{R}r = \frac{2n}{m}\text{AB}.$

Therefore, of a slender pencil of rays, suppose
flow along the physical line Q a, some will be im
diately reflected at a into $a\,p$, belonging to the foc
but the greatest quantity will enter the glass alon
line a b, and being reflected at b along b c, will e
into $c\,k$, belonging to the focus R. Again, so
the rays proceeding along $b\,c$ will be reflecte
$c\,d$, $d\,e$, as if they proceeded from I, K, respec
and then some of them will emerge along e
ceeding directly from s. In like manner, some
rays may be reflected back at e, into $e\,f$, then
and at g, some will be refracted into g m, whil
will be returned back along $g\,h$, $h\,i$, $i\,n$, &c.
of the refracted rays g m, $i\,n$, will be t, v;
GI, IL, LN, &c. HK, KM, MP, &c. are each
2 AB; and R s, $s\,t$, $t\,v$, &c. are each equal t

By making the angles of incidence and
equal to each other, it is manifest that

$$\text{AI} = \text{AH} = \text{AG} + 2\,\text{AB},$$
$$\text{BK} = \text{BI} = \text{BH} + 2\,\text{AB} = \text{BG} + \text{C}$$

therefore, GI = 2 AB = KH.

In the same manner, it may be shown th
LN, MP, &c. are each = 2 AB.

Again,

$$\text{A}s = \frac{n}{m}\text{AK} = \frac{n}{m}\text{AH} + \frac{n}{m}\text{HK} = \text{AR}$$

whence, $\quad \text{R}s = \frac{2n}{m}\text{AB} = \text{R}r;$

and in the same manner it appears that s

219. The phenomena which we have
on theoretical principles, to obtain in

may be exhibited experimentally, by placing a lighted candle in an oblique position with respect to the plane of the mirror, by which a series of images such as we have been describing, will be observed all situated in the same line and at equal distances from each other. In this case, all the images beyond R will continually decrease in brightness, until at length the rays become too faint to exhibit a visible image; the number, however, will be greater or less according to the intensity of light in the illuminating body; and the images will be more distinct and distinguishable from each other, the more obliquely they are viewed; because, then the angles formed by the spaces r R, R s, &c. become so much the larger. In other cases, the images will partly cover each other, and the more so the larger the object is, or the thinner the glass.

It appears from the above phenomenon, that light must suffer a considerable diminution in its passage through glass; that is, a considerable portion of the light which arrives at the surface of the glass is prevented from taking its natural direction, or that which it ought to take according to theory.

220 We have seen (art. 215) that the image of a body viewed at the bottom of a vessel of water, will appear to be at only $\frac{3}{4}$ths of its actual depth below the surface; and on the same principle, an object viewed through a glass plate, will exhibit an image which will lead us to give to the glass only $\frac{2}{3}$ of its actual thickness. Knowing, then, this fact, we have a ready way of estimating the thickness of a looking-glass plate, by observing the distance between a wire or other slender object brought into contact with the fore surface of the glass and its image, as shown by reflection; for, as the apparent place of the image behind the silvered or reflecting surface, is always equal to the apparent distance of the object itself from the same, the distance between the two, therefore, in this case, will be $\frac{4}{3}$ds of the thickness of the glass; or, which is the same, the thickness of the glass will be equal to $\frac{3}{4}$ths of the apparent distance between the object and its image.

221. The course that a ray would take in coming from Q (fig. 45), to an eye at O, is represented by Q a b c O, on the supposition that if a small pencil of rays, impinging obliquely on the looking-glass CF, had its foci, after refraction, at the surface CD in the points G, R, as above determined. But because of the aberrations of refractions at the surface CD (art. 216), the ray Q a will be refracted into a d, whose focus is somewhere in the line AG further from Q than G. Again, the ray a d will be reflected into d e, making the angle B a h = A e d = A a g = D e p; and, lastly, the ray at e will emerge into e o, whose focus is x, lying somewhere between R and r; and the greater the obliquity of the incident ray, the greater will be the aberration R x. When the point of incidence a is nearest of all to A, then A x = AR; and we shall have then

$$Aa : ae :: Ar : rx.$$

For the angle A a g being equal to D e p, the angle D e o = Q a A = A a r; consequently, e x is parallel to a r. If we conceive the line a g to be produced until it meets the perpendicular AG also produced, in a point y, then we shall have

$$yA : AB :: Aa : \tfrac{1}{2} ae.$$

But as the point a recedes from A, the ratio of y A to AB is continually increasing, and therefore the ratio of e a, a A is constantly decreasing, whereby the point x is always approaching nearer to r; with which, however, it can never coincide until A a becomes infinite; for until then the ratio of A a : a e, and consequently, of A r : r x will be finite. The point x is therefore always betwixt R and r; and as the point a is nearer to or more remote from A, so will the point x be nearer to R or r.

In the same manner, the distances between the foci R, s, t, v, &c. which here correspond to those in fig. 44, will, as a recedes from A, be also lessened, so as to be always nearly equal to r x. But the focus of a pencil, whose axis is Q a d e o, will be in the line o e x, nearer to e than x; for a ray diverging from Q upon a point of the speculum, which is nearer to A than a, will have its focus after emersion between x and R; and a ray falling on the other side of a, will emerge in a line proceeding to a point between x and r; and these two rays will intersect the line e x in a point between x and e: so that a line connecting the several successive images made by the reflections within the glass, or those corresponding to R, s, t, v, &c. will be inclined from BA towards d, and also a little curved.

When the angle AQ a = 60°, r x will be only about $\frac{2}{3}$ of AB.

To judge of the perfection in the polish.

222. The degree of perfection in the polish and figure of a plane speculum, may be easily known by observing whether the images, seen in all positions, especially in very oblique ones, and from all parts of the speculum, appear exactly equal and similar to the objects; that is, whether the images (more particularly of the most distant objects) in the room, appear naturally, without having any part of them distorted; when this is the case, the speculum may be pronounced to be a perfect one: the straight edges of the rails of wainscot are among the best objects for this experiment. A mirror must be exceedingly bad, that will distort the face of a person looking into it; because the rays being returned almost directly back to the eye, small aberrations will not be rendered sensible; but let two persons look at each other's image as obliquely as they can, and they will soon perceive whether or not the figure of the speculum be defective.

The grinding of a plate to a true plane is considered to be a very difficult operation, but that of making a good looking-glass is still greater; because, beside each plane being perfect, it is also necessary that they are also parallel. If the images of a candle, seen very obliquely as in fig. 45, and under different degrees of obliquity, and from all parts of the glass, do not always keep pretty nearly at equal distances from one another, it is a proof that the sides of the glass are neither plane nor parallel.

Another method, analogous to the preceding, to try the goodness of a looking-glass, is as follows: stick a pin or slender wire by the point into the upright jamb of a window-frame, or in a bar of the sash, so that the pin may be nearly horizontal, and in the plane of the window. Then hold the looking-glass, and turn it about so as to see the image of the pin very obliquely, and from all parts of the glass. In this case, there will seldom be visible but two images corresponding to R, r, of the two preceding figures; and if these images keep always straight, parallel, and at regular distances one from another, according to the preceding theory, the glass may be considered as being well figured.

The phenomena of the different distances of the images from each other will be more conspicuous, if two pins be stuck parallel to one another, and at a small distance asunder. By this method also, the correctness of a figure of a metallic plane speculum may also be examined. By making such trials, a very considerable difference will be observed in the perfection and imperfection of different looking-glasses: and it is very rare any common glass will be able to bear these tests in every part of its surface.

In all speculums, the better they are polished, other circumstances being the same, the brighter will be the images; that is, the more light an eye will receive from a given object, which will enable us to examine the goodness of speculums, as to their polish, whenever we have an opportunity of comparing several of the same sort and in the same light together. We may also observe, *cæteris paribus*, that the darker the colour of the glass of the speculum is, the better, generally, is the polish; for the glass itself can be no otherwise seen, than by the reflections of those particles which have irregular positions with respect to the rest of the surface. But different glasses, though equally well polished, will not always appear equally dark; generally, however, the above rule may be observed, and the darkest mirror be considered as that which has the best polish.

V. *Of concave and convex mirrors.*

Of concave and convex mirrors.

223. In our first and second sections we have treated of images formed by reflection from concave and convex mirrors, on the supposition that these images were received on a plane. In this particular case the images are always in the right lines produced, drawn from different points of the object to the centre of the mirror, and their distances, as well as their magnitudes, are determinable from the formula

$$f = \frac{dr}{2d \pm r};$$

but when the images are seen without being thus received on an intermediate plane, each point ought to be found in the right line which arrives at the eye, after having been reflected at the surface of the mirror, which right line will vary with the position of the eye.

In order to determine the form and the position of the image, we must trace from many points of the object, the caustics by reflection, and draw from the two extremities of the pupil of the eye, tangents to these caustics: the tangents prolonged to the surface of the mirror, form, with the cathetus, angles of reflection equal to the angles of incidence formed by the points of the object corresponding to these caustics, as in the figure referred to below.

Determination of the direction of the incident ray.

We may determine by analysis the positions of the points of the mirror, where these reflections are made from different luminous points proceeding from the body, and which arrive at the eye after having been reflected. In order to take the simplest case, we shall suppose the body to be a point, and the opening of the pupil indefinitely small.

Fig. 46.

Let us also suppose any three points in a plane, as A, B, C (fig. 46), to be given, of which C is at the centre of the mirror M M′ M″; and let it be required to determine the point M, such that the angle AMC shall be equal to the angle CMB.

224. Conceive a plane to pass through the three points A, B, C, cutting the mirror in the circle M M′ M″; and then supposing the origin of the co-ordinates in C, we may denote the co-ordinates of the point B by $-\alpha$ and $+\beta$, and let the distance AC $= k$. Assume the angle AMT $= m$, the angle MTA $= t$, the angle MAI $= a$, and $180° = \pi$; then we shall have

$$\tan m = \tan(\pi - a - t) = -\tan(a + t).$$

From the equation of the circle, we have

$$x^2 + y^2 = r^2,$$

which gives $\tan t = \frac{x}{y}$:

we have also $\tan a = \frac{y}{x-k}$.

Whence

$$\tan m = \frac{\frac{x}{y} + \frac{y}{x-k}}{1 - \frac{xy}{y(x-k)}} = \frac{r^2 - kx}{ky}.$$

Now the exterior angle OMQ of the triangle TMA, or TMQ, is equal to the sum of the two interior and opposite angles a' and t; denoting the angle TMQ by a', consequently,

$$\tan m = \frac{\frac{x}{y} + a'}{1 + \frac{a'x}{y}} = \frac{x + a'y}{y - a'x};$$

whence we have

$$\frac{r^2 - kx}{ky} = \frac{x + a'y}{y - a'x},$$

or, $$a' = \frac{r^2 y - 2kxy}{k(r^2 - 2x^2) + r^2 x}.$$

The equation of the reflected ray is, therefore,

$$\beta - y = \frac{r^2 y - 2kxy}{k(r^2 - 2x^2) + r^2 x} \times (a + x);$$

or,

$$x^2 + \frac{a}{\beta} xy - \frac{r^2}{2k} x + \frac{r^2(k-a)}{2\beta k} y - \frac{r^2}{2} = o.$$

This equation, together with that belonging to the circle, will enable us to determine the values of x and y: but, it unfortunately ascends to an equation of the 4th degree, and is, therefore, but ill suited to practical computation. The construction of the equation by means of the hyperbola is also not less unmanageable: but, supposing the direction of the incident ray to be determined, the situation of the image may be easily ascertained as follows:—

225. Let GBH (fig. 47) be the mirror; C the centre; A the luminous point; AB the incident ray; and O the eye.

If from the point A we draw the ray AB, and another A b indefinitely near to the former; also, if from the points B b, we draw to the two extremities of the pupil of the eye, the two lines BO, b o, the point E of intersection of these two lines, will be the place of the image.

In order to determine the position of this point, let there be drawn the two perpendiculars CD, CE, to the two right lines AB, BO; and the perpendiculars B ϕ

and $B\partial$ to the lines $A b$, $b o$; also divide the line BD, BE into two equal parts in a and f, then from the similarity of the triangles $AB\partial$, $AD d$, $BF\phi$, $EF e$,

$$AB : AD :: B\partial : D d$$

$$BF : EF :: B\phi : E e.$$

But since $D d = E e$, we have

$$AB : AD :: BF : FE.$$

Wherefore,

$$AB \pm AD : AD :: BF \pm EF : EF$$

and consequently,

$$\frac{AB + AD}{2} : \frac{BF + EF}{2} :: \frac{AB - AD}{2} : \frac{BF - EF}{2}.$$

That is

$$Aa : Bf :: aB : Ff,$$

and alternately,

$$Aa : aB :: Bf : Ff.$$

Whence we have

$$Ff = \frac{aB \times Bf}{Aa}$$

VI. *Of vision by reflection of spherical speculums.*

Convex speculum.

ms.

226. It follows, from what has now been established with reference to the reflection of spherical surfaces and the images thus formed, that, in a convex speculum, the images of objects are always seen erect, a little convexed towards the eye, and diminished as the objects approach the speculum; and, moreover, that the greater the convexity of the speculum, or the smaller its radius, the nearer, the less, and the more convex will be the images of given objects. In (fig. 48) ABC represent a convex speculum, whose centre is O, and of which the principle foci in the axes OP, OQ, OR, are E, F, G; the curved line $p\ q\ r$, is the geometrical image of the straight object PQR; and $s\ v\ t$ is the image of the object SVT. The lines diverging from BA towards PR, represent rays that originally diverged from P, and are now reflected from the focus p; and the lines proceeding from the part AC of the speculum, represent rays belonging to the physical point R, and now diverging from the focus r. In like manner, as rays diverge upon the speculum from every physical point of the object PQR; so after reflection they diverge from every corresponding point in the image $p\ q\ r$: and as it is manifest that some rays, belonging to p and r, will, after reflection from some part or other of the speculum, be mixed throughout the whole space BCKH, so that wherever the eye is placed within that space, it will see the whole image $p\ q\ r$. Thus, if $K a$ be the breadth of the pupil, an eye placed at $K a$, will see the point r, by the pencil of rays whose base is $K a$ and vertex r; and the point p, by the pencil of rays $K p a$. And as this is agreeable to the nature of simple vision, $p\ q\ r$ will appear as a real object of the same colour with PQR, because it is seen by rays propagated from PR, without having undergone any other change than that of being made to diverge more, which is the cause of the diminution and convexity of the image.

48.

If the speculum be a segment of any considerable extent with reference to the sphere, of which it is a part, it will exhibit the images of objects so much the wider as under, so that part of the ceiling, floor, and sides of the room may be seen at the same time, the whole making a kind of picture, very agreeable in its effects; and the nearer the eye is to the speculum, the larger will be the field of visible images. If, while the objects and speculum remain fixed, a spectator move towards or from the speculum, the apparent places and magnitudes of the images will remain invariable; but, if an object move towards or from the speculum, the image will appear to move the contrary way; and as it approaches nearer, or recedes further, it will accordingly appear more and more enlarged or diminished.

Positions sometimes falsely assigned.

227. In certain cases, the mind is very much deceived with respect to the magnitude and place which it assigns to objects seen in speculums, such as we have been describing. Thus, if a convex speculum be placed against a window having an extensive prospect, or facing the end of a street, the great multiplicity of objects that are seen one behind another, and the diminution of their images, will frequently, after looking for a long time into a mirror, lead us to imagine that these images are at a greater distance than they really are; but after a time, and a due comparison of the effects, the mistake will be corrected, and the apparent places of the images will not differ sensibly from their real places, viz. those whence the rays diverge to the eye.

Fig. 49.

228. When the eye sees an image very obliquely, the rays, in that case, diverging from the foci that are nearer to the speculum than $p\ q\ r$, the place and the figure of the apparent image will be different from those of $p\ q\ r$; but the image will not be sensibly the less distinct; because, though the rays that enter the eye, have great aberrations from the axes of the pencils, yet as they make but a slender pencil, they have but small aberrations with each other, and will, therefore, cause no confusion. Thus, to an eye at $b\ c$ (fig. 49), the apparent image of the object PQR, is $s\ t\ v$, lying in a different position from the geometrical image $p\ q\ r$, and in a line between it and the speculum.

Image darker than the object.

229. Again, as the rays diverge much more after reflection from a convex speculum than they did before, this will be an additional reason, besides that which arises from the loss of light by reflection, why images, seen in such speculums, appear much darker than the objects. For the eye at $b\ c$ receives no more light from P, than what is contained in the slender pencil Pd; the two extreme rays of that pencil being reflected respectively to b and c, the edge of the pupil. In like manner, the two extreme rays in $R g$ are reflected respectively to b and c. But the pencil $R g$ is broader than $P d$, because it falls less obliquely on the speculum, and, therefore, the point v of the image will be a little brighter than the point s.

Concave speculums.

Concave speculums.

230. We have here three cases to consider, viz.

1. When the object is between the speculum and its principal focus.
2. When it is between its centre and that focus.
3. When it is beyond or more remote than the centre.

Figs. 50, 51, 52, 53.

In the four following figures, 50, 51, 52, and 53, appertaining to the present inquiry, BAC is supposed to denote the concave speculum, whose centre is O, and whose principal foci are in the arc EFG : PQR is a rectilinear object, placed at right angles to the axis AOQ of the speculum; and $p\ q\ r$ is its geometrical image. The straight lines issuing from the speculum, represent rays which diverge upon it from the respective points P, Q, R, and which now diverge from, or converge to, the respective points p, q, r,; the aberrations of the more remote rays not being considered. The rays issuing immediately from the object to the speculum, are not here represented, in order to prevent the confusion of too many lines.

CASE 1.—*When the object is between the speculum and its principal focus.*

Object between the speculum and principal focus. Fig. 50.

231. It is obvious (referring to fig. 50), that in this case the images of objects will be erect and magnified, and the more so, as the object is nearer to the principal focus F. If the object be a straight line, or a plane, placed at right angles to the principal axis AO, its geometrical image will be always concave, and the nearer the point Q is to F, the greater will be the concavity of the image $p\ q\ r$.

In the figure referred to, the rays diverging from the part BA of the speculum, are the rays belonging to P, and those diverging from CA, are those which issue from R. If the rays belonging to P and R, had also been drawn from the other half of the speculum, they would have intersected throughout the whole space BCKH; and, therefore, to any eye placed anywhere within that space, the images of the points P and R, and consequently the images of every other point of the object PQR, will be visible. If K a be the position and breadth of the pupil, the eye will see the image of the point P, by the pencil of light $a\ b$ AK; and the image of the point R, by the $a\ d$ CK; and in the same manner it will see every point of the image $p\ q\ r$, by pencils of rays reflected from the speculum between b and C.

Phenomena observed in concave speculums.

232. We have seen that in the case above supposed, the image will always appear erect, and larger than the object; and if, while the object and speculum retain the same position, the eye moves a little either way in the axis OA, the apparent place, figure, and magnitude of the visible image will continue nearly the same. Also, if the geometrical image be pretty near the speculum, so that the rays in the pencil entering the eye have sensible divergencies, the visible image will appear nearly in the same place, and of the same figure and magnitude with the true image. But when the geometrical image is remote from the speculum, its apparent place will be in a manner indeterminate and variable according to different circumstances of light, and particularly as the eye happen to have a different conformation for seeing at different distances. But the visible image never appears very remote; and, in some cases, it will seem to approach the speculum when the geometrical image recedes from it, while at others it may appear stationary when the object is actually in motion.

Let us for example, conceive PR to represent the length of the face of a person viewing his own image in the speculum ABC, from different stations in the axis AF; then when PR is near the speculum, its visible image on the other side will be pretty nearly in the same place, and of the same figure and magnitude with the geometrical image $p\ q\ r$, as we have already observed above; the rays in this case having a sensible divergency, and the geometrical image $p\ q\ r$ being not very concave. As PR recedes from A, its image will also in the first instance appear to recede from A, but not so fast as the geometric image $p\ q\ r$; at length the visible image will become stationary; or may be even seen to approach towards the speculum, more especially if PR be an object whose image is viewed by a spectator placed near the centre O. For, as the object advances towards F, the divergencies of the reflected rays become less and less sensible; while, at the same time, the images of the smaller parts of the objects appear pretty nearly equally bright and distinct. Also, as the spectator recedes from A, his image fills a larger part of the speculum; that is to say, a part or segment whose breadth is the measure of half the angle POR. We, therefore, combining together all these circumstances, see why, as the spectator retreats from A towards F, his apparent image at first also retreats from the speculum pretty quickly, then slower and slower, till it becomes stationary, and at last seems to advance forward.

233. With respect to the observed magnitude of the image, it is obvious, that when the apparent distance is less than the geometrical distance, the apparent magnitude will be less than the geometrical magnitude. And while the apparent distance increases but slowly, or remains the same, the apparent magnitude will likewise increase but slowly, or may remain for a time invariable; for the visual angle decreases, though slowly, while the spectator sees his image reflected from a larger part of the speculum, and for this reason the apparent magnitude scarcely ever appears to decrease by retreating from the speculum. For the same reason, the image of a flat and blank surface will not appear so concave as its geometrical image; because the spectator being unable to determine the distance of the several parts of this image, he sometimes imagines it to be nearly flat like the object. But when the geometrical image is not very remote, our prejudice will be overcome, and the image will appear more concave as it is viewed with the greater attention.

234. The same reasons as we have advanced above to account for the apparent magnitudes of objects, will also serve to explain the appearances observed in the following experiment. If a hollow cone be held, with its base towards the speculum, so that its vertex nearly coincides with the principal focus F, its image will appear projected into a prodigious length, in consequence of our having in this case a series of these images, lying orderly one behind another; whereby we are enabled to judge of the places where the rays diverge to the eye more correctly; and the whole visible length is, in consequence, protracted, and the image rendered nearly cylindrical, or at least much more accute than the cone itself: and for the same reason, a stick laid nearly in the direction of the axis, with its further extremity nearly in the focus, will appear of great length and slightly conical, its further extremity being the greatest.

CASE 2.—*When the object is between the principal focus and the centre of the speculum.*

235. We have above examined the phenomena attending a spectator viewing his image from any point between the speculum and its principal focus; and we have seen that, at the focus, his image is erect, magnified, and pretty near to the speculum; let us now follow him in his retreat from the focus to the centre. In this case, as he moves, his apparent image will be

behind him, but advancing towards him, as he approaches nearer to the centre, at which point the image and the object will coincide. During this retreat the object will appear to be still erect, because the rays belonging to each visible point will not intersect before they reach the eye; but the image becomes less and less distinct, at the same time that its visual angle is increasing, so that at the centre, or rather a little before, the whole is lost in confusion; arising from the small parts of the objects subtending angles too large for distinct vision, exactly as in the case of viewing them too near with the naked eye.

In order to illustrate this case, let an object, PQR, be fixed between the principal focus and the centre O (fig. 51), while the eye looks for its image in the speculum, from different parts of the axis Aq. If the pupil of the eye be bc, the images of the points P, Q, R, will be seen by the pencils of rays $bdAc$, $befc$, $bAgc$, whose respective foci are p, q, r. Now, if the constitution of the eye be such, that the rays contained in the above pencils are brought to unite before they reach the retina, the representation of PR will be confused, as remote objects appear to short-sighted persons. But although the rays come to the eye converging, which is a case that never happens in naked vision, yet, as they are orderly reflected accordingly as they come from different parts of the object, they convey an idea of that object, through an indistinct one, both as to its shape and colour. And in the present case, an aged person might see distinct enough, and perhaps more distinct, than he would be able to do by the naked eye. For whatever be the disposition of the rays coming from every visible point of an object, before they enter the eye, as diverging or converging, vision will be still distinct, if the several pencils are not mixed upon the retina; and the apparent places and magnitudes will be according to the general laws of vision in other cases; that is, as some ordinary helps are wanting, others will preponderate.

As the eye recedes further back from the speculum, the confusion of the image will necessarily increase. When the pupil xy is at the centre O, it receives from the points P, Q, R, the pencils $Bxyh$, $dxyg$, K$xy$C, whose extreme rays converge about twice as much as those of the former pencils, and before the eye is got quite as far as q, the whole image will be lost in confusion; while all the way from A to q the image necessarily appears erect, and also its visual angle is continually increasing.

234. As the eye recedes from A, towards q, the field of vision in the given speculum of an indefinite object PR, is continually diminishing; so that beyond K, none of the rays belonging to the points PR, can enter the pupil; and at q, the said field will be diminished to nothing. After the eye has past q, the field of vision of the indefinite object PR, will again begin to open, and the further back the eye is from q, the wider will be the field. All which will be manifest from a simple inspection of the figure above referred to. After the eye has passed q, the image will appear inverted, and for some time continue indistinct, as in the case of objects viewed too near, till the eye has got so far that the rays enter it at a proper divergency for distinct vision. But as the image itself is so much the rarer in proportion as it is magnified, and the pencils of light entering the pupil continually decrease in density as the eye retreats further back, so beyond a certain distance it will be difficult to find the image, unless it be that of a bright object as a candle; and when it is found, the spectator will, for the above reasons, be likely to imagine it further from him than it really is; and conforming his eyes accordingly, will thus furnish another cause for indistinctness.

If the object be a straight stick, or a plane at right angles to the axis of the speculum, its image, to a spectator at a proper distance behind it, instead of appearing concave will be convex. This phenomenon, as at first sight it may seem to involve some difficulty, deserves to be further explained.

Fig. 52.

Let, as before, $p\,q\,r$ (fig. 52) be the geometrical image of the object PQR after reflection, by the speculum BAC, and let $b\,c$ be the breadth and place of the pupil. Now the extreme rays $d\,t\,c$, $g\,t\,b$ of the pencil, belonging to Q, being not remote from the axis AQa, will diverge to the eye from a point t that is very near to q. But the nearest ray $k\,s\,c$ to the axis POD of the pencil belonging to P, that can enter the pupil, being very oblique to that axis, will intersect it at a point s, much nearer to the speculum than p, and the rest of the rays belonging to this pencil will intersect the axis a little further from p than s; the extreme ray C$\,y\,b$ intersecting the ray $k\,s\,c$ in y. In like manner the nearest ray $h\,v\,b$, to the axis ROH of the pencil belonging to R, intersects that axis at the point v; and the extreme ray $B\,x\,c$ intersects the other extreme ray $h\,x\,v\,b$ at the point x. And thus the apparent image is a convex curve $y\,s\,t\,v\,x$, agreeably to the theory illustrated in the preceding sections.

CASE 3.—*When the object is further from the speculum than the centre.*

The object beyond the centre.

235. We have seen that when the object is placed between the speculum and the principal focus, the image will preserve its direct position, as it will also while the object passes from this situation to the centre, at which point it is lost; but, if now, as in the third or last case, the object be supposed to pass beyond the centre, then the image will be inverted; and the further the object is from the centre, the less will be its image, and the further from the said centre, or the nearer to the focus, and the converse; the image and object coinciding when the latter is placed in the centre, as we have seen in the preceding case.

Fig. 53.

Let PR (fig. 53) represent a spectator viewing his own image at a considerable distance beyond the centre O of the speculum; his image will then appear inverted, small, faint, and indistinct. The faintness is occasioned by the few rays that can enter the pupil, and this faintness occasions the apparent distance to be the greater, in consequence of which the image generally appears beyond the speculum; and this misconception of its place adds to the indistinctness. As the spectator advances forward, his image will appear gradually larger and brighter, and also to approach nearer to him; but if he do not see it between him and the speculum, it will continue still indistinct.

Experiment.

236. If an observer situated as above supposed, hold out a stick with its end towards the speculum, it will immediately perceive its image come out, as it were, to meet him; the length of image now assisting him to judge more accurately of its distance. But the image of the stick will appear shorter than the stick itself, and

if it be a cylinder, or a ruler with parallel sides, the image will appear tapering, that end appearing the smallest which is held in the hand, or which is seen at the greatest distance. Both these phenomena, viz. the apparent shortening of the stick, and its conical figure, are exactly conformable to the theory of the geometrical image. If the object be stretched forward nearer to the speculum, its image will be seen to advance further out; and, when the end of it is in O, its image will be seen there also: and, if the spectator advance further, he will see the image and the object pass by each other. After this, stretching out his arm without the stick, the spectator will see its image advance, so that it appears as if he might actually grasp his own hand; and if, at this time, he observes the image of his face, he will see it suspended in the air at its proper place, between himself and the speculum. As soon as the image is seen at its true place, it becomes a determinate and a fixed object, whereas, before, it shifted its place, as it were, casually, and then, and not till then, it will appear perfectly distinct.

ie image, hen most istinct.

After a few trials, a person will find the true place of his image pretty readily, provided he be not too near, or too far distant from it, and it will then continue distinct as he moves backward and forward from his first position for some distance; but when he has got within a small distance of the true place of the image, the latter will begin to appear indistinct, and this indistinctness will increase till he has arrived at the centre O, where the image will be wholly lost in confusion. If he advance beyond the centre, he will begin to have some glimmerings of an erect image, which will become more perfect as he advances forward, as we have already explained.

237. If PQR be an object remaining fixed, while the eye is in motion along the axis OAQ, the image will appear inverted while the eye is behind $p\,q\,r$, and when it is at such a distance that the image appears at the place $p\,q\,r$, it then also is seen distinctly. If the eye be far behind the image, it will appear faint for want of sufficient light, and if it be too near it, it will be indistinct, because of the two great divergencies of the rays. If the eye get before the image, it will appear erect and confused, but this confusion will diminish as the eye is brought nearer to the speculum, because the divergencies of the rays will then be lessened.

Again, if the eye be fixed, while the object PQR moves towards the centre of the speculum, as the real magnitude of the image will be all the way increasing, so will also its apparent magnitude. And if the eye be pretty near the centre O, the above motion of the object will cause an indistinctness which will increase as the image approaches nearer to the eye. If the eye had been placed between the image and the speculum, the indistinctness of the image would have lessened as the object advanced nearer towards the centre. And it would be the contrary in both cases, if we suppose the object, instead of advancing, to be made to retreat further back.

When the eye is out of the principal axis.

238. In the two last cases, if the eye be supposed to be situated out of the principal axis, the figure and position of the image formed by the pencil entering the pupil, will be different from those of $p\,q\,r$, as we have already explained to be the case in convex mirrors (art. 228), and as shown in (fig. 49); but as vision in such positions of the eye is generally indistinct, and the apparent image necessarily [illegible] stances from that formed by the concourse of the [illegible] entering the pupil, it would be needless to dwell upon this subject at great length: we shall therefore only observe, that when the eye is in the axes q Q, at a small distance behind $p\,r$, the apparent image will be less concave than $p\,q\,r$, and may even be convex; because the pencils entering the pupil from the extreme parts, they diverge from the foci that are nearer the speculum than the points p, r, as in art. 234.

239. When the eye is in or near the principal axis, in any of the three last figures, and between the image $p\,q\,r$, of a plane object PQR, and the speculum, the image generally appears concave or hollow, and somewhat less indistinct about the middle, than towards the edges, because the rays belonging to the middle of the object, converge less than the other at their entrance into the pupil; which again seems to be the reason why the middle appears further, or at a better distance for distinct vision; the greater indistinctness, *in this case*, suggesting a nearer distance of the object.

240. We have hitherto supposed the observer to be using one eye only; but it may be proper, before we close this section, to offer a few remarks on the more natural case, where both eyes are employed in viewing the images of objects presented to spherical specula. A small image q, of an object Q (fig. 54), will appear double, as at m, n, if the optic axes $a\,v$ F, $b\,v$ E, be not directed to q, but to some point F beyond it. This phenomenon frequently happens in consequence of our being led to look beyond the speculum for the image; and the cause of it is, that the points r, s, where the visual rays Q $m\,q\,r\,r$, Q $n\,q\,v\,s$, meet the retina, fall on different sides of the points a, b in the optic axes. In like manner, if $p\,q\,r$ (fig. 51) be the image of an upright and slender object, as a wire, this image to a spectator beyond it, will also frequently appear double, and that for the same cause.

When the image q (fig. 55) is behind the spectator, there will likewise appear a double image, if the speculum be broad enough to throw rays diverging upon it, from Q, into both eyes. For in this case also, the points r, s, where the visual rays B $v\,r$, C $v\,s$, meet the retina, as they are tending towards q, are on different sides of the optic axes; and consequently the vision is double. The case here is analogous to that where the object seen is further off than the point towards which the optic axes are directed, which is the reverse of the preceding.

241. The inverted image of a person standing a small distance behind it, appears to him equally bright and distinct with his image in a convex speculum of the same radius as the concave one, supposing him to be at such a distance that his image is of the same size in each.

The above will be found to illustrate most of the phenomena exhibited by concave and convex spherical reflectors, we shall now, therefore, conclude this section by demonstrating the following general proposition:

242. *If the object placed before a spherical reflector be a right line, the image is a conic section.*

Let ACB be the reflector, E its centre, PR the object placed before it. Through E draw DEPN, at right angles to PR; take any point R in the object, join ER and produce it; bisect EC in F, and let r be the image of R; in the line PED, take ED equal to

EP, and draw DH at right angles to PD; again, from *r* draw *r* H, *r* S, respectively parallel to DN and RP.

Then, because *r* is the image of R, we have

$$RF : FE :: ER : Er;$$

and alternately

$$RF : ER : FE : Er.$$

Also in the similar triangles EPR, ES *r*,

$$ER : Er :: EP : ES :: ED : ES;$$

therefore

$$RF : EF :: ED : ES;$$

consequently

$$RF : RF \overset{\sim}{+} EF :: ED : ED \overset{\sim}{+} ES;$$

or, which is the same,

$$RF : ER :: ED : DS, \text{ or } Hr,$$

wherefore

$$ED : Hr :: FE : Er;$$

and alternately

$$ED : FE :: Hr : Er;$$

that is E *r*, bears an invariable ratio to H *r*, and therefore *r* is a point in the conic section, whose focus is E and directrix DH. We may hence draw the following deductions:

1. When EP is equal to EF, H *r* is equal to E *r*, and the conic section is a parabola; it is an hyperbola or ellipse according as EP is less or greater than EF.

2. When the distance of the object is so great that the rays proceeding from every point of it may be considered as parallel, the image is a circle whose radius is EF.

3. If ET be drawn perpendicular to EN, when *r* coincides with T, H *r* becomes equal to ED; and since we have ED : EF :: H *r* : E *r*, E *r* becomes equal to EF: that is, ET (half the parameter of the conic section) is equal to E F, half the radius of the reflector.

4. Since the radius of curvature at the vertex of the figure is equal to half the parameter, the curvature of the image at N is the same, wherever the object is placed.

5. If the radius of the speculum be finite, the evenescent arc *r* N is equal to the ordinate *r* S, and therefore RP : R *r* :: EP : ES :: EP : EN. Also while the angle REP, which the straight line subtends at the centre of the reflector, is small though finite, the arc *r* N will, as to sense, be a right line.

6. In the figure above referred to, where the object is supposed to lie between the principal focus and the surface ACB, TN *x* is the erect image of the line RP, indefinitely produced both ways, and TM *x*, its inverted image. The part ANB is formed by reflection from the concave surfaces A *k*, BL; and TM *x* by reflection from the concave surface *k* GL.

VII. *Of speculums formed by the revolution of conic sections.*

243. It appears from what has been stated in our first and second sections, that unless the focus coincide with the centre, a spherical reflector does not cause all the rays in a pencil either to converge or diverge accurately. This circumstance produces some confusion in vision, when these reflectors are employed; and by increasing the breadth of each pencil, or, which is the same thing, by enlarging the aperture of the reflecting surface, in order to increase the quantity of light, the indistinctness thus produced is increased, instead of being diminished.

In order to remedy this inconvenience, it has been proposed to make use of reflecting surfaces formed by the revolution of conic sections about their axes; and that such surfaces are capable of causing the rays to converge or diverge accurately, in certain cases, may be shown as follow:

244. Theorem I.—Parallel rays may be made to converge or diverge accurately by means of a parabolic reflector. Theorem.

Let ACB (fig. 57) be a parabola, by the revolution of which about its axis QC, a parabolic speculum is generated. Take F, the focus; and let DA, which is parallel to QC, be a ray of light incident upon the concave side of this reflector, and join AF. Draw TAE, in the plane DAF, and touching the paraboloid in A. Then, since the angle TAD is equal to the angle EAF, from the nature of the parabola, the ray DA will be reflected in the direction AF. In the same manner it may be shown, that any other ray parallel to QC, will be reflected to F, and therefore the reflected rays converge accurately to this point. Fig. 57.

If DA, FA, be produced, it is manifest that rays incident upon the convex surface of the paraboloid, parallel to the axis, will, after reflection, diverge accurately from F.

The advantage, however, of a parabolic reflector is not so great as might at first-sight be imagined; for if the pencil be inclined to the axis of the parabola, the rays will not be made to converge or diverge accurately; and the greater this inclination is, the greater will the error become. If F be the focus of incidence, the rays will be reflected parallel to the axis.

245. Theorem II.—Diverging or converging rays may be made to converge or diverge accurately, by a speculum in the form of a spheroid; and to diverge or converge accurately by one in the form of an hyperboloid. Hyperbolic and elliptic reflectors.

Let F and D (fig. 58) be the foci of the conic section, by the revolution of which about its axis, the reflecting surface is formed; F the focus of incident rays, then will D be the focus of the reflected rays. Fig. 58.

For, let FA (fig. 59) be an incident ray; join DA, and produce it to *d*; draw TAE, in the plane DAF, touching the reflector in A; then the angle EAF is equal to the angle TAD, in the ellipse, and to *d* AT in the hyperbola: therefore, AD is the reflected ray in the former case, and A *d* in the latter: consequently, D is the focus of reflected rays. Fig. 59.

If FA be produced to *f*, the figures serve for the case in which rays are incident upon similar convex surfaces.

It may be remarked here, as in the preceding theorem, that if rays fall upon the reflector, converging to or diverging from any other point than one of the foci, they will not converge accurately after reflection.

VIII. *Of pyramidal, conical, &c. mirrors.*

246. Mirrors of the above description are little else than mere subjects of curiosity; they serve to distort regular objects presented to them, or to show those regular which are actually distorted. Cylindrical conical, &c. reflectors.

Prismatic and pyramidal mirrors being nothing more than plane mirrors placed vertically, or inclined, they will not require any particular description, as all the phenomena of them may be referred to the subjects treated of under section III. p. 435.

As to cylindrical mirrors, they ought to be considered as an assemblage of mirrors, in part plane and erect, and in part spherical; while the conical may be referred to a similar assemblage, in part plane and inclined, and in part spherical. In short, by combining the properties of plane mirrors with those that have the spherical figure, we shall be able to conceive the cause of the distortions of regular images produced by such reflectors.

Cylindrical mirror.

For example, a regular object being presented vertically before a cylindrical mirror, placed also vertically, it is obvious that the vertical dimensions of the object, whatever may be its distance, ought not to be distorted; because these dimensions are, in fact, in the same case as if they were presented before plane vertical mirrors; but the horizontal dimensions will be so much the more disfigured as they are less concentric with the mirror, or as the distance of their several parts are more unequal; consequently, the images of the different parts of an object, being in one case regular, and in the other irregular, the entire image will be extremely distorted. On the other hand, a very distorted object being placed before such a mirror, may be sometimes found to exhibit a very regular image; but in this case, the distortion ought to be made according to certain principles, as follows:

Construction of an anamorphosis.

247. Problem.—To draw a distorted figure, or *anamorphosis*, which, when placed vertically before a cylindric mirror of given radius, and at a certain distance from it, shall appear regular, when seen from a given point.

Figs. 60 and 61.

On a separate plane draw the regular object which is intended to be shown in the mirror, observing that its lateral dimension does not exceed the chord of 120° or 130° of base of the cylinder. About this figure draw a square or rectangle, as shown in (fig. 60), which is called the *prototype*. Divide this rectangle into any number of small equal squares, or rectangles, and which will of course divide the picture into a number of small equal parts. On the given plane, fix the place where the base of the cylinder is to be posited, which let be the circle FTK (fig. 61), and draw in it the chord FK, equal to the side FK of the *prototype*, answering to the bottom of the designed figure. Divide also the chord FK in the same manner as the side KF (fig. 60). Through the centre H of the chord FK, draw a perpendicular HO, which terminates in O, the point where a perpendicular let fall from the eye meets the plane. From the point O, draw through the divisions of the chord FK, the indefinite right lines OA′, O*g*′, O*h*′, O*i*, &c. and on one of these, as OF, erect a perpendicular OV, equal to the height at which the eye is proposed to be placed above the point O, that is to say, above the plane of the distorted figure. On the same right line OF, and from the point F, where it cuts KF, erect the perpendicular FA, equal to the side FA of (fig. 60), and divide it in the same manner; through V, and the points of division of FA, draw the indefinite right lines VF′, VE′, VD′, VC′ VB′, VA′, which will meet the right line OA′, in certain points, through which draw parellel to FK, E′*e*, D′*d*′, C′*c*′, B′*b*′, A′*k*′; and we shall have, according to the laws of perspective, a trapezium KFA′*k*′, which is the perspective representation of (fig. 60), seen from the point where the eye is to be placed, in order to see the object in the cylinder, that is to say, from a point raised above the point O, by a quantity equal to OV.

Through the centre Q of the arc FTK of the base
of the cylinder, and through the points of inc
F, S, T, K, where the right lines, or radii OF, C
OK meet this arc, draw the cathetus of inciden
also QP, QR, QX: then the indefinite right lin
S*g*, T*i*, K*k*, which make the angles *a* FL =
g SP = OSP, *i* TR = OTR, and *k* KX = OK
be the reflected rays. On these right lines,
flected radii, carry the divisions of the corresp
right lines of the perspective trapeziums; tha
say, the right lines FA′, S*g*′, M*h*′, T*i*′, K*k*
through all the points found in this manner, draw
which are nearly arcs of concentric circles, of
the centre is in H, and which represent the rig
A*a*, B*b*, C*c*, D*d*, &c. of the prototype, the s
FA′, G*g*′, H*h*′, I*i*′, &c. represent the sides F
H*h*, I*i*, &c. of the same; and finally, each
spaces, or trapeziums, represent the small squ
rectangles of fig. 60. If, therefore, we pla
cylinder on the arc FTK, and the eye at the
point of sight, we shall see in the cylinder a
image of the prototype, and consequently by d
in each trapezium, or mixtilinear space, the
the figure drawn in each square of fig. 60, w
have the distorted figure required.

§ III. Dioptrics.

1. *Of images and foci by a simple refraction*

248. By the term simple refraction, is here to
derstood the refraction of a ray in passing out
medium into another, without considering its en
again from the same; this latter, which is a s
case, will be treated of in a subsequent article.

We have already in our first section, conside
refraction of a ray impinging upon the plane surf
given medium, and have shown that the angle
dence and the angle of refraction are such, th
sines have a constant ratio; we now proceed
amine the phenomena and laws attending the re
of rays impinging on a medium terminated by s
concave and convex surfaces.

249. Theorem I.—Parallel rays refracted at a
spherical surface of a denser, or a concave, of
medium into which they pass, are made to co
and refracted at a concave spherical surface of a
or convex of a rarer medium, they are made to d

1. Let DA, GC (fig. 62), be two rays of a
pencil passing out of a rarer medium into a dens
incident upon the convex spherical surface ACB,
centre is E. Let GCE pass through the centre
surface, and it suffers no refraction. Join E
produce it to H; also produce DA to K; and
be refracted in the direction AF; then DAH
angle of incidence, and EAF the angle of refrac
this ray; and since it passes out of a rarer medi
a denser, the angle EAF, is less than the angle
and, therefore, it is less than the angle KAE;
each the angle AEF, and the two angles FAE
are together less than the two KAE, AEF; and,
fore, they are less than two right angles; conseq
AF and GE, if produced, will meet.

2. When the rays pass out of a denser medi
a rarer, and the surface of the medium into whi
are refracted is spherically concave.

The construction being made as before, since
DA (fig. 63), passes out of a denser medium

rarer, the angle of incidence DAE, or its equal AEC is less than the angle of refraction HAF; add to each the angle EAF and the two EAF, AEF, are together less than the two EAF, HAF; that is, they are together less than two right angles; therefore, AF and EC, if produced, will meet.

3. When the rays pass out of a rarer medium into a denser, and the surface of the medium into which they are refracted, is spherically concave.

The same construction being made, let DA (fig. 64) be refracted in the direction AL; and produce LA to F. Then, since the ray DA passes out of a rarer medium into a denser, the angle DAE is greater than the angle HAL or FAE; add to each the angle AEG, and the two FAE, AEG, are together less than the two DAE, AEG; that is, they are less than two right angles; therefore AF and EG will meet.

4. When the rays pass out of a denser medium into a rarer, and the surface of the medium into which they are refracted, is spherically convex.

In this case, as before, the angle DAH (fig. 65), or its equal AEC, is less than the angle EAL; add to each the angle EAF, and the two EAF, AEC are together less than the two EAF, EAL; that is, they are less than two right angles; therefore AF, and EC, if produced, will meet.

In the two last cases the rays meet, if produced backwards; that is, they diverge.

5. It follows, from what has been shown above, that when parallel rays are incident nearly perpendicularly upon a spherical surface, the distance of the geometrical focus of refracted rays from the surface, is to its distance from the centre as the sine of incidence to the sine of refraction. For, referring to the last four figures, the angle AEF is either equal to the angle of incidence of the ray DA, or to its supplement; therefore, sin AEF = sin of incidence. In the same manner, sin EAF = sin of refraction; and since the sides of triangles are proportional to the sines of their opposite angles, we have

FA : FE :: sin AEF : sin EAF :: sin incidence : sin refraction.

Now, let the point A approximate to C, and FA will ultimately become equal to FC; consequently, the proportion becomes

FC : FE :: sin incidence : sin refraction.

250. Problem.—Given the position of an object, with reference to a spherical refracting surface of given radius, together with the ratio of the sines of incidence and refraction, to find the place of the image formed by the refraction.

Let BAI be the given spherical surface (figs. 66 and 67), of which the radius is AK, and let O be the given object; assume also the ratio of the sines as p to q. Through the object O, and the centre K, draw the indefinite right line OA, to represent the axis of sphericity which passes through the object O. Let OI be an incident ray indefinitely near to the axis OA, and draw from the centre K to the point of incidence I the radius KI, which will be the cathetus of incidence. On the incident ray OI (produced if necessary), let fall from the centre the perpendicular KG, which will be the sine of the angle of incidence OIN, or KIG: then say as $p : q ::$ KG : a fourth term, with which as a radius describe from the centre K an arc; to which draw the tangent IH, and this will cut the axis AO in the point P required.

Dioptrics.

For letting fall the perpendicular KH, to the line PI, it is obvious that it will be the sine of the angle of refraction KIP. And since the same construction will obtain for every ray proceeding from the point O, on the surface of the medium indefinitely near to the point A, it follows that they are all so refracted as to be directed to the point P, which is therefore the focus or image.

Analytical value of AP.

250. In order to obtain an analytical expression for AP (fig. 66), let OA or OI $= d$, the radius of sphericity KI or AK $= r$; and AP or IP $= f$; then, by the above construction,

$$p : q :: \mathrm{KG} : \mathrm{KH},$$

or

$$\mathrm{KG} = \frac{p \times \mathrm{KH}}{q}.$$

Now since, by the supposition, OI is indefinitely near to OA, the arc AI may be considered as a right line perpendicular to the radius OA, and the triangles AOI, OKG, as similar triangles; as also the two PAI and PKH; therefore

$$\mathrm{OK} : \mathrm{OI} :: \mathrm{KG} : \mathrm{AI}.$$

Whence $\mathrm{AI} = \dfrac{\mathrm{OI} \times \mathrm{KG}}{\mathrm{OK}} = \dfrac{\mathrm{OI} \times p \times \mathrm{KH}}{\mathrm{OK} \times q}.$

And $\quad \mathrm{KH} : \mathrm{AI} :: \mathrm{PK} : \mathrm{PI}$ or AP.

Whence, again,

$$\mathrm{AP} = \frac{\mathrm{AI} \times \mathrm{PK}}{\mathrm{KH}} = \frac{\mathrm{OI} \times p \times \mathrm{KP}}{\mathrm{OK} \times q}.$$

We have here OI $= d$, OK $= d + r$, and KP $=$ AP $-$ AK $= f - r$; wherefore

$$f = \frac{dp\,(f - r)}{(d + r)\,q}.$$

Or $\quad fdq + frq = dpf - dpr.$

Wherefore $\quad f = \dfrac{dpr}{d(p-q) - rq},$

for the case in which the rays impinge on the convexity.

In order to reduce this formula to the case in which the rays impinge on the concavity, it is only necessary to make the radius of sphericity r (which has now a contrary direction with respect to the object) *minus*, or equal to $- r$, which gives for this case

$$f = \frac{-\,dpr}{d\,(p-q) + rq}.$$

Remark.

251. The figures referred to in the above investigations, are applicable to the cases in which the passage is made from a rarer into a denser medium, or to the supposition of $p > q$; but the formulæ will apply equally to those, in which $p \angle q$; the sign of f, however, will obviously then be changed, this depending upon the relative value of d, p, q, and r. When the rays are parallel to the axis, d may be considered as infinite, and, consequently, we shall then always have $d\,(p - q) > rq$; that is, the denominators of our fractions will always have the same sign as $p - q$. Therefore, when the rays impinge on a convex surface, f will be positive, or negative, accordingly as p is greater or less than q; and when the surface is con-

cave, it will be negative or positive, under the same circumstances: that is, the rays will converge in passing out of a rarer into a denser medium, when the surface is convex; or from a denser into a rarer, with a concave surface; and they will diverge in the former case when the surface is convex, and in the latter when the surface is concave, which agrees with the enunciation of our theorem (art. 249).

252. The same remarks will apply also to these formulæ, as we have made in the case of reflection (art. 168). We may likewise apply here the theorem, with its demonstration and corollaries, as given in the former case (art. 177). We shall suppose this to have been done in what follows.

I. *Application of the preceding formulæ to the determination of the place of images, as it depends upon the passage of rays from air into glass.*

On the place of images in the passage of light from air into glass.

253. As the most important application of Dioptrics is the determination of the laws observed by light in passing out of air into glass, and the contrary, in order to the forming a due estimation of the effects of telescopes, microscopes, and other optical instruments, we shall examine, with reference to this application, the preceding theorems or formulæ.

We have seen that, in the passage of light from air into glass, the sines of incidence and refraction are to each other as 3 : 2, or, more accurately, as 31 to 20; that is, we have in this case $p = 31$, and $q = 20$, which numbers being substituted, reduce the above formulæ to

$$f = \frac{31\,d\,r}{11d - 20r} \;\ldots\; (a)\ \textit{for convex surfaces,}$$

and $$f = \frac{-31\,d\,r}{11d + 20r} \;\ldots\; (b)\ \textit{for concave surfaces.}$$

Convex surface.

254. Let us now conceive two mediums of indefinite extent, the one of air and the other of glass, each respectively homogeneous, and separated only by a spherical surface.

In the first place, let us suppose this surface to be convex, on the part next the air, and that a luminous object of small extent is placed before it at the surface, and that it is afterwards carried to an indefinite distance from it, in a direction perpendicular to the surface; it is proposed to determine, by means of the formula (*a*)

$$f = \frac{31\,d\,r}{11d - 20r},$$

all the circumstances relative to the motion and position of the image of this object, according to the different values of d; which we shall suppose to be expressed in parts of which r will be assumed as the unit.

In the first place we may observe, that between the values of d, infinitely small, or $d = \frac{1}{\infty} r$, to $d = \frac{20}{11} r$, f will always be negative, and that its value will increase to infinity, which happens when $d = \frac{20\,r}{11}$; the image or focus will therefore, during this time, be without the glass, or on the same side as the object; because, by the construction of our figure, we have supposed the positive sign to indicate a differe rection of f; that is, a direction passing from tl fracting surface from the object: the image als in the above case, be erect, and will continue s its nearest distance to infinity; and the rays whi their concourse, determine the place of the imag be less and less diverging till they become pa

From this value of $d = \frac{20}{11} r$, to $d = \infty r$, f is positive; the image is formed within the glass, reversed; it comes from an infinite distance sponding to the former value of d, and approac d increases nearer and nearer to the spherical su till it arrives within the limit $\frac{31}{10} r$, which corre to the infinite value of d; and during this time tl will acquire a greater degree of convergency.

255. But if the surface which separates the me be concave on that side next the air, then the fo (*b*) (art. 253)

$$f = \frac{-31\,d\,r}{11d + 20r}$$

indicates, that whatever may be the distance object, that is, whatever may be the value of d, always be negative, that the image is witho glass and direct; also, that as d increases from to $d = \infty r$, f will increase from an infinitely quantity to $\frac{31\,r}{11}$, or the image will pass from fracting surface to the above distance, and th which enter the glass will diverge, but the dive will be less and less as d augments.

256. If we had supposed the image to be within the glass, on the surface which separat two mediums, and that it were made to move the glass, then we should have $p = 20$, $q \doteq 3$ our two formulæ (art. 253) would in this case b

$$f = \frac{20\,d\,r}{-11\,d - 31\,r} \;\ldots\; (c)\ \textit{for the convex s}$$

$$f = \frac{20\,d\,r}{11\,d - 31\,r} \;\ldots\ldots\; (d)\ \textit{for the concave}$$

If, therefore, we suppose the object to be pla the convex surface, within the glass, it will be c by a course of argument, similar to the precedin while the object moves from its first situation finity, the image will be direct, and formed wit glass, and will pass from the common surface two mediums to the distance $\frac{20}{11} r$, and the rays pass into the air will become more and more div

257. Final, if the object be placed on the c side, the formula (*d*) viz. $f = \frac{20\,d\,r}{11\,d - 31\,r}$ show f will be negative for all values of d, b the limits $d = \frac{1}{\infty} r$, and $d = \frac{31}{11} r$; that the will be erect, within the glass, and will ulti be formed at an infinite distance from the su

$$\frac{\mathrm{FF}'}{\mathrm{DD}'} = \frac{-f + r}{d + r} = \frac{31\,r}{11\,d + 31\,r};$$

which is, therefore, the ratio of the magnitude of the image to that of the object.

If we here suppose $d = 0$, the ratio reduces to unity, which shows, that while the object is placed at the common surface of the two mediums, the magnitude of the image and object are equal to each other.

From $d = 0$, to $d = \infty\, r$, this ratio decreases from unity to 0; that is, while the object passes from the surface to an infinite distance, the magnitude of the object will diminish, from being equal to the object, to nothing.

4. *The object being still supposed in the glass, and opposite to the concave surface.*

The object opposed to the concave surface. Figs. 71, 72.

261. Let AA′ (figs. 71 and 72) represent the surface which separates the two mediums, its concavity being on the side of the glass; DD′, FF′, the image, &c. as in the preceding cases; then we have, as before,

$$\mathrm{FF}' : \mathrm{DD}' :: \mathrm{CF} : \mathrm{CD};$$

and
$$\frac{\mathrm{FF}'}{\mathrm{DD}'} = \frac{\mathrm{CF}}{\mathrm{CD}}$$

will denote the ratio of the image to the object.

Now $\mathrm{CD} = \mathrm{AD} - \mathrm{AC} = d - r$, figs. 71 and 72,
$\mathrm{CF} = \mathrm{AC} + \mathrm{AF} = f + r$, fig. 71,
$\mathrm{CF} = -(\mathrm{AC} + \mathrm{AF}) = -f - r$, fig. 72;

this change of sign being merely with reference to the direction of CF, and has no influence on the magnitude.

Again (art. 256), $\mathrm{AF} = f = \dfrac{20\,d\,r}{11\,d - 31\,r}$;

and substituting this value, we have

$$= \frac{\mathrm{FF}'}{\mathrm{DD}'} = \frac{\mathrm{CF}}{\mathrm{CD}} = \frac{(f + r)}{d - r} = \frac{\dfrac{20\,d\,r}{11\,d - 31\,r} + r}{d - r}$$

$$= \frac{31\,r\,(d - r)}{(d - r)(11\,d - 31\,r)} = \frac{31\,r}{11\,d - 31\,r};$$

which latter, therefore, expresses the ratio of the magnitude of the image to that of the object.

If now here, we make $d = 0$, the ratio reduces to -1; which indicates that when the object is at the surface and on the side of the glass, the image is equal to the object, and on the interior surface of the glass.

From $d = 0$ to $d = \dfrac{31\,r}{11}$, the ratio will increase from -1 to infinity; that is to say, while the object passes from the surface to the distance $\dfrac{31\,r}{11}$, the magnitude of the image will increase from being equal to the object, to infinite (fig. 71); and will be always in the interior of the glass.

From $d = \dfrac{31\,r}{11}$ to $d = \infty\, r$, the ratio will decrease from ∞ to 0, and consequently the image will decrease from a quantity infinitely great to nothing, it will be situated in the air, and will be direct or reversed, accordingly as the eye receives the impression of the rays before or after their concourse.

We may further observe, that in the passage of from $\dfrac{31\,r}{11}$ to infinity, we shall find the value $d = \dfrac{2 \times 31\,r}{11}$, at which latter distance the magnitude o the image and object will be again equal to each other

262. When the surface which separates the two mediums is a plane, the radius of curvature being the infinite, the formula (art. 250)

$$f = \frac{d\,p\,r}{d\,p - q\,(d + r)} = \frac{d\,p\,r}{d(p - q) - r\,q},$$

becomes

$$f = \frac{d\,p\,r}{-\,q\,r} = \frac{-\,d\,p}{q};$$

that is, all the terms will then vanish in respect of which is infinitely great; therefore, when the two mediums are glass and air, we shall then have $p = 31$ a $q = 20$, which give the value of $f = \dfrac{-31}{20}\,d$.

Therefore, in this case, the image will be found the air, and is more distant than the object in the ra of 31 to 20; and the ratio between the magnitude the image and object is unity, provided only that d not infinite.

If the object be supposed to move in the medium the glass, we shall have $p = 20$, $q = 31$, and

$$f = \frac{-20}{31}\,d;$$

that is, the image is nearer to the surface than the object, in the ratio of 20 to 31; or, as we common assume the ratio of the sine's incidence and refractio as 2 to 3, which corresponds with what has be already stated with reference to plane mirrors (art. 226

IV. *Of the refraction of light at two surfaces.*

263. We have hitherto, in this section, only con dered the simple refraction of the rays of light wh they impinge upon the plane or spherical surface of indefinite medium; but the case which most commo requires investigation, particularly in the constructi of optical instruments, is that in which the rays pa through a finite medium, and are, therefore, subject refraction both at their incidence on, and emergen from, the medium in question, as in the passage light through a glass-plate, or lens. The case of pla surfaces, whether parallel, or forming any given ang with each other, has been already treated of as far relates to the directions of the incident and emerge ray, in our first section; it only remains, therefor for us here to examine that of images formed by t refraction of rays, at entering and emerging from a m dium terminated by a plane and spherical, or t spherical surfaces, particularly as it relates to the ca of lenses.

Definitions.

264. 1. A *lens* is a piece of glass, or other transparent substance, whose surfaces are either both spherical, or one plane and the other spherical.

2. A lens, having one side plane and the other convex, as in fig. 73, is called a *plano-convex lens*; and that having one plane and the other concave, a *plano-concave lens*, as fig. 74. A lens terminated with two

ides, as fig. 75, or with two concave sides, as s called accordingly, a *double convex*, or a cave *lens*; and fig. 77, in which one surface is and the other convex, is called a *meniscus*.

en the particular figure is not considered, a h is thickest in the middle is called a *convex* l when thinnest in the middle, a concave lens, urther distinction.

ight line drawn perpendicular to both sur- any lens, is called the *axis of the lens*. The efore, passes through the centres of the sphe- s; and it is hence manifest that no lens, ex- sphere in which both centres are in one point, more than one axis, and therefore all pencils e, excepting those whose focii are in the axis s.

geometrical focus of a direct pencil of parallel that point in the axis where rays parallel st to it are united after refraction, if the lens st in the middle, or whence they diverge, if e thinnest in the middle), is called the focus of sometimes the principal focus, by way of dis- g it from the focus after refraction, of con- r diverging rays; and the distance of the prin- s from the lens is called the *focal length*.

ROBLEM I.—Having given the dimensions of AB (fig. 78), the position of the point O on non axis of sphericity of the two surfaces, of centres are C and K, to find the point F on , where the radius OI infinitely near to the cut that axis after two refractions, the one at other at T.

$= d$, CB $= \mathrm{R}$, KA $= r$, FB $= x$, PB $= z$; he point in the axis where it is intersected by ion of the incident ray OI, after the first re- t I. Let AB, which is the thickness of the and denote the ratio of the sines of inci- l refraction, when the ray enters the lens, by d on its emerging, by $q : p$. Lastly, let and KG $= n$. Then it is obvious from the at

$$p : q :: \mathrm{KG} \text{ or } n : \mathrm{KH} = \frac{nq}{p},$$

$$q : p :: \mathrm{CD} \text{ or } m : \mathrm{CE} = \frac{mp}{q}.$$

g established, in consequence of the similar AI and OKG, we have

$$\mathrm{OG} = \mathrm{OK} : \mathrm{OA} :: \mathrm{GK} : \mathrm{AI},$$

$$+ r : d :: n : \mathrm{AI} = \frac{dn}{d+r}.$$

from the similarity of the triangles PAI, PKH,

$$\mathrm{PA} : \mathrm{PH} :: \mathrm{AI} : \mathrm{HK};$$

$$+ e : z + e - r :: \frac{dn}{d+r} : \frac{nq}{p};$$

$$+ e)\frac{nq}{p} = (z + e - r)\frac{dn}{d+r}.$$

hich we readily obtain the value of z, viz.

$$z = \frac{deq + eqr + dpr - dep}{dp - dq - qr}.$$

ecause of the similar triangles PCD, PBT, FBT, we have

$$\mathrm{PD} : \mathrm{PB} :: \mathrm{CD} : \mathrm{BT};$$

or $$z + \mathrm{R} : z :: m : \frac{zm}{z + \mathrm{R}}.$$

And $$\mathrm{FC} : \mathrm{FB} :: \mathrm{CE} : \mathrm{BT};$$

or $$x + \mathrm{R} : x :: \frac{pm}{q} : \frac{zn}{z + \mathrm{R}}$$

This equation gives

$$z = \frac{p\mathrm{R}x}{qx + q\mathrm{R} - px}.$$

Now, equating the two values of z thus obtained, we have

$$\frac{deq + eqr + dpr - dep}{dp - dq - qr} = \frac{p\mathrm{R}x}{qx + q\mathrm{R} - px}.$$

Whence we have for a final value of x

$$x = \frac{dpq\mathrm{R}r + deq^2\mathrm{R} - depq\mathrm{R} + eq^2\mathrm{R}}{dp^2\mathrm{R} - dpq\mathrm{R} - pqr\mathrm{R} - deq^2 - dpqr + 2depq - dep^2 + dp^2r - eq^2r + epqr}.$$

This equation, which is rather complicated, becomes much more simple when we apply it to any particular case; that is, when we substitute particular numbers for p and q: thus, supposing the lens of glass and the first medium to be air, we shall have $p = 31$, and $q = 20$, and the above expression reduces immediately to

$$x = \frac{620\,dr\mathrm{R} - 220\,de\,\mathrm{R} + 400\,er\mathrm{R}}{341(d\mathrm{R} + dr) - 620\,r\mathrm{R} - 121\,de + 220\,er}. \quad (b)$$

And if we make $e = 0$, by neglecting the thickness of the glass, we may reduce the same much lower; for in this case we find

$$x = \frac{20\,dr\mathrm{R}}{11(d\mathrm{R} + dr) - 20\,r\mathrm{R}}. \quad (c)$$

Lastly, by supposing the radii of sphericity of the two surfaces equal to each other, or $r = \mathrm{R}$, we have

$$x = \frac{10\,dr}{11\,d - 10\,r}. \quad (d)$$

Dioptrics.

Remark.

266. Having given the arc AI (fig. 78), comprised between the point A of the common axis of the two spherical surfaces, and the point I, where an oblique ray OI falls, that issues from a point O taken on the axis, we may calculate trigonometrically, the place of the point F, where the radius OI meets the same axis after two refractions. For in the triangle OKI, we know OK, KI, and the angle OKI, and the calculation will give OI, and the angle KIO, the supplement of which is KIG. In the right-angled triangle IKG, we have IK, and the angle KIG, whence we readily obtain KG. Afterwards making

$$p : q :: \mathrm{KG} : \mathrm{KH},$$

the latter becomes known.

And in the right angled triangle KIH, having KI and KH, we readily obtain the angle KIH. Then in the triangle KIP, we have KI, and the angles IKP, KIP, we shall have therefore KP, and the angle KPI.

In the triangle PCD, right-angled at D, we have

$$\mathrm{PC} = \mathrm{PK} + \mathrm{KA} + \mathrm{CB} - \mathrm{AB},$$

and the angle CPD, whence we obtain the angle PCD, and the side CD.

Make now $q : p :: \mathrm{CD} : \mathrm{CE}$,

and CE becomes known.

Then in the right-angled triangle CTD, we shall know CT, CD; whence we may find the angle TCD. In the triangle CTE, we know CT, CE, wherefore the angle ETC may be computed, of which the supplement is DTF. Finally, in the triangle CTF we shall have the side CT, the angle CTF, and the angle $FCT = PCD - TCD$; we shall therefore have CF, and consequently BF, because

$$BF = CF - CB.$$

If the object O be supposed at an infinite distance, the calculation becomes a little shorter; for OI, being then parallel to the axis, the angle $KIG = AKI$, is measured by the given arc AI.

267. From the above calculation, or even by a simple geometrical construction, it will be obvious, that when a ray OI falls at any distance from the point A in the common axis of the two surfaces, the curvature of the arc AI, will bring the ray towards the axis, and consequently the point F, where the axis is cut by it, will approach nearer to the point B, in proportion as the arc AI is of a greater number of degrees.

When the object is out of the axis. Fig. 79.

268. Problem II.—Having given the dimensions of any lens AD (fig. 79), the centre of whose surfaces are in C and K, the position of an object O, out of the axis BK, at the same distance from the lens as the point B, which is in the axis; to find the point F, where the rays of light, issuing from the point O, are united after traversing the lens.

Through the point O and the centre K, draw the right line OKP, and compute, according to the principles explained in article 250, the point P, where the image would be formed by simple refraction at the convex surface of a medium of glass, and let this point be P; considering now this first image of O, as an object placed in a medium of glass, join P and the centre C, of the spherical surface ATD, and compute by the proper formula in the article above referred to, the focus F, of rays issuing from P, and this point F will be the place of the image due to the second refraction; the same as the point f, found as in the preceding theorem, will be the image of an object B placed in the common axis of the two surfaces.

Deductions.

From the above construction, the following corollaries are readily deduced:

1. Neglecting the thickness of the glass, and supposing the points O and B to be at equal distances, it is obvious that the points p and P, will be equally distant from the surface, because they are each found by the same formulæ, and with the same data; consequently, the points f and F will, for the same reason, be also at equal distances.

2. What has been stated above with reference to the point O, is equally true of every point in the surface of an object; therefore, the entire image of an object will necessarily be nearly similar to the visible surface of the latter.

3. When all the points of an object of considerable extent, or when many objects are placed at equal distances from a lens, their images ought to be shown distinctly in a portion of a sphere of which the glass is the centre.

4. It will be sufficient, therefore, to compute by (art. 250) the position of the image, from the point of the object which is in the axis of the glass, in order to have that of the entire object itself.

5. We see also by the preceding construction, that when the object OB is sufficiently distant to have its image formed on the other side of one of the convexity of the lens, this image will be reverse is, its parts will have a contrary position to the sponding parts of the object.

269. Experiments show that images of obje sented to a lens, are made to a considerable and very distinctly; for, if in a dark room we hole of 2 or 3 inches diameter, and supply it with vex lens, we shall perceive on a sheet of white pape at a proper distance, that is, at a distance prop to the radius of convexity and the distance of the the images of them reversed, and in colours s the more vivid as the objects themselves are be lightened; but the images will be more distinct, represent objects situated nearer to the axis lens.

270. Theorem.—When the two surfaces of a or concave lens have equal radii of sphericity; rays of light which issue from any point taken the axis, and falling on the lens, that which through that point of the axis, which is in the of the thickness, will emerge, after two refractio right line, parallel to its first direction.

Let O (fig. 80 and 81) be the given point ta of the axis of the lens; OI, the ray which through the point I in the middle of the thick either figure, and FT the emerging ray. Then, the two arcs ALD, ATD, are equal, and are radii, the figure of the lens may be consider symmetrical pylyëdron of an infinite number o of which the centre is I; whence it follows, t ray of light TL, which passes through I, will pass two parallel sides of this figure, the positions will be determined by the tangents GH, HT. fore this ray will be refracted equally, both at dence and emergence; that is, the angle of re ILO, will be equal to the second angle of re ITF, and consequently, the directions MO and parallel to each other. *On account of this prop ray OI is called the principal ray.*

271. If the lens were plane on one side, and or concave on the other, then the principal ra be that which entered or emerged at the summ curvature, according as this curvature was towards, or from the object; for the summit of vature may be considered as an infinitely sma parallel to the plane side of the lens.

272. Neglecting the thickness of the lens, th pal ray will emerge in the same right line as which it enters; or, which is the same, every ray to the lens, tending to the point in its axis cor ing to the centre of the thickness, will pass th in a right line, or it will suffer no refraction.

V. *Of lens formed by the revolution of a conic*

273. It appears from what has been stated in(and the following, that spherical surfaces do n all the rays in any diverging pencil to conver diverge from the same point, except in one p case, and the same will appear more obvious we have to treat of optical instruments. In obviate this imperfection, it has been proposed such refractors as are generated by the revoluti ellipse and hyperbola. But as these are never to in practice, on account of the great difficulty them the exact form, and because the same ef

easure, be produced by the proper adjust-
surfaces of spherical refractors, it will be
explain the geometrical principles upon
'oposed refractors depend.

prolate spheroid be generated by an ellipse,
axis is to the distance between its foci as
ıcidence to the sine of refraction, out of the
ium into the solid, a pencil of rays, incident
ion of its axis, will be refracted, converging
› the further focus.

(fig. 82) be an ellipse, by the revolution of
its major axis DK, the spheriod is generated;
ıd I be its foci, then by the supposition,

$DK : HI :: p : q,$

ı p : *q*, the ratio of the sine of incidence to
refraction). Let AB, which is parallel to
y of light incident upon the spheroid; join
ıw EBC, touching the generating ellipse in
B and H; draw GBL, and HCO at right
3C; let GBL meet DK in N, and produce
ets HCO in O. Then, since the angle HBC
he angle IBE, by the nature of the ellipse,
e OBC is equal to the angle IBE, the angles
are equal; also the angles BCH and BCO
ght angles, and BC is common to both
Wherefore,

BO = BH and IO = DK;

r,
' : IH :: *p* : *q*;
BN is parallel to OH,
ɜ : IN :: IO : IH :: *p* : *q*.
ɜ : IN :: sin INB : sin IBN,
:: sin BNH = sin ABG : sin IBL:
ı ABG : sin IBL :: *p* : *q*.

:e the sin ABG is the sine of incidence, sin
sine of refraction; and because the angle
than a right angle, BI is the refracted ray.
manner it may be shown, that every other
:ncil will be refracted to I.

e centre I, with any radius less than ID, a
PQ (fig. 83) be described, the solid gene-
: revolution of PDQ, about the axis DI,
all the rays incident parallel to DI, accu-

refraction at the surface PDQ, the rays
I, and they suffer no refraction at the sur-
ecause they are incident perpendicularly
is also obvious, that rays diverging from I,
cted parallel to ID.

n hyperboloid, whose major axis is to the
ween the foci, as the sine of incidence to the
ction, out of the solid into the ambient me-
:nerated in a similar manner to the preced-
rallel rays, incident in the direction of the
'racted out of the hyperboloid, will converge
:r focus.

ɜr proof may be applied to this case, and
he same words, referring only to (fig. 84)
ig. 82).

. 85) be drawn perpendicular to the axis of
la, and meet the curve in P and Q, the solid
›y the revolution of PDQ, about the axis
'efract all the rays incident parallel to MI
o I.

rays will suffer no refraction at the plane

Rays diverging from I, and incident upon the surface PDQ, will be refracted parallel to ID.

VI. *Of the motion and situation of images formed by refraction at two surfaces.*

Images seen through lenses.

277. When a lens of glass is equally convex on both sides, if we suppose a luminous object of small extent to be placed on one of its surfaces at the point coinciding with the common axis of sphericity, and that afterwards this object is made to move along the axis to an infinite distance, it is obvious, from the formula (*d*) (art. 265)

$$x = \frac{10dr}{11d - 10r},$$

that the image of this object will remain during all this motion, in the same axis; that in the first instance, that is, while the object is on the surface, the image and object will be confounded, or coincide, and that from the value $d = 0$ or $\frac{1}{\infty} r$, to $d = \frac{10}{11} r$, x will be negative, and when $d = \frac{10r}{11}$, x will be infinitely negative; between these limits, therefore, the image will be on the same side of the lens as the object, and it will be erect. But for all values of d greater than $\frac{10r}{11}$, the value of x will be positive, and decrease more and more till d becomes infinite; in which case we shall have $x = \frac{10r}{11}$, and the image will be on the opposite side of the lens to the object.

Double convex lens.

Plano-convex lens.

278. If the lens be plano-convex, one of its radii of sphericity will be infinite; let this be R, that is, $R = \infty$; then the formula (*c*) (art. 265)

$$x = \frac{20drR}{11dR + 11dr - 20rR},$$

will have all those terms vanish, in which R is not found, in comparison with those in which it is. We shall have, therefore,

$$x = \frac{20drR}{11dR - 20rR} = \frac{20dr}{11d - 20r};$$

and making again the suppositions as above, with respect to the different positions of the object, it will appear, as in the case referred to, that the object and image will coincide when $d = 0$; that x will be infinite when $d = \frac{20}{11} r$; and that between these limits x is negative; or the object and image are on the same side; but that from the value of $d = \frac{20}{11} r$ to $d = \infty r$, x is positive, and the image is on the contrary side; that the rays become more and more converging as d increases; and when d is infinite, the distance of the image, or the value of x, is $\frac{20r}{11}$.

279. In the formula above referred to, the thickness of the glass is not regarded; and consequently, it is indifferent whether the object be placed on the plane or

The objects presented to the different sides.

on the convex surface of the lens; but if we take this thickness into consideration, then there will be some difference in this respect, the image being more distant from the convex surface when the object is placed on the plane side, than it is from the plane surface, when the object is on the convex side; and this difference, when the distance of the object is very great, that is, when d is infinite, is equal to about $\frac{2}{3}$ds the thickness of the lens. For if in the formula in which the thickness of the lens e enters, viz. (a) (art. 265), viz.

$$x = \frac{620\,d\,r\,\mathrm{R} - 220\,d\,e\,\mathrm{R} + 400\,e\,r\,\mathrm{R}}{341\,d\,\mathrm{R} + 341\,d\,r - 620\,r\,\mathrm{R} - 121\,d\,e + 220\,e\,r},$$

we make $d = \infty$, and $r = \infty$, in order to express that AI (fig. 78) is a plane surface turned towards the object O, it will be found to reduce itself to

$$x = \frac{620}{341}\mathrm{R}.$$

But if we make $\mathrm{R} = \infty$, which expresses that the surface BT is a plane, the formula gives

$$x = \frac{620}{341}r - \frac{220}{341}e.$$

And as r, and R, in these two cases are supposed to express the same radius of curvature, it is clear that the difference in the distance of the object is $\frac{220}{341}e$, or $\frac{2}{3}e$ nearly.

This remark is of service in the use of telescopes fitted with micrometers. In refracting telescopes of this kind, the object glass is frequently plano-convex, and when it is taken out, for the purpose of cleaning, &c. it follows, from what is above stated, that it is important to be careful to replace it again in the same position, as otherwise the threads of the micrometer will be found distant from their true places, by a quantity equal to $\frac{2}{3}$ds of the thickness of the object-glass, which may therefore amount to the $\frac{1}{20}$th of an inch.

Double concave lens. 280. If the lens be equally concave on its two sides, then the radius KA (fig. 78) is turned towards the object O, and we must therefore consider r as negative, that is, we must replace it by $-r$, and, for a similar reason, R will become $-\mathrm{R}$. The formula will therefore be the same in this case as in (art. 265), by simply changing the sign of r; that is, it will be

$$x = -\frac{10\,d\,r}{11d + 10r}.$$

Whence it appears that, if we suppose an object to be placed on one of its surfaces, at the point which coincides with the common axis of the lens, and to move along this axis to infinity, its image will move along the same axis, and in the same direction, from the glass, where $x = 0$, to the distance $x = \frac{10}{11}r$; it will always be direct, and formed by rays emerging from the concave surface, which will diverge more and more.

Plano-concave lens. 281. If the lens be plano-concave, the formula will be

$$x = -\frac{20\,d\,r}{11d + 20r}.$$

The motion of the image and of the object will be made

concavity, except

x will always be greater than in the

the greatest possible distance of the image will

$\frac{20}{11}r.$

282. Lastly, if the lens be a meniscus, that is con[illegible]cave on the one side, and convex on the other, we mu[illegible]t change the sign of one of our radii of curvature, [illegible]y writing $-\mathrm{R}$, instead of $+\mathrm{R}$, in equation (c) (art. 26[illegible]), and we shall thus have, neglecting the thickness of t[illegible]he glass, Menisc

$$x = \frac{20\,d\,r\,\mathrm{R}}{11d\mathrm{R} - 11d\,r - 20\,r\mathrm{R}}.$$

Here, by making the different suppositions w[illegible]th respect to the place of the object, or on the value o[illegible], we shall arrive, as above, at the different correspo[illegible]ing distances of the image; but as such lenses are ve[illegible] seldom employed, it would be useless to enter into th[illegible] details, which would require more space than we have bestowed upon the lenses of the more usual figure. For the same reasons, we shall not examine the laws of lenses, concave on the one side and convex on the other, each having different radii of sphericity.

283. We may consider in practice an object to be Ren infinitely distant; if that distance be equal to 1000 times the radius of sphericity, that is, in such cases we may suppose all the terms of a formula to vanish, except those in which d enters. For example, in the formula $x = \frac{10\,d\,r}{11\,d - 10\,r}$.

If we suppose $r = 10$ inches, and $d = 10000$ inches, we shall find

$$x = \frac{10 \times 10000 \times 10}{110000 - 100} = 9{\cdot}102 \text{ inches.}$$

As if d were infinite then, we should have

$$x = \frac{10\,d\,r}{11\,d} = \frac{10}{11}r = 9{\cdot}091 \text{ inches,}$$

which makes a difference only of about $\frac{1}{100}$ of an inch.

VII. *On the magnitude of images, formed by lenses of different figures.*

Convex lens.

284. In order to determine the actual magnitude of the image of an object situated in the axis of a lens, and of which all the points are nearly equally distant from the surface of the lens. Let DD′ (fig. 86) be the object, FF′ the foci of the extremities DD′, these being similarly situated with respect to the point O, and to the axes OD OD′; then FF′ will be the image of the object, and we shall have FF′ : DD′ :: FO : DO; whence

$$\frac{\mathrm{FF'}}{\mathrm{DD'}} = \frac{\mathrm{FO}}{\mathrm{DO}} = \frac{x}{d}.$$

Fig. 86.

Now, neglecting the thickness of the glass, and supposing the curvature of the two surfaces to be equal to each other, we shall have

$$\mathrm{FO} = x = \frac{10\,d\,r}{11\,d - 10\,r}, \text{ (art. 277);}$$

consequently, since $\mathrm{DO} = d$, we obtain

$$\frac{\mathrm{FF'}}{\mathrm{DD}} = \frac{x}{d} = \frac{10\,r}{11\,d - 10\,r},$$

which expresses the ratio of the image to the object.

292. *Lens with equal radii.*

Direction of the rays.	Convex.	Concave.
Diverging	$f = \frac{d r}{2 a d - r}$	$f = \frac{- d r}{2 a d + r}$.
Parallel	$f = \frac{r}{2 a}$	$f = \frac{r}{2 a}$.
Converging	$f = \frac{- d r}{-2 a d - r}$	$f = \frac{d r}{-2 a d + r}$.

293. *Lens with one plane surface, or* R $= \infty$.

Direction of the rays.	Convex.	Concave.
Diverging	$f = \frac{d r}{a d - r}$	$f = \frac{- d r}{a d + r}$.
Parallel	$f = \frac{r}{a}$	$f = \frac{- r}{a}$.
Converging	$f = \frac{- d r}{- a d - r}$	$f = \frac{- d r}{a d - r}$.

294. ***Lens with one convex and one concave surface, or* R *negative.***

Direction of the rays	Radii unequal.	Radii equal, or $R = - r$.
Diverging	$f = \frac{- d r R}{a d r - a d R + r R}$	$f = - d$.
Parallel	$f = \frac{- r R}{a r - a R}$	$f = \frac{- r R}{0} = \infty$.
Converging	$f = \frac{d R r}{a d R - a d r + r R}$	$f = d$.

These formulæ will be found very convenient for references in many cases, as they include all those most commonly required in practice. We have not introduced those which include the thickness of the glass, as this datum may be generally omitted; if it be at any time requisite to retain it, the formula appertaining to this case will be found under (art. 265).

To find the focal length of a lens, or mirror, experimentally.

Foci determined by experiment.

295. If it be either a concave mirror, or a convex lens, present it to the sun, and ascertain the point where the reflected or refracted rays received on a plane, form the most vivid and the smallest circle of pure white light, or where any combustible substance is soonest enflamed; and this point will be the focus sought. Otherwise cover the surface of the mirror, or one of the faces of the lens with a black paper, having first made in it several small pin holes, and ascertain the point where the solar rays, passing through these several holes, unite with each other; and this point will be the focus sought.

Convex lens, or concave mirror.

A third method is to present the mirror or lens to a strong light, as that of a torch, or flambeau, so far distant as at least to be beyond the centre of sphericity; and find the distance that a plane must be situated from the flambeau and mirror, or lens, in order to form a distinct inverted image of the light: and we shall then have the necessary data for calculating, by the preceding formula for convex lenses, or those of (art. 223), for mirrors, the radii of sphericity; and hence the focal length, which, in mirrors, is equal to half radius; in double convex lenses of equal sphericity, to radius itself, and in plano-convex lenses, to double radius.

In the case of a convex mirror, or of a concave lens, cover the surface of either, as above, with a black paper, in which make several pin-holes in the circumference of a circle described upon the paper; then the rays of light reflected from, or refracted through these holes, if received on a parallel plane, will also be found disposed in the circumference of a circle of greater or less radius, according to the distance of the plane from the mirror or lens. If now, by moving the plane to and fro, we at length find that situation where the circle thus formed has its radius double of that on the black paper, the distance of the plane from the mirror, or lens, will be equal to the focal length sought, as is obvious.

This experiment is easier performed with a concave lens, than with a convex mirror; because in the latter case the holes must be of some sensible magnitude, in order to admit the rays of the sun to reach the surface of the mirror, and to be again reflected from it.

We have said nothing of the particular apparatus proper for moving and fixing the lens and plane employed in these experiments; these are of various constructions, and may be made according to the views of the operator.

§ IV. OF VISION.

I. *Description of the eye.*

296. The eye, or organ of vision, may be considered as an optical instrument, composed of different diaphanous mediums, in which the curvature and the refracting properties are so combined, that the aberrations arising from figure and different degrees of refrangibility are made to counteract each other, whereby the effects of either are rendered insensible.

As an optical instrument, however perfect it may be, always concentrates with greater advantage those rays which are nearest to its axis, the eye has been provided with muscles which enable us at all times to direct it to the object we wish to observe; and the same wisdom which has so admirably constructed this organ in the latter respects, has protected it in a thousand ways from the effect of accident and exposure. But however pleasant it might be for us to examine and point out for admiration the various precautions which nature has taken to preserve from injury this most delicate organ, yet, as it is foreign to our subject, we must not indulge in the pleasing digression into which we should thus be led.

Our business in this place, is merely to describe the eye with reference to its mechanical and optical purposes; and in doing this, it will be desirable that we avoid as much as possible the terms introduced into the description of this organ by anatomists, as these would, in this place, tend rather to embarrass than to illustrate its optical properties; the anatomy will, moreover, be given in its proper place in this work, and we shall therefore confine ourselves entirely in this article to such a description of the eye as is sufficient for illustrating its visual properties, and must refer our readers for a more minute dissection of its parts to our treatise on ANATOMY.

Differ parts. Fig. 9

297. A section of the human eye is represented in (fig. 90); this section being supposed to be made by a plane perpendicular to the surface of the coats, which contain its several humours, and also to the nose.

Its form is nearly spherical, and it would be exactly ɛe not the fore-part a little more convex than the der; the parts BFB′, BAB′, being in reality the ıts of a greater and lesser sphere.

humours of the eye are contained in a firm coat , called the *sclerotica*; the more convex, or pro- ıt part of which, BAB′, is transparent, and from sistency and horny appearance, it is called the This coat is represented by the space con- between the two exterior circles BFB′A.

tiguous to the sclerotica is a second coat of a substance, called the *choroeides*. This coat is nted by the next white space, and extends along k part of the sclerotica to the cornea.

n the junction of the choroeides and cornea, the *uvea*, B *a*, B′ *a*, a flat opaque membrane, in e-part of which, and nearly in its centre, is a r aperture called the *pupil*.

pupil is capable of being enlarged or contracted reat readiness; by which means a greater or less r of rays may be admitted into the eye as the stances of vision require. In weak light, too ys might render objects indistinct; and in strong oo many might injure the organ. Whilst the s thus enlarged or contracted, its figure remains ed. This remarkable effect is thought to be ed by means of small fibres which arise from the ircumference of the uvea, and tend towards its ; this circumference is also supposed to be mus- and by its equal action upon the fibres, on each he form of the pupil is preserved, whilst its dia- is enlarged or contracted.

he back part of the eye, a little nearer to the han the point which is opposite to the pupil, the *optic nerve* V, which spreads itself over the of the choroeides, like a fine net; and from this stance is called the *retina*. It is immersed in a ucus which adheres to the choroeides.

se three coats, the *sclerotica*, the *choroeides*, and *tina*, enter the socket of the eye at the same The sclerotica is a continuation of the *dura* a thick membrane which lies immediately under ll. The choroeides is a continuation of the *pia* a fine thin membrane which adheres closely to in. The retina proceeds from the brain.

hin the eye, a little behind the pupil, is a soft rent substance EDE′, nearly of the form of a convex lens, the anterior surface of which is rved than the posterior, and rounded off at the E, E′, as the figure represents. This humour, is nearly of the consistency of hard jelly, de- g gradually in density from the center to the ference, is called the *crystalline humour*. It is n its place by a muscle, called the *ligamentum* which takes its rise from the junction of the ides and cornea, and is a little convex towards a.

cavity of the eye, between the cornea and the line humour, is filled with a transparent fluid ter, called the *aqueous humour*. The cavity be- the crystalline humour and the back part of the also filled with a transparent fluid rather more than the former, called the *vitreous humour*.

. It is not easy to ascertain, with great accuracy, racting powers of the several humours; the re- g powers of the aqueous and vitreous humours, are nearly equal to that of water; the refracting power of the crystalline humour is somewhat greater.

The axis, focal length, &c.

The surfaces of the several humours of the eye are so situated, as to have one line perpendicular to them all. This line ADF is called the axis of the eye, or the *optic axis*.

The *focal centre* of the eye is that point in the axis at which the image upon the retina and the object, subtend equal angles.

This point is not far distant from the posterior surface of the crystalline lens, though its situation is probably subject to a small change, as the figure of the eye, or the distance of the object is changed.

Operation of the eye.

299. From the consideration of the structure of the eye, we may easily understand how the notices of external objects are conveyed to the brain.

Fig. 91.

Let PQR (fig. 91) be an object, towards which the axis of the eye is directed; then the rays which diverge from any point, Q, and fall upon the convex surface of the aqueous humour, have a degree of convergency given them; they are then refracted by a double convex lens, denser than the ambient mediums, which increases the convergency; and if the extreme rays QH, QI have a proper degree of divergency before incident, the pencil will be again collected upon the retina at *q*, and there form an image of Q. In the same manner, the rays which diverge from any other points P, R, in the object, will be collected at the corresponding points *p*, *r*, of the retina, and a complete image *p q r*, of the object PQR, will be formed there. The impression thus made, is conveyed to the brain by the optic nerve, which originates there, and is evidently calculated to answer this purpose.

Direct objects inverted on the retina.

300. Much has been written to explain a circumstance attending vision, which it is now a proper place to notice; that is to say, that notwithstanding, according to what we have already stated, the image of an erect object must appear inverted on the retina, it still produces in us the sensation of the object being erect. In order to account for this phenomenon, some writers have imagined a different internal structure of the eye, and have supposed that the image is again reflected after being received on the retina, to another plane, by means of which, the position of the image is corrected, at least this is the opinion of Mr. Horn, who has not long since published an essay on this subject; but it seems more probable, that the correction is due to our own experience; the rays refracted by the humours of the eye have, when they impinge upon the retina, directions wholly different from what they had in coming from the object to the cornea; yet it is always on the prolongation of their primitive direction that we refer, by our senses, the place of the object. This, as we have said above, appears to be the result of experience, which from our infancy, has always shown the objects to have such or such positions. This experimental result at length connects itself so intimately with our sensations, that we never fail of giving mentally, certain directions to objects according to the sensation they have impressed. And it is to this we must attribute certain optical deceptions, where we are viewing objects under circumstances with which we are not familiar. Thus, objects seen by reflection in a plane mirror appear to be beyond the surface, although reason and experience, but an experience much later than our first associations, apprize us of our error. And

on similar principles we may account for most other optical illusions.

Distinct and indistinct vision.

Distinct vision.

301. Having said thus much with respect to the manner in which vision takes place, let us now examine a little more minutely the sensations produced, and the phenomena that are observed, according to the circumstances under which objects are viewed, particularly as regards distinct and indistinct vision. If the form of the eye, the situation of the several humours, and their respective surfaces remain unaltered, it is manifest that those rays only, which diverge from points at a particular distance, can be collected upon the retina. Thus, if the image Q (fig. 92) be formed exactly upon the retina, the image S, a point further from the eye than Q, will be formed within the eye; therefore the rays which proceed from this point will be diffused over some space upon the retina; and if they are mixed with rays which diverge from other points in the object, necessary to be distinguished from the former, the vision will be indistinct. On the other hand, the rays which diverge from T, a point nearer to the eye than Q, will, after refraction, converge to *t*, a point behind the retina, and will therefore, in this case, also be spread over some part of it, and the vision will be indistinct as before.

Fig. 92.

Change of conformation of the eye.

302. In what manner we change the conformation of the eye, so as to enable us to see objects distinctly, at different distances, is not satisfactorily ascertained; but the fact itself is sufficiently manifest. It is supposed by some, that the general figure of the eye is altered; that when the object to be viewed is near, the length of the eye, measured along the axis, is increased by the lateral pressure of external muscles; and, on the contrary, when the object is remote, that the length of the eye is diminished by the relaxation of that pressure. Others suppose the effect to be produced by a change in the place, or figure of the crystalline humour. Others, by an alteration in the diameter of the pupil, while some ascribe the effect to a change in the curvature of the cornea.

Much stress cannot be laid upon the first of these causes, as distinguished from the last, since its existence is not proved by experiment; and there is no necessity for recurring to a bare hypothesis of this kind. With respect to the second, the *ligamentum ciliare* does not appear sufficiently strong to produce any considerable change in the form or situation of the crystalline humour. And as it is clearly ascertained that persons couched, can see distinctly at different distances, we must conclude, that the effect is not to be ascribed to any change in this humour.

Change in the aperture of the pupil.

303. That a change in the aperture of the pupil has some effect in rendering objects distinct at different distances, may be shown as follows:

If the eye of a spectator be directed first to a distinct object, and then to one which is nearer, the diameter of the pupil is observed to decrease. Let HI (fig. 93) represent the diameter of the pupil, QH, QI, the extreme rays of a pencil diverging from Q, and which the humours of the eye are capable of collecting upon the retina, then, when Q is at a less distance, Q *h*, from the eye than before, if *h i*, the diameter of the pupil be equal to HI, the extreme rays Q *h*, Q *i*, of the pencil, which now enters the pupil, diverge more than QH, QI, and therefore, they will not be collected upon the retina (art. 301); but if QH, QI, cut *h i* in *m* and *n*, and the diameter of the pupil be contracted to *m n*, then the extreme rays Q *m*, Q *n*, coincide with QH, QI, and will (as those rays are) be collected upon the retina. Thus, if the diameter of the pupil, *m n*, be proportional to the distance Q *m*, and the object be seen distinctly at any one distance, it will be seen distinctly at all other distances, by the variation of the pupil alone, as far as this variation extends. It is on this account, that a small hole in a thin plate, enables us to view objects at a less distance than we could with the naked eye, as it answers the purpose of a further contraction of the pupil, and excludes those rays in each pencil, which diverge too much. This assistance cannot be made use of to any great extent, because the image upon the retina will soon become indistinct for want of light, and the reflection of the rays at the side of the hole will render it confused. The diameter of the pupil, however, appears to be more affected by the quantity of light, than by the distance of an object from it; therefore, though its variation tends to correct the different degrees of divergency of the extreme rays, when objects are at different distances, it will not produce the effect so constantly and regularly, as is necessary for distinct vision.

Fig. 93.

Chan the cu ture of cornea.

The principal change by which the effect is produced, seems to be an alteration in the curvature of the cornea. In order to show that such a change takes place, Mr. Ramsden fixed the head of a spectator so securely, that no deception could arise from its motion, and directed him to look at a distant object. Whilst the eye was in this situation, he placed a microscope in such a manner, that the wire with which it was furnished apparently coincided with the outer surface of the cornea, and then directing the spectator to look at a nearer object, he found that the cornea immediately projected beyond the wire of the microscope.

Now when the distance of an object is diminished, supposing no alteration to take place in the eye, the divergency of the extreme rays of the pencil incident upon the pupil, is increased; and therefore, if the image of the object in the first situation be formed upon the retina, in the latter it will be behind it (art. 301); but an increase in the curvature of the cornea will increase the convergency of the refracted rays, or bring them sooner to a focus; and thus, by a proper change in this coat of the eye, the rays will again be brought to a focus upon the retina, and the object be still seen distinctly.

304. The least distance at which objects can be seen distinctly by common eyes, is about 7 or 8 inches. The greatest distance cannot be so easily or accurately ascertained. It seems that the generality of eyes are capable of collecting parallel rays upon the retina, or so near to it as to produce distinct vision; and thus, the greatest distance at which objects can be distinctly viewed is unlimited. For this reason, in adapting optical instruments to common eyes, and caculating their powers, we suppose the parts to be so arranged, that the rays in *each pencil* may, when they fall upon the cornea, be parallel.

Cause of short- sighte

305. Admitting the above hypothesis, it appears, that if, in consequence of a defect in the eye, the sight

ɛs have either not the strength or pliability ne-
y for producing a sufficient change in the con-
ion of that organ, we shall only be able to see
tly such objects as are placed within certain
more or less extended, according to the particu-
ıstitution of the eye and its parts. For example,
crystalline, or even if the front of the cornea be
nvex, the true place of the images of distant ob-
s very near the crystalline, and consequently
the retina; we can see them, therefore, only
ıdistinctly, and we must bring the eye nearer to
ects, in order that their images may fall on the
This is the defect in the eyes of short-sighted
s, who are called *myopes*.

On the contrary, if the front segment of the
, or if the crystalline have only a sufficient con-
to admit the images of very distant objects to
on the retina, those of near objects will have a
cy to form beyond it; and, consequently, the
ing intercepted by the retina before their union,
ll also cause a confusion and prevent distinct
This is the defect of those who are too long-
, and who are called *presbytes;* a defect which
ly attends the eyes of persons advanced in
being occasioned by the absorption of a part of
nours which causes a flattening of the crystalline
ks the anterior part of the cornea.

Myopes, therefore, are those which can see
ly only near objects, or those which emit the
ry diverging; and *presbytes* are those which can
e distinctly distant objects from which the rays
early parallel. Now it is evident, from the
of lenses, that concave glasses will cause such
enter them nearly parallel to diverge. An eye
ly short-sighted may therefore be enabled to
ıish distant objects, by means of the artificial
ncy produced by such a lens, the concavity of
proportional to the state or figure of the eye.
a similar reasoning, it will be equally evident
esbytes will be enabled to correct the defect of
es by convex lenses, these having a tendency
uce a parallelism in rays which enter them di-
.

It sometimes happen that the two eyes of a
are very different in their powers, the one being
and the other imperfect, viz. either too long or
rt-sighted. In this case, such a person will
y turn towards objects the eye most adapted to
m distinctly, and turn away the other from
est the imperfect image thus formed should
n and confound the image formed by the perfect
uch persons are said to *squint*.

II. *Vision by means of lenses or mirrors.*

Since we see an object only by means of the
hich is formed in our eye, it is obvious,
ıat we ought to see an object only in the direc-
ording to which the rays enter our eye to form
ge. If, therefore, these rays enter after many
ns and reflections, which have considerably
the primitive direction of the rays, we shall
er see it in the direction indicated by that of
joining the object and the eye.

hen we view an object by means of a lens,
r, &c. which, by its figure, changes the di-
of the rays, this object cannot be judged of either as to its magnitude or situation, according to the common laws of naked vision. We naturally estimate its extent by that of the picture formed on the retina, and this depends upon the angle under which it becomes visible, or the angle formed between the two extreme rays; if, therefore, the refraction or reflection have rendered this angle greater or less than it would have been with the naked eye, or, which is the same, if the angle at the eye between the two rays which pass through the extremities of the last image of the object, formed by refraction or reflection, be greater or less than the angle at the eye, between the extremities of the object, with naked vision, the object will appear greater or less in the same proportion. So that if the eye approach, or is carried further from this last image, the object will appear to be augmented or diminished accordingly, without any consideration whether, in this motion, the eye really approaches or recedes from the object itself. But, notwithstanding, if an object, or the image of an object, were so placed with regard to an extremely thin glass or a mirror, that its rays were refracted or reflected, so as to become afterwards parallel, the eye which is placed in their direction will see this object, or image of the same magnitude, whatever may be its distance from the lens or the mirror: and this magnitude will be the same as if the object were seen by an eye placed in the situation of the lens or mirror.

Let RS (figs. 94, 95) denote one of the dimensions of an object or an image, placed, with regard to the lens CB, so that the rays which issue from the point R, or which tend to that point, issue parallel to each other, supposing the lens indefinitely thin. Among these rays there is one, viz. the principal ray, which traverses the glass without refraction. Let SC be the ray which issues from the centre of the object or image, and which is in the axis of the lens. It is evident that the angle SCR is that under which the image or object SR is seen by an eye at C, which is the place of the lens; and in whatever point E of the axis the eye is situated, provided it be found in the direction of some one of the rays leaving the point R, or which tend to that point, it will see this object, or this image, under the angle CEB = SCR. It would be the same if the eye were placed at the focus of a lens or mirror, on which the rays of an object or of an image were to fall parallel, viz. at whatever distance the object or the image was placed, with regard to the mirror, the eye would see it always of the same magnitude. **Figs. 94, 95.**

310. With regard to the distance of the eye from the place where the object or image appears to be, it is not measured by the actual distance of the eye from the last image. But since the apparent distance of objects is estimated principally by the idea we have of their magnitude, it follows, that when we see objects, the images of which are magnified or diminished by refractions or reflections, we naturally judge them to be nearer to or more remote from the eye than they really are, in proportion to the magnitude under which they appear, compared with that which we attribute to them from our actual knowledge. **Apparent distance of objects.**

Now, as the visible surface of objects seen directly is the base of a pyramid of light, of which the vertex is in the eye, if the point of this vertex become more obtuse by means of one or more refractions or reflections, the object, which is always the base, ought in consequence to appear to be so much nearer to the eye. On the contrary, if the point of the vertex be rendered more acute, the object will, for a similar reason, appear more distant; and hence we have the following construction for determining the apparent place of objects seen by means of lenses or mirrors.

Determined by construction.
Fig. 96.

311. Let QR (fig. 96) be one of the dimensions of any object, O the eye, OR the axis of the pyramid of rays, by means of which the object is rendered visible. OT the direction of the ray, which comes from the extremity Q of the object, after any number of reflections or refractions by spherical surfaces, whose axis are all placed in the line OR. Draw Qq parallel to OR, till it meets the ray OT, and the point q will be the apparent place of Q, or of the last image of the point Q, and Or will be the apparent distance of the eye from the object, $q\,r$ being the apparent place of the last image seen by the eye situated in O.

Hence we see obviously, why convex glasses magnify, and diminish the apparent distance of objects, while concave ones reduce them, and make the apparent distance greater.

General illustration.
Fig. 97.

312. All that has been above stated, may be illustrated generally thus. Let GR (fig. 97) be the common axis of any number of lenses, A, B, C, &c. and QR one of the dimensions of any object; E the place of the eye which receives the ray QKIHE, issuing from the point Q, and which falling on the lens AK at the point K, is so refracted as to be directed towards F, but meeting in I another lens BI, it is refracted again towards the point f, where meeting still another lens HC in H, it is refracted to the eye at E. Now it is evident,

1. That because ET is the last direction of the ray which arrives at the eye, if we draw Q q parallel to GR, and $q\,r$ parallel to QR, the last image of the object QR will appear to be in $q\,r$.

2. That the apparent distance of the eye from the object is Er.

3. That the apparent magnitude of the object is to its actual magnitude as ER, Er, for the angles qEr, QER being small, they will have this ratio, nearly.

4. That the situation of the object will appear direct, or reverse, according to the position of the object beyond, or on this side the centre of sphericity of the last glass or mirror, with regard to the object, or to the image, which shall precede this last, and which served for an object—these conditions may be calculated on the principles illustrated in (art. 284) *et seq.*

By merely changing the words refraction of the lens, for the reflection of the mirrors, all that has been said above will equally apply to the latter case; see also with reference to mirrors our concluding section on Catoptrics.

III. *Of spectacles.*

Theory of spectacles.

313. Spectacles are so well known as to their general form, and their purposes so well understood, that it seems scarcely necessary to attempt any formal definition of them, or description of their general form in this place; we shall, therefore, merely offer a few remarks relative to the theory of these glasses, which are certainly the most common and by far the most useful of any optical instruments.

Convex lenses magnify, and concave lenses diminish the objects seen through them; but these must be considered as accidental or secondary properties of spec-
tacles, the design of them being neither to magn
diminish objects, but to alter the disposition o
tion of the rays at their entrance into the eye, i
to render them more suitable for distinct vision ;
is an object of so much importance, that the a
thereby put as it were upon a level with the you
enabled to read a common print, when with the
sight they could scarcely distinguish one lett
another. By the help of proper glasses, or spe
the short-sighted, whose sphere of distinct visi
bably does not much exceed two or three fee
their sight extended to the greatest distances : i
owing to similar aids, that many of the arts ha
brought to that high state of perfection to whi
have now attained; amongst others, we may
the accurate division of astronomical instrum
which the artist is much indebted, if not to sp
at least to eye-glasses.

The long-sighted eye receives greater benefit
convex lens than the short-sighted does from o
is concave. For an object seen through a conv
is not only magnified, but it is seen brighter by
receiving a larger pencil of light from each
point, on account of the rays entering it less
ing. On the contrary, a concave lens not onl
nishes objects, but also renders them darker, t
the rays are thus transmitted more diverging
eye, and consequently the pupil cannot receive s
of those rays at it otherwise would. It is on
count that short-sighted persons cannot see
objects, unless they be very large and bright,
through a concave lens, as the theory of
formed on the retina may seem to promise.
principal impediments to the distinct vision of
objects, are their want of light and magnitud
both these defects are further increased by c
lenses.

The two following problems embrace nearl
thing connected with the theory of spectacles.

314. Problem I.—Given the distance at w
short-sighted person can see distinctly, to find t
length of a glass which will enable him to see
other given distance.

Let Eq (fig. 98) be the distance at which he
distinctly, and QE a greater distance at which he
to view objects; let AB be a concave lens whos
length is such, that the rays which are incide
it diverging from Q may, after refraction, diver
q; then they will have a proper degree of conv
for the eyes of the myope.

Take F the principal focus of rays incident
contrary direction; then

$$\mathrm{QF} : \mathrm{QE} :: \mathrm{QE} : \mathrm{Q}q;$$

consequently,

$$\mathrm{QF} - \mathrm{QE} : \mathrm{QE} : \mathrm{QE} - \mathrm{Q}q : \mathrm{Q}q;$$

whence $$\mathrm{FE} = \frac{\mathrm{QE} \times \mathrm{E}q}{\mathrm{Q}q}.$$

If QE be indefinitely great, then QE = QF and
Eq.

315. Problem II.—Having given the dist
which a long-sighted person can see distinctly,
the focal length of a convex lens, which will
him to see distinctly at any other distance.

If qE be the distance (fig. 99) at which he
distinctly, and QE the distance at which he wi

view objects, and AB be a convex lens, whose focal length FE is such, that the rays which diverge from Q may, after refraction, diverge from q. Take F the principal focus of rays incident in the contrary direction, then as above,

$$QF : QE :: QE : Qq;$$

consequently,

$$QF + QE : QE :: QE + Qq : Qq;$$

whence
$$FE = \frac{QE \times Eq}{Qq}.$$

If Eq be indefinitely great, or the eye require parallel rays, then, $Eq = Qq$ and $FE = QE$.

If the eye require converging rays, q falls on the *other* side of the lens, in which case FE is less than QE.

In the choice of glasses for long or short-sighted persons, care should be taken to select such as have the least refracting power that will answer the purpose. For the eye has a tendency to retain that conformation to which it is accustomed, and therefore, by the use of improper glasses, its imperfections may be increased.

316. In all that we have hitherto advanced respecting the nature of vision, we have only considered one eye, and this is sufficient; for in whatever manner rays are refracted or reflected, and images formed, by the humours of one eye, in the same manner will the effects be produced in the humours of the other; we have only to account for the circumstance that we do not perceive two images, as there are doubtless two formed, and this is a question perhaps not readily decided; it is most likely to be attributed to habit, the same as we have before endeavoured to account for an inverted image conveying to the mind the idea of an erect object; although some authors are of opinion, that it is due to the similarity of the corresponding parts of the optic nerves and their union with the brain.

In support of the former hypothesis, we have a strong fact stated by Mr. Chesselden. A person had one of his eyes distorted by a blow, and for some time after every object appeared to him double; but by degrees the most familiar ones became single, and in time all objects became so, although the distortion remained. To this we may add, that children sometimes learn to squint, and by proper attention, this habit may again in a great measure be corrected. Under both circumstances objects appear single, and it is manifest that the images cannot, in both cases, fall upon corresponding parts of the retinas.

317. We have only further to observe, that whatever be the cause of single vision, when an object is viewed attentively, the axes of both eyes are in general directed to it: thus, if P (fig. 100) be the object, the eyes are moved till the optic axes, AP, a P, meet in P, and the images A, a, are formed on corresponding points of the retinas. In this position of the images, whether from the correspondence of the nerves, or from experience, the idea of a single object is suggested to the mind scarcely differing from the idea excited by one image only, excepting that the images appear somewhat brighter when seen with both eyes, than when they are seen with one only.

While the eyes remain in the same position, the images B, b, of Q an object near to P, and at the same distance from the eyes, will be formed on the retinas, the eyes having assumed proper conformation for distinct vision at that distance; and B, b, which are both on the right or both on the left of the respective axes, are corresponding points, and suggest the idea of a single object as before.

§ V. *Optical instruments and phenomena.*

Optical instruments and phenomena.

318. Having in the preceding sections of this treatise, illustrated all the fundamental principles of the theory of Catoptrics and Dioptrics, we shall conclude by adding a description of all the more popular and useful optical instruments, with an illustration of such optical phenomena as are dependant upon the principles treated of in the foregoing section; others, which are more intimately connected with the physical properties of light, will be best examined in our seperate treatise on that subject. We propose here, to follow the same method and arrangement as in our treatise of Hydrodynamics and Pneumatics; that is, we shall not in this chapter describe the instruments, of which we treat at such a length, as is necessary for explaining the various processes followed in their construction, our views being merely to illustrate the theory of what may relate to the practical application,—the different forms given to them by different artists, will be reserved in those cases where such detail is required, for the proper places in our alphabetical arrangement.

Of telescopes.

Of telescopes.

319. The invention of telescopes is certainly one of the noblest monuments of human genius; by means of this instrument we have extended our views and researches far beyond the limits of our own globe, or that sphere which nature seems to have designed for our inquiries. Telescopes have enabled us to penetrate into the immensity of space, and to become as it were familiar with other worlds, and other spheres, placed almost at immeasurable distances from us, and to discover an infinitude of celestial bodies, whose existence must for ever have remained unknown to us, but for this invention.

Such being the high importance of telescopes, it is not astonishing that so much talent and attention have been directed towards bringing it to its greatest possible perfection, a state which it has now perhaps attained; but, as we have observed above, it will not be our province in this place to enter into a minute description of the several successive changes and improvements which it has undergone in the hands of different philosophers; this is a part of the subject which will be treated of hereafter; we shall here merely examine the theory of these instruments, under their two principal divisions of catoptric and dioptric telescopes.

Of dioptric, or refracting telescopes.

Refracting telescopes.

320. We may distinguish three different kinds of refracting telescopes, depending upon the figure, position, and the number of its lenses, viz. the Galilean, the astronomical, and the common, or terrestrial telescope.

1. *Telescope of Galileo.*

Galileo's telescope.

321. Of these, the first that we shall consider, being the first in the order of invention, is the Galilean telescope, thus called from its being supposed to have been first invented by Galileo, about the year 1609. In this instrument, the eye-glass PQ (fig. 101) is concave, or plano-concave, and is placed between the

Fig. 101.

object-glass MN and its focus O, in such a manner that the axes of the two glasses may be in the same right line AO, and their foci in the same point O.

From this construction, it appears,

Properties of.

1. That because the surface of the object-glass is much greater than the opening of the pupil of the eye, there will necessarily be a much greater quantity of the rays issuing from any point of an object, conducted to the eye in this manner, than could possibly take place in naked vision.

2. That the object being supposed infinitely distant, the incident and parallel rays falling upon the object-glass, as, for example, AD, A′D′, A″D″, being first rendered converging towards O by their refraction in traversing MN, and afterwards parallel in passing through PQ, they will be much more dense after this second refraction than before the first, and will therefore, when received by an eye placed near O, paint an image of the object so much the more vivid, as the density of the pencil of rays is then greater than before they fell upon the object-glass.

3. With regard to the points B of the object OB, which are situated out of the axis OA of the telescope, it is obvious that they will send forth parallel rays, represented in the figure by CD and its two parallels, which in traversing the object-glass, are refracted towards some point *b*, near to the point *o*; they are then rendered nearly parallel in traversing the eye-glass PQ, whence they emerge much denser than when they first impinged on CD, and will consequently impress on an eye, properly situated, a vivid image of the point B. But since this pencil in emerging from the eye-glass diverges from the pencil formed by the point O, the eye may not receive at the same time impressions of both these objects; that is, if it be not sufficiently near to the point of concourse F of both these pencils. It follows, therefore, that in viewing an object by means of this telescope, we shall see a greater number of its parts as the eye is placed nearer to the eye-glass, and as the pupil is more dilated. But since the dilation of the pupil is naturally very small, and that it is involuntarily contracted proportionally to the quantity of light which impinges on the eye, it is obvious, that the field of view of these telescopes will be so much the less as the focus of the eye-glass is greater.

Imperfections in this instrument.

4. Lastly, because the nature of light will not admit of our using eye-glasses whose foci are diminished beyond a certain limit; but that on the contrary, the foci ought to be longer in proportion to the length of the foci of the object-glasses, as we shall see in a subsequent article; it follows, that the field of view in telescopes of this construction, will be less, the greater the length of the instrument, which inconveniences, together with those above described, soon led astronomers to reject this instrument for the purposes of astronomical observation; but the principles of it are still employed in the construction of *opera-glasses.*

Objects are seen erect.

5. It is obvious from the figure, that the objects seen through this instrument will be erect; for the pencil of rays *c*, which exhibits the point B of the object under the axis, will meet the eye in the direction *c* E, which answers to a point also below the axis; and in the same manner, a ray issuing from any point of the object above the axis, will have a similar direction when it emerges from the eye-glass; consequently, objects and their images will have like positions with respect to direction, when viewed through a telescope of this construction.

6. The nearer an object approaches to the object-glass, the more distant will be its focus from the same, and consequently, the more distant must also be the eye-glass, in order that the two foci may coincide.

7. We have seen that the rays which emerge from the eye-glass will be parallel, when the object and eye-glasses are so adjusted that their two foci coincide; in this state the instrument will answer to an eye of perfect sight. If the observer be near-sighted, the eye-glass must be adjusted a little nearer to the object-glass, in order to change the parallelism of the emerging rays into a slight degree of divergency; that this approximation of the glasses will produce such an effect is obvious, by referring to (art. 286), where it has been shown, that as the object recedes from the focus of a concave lens, the refracted rays will be rendered convergent on the side opposite to the eye, or on the side of the object, and consequently diverging towards the observer.

322. Problem.—To determine the magnitude of the field of view in Galileo's telescope when the eye is placed close to the eye-glass.

Let MN (fig. 102) be the diameter of the object-glass, AB, the diameter of the pupil, whose centre is in the axis of the telescope; join MB, the opposite extremities of these diameters; and let MB meet the axis in x, and the image in p; join LB, Lp, and suppose p L, and q L to be produced, till they meet the object in P and Q; join also MP. Then will PM be refracted to the pupil in the direction MB; but any other ray in the pencil, as PL, and every ray which flows from a point more distant from the axis of the telescope, which is refracted by MN, will fall below the pupil; therefore PQ is half the linear magnitude of the greatest visible area. Now QP is measured by the angle QLP, or by its equal q L p, and

$$q\,\mathrm{L}\,p = \mathrm{ELB} + \mathrm{BL}\,p =$$
$$\mathrm{ELB} + \mathrm{LBM} - \mathrm{L}\,p\,\mathrm{M}.$$

By doubling these quantities, 2 QP is measured by $2\,\mathrm{ELB} + 2\,\mathrm{LBM} - 2\,\mathrm{L}\,p\,\mathrm{M}$.

That is, the linear magnitude of the field of view when the eye is placed close to the concave lens Galileo's telescope, is measured by the angle which the diameter of the pupil subtends at the centre of the object-glass, increased by the difference between the angles which the diameter of the object-glass subtends at the pupil, and at the image.

323. In order to determine the brightest part of the visible area, join NB, and let it meet the image in [illegible] join BL, s L, and produce s L till it meets the object in S. Then all the rays which flow from S, or from any point, between S and Q, and fall upon the object-glass, will be refracted to the eye; and QS will be the linear magnitude of half the bright part of the field of view. Now QS is measured by the angle SLQ, which is equal to

$$s\,\mathrm{L}\,q = \mathrm{BLE} - \mathrm{BL}\,s = \mathrm{BLE} - \mathrm{LBN} + \mathrm{BLN} + \mathrm{L}\,s \text{ [illegible]}$$

and $$2\,\mathrm{QS} = 2\,\mathrm{BLE} - 2\,\mathrm{LBN} + 2\,\mathrm{L}\,s\,\mathrm{N}.$$

II. *Astronomical telescope.*

324. The second description of telescope, and nearly the only one of the dioptric kind employed for celestial observation, and therefore called the astronomical telescope, is represented in (fig. 103); it has, like the

preceding, only one eye-glass, but this is a convex or plano-convex lens, and so placed with regard to the object-glass, which is a similar lens, that their two foci may coincide in the axis of the telescope, between the two glasses, instead of being beyond the eye-glass, as in the instrument last described. In the figure, PQ is the eye-glass, MN the object-glass, KD the axis, and o the common focus, or rather the point of concourse of the two foci.

According to this construction, it is obvious,

1. That the rays emitted from the point O, of an object OB, infinitely distant, represented in the figure by AD, $A'D'$, $A''D''$ (O being in the axis), after traversing the object-glass, are refracted to the focus o, where therefore they will form an image of the point O.

2. This image o may be considered as an object placed in the focus of the eye-glass PQ, and consequently the rays diverging from o, and falling on PQ, will emerge from it in a direction parallel to the axis and to each other, and will be so much the more dense as the focal length of the latter lens is less than that of the former. They will therefore paint on the retina of a common eye, or one of long sight, a new image of the point O, and which will be so much the more vivid, as the surface of the object-glass is greater, that is as it admits a greater quantity of light.

3. At whatever distance the eye is placed from the eye-glass, provided it be in the direction of the emerging ray, it ought to see the image with the same distinctness.

4. The parallel rays emitted from the extremity B of the object OB, ought to form in b, near the focus o, an image of this extremity, and which falling afterwards on the eye-glass PQ, they will emerge parallel to each other, but so much the more inclined to the axis AF as the curvature of PQ is greater; so that the axis of the pencil which they form will cut the common axis of the two lenses in the focus F of the eye-glass; and consequently, in order that an eye may see all the image $o\,b$ at the same time, it is necessary that it be placed at the point F, the common intersection of all the rays, emitted from each point of the image $o\,b$, or of the object OB.

5. It follows also, from what has been above stated, that the image of the object will be reversed; for the point b, in which is formed the image of the point B, will be refracted to the eye in the direction PF, and will consequently have a different direction from that with which it is emitted from B, and in the same manner it is obvious, that the ray which renders visible the point A above the axis, will reach the eye as if it proceeded from below the axis; whereby the entire image will be reversed, as above stated.

7. It is obvious, by referring to the figure, and to what has been above stated, that the field of view of this telescope depends principally upon the magnitude we give to the instrument towards $o\,b$, which may be considered as the common focus of the two glasses; for the eye placed at the point F may see all the points whose images have reached, or very nearly reached, that focus. This is an advantage that does not appertain to the Galilean telescope, and it is that which has induced astronomers to adopt the former in preference to the latter.

Astronomical Instrument.

8. In this telescope, if the object be brought nearer to the object-glass, its image will necessarily be carried further from it on the opposite side, and therefore the eye-glass must also be carried further back, in order to preserve the coincidence of the two foci. The instrument being therefore formed of two parts, moveable one within the other, or any other contrivance being had recourse to, which will enable us to move the eye-glass within certain limits, it will answer equally well for near or remote objects.

Adjustment to different eyes.

9. What is stated above, is all given with reference to common eyes; but we may observe, that if the eye applied to this instrument be short-sighted, the object glass must be brought nearer to the eye-glass, or which is the same, towards the image $o\,b$, in order that this image, being then placed between the eye-glass and its focus, the rays falling upon it may, after passing through, be rendered diverging.

The greatest field of view.

Fig. 104.

325. In this telescope, the field of view is the greatest when the eye is so placed as to receive the extreme rays, refracted from the object-glass to the eye-glass. Let MN (fig. 104) be the diameter of the object-glass, AB the diameter of the eye-glass; join NB, the corresponding extremities of the glasses, and let NB cut the image $p\,q\,r$, in p; join p L, and conceive p L, EL, to be produced until they meet the object PQ in P and Q; join also p E, NP; and draw BO parallel to p E. Then if the eye be placed at O, it will receive the ray NBO; and it is this ray which comes from a point in the object at the greatest visible distance from the axis of the telescope; for the rays which flow from any point in the object above P, will, after the first refraction, converge to a point p below the glass AB, and consequently P is the extreme visible point in the object, and QP is half the linear magnitude of the visible area. Now when the diameter of the object-glass is greater than that of the eye-glass, which is the case in those instruments used for astronomical purposes, the rays NB, LE will converge to some point x beyond the eye-glass; therefore, the point O to which they converge after the second refraction, and where the eye must be placed to receive them, is between the glass E and its principal focus.

If NB and LE be parallel, O is the principal focus; if they diverge, then O will be beyond that focus.

Of the bright part of the field.

326. Referring to the above figure, it is obvious that NBO is the only ray which comes to the eye from P; for any other ray of the pencil, as PLp, after refraction at the object-glass, crosses NB in p, and falls below the eye-glass. Hence it follows, that the extremity of the visible area is very faint. If the point be taken in the object nearer to the centre of the field of view, more of the rays which flow from it will be refracted to the eye; and thus the brightness will continually increase till all the rays in each pencil incident upon MN are received by the glass AB; when this takes place, the brightness will become uniform; because the same number of rays, or very nearly so, is received by the glass MN from every point in the object.

The brightest area determined

If it be required to determine the bright part of the field of view, we may join MB, which will cut $p\,q\,r$ in s, and the axis LE in x'; join also Ls, and conceive it to be produced till it meets the object in S. Then if x'E be greater than qE, all the rays which flow from S, and fall upon MN, will be refracted to the eye-glass; for MS is refracted in the direction MsB, and any other ray of the pencil, as SL, crosses MB, at s, and falls somewhere between A and B.

In the same manner rays which flow from any point

Optics.

between S and Q, and fall upon MN, will be refracted to the eye-glass AB.

If x' and q coincide, that is, if the linear apertures MN, AB, of the glasses be proportional to their focal lengths, the brightness of the field increases to the centre. And if q fall between L and x', the whole of the rays belonging to any one pencil incident upon MN, will not be received by the eye-glass. In this case a less aperture of the object-glass would produce the same brightness, with less aberration.

Greatest visible area determined.

327. Problem.—To determine the greatest visible area of the field of view in the astronomical telescope. Referring still to (fig. 104), let pqr be the image formed by the object-glass, and join NB, cutting pqr in p; join also LB, Lp; and suppose pL, EL, to be produced, till they meet the object in P, Q; then it is obvious that QP will be half the linear magnitude of the greatest visible area (art. 325), and it will be measured by the angle PLQ, or its equal pLq; that is, by BLE + BLp, or by BLE + LpN − LBN; consequently 2 QP is measured by

$$2\,\mathrm{BLE} + 2\,\mathrm{L}p\mathrm{N} - 2\,\mathrm{LBN}:$$

that is, by the angle which the diameter of the eye-glass subtends at the centre of the object-glass, increased by the difference between the angles which the diameter of the object-glass subtends at the image, and at the eye-glass.

Angle subtended by the brightest part.

328. In like manner it may be shown, that the linear magnitude of the brightest part of the visible area is measured by the angle which the diameter of the eye-glass subtends at the centre of the object-glass, diminished by the difference between the angles, which the diameter of the object-glass subtends at the image and at the eye-glass.

III. *Telescope for terrestrial observations.*

Terrestrial telescope.

Fig. 105.

329. The third sort of refracting telescopes are those which, by the addition of two other eye-glasses, bring the objects observed by them into their natural position, instead of inverting them as in the instrument last described. The nature of this construction will be readily comprehended by a reference to (fig. 105). The four lenses MN, PQ, RS, TV, have all one common axis Af, each contiguous; two of which are so posited that their foci coincide in the same point.

Let OB be an object at an infinite distance; then the parallel rays issuing from the point O of the object, which is in the axis produced, will, after traversing the object-glass, form an image in the focus o, whence falling on the eye-glass PQ, they will emerge in directions parallel to each other, and to the common axis of the telescope, till they fall upon the second eye-glass RS, and traversing this, they will be refracted to the focus o', where a second image will be formed, and the rays then diverging from this focus, and falling on the last eye-glass TV, will, after refraction, emerge from it and pass to the eye in directions parallel to the common axis, and impress upon the retina a lively image of the object.

In the same manner the rays issuing from the point B of the object OB, after traversing the object-glass, will form in b an image of that point; whence, falling on PQ, the rays will be rendered parallel to each other, but oblique to the axis, till they meet the lens RS, at which they will be refracted to the second focus b'; whence, falling on the last eye-glass TV, they are again rendered parallel to each other, but so inclined to the axis as to meet it again with all the other rays in the focus f, where, therefore, the eye ought to be placed to receive the final impression, which will obviously be that of an image corresponding, as to its direction, with the object itself; for the ray b'Vf which conveys the image of the point B, has a similar direction with respect to the axis, as the ray proceeding immediately from B.

330. The properties of this instrument are analogous to those of the astronomical telescope described in the last section; but, for terrestrial observations, it is much more pleasant, on account of its preserving the direct position of objects; whereas, the latter is better suited to astronomical purposes, because it admits of a larger field of view; will carry an eye-glass of a shorter focus; and may be shorter in proportion to its diameter. There will, moreover, be less light lost by two than by four refractions.

331. In order to ascertain the change which takes place in the visual angle, by the addition of the two intermediate lenses, let PS (fig. 105) be the extreme pencil of rays refracted by PQ, draw Lb' parallel to PS, and let it meet $a'b'$ in b'; join b'S, b'D', and produce Sb', till it meets the lens TV in V; draw Vf parallel to D'b', f being the place of the eye. Then BPSVf being the course of the pencil of rays, which flows from B, and OD, o'D', &c. the course of the pencil which flows from O, the angle which OB subtends at the centre of the eye, when seen through the four glasses, is to the angle which it subtends there, when seen only through the first two, as the angle VfD, angle to $o\,o'$P, or as $\angle\, b'\mathrm{D}'a'$ to the $\angle\, b'\mathrm{L}a'$, that is, as $\mathrm{L}a' : a'\mathrm{D}'$.

Since the angle subtended by OB at the centre of the eye, when seen through the first two glasses, is to the angle which it subtends at the naked eye as Do to Lo, by compounding this proportion with the last, we have the angle which the object subtends at the centre of the eye, when viewed through the four glasses, to the angle which it subtends at the naked eye, as

$$\mathrm{D}o \times \mathrm{L}a' : \mathrm{L}'o \times \mathrm{D}'a'.$$

Therefore, if TV and RS have equal focal lengths, the angle subtended by the object will be the same in either case, that is, with two or with four glasses. The adjustment of the lens to different eyes is the same as in the preceding instrument.

Of reflecting, or catoptric telescopes.

332. There are three distinct constructions of reflecting telescope in common use, besides others of less general application. These three are distinguished by the names of their respective inventors, as the Newtonian, the Gregorian, and Cassegrainian telescopes; they all depend fundamentally on the same principle, although there is a considerable difference in the manner of applying it.

IV. *Newtonian telescope.*

333. Let DAD (fig. 106) represent a concave spherical reflector, whose middle point is A, and centre F, fixed at the bottom of the open brass tube, or cylinder, MN M'N'. Then it is obvious, from what has been already stated with reference to concave speculums, that an image of any object presented to the open end of

this tube, will be formed in some point F in the axis. The situation of this image, between the speculum and the object, prevents its being seen by means of the common application of eye glasses; for it would then be necessary for the observer to place his head between the image and object, which would of course, in any instrument of moderate dimensions, prevent the light from arriving at the mirror in sufficient quantity and sufficiently near the axis.

334. In order to prevent this, a small plane mirror IH is placed, inclined to the axes of sphericity, at an angle of 45°: this plane mirror reflects to o the point of the cone of the first reflected rays, and where therefore the image of the point O will be formed, and may be observed by an opening made in the side of the principal tube, directly opposite this reflector, and furnished with an eye-piece, containing one or three eye-glasses, according as we wish the image to be reversed or erect.

335. The principle advantage of this telescope is, that it produces the same effect as the refracting telescope, although it be much shorter, the image being formed by the speculum itself, at a distance equal to only one fourth of the radius of sphericity (the rays being supposed parallel, or the object being supposed at an infinite distance), whereas in refractors, the image is distant from the object-glass by half the radius of sphericity, which distance is still necessarily extended by the length of the focal distance of the eye-glasses. But the most important advantage is, that the same mirror will bear the application of eye-glasses of various powers; whereby the same tube and speculum may be made to answer the purpose of various refractors, because, in the latter, it is necessary that the focal length of the eye-glasses bear a certain relation to those of the object-glasses: and the limits of this ratio are very confined, as we shall see in what follows.

336. In applying this telescope to observations, it is obvious that the mirror IH ought to be moveable, in order to ensure the image of the object falling in the focus of the eye-glass; the first image being formed at different distances from the mirror, according to the distance of the object from the same. It is also requisite that the eye-glass itself may be moveable in the direction of the side of the tube MN of the telescope, while the mirror itself moves within the tube in order that the former may have its focus placed in the summit of the cone of rays reflected by the mirror IH.

337. With respect to the adjustment of this instrument to different eyes, it is evident that a short sight will require the mirror IH to be brought up nearer to the speculum DA, in order that, by placing the image between the eye-glass and its focus, the rays may emerge from the eye-glass more or less diverging as the state of the eye may require, whereby to produce distinct vision.

337. PROBLEM I.—To determine the angle subtended by the image of an object viewed by a Newtonian telescope.

Let PQ (fig. 107) be the object, $p\,q$ its image in the principal focus of the reflector; $n\,m$ the image formed by reflection, at the plane surface I c H; QEC the axis of the telescope, cutting I c H in c, draw $c\,n\,l$ O perpendicular to CE, and at l let a convex eye-glass $k\,l\,r$, be placed, whose focal length is $l\,n$, and whose axis coincides with that line; join $l\,m$. Then the image $m\,n$, which is equal to $p\,q$, and corresponds to QP in the object, is seen through the glass $k\,l\,r$, under an angle, which is equal to $m\,l\,n$; and QP is seen under an angle (with the naked eye placed at E), equal to q E p; now since these angles have equal subtenses $m\,n$, and $q\,p$, they are to each other inversely as the radii $l\,n$, q E; consequently the angle which the object subtends at the centre of the eye, when viewed with the telescope, is to the angle when viewed with the naked eye, as E q : $l\,n$; that is, as the focal length of the reflector to the focal length of the eye-glass. We call on this account the fraction $\frac{\text{E}\,q}{l\,n}$ the measure of the magnifying power of the telescope.

The magnifying power of the same instrument is, therefore, not the same for objects at all distances, for the length E q will be greater or less according to the distance of the object, the magnifying power will therefore vary accordingly, being greatest when E q is greatest, that is, as the object is more distant.

Of the field of view.

338. PROBLEM II.—To determine the field of view in the Newtonian reflector.

Referring still to the same figure, join H k, and let it cut the image $m\,n$, in m; take $p\,q = m\,n$, draw p E, and produce it till it meets the object in P. Then a small pencil of rays flowing from P, will, if I c H be not so large as to intercept them, before they fall upon the spherical speculum, be reflected at H, to the extremity of the eye-glass, and reflected thence to the eye at O; the point P will therefore be visible, and it will obviously be the extremity of the field of view; for if a point be taken in $n\,m$, further from the axis of the lens, than m, a right line drawn through it, will either fall beyond IH or beyond $k\,r$; that is, no ray belonging to a point in the object QP, beyond P, can be reflected from the mirror IH to the eye-glass.

V. *Of the Gregorian telescope.*

Gregorian telescope.
Fig. 108.

339. The general principle of this instrument is shown in (fig. 108). The speculum is here represented by AB. A little beyond the image T, which is formed on the axis OT of this mirror, is fixed another small spherical concave mirror CD, of a shorter focal length, the axis being in the same right line as that of the larger speculum AB. The image T is, with regard to the mirror CD, as an object placed between its focus t and its centre c; and consequently it will form on the same axis a second image q, which will be more or less distant from the centre c, as the first image T, is nearer to or more remote from the focus t of the small mirror. By approximating this small mirror towards the image T, or in removing it from the same, we may carry the second image q to any distance at pleasure; this is commonly placed a little in front of the great mirror or speculum AB; in the latter is an aperture, through which, by means of a proper eye-glass, the image q may be seen. And it is evident that this image ought to appear erect, being reversed with regard to the image T, which is itself reversed with respect to the object. When the object is very luminous, we magnify the second image by making it fall towards O, beyond the speculum, and placing in O, the focus of an eye-glass FH, in order that the rays which tend towards the point O, when they meet FH, may be made to emerge parallel to the axis, till

ness of vision, supposing the eye to be one of perfect sight.

Of the apparent diameter of objects.

350. According to the same hypothesis, the apparent diameter of the object seen through a lens, is to the diameter which would have place at the distance at which the eye perceives ordinary objects distinctly, as the angle under which the same eye sees the object placed in the focus of the lens, is to the angle under which it would see it at the distance we have stated, that is, as the angle *a c b* to *a l b*, *x l* being the distance of distinct vision. Now, the two angles subtend equal chords *a b*, either of which measures the real diameter of the object, in such manner that each of them forms with this diameter a triangle of which the diameter is the base. Therefore, these angles are to one another nearly in the inverse ratio of the heights of the triangles to which they belong. Now, the height of the shortest triangle is the focal distance of the lens *x c*, and the height of the longest the distance requisite to distinct vision *x l*; whence it follows, that the diameter of the object appears magnified in the proportion of this last distance to the distance of the lens. For example, if we suppose as before, the distance requisite to the distinct vision of objects of a sensible magnitude to be 8 inches, and the focal distance of the lens to be $\frac{1}{10}$ of an inch, the diameter of the object will appear increased in the proportion of 80 to 1.

251. We may substitute for a lens a glass globule, which is easily made, by melting a small piece of that substance in the flame of an alcoholed wick, to prevent the mixture of fuliginous matter, which would injure the transparency of the glass. Simple microscopes also may readily be formed, by making a small hole through a thin plate of metal, and conveying to the hole a drop of water suspended to the point of a pin, so that the drop may assume a perfect round form. But these drops have less effect than the glass globules, because their refractive power is less. We shall not, however, as we have before observed, enter at length in this article into the various forms under which microscopes may be constructed; enough has been said to illustrate the general theory of them, which is all that appertains to our present inquiry.

VIII. *Of the double compound microscope.*

Compound microscope. Fig. 111.

353. The compound microscope, with two glasses, has a considerable analogy to the astronomical telescope. The object-glass GH (fig. 111) is very small and very convex, the object AB, is placed a little beyond the focus of this glass, whence it follows, that rays of each of the pencils that pass from A to A′, and from B to B′, which rays would emerge parallel if the object were exactly in the focus, are but little inclined to one another; so that they form a reversed image A′ B′ of the object, at a considerable distance from the object-glass, and which, consequently, is already of much greater extent than the object. The eye-glass KN is so situated, that its focus concurs nearly with the middle *x*, of the image A′B′, and thus the rays LO, S *y* on the one part, and T *y*, RO on the other, being very little divergent, and, besides the two pencils to which these two rays belong, acquiring, on the contrary, a considerable degree of convergency, the eye placed at O will see the object at A′B′ very much enlarged, for two distinct reasons: first, the image A′B′, if observed with the naked eye, would already appear sensibly larger than the object AB; secondly, that image becomes, in its turn, the object which the eye perceives through the eye-glass; and as this glass performs here the office of a powerful lens, the angle ROL in which the eye will see this same object distinctly in A″B″, will be much greater than that under which it would see it with the same clearness without any intermediate aid. Therefore, the magnitude of the image being the result of two combined effects, of which each one tends of itself very perceptibly to enlarge its dimensions, will increase in a very considerable proportion.

354. In order to compute the actual ratio of the angle which an object subtends at the centre of the eye, when seen through the double microscope to the angle it subtends at the naked eye, when viewed at the least distance of distinct vision. Let A′B′ (fig. 112) be the image of AB, and terminated by the lines BLB′, ALA′. Then the visual length will be increased, as we have said above, first, becasue A′B′ is greater than the object, and, secondly, because this image is seen under a greater angle when viewed through the glass, than when viewed with the naked eye. Now, supposing AB and A′B′ to be viewed with the naked eye, at the least distance of distinct vision, the visual angle of AB is to the visual angle of A′B′, as AB : A′B′, or as BL : B′L. Also the visual angle of A′B′, when thus viewed, is to its visual angle, when seen through the glass KN, as B′E to the least distance of distinct vision. Therefore, by compounding these two proportions, the visual angle of AB, when viewed with the naked eye at the least distance of distinct vision, is to the visual angle when it is viewed with the microscope, as BL × B′E is to LB′ × the least distance of distinct vision.

When the lenses are thus combined, the objects appear inverted.

355. When a great magnifying power is not required, the object is placed between the glass GH (fig. 113) and its principal focus; thus an erect image A′B′ is formed on the same side of the lens as the object; and if EB′ be the focal length of the eye-glass GH, the image may be seen distinctly. The visual angle is determinable as in the former case.

One advantage of this construction is a greater field of view; and another, that the objects appear erect; there is likewise less confusion produced by the spherical surfaces of the glasses, than would be caused by a single glass with the same magnifying power.

The above theory will apply equally to microscopes with three glasses; the construction of which will be described under the article Microscope.

IX. *Solar microscope.*

The solar microscope differs entirely, both in its theory and construction, from those above described, differing from the *magic-lantern* (which see) only in this, that the object is enlightened by the solar ray admitted into a dark room by means of a plane mirror, which reflects it horizontally, the rays passing through a lens fixed into a hole in the window-shutter. To the bright light which proceeds from this lens a small white glass is presented, and on this glass small insects, objects, &c. are placed.

A second lens, intended to produce the image, is lined on the side next the objects with lead or tin foil, having a pin hole in the centre, and it is through this aperture that the streams of light from the different

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also in the principal foci of the segments. Let the glasses be moved till these images coincide, as at E; then the angle PEQ, which the objects subtend at E, the principal focus of C or D, is equal to the angle which CD, the distance of the centres of the two segments subtends at the same point; and therefore by calculating this angle, we may determine the angular distance of the objects P and Q, as seen from E. Draw EG perpendicular to CD; then because the triangle CED is isosceles CG = CD, and the angle CEG = GED; also GD is the sine of the angle GED, to radius ED; therefore knowing ED and GD, the angle GED may be found by the proper tables: consequently 2 GED or CED may be determined.

The angle CED is generally so small, that it may, without sensible error, be considered as proportional to the subtense CD; and being determined in one case by observation, it may be found in any other by a simple proposition.

If the objects be at a given finite distance, the angle PEQ will still be proportional to CD; for on this supposition the distance CE or DE, of either image from the corresponding glass, will be invariable; therefore the angle CED will be proportional to CD. The divided object glass is supplied both to reflecting and refracting telescopes, and thus small angular distances in the heavens are measured with great accuracy.—"Wood's Optics."

XII. *Camera obscura.*

Camera obscura.

360. The camera obscura, or *dark chamber*, is a machine, or apparatus, so constructed, that principally by means of a convex glass, or a convex glass and plane mirror, the images of external objects are represented on a rough ground plane glass, white paper, white wall, or other surface, in the most vivid and distinct manner, with all their natural motions, colours, shades, &c. The first invention of the camera obscura has been ascribed to Baptista Porta, (see his Magia Naturalis, first published at Frankfort, about the year 1589). But Dr. Freind, in his History of Physic, observes, that Friar Bacon, who flourished in the beginning of the thirteenth century, describes this instrument and various glasses, which magnify or diminish any object, bring it nearer to the eye, or move it further off. See Bacon's Opus Majus, by Dr. Jebb.

Invention of.

In fact, it seems highly probable, that a similar effect to that produced by what is now called a camera obscura, could not remain long unnoticed by optical writers or observers, as it is produced in a certain degree in every dark room in which a hole is found sufficient to admit a small pencil of rays, without the assistance of any lenses whatever; for, in the case we have supposed, the great masses of light and shade before the window-shutter, or other screen, will be represented in an inverted order in the parts of the room diametrically opposite to them, and be illuminated in different degrees, according to the quantity of light which can reach them in straight lines from the external objects.

The effect is increased by a lens.

361. This effect, however, is increased by means of a convex lens, of a focal length somewhat less than the distance of the surface on which the picture is projected, as it renders the images much more vivid and distinct. Still, however, some of them are necessarily imperfect and ill-defined, unless the objects happen to be situated at the same distance from the aperture; for it is impossible that the focus of the lens can be at once adjusted to near and remote objects. Nor would the picture be rendered more perfect, if such an adjustment were possible; for we should then have presented to the eye at one view, with equal distinctness, objects that never can be seen at one time without some degree of confusion.

Op st

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362. Sometimes the picture is intercepted by a plane speculum, placed obliquely, and is thence thrown upwards on the surface of a plate of ground glass; and sometimes the lens is placed horizontally, with the speculum above it, which throws the image through the lens, upon a flat surface placed below, on which the objects are delineated in their natural position.

Th the me Fig

363. In order to demonstrate the truth of the preceding observation with reference to the delineations of objects seen in this manner, let C (fig. 115), represent a small aperture in a perfectly darkened room, through which light radiates upon a white paper or screen opposite to it, and by means of which the inverted image is formed. The aperture C being supposed very small, the rays issuing from the point B, will fall on *b*; those from the points A and D, will fall on *a* and *d*, &c. Whence, since the rays issuing from the several points are not blended, they will by reflection exhibit its appearance on the screen. But since the rays AC and BC intersect each other in the aperture, and the rays from the lowest points fall uppermost and those from the highest below, the situation of the object is necessarily inverted. Hence, since the angles at D and *d*, and the vertical angles at C are equal, the angles B and *b*, A and *a* are also equal. Consequently, if the screen on which the object is delineated be parallel to it, we shall have

$$ab : AB :: dC : DC;$$

that is, the height of the image will be to the height of the object, as the distance of the aperture from the former is to its distance from the latter.

This proposition, of course, will not obtain when lens is introduced into the aperture, the purpose which is to produce a more rapid convergency of th rays, to furnish a greater quantity of light and a mo vivid image; but the image is necessarily inverted this case the same as in the above.

In the choice of a lens it should be observed, th the shorter the focus the smaller and brighter t images will appear; and the longer it is, the larger a less vivid; in consequence of the light being in t latter case more dilated or spread over a larger surfa

364. A serious disadvantage attends all the co structions of cameras obscura to which we have hither adverted, in consequence of their being necessari fixed to the same objects, and the trouble of darken the room, adjusting the apparatus, &c. This may averted by means of a roof in the form of a dome cupola, placed over a building, prospect room, or te porary room erected for the purpose. This dome m be made to turn on friction wheels in a groove ma in the roof, and to carry round with it glasses in a b above, or, which is more manageable, the box with t glasses may be made moveable, so as to be turne round at pleasure by a proper contrivance for such motion. The box should approach to a cubical form 6 or 7 inches on the side, in which is to be placed dis-

gonally a true ground mirror in a frame moveable somewhat on an axis below, to reflect the rays from objects at various distances; under the mirror is a round cell, and at the bottom of the box is fixed a double convex lens, of about 6 or 8 feet focus, and 4 or 4½ inches in diameter: this lens will form upon a white table placed on the floor below, the images of the objects reflected by the mirror above at the proper focal distance of the lens.

The diameter of the table ought to be about 2½ feet or 3 feet, and excavated on its surface to a small degree of concavity, the radius of which being about equal to that of the sphericity of the lens, in order that the inequality of the distance of its surface from the centre of the lens, presenting the images indistinctly at the circumference, when they are clear in the middle, a defect, necessarily attending a plane surface, may be avoided.

This surface should be painted perfectly white, or, which is better, covered with a thin coating of plaster stucco. The pillar of the table should be made with an internal screw, working in an external one cut in the pedestal, so that by turning round the table and screw, its surface may be either elevated or depressed, as may be necessary to admit the clearest and best defined picture of the objects possible.

XIII. *Portable camera obscura.*

365. The glasses of cameras obscura are frequently fitted to a machine shutting up in the form of a chest, or box, so as to be easily transported from place to place, and carried about by the artist. The apparatus within is contrived to fold outwards, and form a machine, as shown in (fig. 116), where it is represented as placed together for use. The lid from A, and the slides (one of which is shown at B), by means of hinges, turn up to the height of about 2 feet from the case CDE, and are fastened together by small brass hooks. The head and sliding box, with glasses F, are also fastened on by hooks within. The lens, of about 25 inches focus, is placed under the true parallel glass mirror, and forms the images on a white sheet of paper, placed in the bottom of the chest. To view the images the face is applied close to a piece under A, for that purpose; and to trace the outline, or copy them, the arm at the same time is introduced into the cloth sleeve under H. The box F slides on a square tube, and, by means of a brass rack and pinion G, the lens is adjusted, while the images are viewed, to its proper focal distance from the white paper below. The images formed on the paper have a correct and natural resemblance to the original objects; no inversion takes place, and even names and letters, as well as objects, are in their direct order.

366. This camera may be converted into an instrument for magnifying perspective prints and drawings, and forms the best possible apparatus for that purpose. When used in this way, the head FG and tube are to be entirely removed, as well as the front A, and the cloth CD, and another head, with a diagonally placed mirror and large convex lens (fig. 117), substituted in their places. The prints are to be placed at the bottom of the chest, and as the camera-case is open, the print will be illumined either by day-light or candle-light, as required. The print is viewed by reflection in a horizontal direction, by the eye being placed before the large convex lens. When the sides and front are unhooked and folded down into the chest, they all lie close, and admit the head to lie under them; the dimensions of the chest, when shut up, may be such as not to exceed 2 feet in length, 20 inches in breadth, and 5 inches in depth.

Fig. 118.

367. But the most portable kind of camera obscura is that represented at (fig. 118), and is the form most frequently used among artists, on account of its convenient dimensions. The images are reflected on a rough ground plane glass, and are more vivid than those formed on paper by the camera above described. It is made of mahogany, or other wood, and of various dimensions, some so small as to be carried in the pocket. The lens at the front A is fixed in the cell, in the front of a square draw-out tube, its focus being equal to the length of the box when the drawer is half out; a plain mirror is placed diagonally at an angle of 45°, at the end of the box, as shown by the dotted line *a b*, which reflects the rays transmitted by the lens up to the upper side of the plane rough ground glass, the rough side being best placed above, under the folding darkening cover, where are formed the images of the objects before the lens at A. The use of the draw is to adjust the proper distance of the lens from the mirror, according to the variable distances of the objects. The images on the rough glass exhibit a beautiful perspective picture; also the profile of a person seated in a room, in a strong light, before the camera, and more particularly if the sun illuminate the object. The image thus formed may be readily traced on the rough surface of the glass by a blacklead pencil, or by what is preferable, red French chalk. White paper being then gently placed on the glass, the lines will be taken correctly off. If very thin white paper be merely placed upon the glass, the images may still be discerned, and though faintly, yet sufficiently strong to afford the means of tracing correctly. The nearer the object or features are to the camera, the larger will be the image; and an additional lens, of a shorter focus, is sometimes fitted on, in order to be substituted for the other, when the images of very near objects are wanted. Some artists who draw profiles take out the rough glass from its cell, invert the camera, and, by a stand, support it about 10 or 12 inches above the white paper on the table; but the image will then be inverted on the paper. It may, however, be traced with a pencil in a correct manner, and with less trouble than by the other method.

368. In the second edition of Ferguson's Lectures, by Dr. Brewster, the editor describes a very ingenious camera obscura, invented by Mr. Thomson, of Duddingston, which is so contrived that it may be shut up and carried like an umbrella.

XIV. *Magic lantern.*

Magic lanterns.

369. The magic lantern is a highly amusing optical toy, or instrument; first, we believe, described by Kircher, in his "Ars. Magna Lucis et Umbræ." It is intended to represent images on the wall of a darkened room, magnified to any size at pleasure, and exhibited in their natural and vivid colours. The construction may be described as follows: In (fig. 119), ABCD represents a common tin lantern, to which is added a draw-out tube FG. In H is fixed a metallic concave speculum, of a foot diameter at most, or four inches at

Fig. 119.

this there is no distortion; so that every line, even the most remote from the centre of view, is as straight as those through the centre.

3dly, In that, the field of view does not extend beyond 30°, or, at most, 35°, with distinctness. But in the camera lucida, as much as 70° or 80° might be included in one view.

Dr. Wollaston remarks, further, that by a proper use of the same instrument, every purpose of the pentagraph may also be answered; as a painting may be reduced in any proportion required, by placing it at a distance, in due proportion, greater than that of the paper from the instrument. In this case a lens becomes requisite for enabling the eye to see at two unequal distances with equal distinctness; and in order that one lens may suit all these purposes, there is an advantage in varying the height of the stand according to the proportion in which the reduction is to be effected.

XVII. *Dr. Hooke's camera lucida, or megascope.*

Dr. Hooke's camera lucida.

380. Dr. Hooke's apparatus, which we have seen described under the name of a camera lucida, but which appears to us to answer better to the second denomination of megascope, the invention of which M. Biot, in his "Précis Elémentaire de Physique," attributes to M. Charles, is thus described in the "Philosophical Transactions abridged," by Hutton, &c. vol. i.

Fig. 123.

"This optical experiment is new, though easy and obvious, and has not, that I know of, been ever made by any other person in this way. It produces effects not only very delightful, but, to such as know not the contrivance, very wonderful; so that spectators not well versed in Optics that should see the various apparitions and disappearances, the motions, changes, and actions, would readily believe them to be supernatural and miraculous.

"Opposite to the place or wall where the apparition is to be, let a hole be made about a foot in diameter, or larger; or if there be a high window that has a casement in it, it will be so much the better. Without this hole place the picture or object which you would represent, inverted, and by means of looking-glasses set behind, if the picture be transparent, reflect the rays of the sun, so as they may pass through it towards the place where it is to be represented, and let the picture be encompassed on every side, with a board or cloth, that no rays may pass beside it. If the object be a statue, or some living creature, then it must be very much enlightened, by casting the sun-beams on it by reflection, or refraction, or both. Between the object and the place where it is to be represented, there must be placed a broad convex lens, so that it may represent the object distinct. The nearer the lens is placed to the object, the more the latter will be magnified on the wall; and the further off, the less; which diversity is effected also by glasses of different sphericities. If the objects cannot be inverted, as is difficult in the case of living animals, candles, &c. then let two large glasses, of convenient sphericities, be placed at proper distances, to be found by trials, to make the representations erect as well as the objects.

"These objects, reflecting and refracting glasses, and the whole apparatus, as also the person employed to manage them, must be placed outside the window or hole, so that they may not be perceived by the spectators in the room. Whatever may be done by means of the sun-beams in the day-time, the same may be done with much more ease in the night, by the help of torches, lamps, or other strong lights, placed about the objects, according to the several sorts of them."

M. Biot, who, as we have stated above, describes this apparatus under the denomination of the *mégascope*, and attributes its invention to M. Charles, has given us a representation of it, which Dr. Hooke has omitted. This figure is the same as our (fig. 123), which see.

XVIII. *Opera-glass.*

381. The opera-glass is nothing more than a short Galilean telescope, but is thus called in consequence of its use in playhouses; it is also sometimes called a *diagonal perspective*, from its construction, which is as follows:

Let ABCD (fig. 124), represent a wooden tube, about four inches long; on each side of which there is a hole EF, GH, exactly against the middle of IK, a plane mirror which reflects the rays falling upon it to the convex lens ML, through which they are refracted to the concave eye-glass NO, whence they emerge parallel to the eye at the hole $r\,s$, in the end of the tube.

Let PAQ be an object to be viewed, from which proceed the rays Pc, $a\,b$, Qd; these rays being reflected by the plane mirror IK, will show the object in the direction $c\,p$, $b\,a$, $q\,d$, in the image $p\,q$, equal to the object PQ, and as far behind the mirror as the object is before it; the mirror being placed so as to make an angle of 45° with the sides of the tube. And, as in viewing near objects, it is not necessary to magnify them, the focal lengths of both the glasses may be nearly equal; or if that of LM be three inches, that of N O one inch, the distance between them will be but two inches, and the object will be magnified three times, which is sufficient for the purpose to which the glass is applied. If the object be very near, as X Y, it is viewed through a hole $x\,y$, at the other end of the tube AB, without an eye-glass, the upper part of the mirror being polished for that purpose, as well as the under. The tube unscrews near the object-glass LM, for the convenience of taking out and cleansing the glasses and mirrors.

The peculiar artifice of this glass is to view a person at a small distance, so that no one shall know who is observed, for the instrument points to a different object from that which is viewed, and as there is a hole on each side, it is impossible to know on which hand the object is situated which you are viewing.

XIX. *Of the polemoscope.*

382. This instrument in a great measure resembles that last described, and is said to have been invented by, and received its name from Hevelius, in 1637. Any telescope will become what is here understood by the term polemoscope, if the tube be bent at right angles, as ABDM (fig. 125), and between the object-glass AB, and the eye-glass GH, be placed a plane mirror, inclined to the axis of the two branches of the tube at an angle of 45°, and so that the reflected image of the object be formed in the focus of the eye-glass GH. By this means, objects situated opposite the lens AB, will appear the same as if the mirror k

were away, and the object-glass, with the objects, were directly opposite to the eye-glass.

On a similar principle, a well-known optical toy is constructed. Let ABCD (fig. 126) represent the horizontal section of a close box, *a a*, *b b*, *c c*, *d d*, four mirrors, the two former and the two latter parallel to each other, and forming angles of 45° with the side of the box or tube: O and P two holes, furnished or not with plane glasses; then an eye looking in at O, will see any object placed at P, as if the vision were direct; and if between the two branches AB, CD, any solid body as a piece of wood, a book, &c. be placed, an observer, who is not acquainted with the principles of the construction, will be very much astonished at his apparently seeing through the intervening substance.

XX. *Of the polyscope, or multiplying glass.*

283. This is nothing more than a lens contained under several plane surfaces, disposed in a convex form, through each of which the object is seen. If several rays, as EF, AB, CD, (fig. 127) fall parallel on the surface of a polyscope, they will continue parallel after refraction. If then the polyscope be supposed regular, LH, HI, IM, will be as tangents cutting the spherical convex lens in F, B, and D; and consequently rays falling on the points of contact, intersect the axis. Wherefore, since the rest are parallel to these, they will also mutually intersect each other in G. Hence, if the eye be placed where parallel rays decussate, rays of the same object will be propagated to it, still parallel from the several sides of the glass. Wherefore, since the crystalline humour, by its convexity, unites parallel rays, the rays will be united in as many different points of the retina *a*, *b*, *c*, as the glass has sides. Consequently the eye, through a polyscope, sees the object repeated as many times as there are sides. And hence, since rays coming from every remote object are parallel, such an object is seen through a polyscope, as often repeated as the latter has sides.

If rays AB, AC, AD, (fig. 128) coming from a radiant point, fall on several sides of a regular polyscope, after refraction they will intersect in G, and proceed on a little diverging.

Hence, if the eye be placed where the rays intersect, after coming from the different planes, the rays will be propagated to it, by the several planes a little diverging, or as if they proceeded from different points. But, since the crystaline humour, by its convexity, collects rays from different points into the same point, the rays will be united in as many different points of the retina *a*, *b*, *c*, as the glass has sides; and consequently the eye, being placed in the focus G, will see even a near object through the polyscope, as often repeated as it has sides. Thus may the images of objects be multiplied in a camera obscura, by placing a polyscope at its aperture, and adding a convex lens at a due distance from it. And it makes a very pleasant appearance, if the prism be applied so that the coloured rays of the sun, refracted from it, be received on the polyscope; for, by this means, they will be thrown on a paper or wall near at hand in little lucid spots; and in the focus of the polyscope, where the rays decussate (for in this experiment they are received on the convex side), a star will be formed of surprising lustre. Further, if images be painted in water colours in the *areolæ*, or little squares of a polyscope, and the glass be applied to the aperture of a camera obscura, the sun's rays passing through it will carry with them the images, and project them on the opposite wall. This artifice bears a resemblance to that by which an image on paper is projected on the camera; namely, by wetting the paper with oil, and straining it tight in a frame, then applying it to the aperture of the camera obscura, so that the rays of a candle may pass through it upon the polyscope.

Anamorphosis.

284. *To make an anamorphosis, or deformed image, which shall appear regular and beautiful through a polyscope, or multiplying glass.*

At one end of a horizontal table erect a plane perpendicularly, upon which a figure may be designed; and on the other end erect another, to serve as a fulcrum or support, moveable on the horizontal one. To the fulcrum apply a plane convex polyscope, consisting, for example, of 24 plane triangles; and let the polyscope be fitted in a draw-tube, of which that end towards the eye may have only a very small aperture, and a little further off than the focus. Remove the fulcrum from the other perpendicular plane, or table, till it be out of the distance of the focus, and the more so as the image is to be greater. Before the little aperture place a lamp, and trace the luminous areolæ, projected from the sides of the polyscope, with a black lead pencil on the vertical plane, or a paper applied upon it. In the several areolæ, design the different parts of an image in such manner that, when joined together, they make one whole, looking every now and then through the tube, to guide and correct the colour, and to see that the several parts match and fit well together. As to the intermediate space, it may be filled up with any figures, or designs, at pleasure; contriving it so that to the naked eye the whole may exhibit some appearance very different from that intended to appear through the polyscope. The eye now looking through the small aperture of the tube, will see the several parts and members dispersed among the areolæ, exhibiting one continued image, all the intermediate parts disappearing.—"Hutton's Math. Dict."

XXI. *Of the kaleidoscope.*

Kaleidoscope.

285. The kaleidoscope is the name given to a newly-invented optical instrument by Dr. Brewster. The word is derived from καλος, *beautiful;* ειοδος, *a form;* and ςκοπεω, *to see.*

Perhaps no discovery, either of utility or amusement, ever attracted so much of the public attention, or equally excited the curiosity both of the scientific and the vulgar, as that of the instrument to which we now allude. In the course of a few months many thousands, perhaps many hundreds of thousands were constructed, and in the hands of persons of all ages and classes; every one was enraptured with the beauty, symmetry, and novelty of the pictures presented to his eye by this simple and amusing instrument; it is only to be regretted, that in this general enthusiasm, so little respect was paid to the property of the inventor, who had previously secured the exclusive right of constructing and selling the instrument by taking out a patent according to the process established by law.

Dr. Brewster complains, and very justly complains, of this invasion of his right; he states, that after the patent had been signed, and a number of the instruments were in a state of forwardness, the gentleman who was employed to manufacture them under the patent, brought a kaleidoscope to London, for the purpose of taking orders for them from the principal opticians. These gentlemen naturally made one for their own use, and for the amusement of their friends, and the character of the instrument being thus made public, various persons began to manufacture the detached parts of it, without putting them together, in order thus to evade the patent; while others manufactured and sold the instrument complete, without being aware that any patent had been granted; and thus a general invasion was practiced upon the property of the inventor. These depredations were, however, generally confined to the lower classes of workmen; all the more respectable instrument-makers and opticians refusing to participate in this outrage of private right.

Originality disputed.

286. Under the apprehension of having by the above proceedings subjected themselves to a prosecution for an invasion of the patent, some of those who had injured the pecuniary interest of the inventor, now sought to justify their piracy by robbing him also of the honour of the discovery, and various old books and old instruments were brought forward to prove that the principle of the kaleidoscope had been long known, and consequently that no patent right attached to it; but certainly a more absurd pretence was never made to justify an illegal act. It is true, that in the latter respect Dr. Brewster might very well have spared the trifling honour that, in a scientific point of view, attached to the accidental discovery of a particular combination and position of two pieces of glass; he is too rich in optical discoveries to be the poorer for being deprived of this; and it is, we conceive, rather for securing the right of the patent, than that of the honour of the discovery, that he has been at the pains of repelling the feeble attempts of those who wished to deprive him of it.

The principle previously known.

287. That the optical principle of the repeated reflections of two mirrors inclined at an angle was known long ago, no one will attempt to deny; we have investigated this principle in (art. 222) of the preceding treatise; but this has little to do with the construction of the instrument in question; and as to Bradley's instrument, respecting which much has been said, it has scarcely any thing in common with the kaleidoscope; it consisted merely of two pieces of looking-glass, five inches wide and four inches high, joined together with hinges, and opening like a book. These plates being set upon a geometrical drawing, and the eye being placed in front of the mirrors, the lines of the drawing were seen multiplied by repeated reflections. It has been often made by opticians, and was principally designed by the author for producing symmetrical figures for garden-plats, for which purpose it was given in his work on gardening, published in 1717. We cannot spare the room, nor do we think it would be interesting to our readers, to give a more detailed account of this apparatus; we shall merely observe, that all that could be effected with it may be done equally well by two small looking-glasses forming any proposed angle with each other.

It has been very justly observed, that "no proof of the originality of the kaleidoscope could be stronger than the sensation which it excited in London and Paris. In the memory of man no invention, and no work, whether addressed to the imagination or the understanding, ever produced such an effect. A universal mania for the instrument seized all classes, from the lowest to the highest, from the most ignorant to the most learned; and every person not only felt, but expressed the feeling that a new pleasure had been added to his existence. If such an instrument had been known before, a similar sensation ought to have been excited, and it would not have been left to the ingenuity of the half-learned, and half-honest to search for the skeleton of the invention among the rubbish of the 16th and 17th centuries."

Having said thus much with reference to the invention and originality of the instrument, and referring the reader to (art. 222) for the principle of its operation, we shall proceed to describe its general construction in the words of the inventor.

XXII. *Construction of the simple kaleidoscope.*

288. When two reflecting planes are placed in a tube at certain angles, according to the principles explained in the article above referrred to, and the eye applied to one end of it, it will perceive at the other end a circular field, composed of as many luminous sections as the number of times that the angle of inclination of the plates is contained in 360°.

If we now fix upon the edges of A *o*, B *o* (fig. 129), some pieces of differently coloured glass, so as to project within the aperture AOB, the portions of the glass included in the aperture will also be included in all the images of the apertures; the images of these portions in the inverted sectors will join those in the direct sectors, and the whole will be arranged into a fixed symmetrical pattern, far surpassing what can be produced by the hands of the most skilful artist. If we now turn the tube to different parts of the room, the light will fall in different directions upon the fragments of coloured glass, and a variation of the patterns will thus be produced to a very considerable degree from fixed objects. This was the first state in which Dr. Brewster constructed the kaleidoscope; and it was not for some time afterwards that the idea of varying the pattern by the motion of the objects occurred to him. In executing the kaleidoscope in this new form, Dr. Brewster first placed the coloured glass, or other objects, between two plates of the thinnest glass, and holding this object-plate in the hand, it was moved in various directions across the angular aperture of the instrument. The instrument was still further improved, by sliding the object-plate in a grove, as in the magic lantern, by placing the objects in a cell of the reflectors; or by giving the object-plate a circular form, so as to have a rotatory motion round the axis of the tube. By means of these, the power of the kaleidoscope, and the beauty of the pattern, were considerably increased; and from being a mere toy, it now promised to be an useful and amusing instrument.

289. The kaleidoscope, in this simple form, is shown in (fig. 130), where AB is the tube, A its object-end, B the eye-end, and *c* a cell, which holds the circular object-plates D, this having a groove for admitting also rectangular object-plates. This cell is placed on the end A of the tube, and turned round by the hand, whilst the eye is applied at B. When the rectangular object-plate is used, it is applied to the groove of the cell *c*, and moved backwards and forwards in this grove, while the cell itself may be either stationary or moved round its axis. By this means the most splendid

are produced, and the singular effects arising ‑ different directions in which the objects cross the angular aperture, are displayed in a striking

atterns may likewise be varied, by placing the sides of the object-plates next to the instru‑ hough in general the clear glass should be the eye. Other changes of forms may be pro‑ y a pencil or ink, or by varnish or transparent id at random on the ground side of the plate.

of the object-plates is made less in diameter other, in order that it may be loose in the cell vibrate by the slightest motion of the hand: :t of this vibration is singularly fine, particu‑ en it is combined with the motion of the co‑ ragments of the object-plate. The best method icing the vibration is to strike slightly with the 1 the end N of the tube.

ier of the object-plates, in several of the in‑ ts, contains either fragments of colourless glass, regular surface of transparent varnish, or in‑ Canada balsam. This object gives very fine ss figures when used alone; but its principal be placed in the cell between an object-plate ght colours, and the end of the instrument. iis is done, the outline of the pieces of coloured softened down by the refraction of the trans‑ ragments, and the pattern displays the finest f soft and brilliant colouring. The colourless late supplies the outline of the pattern, and the colour behind fills it up with the softest tint.

the instrument is used in candle-light, the end , should be held within an inch or two of the and a little to one side of it; or it should be to a point half way between two candles, s near each other as possible. The light of an lamp is, however, preferable to that of a candle.

observer wishes to look at any object not con‑ n the object-plates, he has only to remove the and hold the object close to the end N of the ent; for the symmetry of the figure is destroyed kept at a distance, or if the tube MO be pulled ie slightest degree. Hence it will be found pattern is most perfect when the object is placed upon the fixed glass at the end N of OM.

In looking at opaque objects, as a seal, a hain, the second hand of a watch, coins, pic‑ gems, shells, flowers, leaves and petals of impressions of seals, &c. the object, instead held between the eye and the light, must be n the same manner as we view objects through cope, being always placed as near the instru‑ possible, and so as to allow the light to fall icularly upon the object; the object-plates , and all transparent objects may be viewed in ner; but after all, the most splendid exhibition ind is to view the minute fragments of coloured stated in the preceding part of this article.

XXIII. *Of the polycentral kaleidoscope.*

What has been stated above is only with re‑ to two reflecting mirrors, but it will equally the case where three such mirrors are em‑ except that the multiplication of the images will of course now be different, and differently arranged.

Whatever may be the angle that the three reflecting planes form with each other, any object placed between them will be multiplied indefinitely, till the image is lost for want of light; but in order that a due symmetry may be observed in the field of the instrument, it is necessary that each angle of the triangular prism thus formed, should be such as of itself to produce a symmetrical figure; that is, each angle must be some even submultiple of 360°. This condition limits the arrangement of the plates to the three following, viz. the angles formed by the planes must be either 30°, 60°, 90°; 60°, 60°, 60°; or 45°, 45°, 90°, and the figures of the field of the instrument will then be as represented in figs. 131, 132, 133, respectively.

Figs. 131, 132, 133.

By a similar reasoning it may be shown, that if four mirrors be employed, they must be at right angles with each other; but they may either form a rectangle or a square.

If the number of planes exceed four, some one or more of the angles will be greater than a right angle, and will therefore not be an even submultiple of 360°; consequently no symmetry is obtainable with a greater number than four planes.

XXIV. *Kaleidoscopes with moveable mirrors.*

Kaleidoscopes with moveable mirrors.

292. We have already observed that the kaleidoscope was no sooner known in London, than the ingenuity of different artists gave it a great variety of forms, and highly improved the pleasure and amusement furnished by this instrument in its simplest form. Amongst others, Mr. Bate's construction seems highly deserving of a minute description. Hitherto we have supposed the planes fixed to some determinate angle, and consequently all the figures exhibited by the same instrument, however various they may be, will have one general character, viz. the polygon, or field of view, will always have the same number of angles; but by the following contrivance the angle of the instrument may be varied at pleasure, and consequently a greater variety of figures exhibited.

Fig. 134.

This construction of the kaleidoscope is exhibited in (fig. 134). It is composed of two cones, M, N, connected together by a middle-piece or ring below R, into which they are both screwed. These lines enclose two highly polished metallic reflectors AO, BO, one of which, BO, is fixed to the same ring below R, to which the two cones are screwed, and so adjusted by screws that its reflecting surface passes through the axis of the cones and rings. The other reflector, AO, is fixed to the outer ring R, by an arm passing through an annular space or arch of a little more than 90°, cut in the circumference of the inner ring; and while its reflecting surface is adjusted so as to pass through the axis of the cones and rings, its lower edge OMNE is finely ground to a perfectly straight line, free of all roughness; and this edge is so adjusted by screws that it coincides with the axis of the cones and rings. The lower edge of the reflector AO comes a little below the same axis, so that the edge EO of the reflector BO just touches a line in the reflector AO, which coincides with the axis of the cones and rings, and forms a junction in every part of the two meeting edges.

If we now fix the outer ring R into the ring of a stand

ST, so as to be held fast, and turn the cones MN with the hand, we shall give motion to the reflector BO, so as to place it at any angle we please, from 0° to 90°, and during its motion through this arch, the junction of the two reflectors must remain perfect, if the touching lines are adjusted, as we have described them, to the axis of motion, which must be also the axis of the cones and rings. If, on the contrary, we take away the stand, and, holding the instrument in the hand by either of the cones MN and turn the ring R with the other, we shall give motion to its reflector AO, and produce a variation in the angle in the same manner as before.

In order to enable the observer to set the reflectors at once to an even aliquot part of a circle, or so as to give pairs of direct and inverted images, the most convenient of the even aliquot parts of the circle are engraven upon the Ring R, whereby we have only to set the index I to any of these parts, to No. 20 for example, and the reflectors will then be set at an angle of 80°, and will form a circular field with twenty luminous sectors, or a star with *ten* points, and consequently a pattern composed of ten pair of direct and inverted images.

As the length of the plate is only 5 inches, it is necessary for some eyes to have a convex lens placed at E. A brass ring containing a plain glass screw into the outer ring CD, and there is an object plate accompanying the instrument, and containing fragments of differently coloured glass. This object plate consists of two plates of glass, one ground and the other transparent, set in brass rims. The transparent one goes nearest the reflectors, and the brass rim, which contains it, screws into the other, so as to contain between them the coloured fragments. A loose ring surrounds the object-plate; and when this ring is screwed into the circular rim CD, the object-plate can be turned round so as to produce a variety of patterns, without any risk of its being detached from the outer cone.

Some of the preceding instruments have been fitted up also with plates of glass ground perfectly flat, and highly polished, which reflect a great deal of light, and are less liable to be injured by the operation of removing the dust, or by other causes. Flint-glass, from its high reflective power, is preferable to any other. The polyangular kaleidoscope, as made by Mr. Bate, with glass plates, is constructed on different principles from the one now described. The reflectors are placed in a cylindrical tube, and motion of the moveable reflector is produced by a contrivance extremely simple and ingenious.

XXV. *Kaleidoscope for projecting figures on a wall.*

293. The pictures created by the kaleidoscope are visible only to one person at a time; but it is also possible to fit it up for the express purpose of exhibiting the same to a number of spectators. The necessary limitation of the aperture at the eye-end of the instrument is, however, very hostile to this species of exhibition, and renders it necessary that the objects should be as transparent as possible, and very strongly illuminated.

The apparatus requisite for this purpose is shown in (fig. 135), where FG is the kaleidoscope, having its reflectors inclined at an angle of 22°½, 25°$\frac{10}{14}$ 30°, 36°, 45°, 60°, or 90°; for if the angle be made smaller, the last reflections will not be easily seen. The objects out of which the picture is created are placed on the object-plate DE, and are illuminated by means of a lens AD, which concentrates upon it the direct light from the lamp or candle C, and also the light reflected from the concave mirror *m*. At the eye-end of the kaleidoscope is placed a lens L, close to the end of the reflectors; this lens should have a focal length about an inch less than the length FC of the reflectors, in order that the image of the pattern created at F may be thrown upon the wall W, at a convenient distance from L.

When the instrument is thus fitted up, an enlarged image of the pattern will be seen upon the wall W, and this image will undergo every possible transformation, and exhibit to the spectators every variety of tint and form by the motion of the object, either through a groove or round the axis of the instrument. The same effect may be produced by the light of the sun, but in this case the mirror MN is unnecessary. One of the polyangular kaleidoscopes, as made by Mr. Bate, has been fitted up in this manner, for exhibition at the lectures on Natural Philosophy delivered at Guy's-hospital, by William Allen, Esq. F. R. S. The light is derived from an oxygen lamp, which produces the most intense illumination.

A distinct treatise on the kaleidoscope has been advertised by Dr. Brewster.

XXVI. *Explanation of certain optical phenomena.*

294. In the preceding treatise on Optics, it has been our object to enter as concisely as possible upon the various subjects on which we have treated, and we have therefore necessarily passed over several optical phenomena, whose explanation would have carried us somewhat out of the line and limits we had prescribed; we propose, however, now to bestow a few columns to the examination of those which more commonly occur, and to give the solution of them as briefly as is consistent with perspicuity.

1. *What is the cause of the lateral radiations which seem to adhere to a candle viewed with winking eyes?*

To this question different answers have been returned. Dr. Young says, the most conspicuous radiations are those which, diverging from below, form each, with a vertical line, an angle of about 7°; this angle is equal to that which the edges of the eye-lids when closed make with a horizontal line; and the radiations are obviously caused by the reflection of light from those flattened edges. The lateral radiations are produced by the light reflected from the edges of the lateral parts of the pupillary margin of the uvea; while its superior and inferior portions are covered by the eyelids. The whole uvea being hidden before the total close of the eye-lids, these horizontal radiations vanish before the perpendicular ones.

2. *Whence arises the luminous cross which seems to proceed from the image of a candle in a looking-glass?*

This is produced by the direction of the friction by which the glass is polished: the scratches placed in a horizontal direction, exhibiting the perpendicular part of the cross, and the vertical scratches the horizontal part, in a manner that may be easily conceived.

3. *Why do sparks appear to be emitted when the eye is rubbed or compressed in the dark.*

When a pressure of a certain breadth, as that of the finger, for example, is made on the opaque part of the eye in the dark, an orbicular spectrum appears on the part opposite to that which is pressed; the light of the disc is faint, that of the circumference much stronger; but when a narrow surface is applied, as that of a pin's head, or the point of a nail, the image is narrow and bright. This is obviously occasioned by the irritation of the retina at the part touched: referred by the mind to the part whence light, coming through the pupil, would fall on this spot, the irritation is greatest where the flexure is greatest, viz. at the circumference, and sometimes at the centre of the compressed part. But in the presence of light, whether the eye be open or closed, the circumference only will be luminous, and the disc dark; and if the eye be viewing any object at that part where the image appears, that object will be totally invisible. Hence it follows, that the tensions and compressions of the retina destroy all the irritation, except that which is produced by its flexure, and this is so slight on the disc, that the apparent light there is fainter, that is, that of the rays arriving at all the other parts through the eye-lids. This experiment demonstrates a truth which may be inferred from many other arguments, and is, indeed, almost an axiom; that is to say, that the supposed rectification of the inverted image on the retina does not depend on the direction of the incident rays.

If the anterior part of the eye be repeatedly pressed, so as to occasion some degree of pain, and a continued pressure be then made on the sclerotica, while an interrupted pressure is made on the cornea, we shall frequently be enabled to observe an appearance of luminous lines, branched and somewhat connected with each other, darting from every point of the field of view towards a centre a little exterior and superior to the axis of the eye.

This centre corresponds to the insertion of the optic nerve, and the appearance of the line is probably occasioned by that motion of the retina, which is produced by the sudden return of the circulating fluid into the veins accompanying the ramifications of the arteria centralis, after being detained by the pressure which is now intermitted. But as such an obstruction and such a re-admission must require particular circumstances, in order to be effected in a sensible degree, it may naturally be supposed that this experiment will not always easily succeed.

4. *Why does a person in a room see another through a glass window passing in the street much better than the latter will see the former in the room?*

The person in the street or open air, being in full day-light, the few rays which pass out of the room through the glass will make but a very small impression, whereas the case is exactly the reverse with respect to the person in the room. If it be but imperfect day-light, and a candle or candles be lighted in the room, then the person out of doors will see him in the room more perfectly than the latter will see the former, for a similar reason.

5. *What is the reason that, looking by day-light at the end of a needle placed near the eye of the observer, and between the eye and a card having a small hole pricked in it by the needle, the needle appears to be behind the card, and reversed?*

We see nothing on the eye side of the card, because it is too little distant, and we see the object beyond it reversed, for the same reason, as when a spectator placed in a dark chamber, and who looks through a small aperture, without preventing the light from entering, will see exterior images reversed, that is, the rays crossing in the aperture in this case, and in the hole in the card in the former, will cause the inversion of the images in question.

6. *Why does a burning coal or other light, when turned rapidly round, appear as a ring of fire?*

We have already referred to this experiment in the preceding pages; as to the cause of the appearance it is, that the impression of the light on the retina produces certain vibrations, which continue for a time, during which the sensation remains the same; if, therefore, the coal be made to perform its revolution before the vibration caused by the first impression ceases, it obviously follows that the consequence will be the sensation of a ring or circle of fire such as we have described. Newton supposes the sensation produced by the impression of light on the retina to continue for a second, others state it at $7'''$; assuming the latter number, it follows that if the revolution be made at the rate of about nine revolutions per second, the luminous ring will be entire.

7. *When a light is suspended by a long cord, and is made to turn on it as an axis, what is the reason that two different persons viewing it from the same point, or the same person, at different times, will suppose it turning sometimes in one direction, and at other times in another?*

The extreme edges of the light forming a circle, we refer the motion of the light to the diameter which passes through the eye; and at a very inconsiderable distance we are unable to assure ourselves which extremity of the diameter is nearest to us, particularly if the plane of the circle pass very near the eye, and that we do not pay a sufficient attention to the effect of perspective. If, therefore, one person take the most distant extremity of the diameter for the nearest, while the other forms a correct estimate in this respect, the one will see the candle turning to the left and the other to the right; and the same person at different times, and even without removing his eye from the candle, may be subject to the like deception. The same thing may happen to persons viewing very obliquely the sails of a windmill in motion. See art. 121.

8. *Whence proceeds that dazzling effect which takes place when we come out of darkness into a great light, and the temporary blindness in going out of great light into a medium darkness?*

This arises from the natural construction of the organ of vision, which is such, that we imperceptibly contract or dilate the pupil, in order to admit as many rays, and no more, than are necessary for distinct vision. A person having been, therefore, for some time in a great or medium obscurity, has dilated the pupil to its utmost, in order to admit the greatest possible quantity of rays. If, now, he go suddenly into a great light, the motion of the iris, by which the required contraction is to be performed, not being very rapid, too many of these intense rays enter, and cause that dazzling effect, which every one must have observed in this case. The contraction, however, is ultimately accomplished, and the vision becomes distinct.

Again, when a person has been for some time in a bright light, the pupil is contracted, and it remains so for a time after he has left the light; the consequence is, that a sufficient quantity of rays cannot immediately enter, and he feels himself, for a short interval, in a state of temporary blindness.

Those animals who can see distinctly in a strong light, and in a medium obscurity, have a greater power of contracting and dilating the pupil than men; a cat, for example, towards evening, has the pupil full and round, but at mid-day it is commonly contracted into a fine line. The eagle has, perhaps, a power of contraction far exceeding other animals.

Near-sighted persons can commonly see better in the dark than those who have a good common sight; and the reason appears to be, that the former see without effort near objects, whereas those who have ordinary sight are obliged to close the eye, and consequently to contract the pupil to obtain the same effect; they consequently receive much less light in a medium obscurity, and when the eye is directed to near objects, than persons naturally short-sighted.

9. *Why do near-sighted persons commonly attribute to distant objects a greater magnitude than those who have a good common sight?*

The distinct images of objects are made on the eye only at the point of intersection of the rays of light issuing from the same point. The eye of short sight receives on the retina all these rays beyond the point of their intersection, and consequently at a point where they are more extended.

10. *Why can those persons who, after a certain age, become presbytes, read a small print by exposing it to the rays of the sun, or other strong light?*

The intensity of the light thrown upon the object, and thence transmitted to the eye, causes them to contract the pupil in an extreme degree, which thus produces the necessary convergency of the rays for rendering vision distinct.

11. *What is the reason that a small object, placed very near the eye, but seen through a small pin-hole in a black card, or leaf of black paper, appears larger as it is nearer the eye; while, if we observe it without the card, it appears sensibly of the same magnitude at all distances?*

The reason of this phenomenon doubtless is, that we judge of the magnitude of objects in most cases, independently of the angle formed by the rays proceeding from its extremities; and in all cases the idea only in part depends upon the convergency of the rays, unless all other means of estimation be taken away, as in the experiment in question, where we view the object through a small aperture. Thus, however near the object may be, the hole will correct, or render the vision perfect; while the card, being interposed between the eye and the object, we are unable to call to our assistance our common criterion, the distance at which the object is placed; and the estimate is therefore formed of its magnitude from that of the image only; which is necessarily greater the nearer the former approaches the card and the eye.

12. *Why does moistened paper appear more transparent than when in its natural state?*

A piece of dry paper has its pores embarrassed with fine interwoven threads: the liquor, which penetrates its pores, breaks the connection of these threads, and the pores become as so many small tubes filled with liquor, proper to transmit the light, and give to the paper, in this state, its transparency, by permitting the light to pass, which was before prevented by the intervention of the threads above alluded to.

13. *What is the reason, when the sun, or any other strong light, illuminates the inside of a round vessel, that we see two species of luminous semicircle, which unite somewhat in the form of a heart, of which the point of union approaches so much nearer to the centre, or axis of the vessel, as the light itself approaches the same?*

The luminous curves in question are the effect of the consecutive intersections of the rays of light reflected from each of the contiguous points of that concave semi-circumference of the vessel which is enlightened, as shown in (fig. 22). Where the point B is the focus of this semi-circumference, and its distance from the centre depends on that of the luminous object from the enlightened circumference, the two curves AB, BC, are called *caustics* by *reflection*.

14. *What is the cause of those long tremulous luminous trains which are observed in a stream of running water when enlightened by the sun or moon?*

The particles of the running water, gliding one over the other in slight waves, or lamina, produce the effect of so many small plane mirrors differently inclined, which changing both their direction and place every instant, as well as their velocity and inclination, they present their reflections sometimes on one side and sometimes on the other, and thus produce the tremulous appearance above noticed.

15. *What is the cause that vision under water is so extremely indistinct?*

The refraction of the rays which enter the water from the air is nearly as great as that of those produced by the humours of the eye; therefore, when the eye is in the water, it makes only a very small refraction of the rays, and consequently can only produce very indistinct images of objects seen by it, the union of the rays falling considerably beyond the retina.

16. *Why are the crystallines of fishes sensibly globular and solid?*

The aqueous humour would have been useless in the eyes of fishes; and if their crystalline humour had been sunk as deep as in quadrupeds, or other land animals, they would not have had a sufficient field of vision; the crystalline is therefore placed under the pupil, in order thereby to give a greater opening to the latter; the crystalline is made more dense to render the refraction greater; and it is made spherical, that as small an interval as possible might be formed between the anterior surface and the bottom of the eye.

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ay. only from this period that we shall commence our historical sketch, and attempt to trace the rise and progress of astronomy.

ive in tory ono- We shall follow it, from the state above described, when every thing depended upon observation only, unaided either by a calculus or instruments; through that in which the latter began to be employed, and some assistance was derived from the more elementary positions of geometry. We shall next examine it from the period when astronomical science was enriched and extended by the invention of the telescope, but while the principles of computations were still founded on the elements of pure geometry; and lastly, we shall exhibit the science as it now exists, supported by every aid that can be derived from the present high state of practical and theoretical mechanics and optics; when the effect of every celestial motion, and every disturbing force is made to depend upon one universal law; and the amount of each investigated and submitted to computation, by means of the powerful assistance derived from the modern analysis.

The first period above alluded to, comprehends a long series of ages, during which the science of astronomy passed into the hands of different people and nations. First, to the Greeks, then to the Arabs; from which latter it seems probable that it found its way to India and China, about the same time that it was also brought into Spain; whence it afterwards spread throughout all parts of civilized Europe. We shall therefore divide this period into the following minor sections. The astronomy of the Greeks; of the Arabs; of the Indians and Chinese; and of modern Europe; which latter will bring us up to the time of Copernicus and Galileo; including, in all, about twenty-one centuries.

Of the Astronomy of the Greeks.

Thales. B. C. 600.

Thales is generally considered as the founder of astronomy amongst the Greeks. This philosopher, who must have flourished about 600 years before the commencement of the Christian æra, is said to have taught that the stars were fire, or that they shone by means of their own light; the moon received her light from the sun, and that she became invisible in her conjunctions, in consequence of being hidden or absorbed in the solar rays, which it must be acknowledged is but an obscure way of saying that she then turned towards us her unenlightened hemisphere. He taught farther that the earth is spherical, and placed in the centre of the world; he divided the heavens, or rather found them divided into five circles, the equator, the two tropics, and the arctic and antarctic circles. The year he made to consist of 365 days; and determined "the motion of the sun in declination." What is meant by this expression is not very easy to comprehend; if it only means that he discovered such a motion, it can scarcely be considered as correct, as it must have been known prior to his time; viz. to the first observers; and it cannot mean that he laid down rules for computing it, as we have every reason to know that the most simple principles of trigonometry were not propagated till many centuries after his time.

Predict an eclipse.

Thales is also said to have first observed an eclipse, and to have predicted that celebrated one which terminated the war between the Medes and the Lydians; an eclipse on which much has been written, but from which very little satisfactory information has been obtained. Herodotus says, "it happened that the day was changed suddenly into night, a change which Thales the Milesian had announced to the people of Ionia, assigning for the limit of his prediction, the year in which the change actually took place." Thales had therefore neither predicted the day nor the month; and in all probability he had no other principle to proceed upon, than the Chaldean period of eclipses already alluded to in the preceding part of this article.

The pointed declaration of the historian, that the limits assigned by the astronomer for the appearance of this phenomenon, was the year in which it happened, is a pretty obvious proof of the low state of astronomical science at this time, and it would be of little importance whether the eclipse was itself partial or total; but as there is little doubt that such an event actually took place, it becomes a matter of high importance in chronology, to ascertain whether it was such as it is described, viz. a total eclipse; for no partial obscuration of the sun's light would accord with the description of Herodotus, *of the day being suddenly changed into night*; and such a phenomenon in any particular place being an extremely rare occurrence, it would, if correct, enable us to determine not only the year, but the very day and hour at which it happened, and thus furnish at least one indisputable period in chronology and history.

Dates a[illegible] signed t[illegible] this ecl[illegible]

Various dates have been assigned to this eclipse. Pliny places it in the fourth year of the forty-eighth Olympiad which answers to the year 585 B. C. (*Hist. Nat.* lib. 2. cap. 12.), a similar opinion has been advanced by Cicero (*De Divinat.* lib. 1. § 49.) and probably by Eudemus (*Clement. Alex. Strom.* lib. 1. p. 354.); by Newton (*Chron. of Anc. Kings amended*); Riccioli (*Chron. Reform*, vol. 1. p. 228.); Desvignoles (*Chronol.* lib. 4. cap. 5. § 7, &c.); and by Brosses (*Mém. de l'Acad. des Belles Lettres*, tom. 21. Mém. p. 33.)

Scaliger, in two of his writings, (*Animad. ad Euseb*[illegible]b. p. 89.) and in (Ολυμ. ἀναγραφῃ) has adopted also the opinion of Pliny; but in another work (*De Emen*[illegible]. *Temp. in Can. Isag.* p. 321.) he fixes the date of this eclipse to the 1st of October, 583, B. C. Calvisius states it in his (*Opus Chron.*) to have taken place in 607 B. C. Petavius says it happened July 9th, 597 B. C. (*De Doct. Temp.* lib. 10. cap. 1.) which date has likewise been adopted by Marsham, Bouhier, Corsini and by M. Larcher the French translator of Herodotus, (tom. i. p. 335.) Usher is of opinion that it happened 601 B. C.; and Bayer, May 18, 603, B. C.; which latter opinion has been supported by two English astronomers, Costard and Stukeley, (*Phil. Trans.* for 1753.) But Volney attempts to show in his (*Chronologie d'Herodote,*) that it could be no other than the eclipse which happened February 3d, 626 B. C.

Mr. F. Bailly has examined with great care and labour the probability of these several statements, from which it appears, that most of the eclipses above alluded to happened under circumstances which render it absolutely impossible any of them should be that alluded to by Herodotus; most of them were hidden, will never rise; and those that are visible will never set. This is what we now denominate a parallel sphere; the great fixed circle corresponding with our equator.

5. If a great circle pass through the poles, all the points of the surface will rise and set alternately. This corresponds to our horizon, and to our right sphere.

6. If the great circle be oblique to the axis, it will touch two equal parallel circles; of which, that adjacent to the one pole will be always apparent, the other always invisible.

The first of these circles was called by the Greeks, (although not by this author,) as we still denominate it, the *arctic* circle, and the other the *antarctic* circle.

7. If the horizon be oblique, the circles, perpendicular to the axis, will always have their points of rising and setting in the same points of the horizon, to which they are all equally inclined.

8. The great circles which touch the arctic and antarctic circle, will, during the complete revolution of the sphere, twice coincide with the horizon.

9. In the oblique sphere, of all the points which rise at the same instant, those which are nearest to the visible pole will set last; and of the points which set at the same instant, those that are nearest the same pole will rise first.

10. In the oblique sphere, every circle which passes through the poles, will be perpendicular to the horizon twice in the course of one complete revolution.

Remarks on this author.

We omit some other propositions of this author, which are of less importance than the above; and even those which we have given, are such as one would imagine could not have escaped the observation of any one who would think of employing an artificial sphere to represent the celestial motions; yet, from the tenor of the work in question, it would seem, that if they were known, they were never before, at least, embodied in the form of a regular treatise.

Here then we may begin to date the first scientific form of astronomy; because in this work, however low and elementary, we have an application of geometry to illustrate the motions of the heavenly bodies; but we shall still find two other centuries pass away, before the same principles were applied to actual computation.

Euclid. B. C. 300.

Contemporary with Autolycus, was Euclid; whose elements of geometry, after so many ages, still maintain their pre-eminence; and in which we find all the propositions that are necessary for establishing every useful theorem in trigonometry; yet it is perfectly evident that no ideas were yet conceived of the latter science. Neither Euclid nor Archimedes, great as were their skill and talents in geometry, had any idea of the method of estimating the measure of any angle by the arc, which, the two lines forming it, intercepted; nor does it appear that they knew of any instrument whatever for taking angles; a very convincing proof of which appears in the process adopted by the latter justly celebrated philosopher, in order to determine the apparent diameter of the sun.

Aristarchus B. C. 264.

Passing over the poet Aratus, who is supposed to have embodied in his poem all the astronomical knowledge of the time in which he wrote, viz. 270 B. C.; but who had not himself made any observations, we come to Aristarchus, who has left us a work, entitled *Of Magnitudes and Distances*; in which he teaches, that the moon receives her light from the sun, and that the earth is only a point in comparison with the sphere of the moon. He likewise added, that when the moon is dichotomized, we are in the plane of the circle which separates the enlightened part from the unenlightened, which is the most curious and original remark of this author: in this state of the moon, he also observes, that the angle subtended by the sun and moon, is one-thirtieth less than a right angle; which, in other words, is saying, that the angle is 87°, whereas we now know that this angle exceeds 89° 50′. In another proposition he asserts, that the breadth of the shadow of the earth is equal to two semi-diameters of the moon, whereas these are to each other as 83 to 64. In his sixth proposition, he states the apparent diameter of the moon to be one-fifteenth part of a sign, or 2°; whereas we know that it is only about half a degree. Again, the distance of the earth from the moon being assumed as unity, its distance from the sun was said to be 17·107, and the distance of the earth from the sun 19·081. Such was the astronomical knowledge in the time of Aristarchus, who lived about 264 years before the Christian æra.

Eratos nes. B. C.

In order of time we pass now to Eratosthenes, who may, perhaps, with more propriety than Autolycus, be considered as the founder of astronomical science; particularly if it be true that he placed in the portico of Alexandria certain armillary spheres; of which so much use was afterwards made, and which, it is said, he owed to the munificence of Ptolemy Euergetes, who called him to Alexandria, and gave him the charge and direction of his library.

Ancie millar sphere

According to the description given of these armillaries by Ptolemy, they were assemblages of different circles; the principal one of which served as a meridian; the equator, the ecliptic, and the two colures, constituted an interior assemblage, which turned on the poles of the equator. There was another circle, which turned on the poles of the ecliptic, and carried an index to point out the division at which it stopped. The instrument of which the above appears to be the general construction, was applied to various uses; amongst others, it served to determine the equinoxes, after the following manner:—The equator of the instrument being pointed with great care in the plane of the celestial equator, the observer ascertained, by watching the moment when neither the upper nor the lower surface was enlightened by the sun; or rather, which was less liable to error, when the shadow of the anterior convex position of the circle completely covered the concave part on which it was projected. This instant of time was evidently that of the equinox. And if this did not happen, although the sun shone, two observations were selected, in which the shadow was projected on the concave part of the circle in opposite directions; and the mean of the interval between these observations was accounted the time of the equinox. At this time we find enumerated five planets, viz. Φαινων, Φαεθων, Πυροειςηρ, which appear to indicate Jupiter, Saturn, and Mars; and to which were added Venus and Mercury.

Deter tion of equin

Magnit of the

Eratosthenes not only taught the spherical figure of the earth, but attempted to ascertain its actual circumference, by measuring, as exactly as could be done in his time, the length of a certain terrestrial arc, since he found it necessary to devise means at which he stopped, after an attempt not very satisfactory. We see, further, that he carried his angles, or their chords, over a quarter of a circle; but he does not say expressly that his arc had been divided; to render his language accurately, it is simply requisite to say, having carried one of the chords 200 times over upon the arc, he found it exhausted; and that the other chords could only be applied 164 times upon the quadrant.

We see, also, that Archimedes had not the means of computing the angle at the vertex of an isosceles triangle, of which he knew the base and the two equal sides. He was obliged to recur to a graphical operation as uncertain as the observation itself. Thus he was entirely ignorant even of rectilinear trigonometry, and he had not any notion of computing the chords of circular arcs.

Hipparchus B. C. 135.

Finds the length of the year.

We come now to the great father of true astronomy, Hipparchus; but our limits will not admit of our entering very deeply into his discoveries and improvements. One of his first cares was to rectify the length of the year, which before his time we have seen had been made to consist of 365 days and 6 hours. By comparing one of his own observations at the summer solstice with a similar observation made 145 years before by Aristarchus, he shortened the year about 7 minutes; making it to consist of 365 days, 5 hours, 53 minutes; which, however, was not sufficient: but the cause of the mistake is said to have rested principally with Aristarchus, and not with Hipparchus; for the observations of the latter, compared with those of modern times, give 365 days, 5 hours, 48 min. 49½ sec. for the duration of the year; a result, which exceeds the truth very little more than a second. It is to be observed, however, that this is no very exact criterion, unless the same be compared with the observation of the more ancient observer; for supposing all the error on the side of Hipparchus, it is more divided by comparing it with others at the distance of 19 or 20 centuries, than in comparing it with one, where the distance of time is only 145 years.

Introduction of trigonometry by chords.

Establishes the theory of the sun's motion,

and the first lunar inequality.

Hour of the night found by the stars.

One of the greatest benefits, which astronomy derived from this philosopher was his enunciation and demonstration of the method of computing triangles, whether plane or spherical. He constructed a table of chords, which he applied nearly in the same manner as we now do our tables of sines. As an observer, however, he rendered great service to the doctrine of astronomy, having made much more numerous observations than any of his predecessors, and upon far more accurate principles. He established the theory of the sun's motion in such a manner, that Ptolemy 130 years afterwards, found no essential alteration requisite; he determined also the first lunar inequality, and gave to the motions of the moon those of the apogee and of its nodes, which Ptolemy afterwards very slightly modified. Hipparchus also prepared the way for the discovery of the second lunar inequality, and from his observation it was, that the fact of the precession of the equinoxes was first inferred. He employed the transit of the stars over the meridian to find the hour of the night, and invented the planisphere, or the means of representing the concave sphere of the stars, on a plane, and thence deduced the solution of problems in spherical astronomy, with considerable exactness and facility. To him also we owe the happy idea of making the position of towns and cities, as we do those of the stars, by circles drawn through the poles perpendicularly to the equator; that is, by latitudes and by circles parallel to the equator, corresponding to our longitudes. From his projection it is, that our maps and nautical charts are now principally constructed; and his rules for the computation of eclipses were long the only ones employed for determining the differences of meridian.

Another most important work of Hipparchus, was his formation of a catalogue of the stars. The appearance of a new star in his time, caused him to form the grand project of enabling future astronomers to ascertain, whether the general picture of the heavens were always the same. This he aimed to effect, by attempting the actual enumeration of the stars. The magnitude and difficulty of the undertaking did not deter this indefatigable astronomer; he prepared and arranged an extensive catalogue of the fixed stars, which subsequently served as the basis of that of Ptolemy. So great, indeed, is the merit of this prince of Grecian astronomy, that the enthusiastic language in which Pliny speaks of him in his Hist. Nat. (lib. ii. cap. 26.) may rather be admired than censured.

After Hipparchus, we meet with no astronomer of eminence amongst the Greeks till the time of Ptolemy, who flourished between the years 125 and 140 of the Christian æra; which, therefore, includes a space of nearly three hundred years. There were, however, some astronomical writers, both Greeks and Romans in the course of this time, whom it may not be amiss to enumerate, although the little progress that the science made in their hands will exempt us from the necessity of entering minutely into an analysis of their several works: these were, Geminus, who lived about 70 years B. C., whose book is entitled *Introduction to the Phenomena*; Achilles Tatius, of about the same period; Cleomedes, who lived in the time of Augustus; Theodosius, Menalaus, and Hypsicles, who are supposed to have written about the year 50 B. C.; Manilius, Strabo, Posidonius, and Cicero, who were about half a century later; after which, we meet with no one to whom it is at all necessary even to refer, till we come to Ptolemy, who was born in the year of Christ 70; and who made, as we have stated above, most of his observations between the years 125 and 140 of our æra.

Ptolemy has rendered all succeeding astronomers indebted to him, both for his own observations, which were very numerous, and his construction of various tables; but most of all for the important collection which he made of all astronomical knowledge prior to his time, and which he entitled, or the Arabs after him, the *Almagest, or Great Collection*. Of his own labours, we may mention his theory and calculation of tables of the planets, and his determination, with a precision little to be expected in his time, of the ratio of their epicycles to their mean distances; that is to say in other terms, the ratio of their mean distances to the distance of the earth from the sun. This theory, imperfect as it was, was adopted and generally admitted, for the space of fourteen centuries; during which time, it was transmitted to the Arabs, the Persians, and the Indians, and with whom it is still held sacred. The equi-distant centres of the earth, from the excentric and the equant, an hypothesis of Ptolemy, led, in all probability, Kepler to the idea of an ellipse and its foci. Ptolemy thus preparing the way for Kepler, as the laws of the latter may be considered as the precursors of the theory of Newton.

To this celebrated Grecian we also owe the substitution of the sines of arcs instead of their chords; as also the first enumeration of some important theorems in trigonometry.

Ptolemy was the author of that system of astronomy which still bears his name; or if he did not entirely invent it, (as there is great reason to suppose he did not,) he enforced it by such arguments as led to its establishment; and it was afterwards rendered sacred through the stupid bigotry and intolerance of the Romish church. He endeavours to prove the absolute immobility of the earth, by observing, "If the earth had a motion of translation common to other heavy bodies, it would, in consequence of its superior mass, precede them in space, and pass even beyond the bounds of the heavens, leaving all the animals and other bodies without any support but air; which are consequences to the last degree ridiculous and absurd." In the same place he adds, "Some persons pretend, that there is nothing to prevent us from supposing that the heavens remain immovable, while the earth turns on its own axis from west to east, making this revolution in a day nearly; or, that the heavens and the earth both turn, it is in a ratio corresponding with the relations we have observed between them. It is true, that as to the stars themselves, and considering only their phenomena, there is nothing to prevent us, for the sake of simplicity, from making such a supposition. But these people are not aware how ridiculous their opinion is, when considered with reference to events which take place about us; for if we concede to them that the lightest bodies, consisting of parts the most subtle, are not possessed of levity, (which is contrary to nature,) or that they move not differently from bodies of a contrary kind, (although we daily witness the reverse); or, if we concede to them that the most compact and heaviest bodies possess a rapid and constant motion of their own (while, it is well known, that they yield only with difficulty to the impulses we give to them), still, they would be obliged to acknowledge, that the earth, by its revolution, would have a motion more rapid than any of those bodies which encompass it, in consequence of the great circuit through which it must pass in so short a period; wherefore such bodies as are not supported on it, would always appear to possess a motion contrary to itself; and neither clouds, nor any projected bodies, nor birds in flight, would ever appear to move towards the east; since the earth, always preceding them in this direction, would anticipate them in their motion; and every thing, except the earth itself, would constantly appear to be retiring towards the west."

If we did not feel convinced that, in certain cases, even the errors and false reasoning of such a man as Ptolemy, possess a greater interest than the more correct and refined arguments of minor philosophers, we should certainly not have laid before our readers this extract from the introduction to the *Almagest*; but considering it as the defence of an hypothesis, which maintained its ascendancy for fourteen centuries amongst all nations, and which is still held sacred throughout every part of Asia, it is impossible to divest it of its interest and importance.

History.

Analysis of the *Almagest*.

The other part of this great work is more worthy of the talents of its author, and is more deserving of our attention; but the limits of this article will not admit of our giving more than a very concise abstract of its contents. The first book, beside what we have hitherto mentioned, exhibits a highly interesting specimen of the ancient trigonometry; and the method of computing the chords of arcs, which, in fact, involves our fundamental theorems of trigonometry, though expressed in a manner totally different.

Theorems in trigonometry.

Ptolemy first shows, how to find the sides of a pentagon, decagon, hexagon, square and equilateral triangle, inscribed in a circle, which he exhibits in parts of the diameter, this being supposed divided into 120. He next demonstrates a theorem equivalent to our expression $\sin(a-b) = \sin a \cos b - \sin b \cos a$; by means of which he finds the chords of the difference of any two arcs, whose chords are known. He then finds the chord of any half arc, that of the whole arc being given, and then demonstrates what is equivalent to our formula for the sine of two arcs; that is, $\sin.(a+b) = \sin a \cos b + \sin b \cos a$; and by means of this he computes the chord to every half degree of the semicircle. These theorems it may be said belong rather to the history of trigonometry than to that of astronomy; but we trust that the obvious dependence of the latter science upon the former, will be found to justify us in introducing them to the reader in this place.

Climates.

We are next presented with a table of climates nearly equivalent to our nonagesimal tables, and it is not a little singular, that amongst them, we find none appertaining to the latitude of Alexandria; because, without such an auxiliary, Ptolemy must have contented himself with interpolations, which were not only difficult to make, but attended at the same time with great inaccuracy; a circumstance from which it has been concluded, that Ptolemy himself made few observations, or that he was not very particular concerning the accuracy of his calculations. The examination of this question would carry us too far out of our track to admit of our entering upon it in this place; but the reader may see it developed in all requisite detail, in the learned *History of Astronomy*, lately published by Delambre.

Length of the year, &c.

Having passed over the above preliminary details, the author treats of the length of the year, the motion of the sun, the mean and apparent anomaly, &c. &c. The length of the year, according to the sexagesimal notation, he makes 365d. $14' 48''$, which answers to 365d. 5h. $55' 12''$; the diurnal motion of the sun is stated to be $0^\circ 59' 8'' 7''' 13^{iv} 12^{v} 31^{vi}$, and the horary motion $2' 27'' 50''' 43^{iv} 3^{v} 1^{vi}$. To this is also added two tables, one of the mean motion of the sun, and the other of the solar anomaly. The fourth book of the *Almagest* is employed in treating of the motion of the moon, being prefaced by a few remarks respecting the observations which are most useful for that purpose: he then gives an abstract of all the lunar motions, with a table of them; in the first of which the motion is exhibited for periods of eighteen years;

in the second for years and hours; and in the third for Egyptian months and days. Four other columns of the same table present the number of degrees which belong to each of the times indicated in the first column; viz. the second, the longitude; the third, the anomaly; the fourth, the latitude; and the fifth, the elongation.

Lunar motion.

The author next treats of various subjects connected with the lunar motion; as, for instance, its general anomaly; its eccentricity; the lunar parallax; the construction of instruments for observing the parallax; the distance of the moon from the earth, which he states at 38·4⅓ terrestrial radii, when in the quadratures; the apparent diameters of the sun and moon; the distance of the sun from the earth, which is stated at 1210 radii of the latter; and the relative magnitudes of the sun, moon, and earth. The diameters of these are stated to be to each other, as the numbers 18·8, 1, and 3⅖; also their masses as 6644½, 1 and 39¼.

The next book is entirely occupied with the doctrine of eclipses of the sun and the moon; the determination of their limits and durations; tables of conjunctions; and methods of computation and construction, &c.

Particular deductions of Ptolemy.

We cannot extend the analysis of this important work to a greater length; but must content ourselves with a few remarks relative to some of the deductions to which we have referred. We have seen that Ptolemy made the length of the year to be more than 365 days, 5h. 55m., which is about 6 minutes longer than it really is; but considering that the observations before his time, with the exception of those of Hipparchus, were very imperfect; and that the distance of time between these two celebrated astronomers, was not sufficient to determine such a question, with the means they possessed, to the greatest nicety, we may rather admire the near approximation to the truth, than be astonished at the difference between his result and that deduced from numerous and long continued observations.

The evection discovered

His researches on the theory of the sun and moon were however attended with better success. Hipparchus had shown that these two bodies were not placed in the centre of their orbits; and Ptolemy demonstrated the same truths by new observations. He moreover made another important discovery, which belongs exclusively to him, except so far as relates to the observations of Hipparchus, by a comparison of which with his own, his conclusion was deduced,—we allude here to the second lunar inequality, at present distinguished by the term *evection*. It is known, generally, that the velocity of the moon in its orbit, is not always the same, and that it augments or diminishes, as the diameter of this satellite appears to increase or decrease; we know, also, that it is greatest and least at the extremities of the line of the apsides of the lunar orbit. Ptolemy observed that from one revolution to another; the absolute quantities of these two extreme velocities varied, and that the more distant the sun was from the line of the apsides of the moon, the more the difference between these two velocities augmented; whence he concluded that the first inequality of the moon, which depends on the eccentricity of its orbit, is itself subject to an annual inequality, depending on the position of the line of the apsides of the lunar orbit with regard to the sun.

When we consider Ptolemy's system of astronomy, as founded upon a false hypothesis, the complication of his various epicycles, in order to account for the several phenomena of the heavenly bodies; and the rude state of the ancient astronomy, it is impossible to withhold our admiration of the persevering industry and penetrating genius of this justly celebrated philosopher; who, with such means, was enabled to discover an irregularity which would seem to require the most delicate and refined aid of modern mechanics to be rendered perceptible.

The work of this author to which we have hitherto confined our remarks, is the *Almagest*;* but Ptolemy also composed other treatises; which, if not equal to the above in importance, are still such as to be highly honourable to his memory and talents, particularly his geography.

This work, although imperfect as to its detail, is notwithstanding founded upon correct principles; the places being marked by their latitude and longitude agreeably to the method of Hipparchus. As to the inaccuracies of their position, although they cannot be denied, they will readily be pardoned, when we conside that he had for the determination of the situation o cities and places of which he speaks, only a small numbe of observations subject to considerable errors; and th mere report of travellers, whose observations we ma readily grant were still more erroneous than those of h own. It requires many years to give great perfection geography: even in the present time, when observatio with accurate instruments have been made in every pa of the globe, we are still finding corrections necessar a remarkable instance of which seems to have occurr lately (1818) to Captain Ross, in his voyage i Baffin's Bay, where he is said to have found so parts of the land laid down nearly a degree and a h out of their proper places. Many other minor pie on astronomy and optics are also attributed to t author; but we have already extended our account his works to a greater length than we had intend and must now therefore pass on to his successors.

After the time of Ptolemy we find no Greek auth of eminence, although we have some few writers this subject. The science of astronomy had n obviously passed its zenith, and began rapidly decline. The Alexandrian school, it is true, still su sisted; but during the long period of 500 years, that can be said is, that the taste for, and the traditi of the science was preserved, by various commentato on Hipparchus and Ptolemy; of whom the most di tinguished were Theon and the unfortunate Hypati his daughter. The latter is said to have herse computed certain astronomical tables, which are lost

We now arrive at that period, so fatal to the Grecian sciences. These had for a long time taken refuge in the school of Alexandria; where, destitute of support and encouragement, they could not fail to degenerate. Still, however, they preserved, as we have said above, at least by tradition or imitation, some resemblance of the original; but about the middle of the seventh century, a tremendous storm arose which threatened their total destruction. Filled with all the enthusiasm a military government is calculated to inspire, the successor of Mahomet ravaged that vast extent of country, which stretches from the east to the southern confines of Europe. All the cultivators of the arts and sciences who had from every nation assembled at Alexandria, were driven away with ignominy: some fell beneath the swords of their conquerors, while others fled into remote countries, to drag out the remainder of their lives in obscurity and distress. The places and the instruments which had been so useful in making an immense number of astronomical observations, were involved with the records of them, in one common ruin. The entire library, containing the works of so many eminent authors, which was the general depository of all human knowledge, was devoted to the devouring flames, by the Arabs; the caliph Omar observing, "that if they agreed with the Koran, they were useless; and if they did not they ought to be destroyed:" a sentiment worthy of such a leader, and of the cause in which he was engaged. In the midst of this conflagration, the sun of Grecian science, which had long been declining from its meridian, finally set; never perhaps again to rise in those regions once so celebrated for the cultivation of every art and science that does honour to the human mind.

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* The first printed edition of this celebrated performance, was a Latin translation from the Arabic version of Cremoneus; which, however, abounds so much in the idiom of that language, as to render it nearly unintelligible, without a constant reference to the Greek text. This was published at Venice in 1515; and in 1538 the collection appeared in its original language, under the superintendence of Simon Grynaeus, at Basil, together with the eleve books of the Commentaries of Theon. The Greek text was again republished at the same place, with a Latin version, in 1541; and again, with all the works of Ptolemy, in 1551; and lastly, splendid French edition with the Greek text, by M. Halma, i three beautiful volumes, royal quarto, Paris, 1813.

Astronomy of the Chinese and Indians.

If we were to adopt the opinions of some authors who have written on the subject of the Chinese astronomical knowledge, we should have now to commence at a much earlier period than we did in giving our account of this science amongst the Greeks; as it is stated that the former possess records of eclipses and other celestial phenomena, so far back as the year 2159 B.C., and that even in the year 2857 B.C. the study of astronomy and the desire of propagating a knowledge of that science amongst his people, were objects of great moment with the emperor Ion Hi; such at least is the doctrine supported in the *Histoire Générale de la Chine, ou Annales de cet Empire*, translated into French from Tong-Kien-Kang-Mou, by the Pere De-Mailla, a French Jesuit, one of the missionaries to Pekin.

We cannot attempt to enter here upon a refutation of the ideas supported in this work, relative to the antiquity of the Chinese astronomy; it will perhaps be sufficient to observe, that of 460 eclipses reported to have been predicted and recorded in the Chinese annals, the first is dated 2159 B.C., and the second, 776 B.C., leaving a great blank of 1383 years, during which no such phenomenon is noticed. That from the latter date to the year 1699 of our æra, they run on in pretty regular succession; but that of this number, the Pere Gaubil, who was at the labour of computing them, found only twelve which answered to the year, month, and day stated in the annals; and that of these twelve, only one of them was anterior to the time of Ptolemy, and even this one is doubtful.

It appears then that very little credit is to be given to the assumed antiquity of Chinese science; but perhaps the following abstract, which Delambre has made from the annals above referred to, will give the reader a better idea of the state of astronomy amongst this singular people, and place their pretensions in a more tangible form than any thing we can advance respecting the improbability of the notions advanced by Mailla.

In the year 687 B.C. we find recorded a night without stars, and without clouds; and that towards midnight, there fell a shower of stars, which vanished before they approached the earth.

141 B.C. The sun and moon appeared of a deep red colour, which produced great alarm among the people.

74 B.C. There appeared a star as big as the moon, followed by many other stars of the ordinary magnitude.

38 B.C. A shower of stones as big as nuts.

88 A.D. Another shower of stars.

321. Spots in the sun visible to the naked eye.

522. The astronomy of Hiuen-Chi-Ly was replaced by the astronomy of Tching-Kouange-Ly. (We do not understand these to be the names of astronomers but of systems.) In the year 892, another change is recorded; and again in 956.

In 949, in the fourth moon, there appeared a star in the mid-day, which was regarded as such a dreadful omen, that the people were forbidden to look at it; and many were put to death for disregarding the injunction.

These facts alone, independent of the superstitious fears and ceremonies, which even to this day are observed during the time of an eclipse, would alone give us a sufficient contempt for the astronomy of the Chinese; and lead us to reject as mere fables their pretensions to ancient observations. All that we can concede to them, and to the Indians, whose claims rest upon nearly the same grounds, is what we have already attributed to the Chaldeans, viz. that they had become, very early, observers of the motions and phenomena of the heavenly bodies; that they registered certain events, and thence were able to discover periods at which these phenomena would return again; sufficiently approximate to admit of their predicting an eclipse or occultation, within certain limits; but frequently, their prediction has been belied by the non-appearance of the expected phenomenon, while others have happened that were not foretold; which omissions, as well as a false prediction, have cost some few unfortunate astronomers the forfeiture of their heads: of which a particular instance is recorded in the case of Hi and Ho.

Upon the whole, therefore, we may conclude, that however ancient may be the rude observations of the Chinese and Indians, they possessed no science, properly so called, but what they obtained from the Greeks, through the medium of the Arabs; which people, after deriving it from the former source, carried it to Persia, whence it was transmitted to India and China. Such at least is the conclusion drawn by M. Delambre from a dispassionate examination of all the claims of these nations. We are aware that this notion is totally at variance with that of Bailly, who has also examined the case in point with great labour and attention, but certainly not without great enthusiasm and prejudice.

Astronomy of the Arabs.

Astronomy of the Arabs

In our last mention of this people, we saw them acting the part of the most savage barbarians, burning and destroying every thing, the most distantly connected with scientific research; we have now to exhibit them in a more honourable and dignified point of view. We stated, in the passage referred to, that some of the philosophers of Alexandria, escaped the vengeance of their barbarous conquerors, and these of course carried with them some remnant of that general learning, for which the school was so deservedly celebrated. Still, however, destitute of books, of instruments, and probably also of the means of subsistence without manual labour, very little farther knowledge could be accumulated, and still less propagated; so that in a few years, every species of knowledge connected with astronomy and mathematics, must have become extinct, had not the Arabians themselves within less than two centuries of the dreadful conflagration of the Alexandrian library, become the admirers and supporters of those very sciences they had before so nearly annihilated. They studied the works of the Greek authors which had escaped the general wreck, with great assiduity; and if they added little to the stock of knowledge these works contained, they became sufficient masters of many of the subjects to enable them to comment upon them, and to set a due estimation upon these valuable relics of ancient science.

Almansor. A.D. 754.

The destruction of the Alexandrian school occurred in the year 640; and it is about a century after, before we find any author worthy of particular notice amongst the Arabs; this, therefore, brings us to the middle of the eighth century; and from Ptolemy to this date there had passed away at least six hundred years, during which time the science had rather retrograded than advanced. The caliph Almansor, A. D. 754, is the first to claim our attention; but rather, perhaps, for the impulse, which, as a philosophical prince, he gave to the science amongst his people, than for any actual improvements which he had personally made. This impulse was, however, so great, that most of his successors seem to have thought it their duty to support and to study the different sciences, particularly astronomy.

Haroun. A.D 786

Almamon. A.D. 813.

Translate the Almagest.

Haroun, the grandson of Almansor, is particularly noticed as following in the steps of his great predecessor; and one of his sons, Almamon, pursued the same path with still greater enthusiasm. He caused to be translated all the Greek works that he could procure; and in particular the *Almagest* of Ptolemy. He is even said to have made the delivery of certain manuscripts deposited at Constantinople, one of the conditions of the peace which he concluded with the Greek emperor Michael III. He himself made numerous observations, and employed and instructed others to supply his place, when public business prevented him from his favourite pursuit. He ordered the obliquity of the ecliptic to be observed at Bagdad and at Dumas; whence it was found to be 23° 35′, which is less than some preceding observations had indicated.

Measures a degree of latitude.

Another important operation performed under the orders of Almamon, was the measure of a degree of the terrestrial meridian; but the unit of the Arabic measure in which it is expressed, is too uncertain for us to attempt to form from the reported result any decisive conclusion; nor are we to expect that any very great accuracy could be expected, seeing the great discrepancy between some modern measures made with the assistance of the most perfect instruments. Almamon also directed certain of his philosophers to compose a work for the purpose of facilitating the study of astronomy amongst his people, entitled, according to the Latin translation, *Astronomia elaborata a compluribus D. D. jussu regis Maimon*. which is still preserved in many libraries.

In the reign of this prince, there were many other celebrated Arabian astronomers, particularly Alfraganus, Thebit-Ibn-Chora, and Albatenius. The former composed a work, many editions of which have been made since the invention of printing, besides some other works more or less connected with this science.

Thebit was an annalyst, a geometer, and astronomer. He observed the obliquity of the ecliptic and reduced it to 23° 33′ 30″. He also determined the length of the year, very nearly the same as it is now established by modern observations.

Albatenius was one of the greatest promoters of Arabian astronomy. His numerous observations and important knowledge in all the sciences of his time, were the cause of his being surnamed the Ptolemy of the Arabs; an honour by no means ill merited. By a comparison of many of his own observations with those of Ptolemy and others, he corrected the determination of the latter respecting the motion of the stars in longitude, stating it to be one degree in 70 years instead of 100 years: modern observations make it one degree in 72 years. He determined very exactly the eccentricity of the ecliptic, and corrected the length of the year, making it consist of 365 days 5 hours 46 minutes 24 seconds, which is about 2 minutes too short, but 4 minutes nearer the truth than had been given by Ptolemy. He also discovered the motion of the apogee; and rectified the theories of Ptolemy respecting the motion of the planets, and formed new tables of them.

The works of this author have been collected, and published in two volumes 4to., under the title of *De scientia stellarum*, of which there are two editions, one in 1537, and the other in 1646.

Montucla, in his valuable history of mathematics, enumerates a long list of Arabian astronomers which followed Albatenius; but we meet with none deserving of particular notice till we arrive at Ebn Iounis, who wrote in the year 1004, and even he is rather celebrated for his having collected and embodied the knowledge of his time, than for his discoveries, although he made numerous observations. The work of this author is still extant, a concise notice of which Delambre has given in the *Mém. de l'Institut*, vol. 2. p. 5, where we learn, that it contains 28 observations of eclipses of the sun and moon, made between the years 829 and 1004; seven observations on the equinoxes; one on the obliquity of the ecliptic: and some others highly important in the determination of certain data, particularly as regards the acceleration of the moon.

Let us now turn our attention to Spain, where the Arabs, who had long been masters of that country, pursued the sciences with the same ardour as in the east. The most distinguished, however, of the astroel. 020.

nomers in this country were Arsachel and Alhazen; the former of whom is celebrated for having added greater accuracy to the theory of the sun, by employing a principle different to that of Ptolemy and Hipparchus, and susceptible of more accuracy. He made some fortunate changes in the dimensions of the solar orbit, and discovered certain inequalities in the sun's motion, which have since been confirmed by the Newtonian theory of gravitation.

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Alhazen is also esteemed a philosopher and astronomer of high reputation; he is said to have first discovered the laws of refraction, and the effect of it in astronomical observations. He explained the phenomenon of the horizontal moon, and indicated the true cause of the crepuscula in the morning and evening; beside various other minor discoveries highly honourable to his memory.

From this time the science of the Arabians seems to have begun to decline; we meet with very little after this period deserving of particular notice; a general shade appears to have been cast over every species of human knowledge, and nearly four hundred years are again lost in darkness and obscurity. We then find the Greeks making some feeble efforts to re-establish astronomy in its original empire, where some faint glimmerings of the genius which animated Archimedes, Hipparchus, and Ptolemy, once more began to discover itself; but which, alas! like the lustre of a passing meteor, was soon extinguished, and darkness and barbarity once more assumed their reign.

Astronomy of modern Europe.

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We may without impropriety refer the revival of modern astronomy to the time of Copernicus, although he was preceded by some others, who prevented all traces of the Grecian and Arabic science from being lost and forgotten, by their reading and studying such works as had been preserved, during what are commonly denominated the dark ages. Copernicus was born at Thorn, in Poland, in the year 1473, but he did not commence his studies till about the year 1507; when after having well "fathomed every depth and shoal" of the ancient doctrine of astronomy; made numerous observations, and various comparisons; he became at first a convert, and afterwards the most strenuous advocate of a system of astronomy, commonly attributed to Pythagoras, and which we have seen Ptolemy using so many ingenious but false arguments to refute.

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According to Copernicus, the sun is placed in the centre of the planetary world, about which the several bodies of our system revolve from west to east; viz. 1st, Mercury, 2d, Venus, 3d, the Earth, 4th, Mars, 5th, Jupiter, and 6th, Saturn; the moon revolves about the earth in the same direction, while the latter body itself is carried in its orbit round the sun. In the next place, he taught that the earth turns on its own axis from west to east, in a little less than 24 hours, and that this axis is always preserved parallel to itself, making an angle of about 23½° with the ecliptic.

The orbits of the several planets he supposed to be circular; but he did not make the sun their common centre. We have seen that what was anciently called the solar orbit had been long known to be eccentric, as well as those of the planets; and to account for the phenomena produced by these eccentricities, Copernicus rendered them still greater, by giving to each a different centre, and to the sun such a position with regard to them all, as with the addition of certain epicycles, best agreed with the appearances already observed. This part of the Copernican system is not often well illustrated in our elementary works on astronomy, where it is generally asserted, that the sun was placed in the common centre, and the several planetary orbits were circles concentric with it; by which means, the discoveries of Kepler are rendered more astonishing than they actually are; for he had at least this much to proceed upon; too little, it is true, for any man possessed of less genius and perseverance than himself. This is by no means the only service which Copernicus bestowed upon astronomy and trigonometry, but it is the principal one, and that which has added most to the celebrity of his name; we shall therefore not stop to report his other discoveries and improvements, as we shall find numerous important points to refer to before we conclude this sketch, already considerably extended. The work containing his new doctrine was composed in the year 1530; but it was not printed till the year 1543; the author having, it is said, received the last sheet a few hours before he expired.

Copernicus was followed by a great number of excellent astronomers, some of whom were firm supporters of his system; others endeavoured to refute it, as Ptolemy had formerly done a similar doctrine; while some few who saw its beauties and advantages, but who by giving too literal a signification to some scriptural passages, wished to modify it so as to retain as many as possible of its advantages, while the system should still be such as to correspond with the passages in question.

Tycho Brache died 1601.

Of this number was Tycho Brache, a noble Dane, one of the greatest observers perhaps, if we except Kepler, that ever lived. His system consisted in depriving the earth of its orbicular and diurnal motion; he places the earth in the centre, and made the moon and sun revolve about it, agreeably to the doctrine of Ptolemy; but he made the sun the centre of the other planets, which, therefore, he supposed to revolve with the former about the earth. By this means, the different motions and phases of the planets may be reconciled, the latter of which could not be by the Ptolemaic system; and he was not obliged to retain the epicycles, in order to account for their retrograde and stationary appearances. This theory was extremely complicated, and did not long survive its author. Many of his other labours were attended with much more important consequences; particularly his discovery of the variation and annual equation of the moon; the greater and less inclination of the lunar orbit; the correction for refraction in astronomical observations; his astronomy of comets; his account of the appearance and disappearance of a great star which happened in his time: his reformation of the calendar, and some other subjects which might be enumerated: these have contributed most to establish his name as a great astronomer, and will not fail to hand it down to the latest posterity.

His system.

Other discoveries of Tycho

Kepler died 1631.

Kepler was about twenty years younger than Tycho, but by no means inferior to him in genius and perseverance; the latter quality is proved by the numerous

His observations on Mars.

observations that he made on all the planets, particularly on Mars; and the former from the memorable consequences which he drew from them. Kepler first determined the line of the apsides by a method independent of the form of the orbit of Mars, and ascertained the ratio of the aphelion and perihelion distances; the former he found to be to the latter, as 166,780 to 138,500; or calling the distance of the earth from the sun 100,000; the above numbers will express the actual distances. Hence the mean distance of Mars was 152,640, and the eccentricity of its orbit 14,140. He then determined, in like manner, three other distances, and found them to be 147,750; 163,100; 166,255. He next computed the same three distances upon a supposition that the orbit was a circle, and found them to be 148,539; 163,883; 166,605; the errors, therefore, of a circular hypothesis, were 789, 783, 350. But he had too good an opinion of Tycho's observations, (upon which he founded all these calculations), to suppose, that the differences arose from their inaccuracies; and as the distance between the aphelion and perihelion was too great according to the hypothesis in question, he conceived that the orbit must be an oval. And, as of all ovals the ellipse is the most simple, he naturally made trial of this figure, placing the sun in one of its foci; and upon making the requisite calculation, he found the agreement complete. He did the same for other points of the orbit, and still found the same accurate agreement; and hence he pronounced the orbit of Mars to be an ellipse, and that the sun occupied one of its foci. Having established this important point for the orbit of Mars, he conjectured the same to have place in the other planets; and upon trial he found his conjecture fully verified: and, hence, he concluded, that the six primary planets revolve about the sun in elliptic orbits, that body occupying one of the foci.

Kepler's first law.

Kepler's second law.

Having thus discovered the relative mean distances of the planets from the sun, and knowing their periodic times, he next endeavoured to find, whether there were any relations between them; and having naturally a strong turn for numerical analogies, he began by comparing the powers of those quantities with each other; and even in the first instance (March 8th, 1616,) he assumed the correct law; viz. *that the squares of the times of revolution are proportional to the cubes of the mean distances;* but, in consequence of some error in his calculation, his comparison did not appear to be complete. Nor did he discover it till the following May. He then found his first conjecture to be correct, and thus established this celebrated law, which Newton afterwards demonstrated to be the necessary consequence of a body revolving in an orbit about a central attracting point coinciding with one of its foci. *Prin. Phil.* lib. i. sec. ii. pr. 15.

Kepler's third law.

Kepler also discovered from observation, that the velocities of the planets, when in their apsides, are inversely as their distances from the sun; whence it followed, that they described at these points equal areas in equal times; and although he could not prove the same for every point of their orbits, he had still no doubt that it was so. He therefore applied this principle to determine the equation of the orbit; and finding that his calculations agreed with observation, he concluded that it was true in general; "that the planets describe about the sun equal areas in equal times." This discovery was perhaps the foundation of the *Principia*, probably suggesting to Newton the idea, that all the planets of our system were governed by one general law! and that the sun was the general focus of action; a proposition which he afterwards succeeded in demonstrating, and thus founded the basis of physical astronomy.

Kepler also speaks of *gravity* as a power which is mutual between all bodies; and tells us that the earth and moon would move towards each other, and meet at a point so much nearer the earth than the moon, as the former is greater than the latter, if these motions did not prevent it. And he farther adds, that the tides rise from the gravity of the waters towards the moon.

Inve of th scope

The beginning of the 17th century was distinguished by two of the most important events for astronomy that we have yet recorded; viz. the invention of the telescope and logarithms; by means of the one, we are enabled to penetrate into the remotest parts of space, and to bring under our immediate view phenomena, which the most sanguine minds could never, if they had suspected their existence, have hoped to bring within the limits of human observation: by means of the other, all the laborious calculations of former astronomers were reduced to mere operations in addition and subtraction; and what was before the labour of a month, became, as it were, the amusement of an hour. With two such powerful auxiliaries, the progress of astronomy could not but be extremely rapid; but still the extent to which it has been since carried, must certainly far have exceeded what the boldest of the astronomers and philosophers of that day could have dared to hope for. It never could have been contemplated that this science, apparently so far beyond the reach of all human power, would become the most perfect of all the physical sciences; that every disturbing force, and every celestial phenomenon of our system, should be submitted, with the greatest precision, to one general simple principle, at that time wholly unknown; and that an analysis would be discovered, to enable us to investigate and compute the motion of bodies of immense magnitudes, and at almost immeasurable distances, with greater accuracy than we can calculate the motion of a projectile from a piece of ordnance. Yet such is actually the case. We know, to a greater nicety, the moment when a planet will arrive at a certain point in the heavens, than we can tell the time that a cannon ball will employ in passing from the gun to the extremity of its destined range, the moment of explosion being given.

Origin physic astrono

As far as we have hitherto traced the progress of astronomy, all our knowledge has been supposed to consist in observations on the motions of the heavenly bodies; and every species of computation relating to it, rested on no other foundation. We are now arrived nearly at that period, when our illustrious Newton, by his discovery of the law of universal gravitation, added an entire new branch to this important science, commonly denominated physical astronomy. From this time a greater range was given to celestial researches; our knowledge was not confined to the mere observation of phenomena, and the return of certain bodies to certain parts of space: the causes of these phenomena, and of these motions, were now to be investigated, and the amount of them computed from the pure principles of celestial mechanics. Every minute inequality was henceforward to be traced to the same source—the action of gravitation: while, on the other hand, every minute variation indicated by this theory, ought to be observable in the heavens. And nothing, perhaps, is better calculated to demonstrate the generality of this law, and the beauty and profundity of the modern analysis, than the fact, that certain small inequalities were indicated by the theory, before the accuracy of instruments, and the delicacy of observation, were sufficiently attained to render them appreciable; but the existence of which, the present high state of practical mechanics, and a corresponding improvement in optics, have confirmed in the most satisfactory manner. We shall, therefore, now divide the remaining part of this historical sketch under two distinct heads; viz. practical and physical astronomy; and slightly glance at the successive steps that have been made in each; but a simple indication of them is all that can be expected, and indeed it is all that is requisite, since we must necessarily meet the same subjects again in the course of the following treatise; where we shall enter upon them as much at length, as is consistent with the limits allotted to this department of our work.

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ope.

Our business here, is neither to give the history of the invention of the telescope nor of logarithms; the former has already been treated of in our treatise of optics, and the other will be found in its appropriate place in this work. We have seen in the article above referred to, that the telescope is perhaps an earlier invention than it is commonly supposed to be; and that Harriot, in a very early part of the 17th century, viz. between the years 1610 and 1613, had actually observed the spots on the sun's disc with telescopes of various magnifying powers; a fact, which has been but lately ascertained by Baron Zach, of Saxe Gotha, who, in a visit to this country, had access to certain MSS. of this English author, in the possession of the Earl of Egremont, a descendant of the Earl of Northumberland, who was Harriot's patron. This, after Galileo's, is one of the earliest well attested facts of the application of this instrument to astronomical purposes.

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1610.

Galileo, as we have seen in our history of optics, was informed of the accidental discovery of Jansen's children, in the year 1609; and on the 8th of January, 1610, he perceived, by directing his new instrument to the heavens, three small stars near the planet Jupiter; two on the one side, and the third on the opposite side; which he took at first for fixed bodies. But having continued to observe them on the next and the following nights, he found that they changed their places and positions; and that they performed their revolutions about that planet, and were, in fact, satellites to it, as the moon is to the earth; and some days after he discovered a fourth. These four satellites he named, in honour of the house of Medici, the Medicean stars; but the appellation was soon lost in the more general and appropriate name of Jupiter's satellites. This discovery was published in the month of March following, in a writing, entitled *Nuncius Sydereus;* he undertook even to investigate the theory of their motion; and, in the year 1613, to predict their configurations for two consecutive months.

Irregularity in the form of Saturn.

Directing his telescope to Saturn, a new surprize, and new pleasure presented itself. After a few imperfect observations on this planet, he was led to believe that Saturn was not a simple globe like the other bodies of our system, but was compounded of three stars touching each other, immovable with regard to themselves, and so disposed, that the largest occupied the centre, with a smaller one on each side of it; but it was not long before he discovered this idea to be erroneous, so far at least as regarded their immobility, or invariable appearance: he found that the figure was changeable, but his telescope had not sufficient power for him fully to unravel the mystery; he found that Saturn appeared irregularly formed, and supposed the extreme parts, of what was afterwards found to be a ring encompassing the body of the planet, were attached to it, forming *ansæ*, or handles; and it was some years after, that Huygens discovered the actual conformation of this beautiful telescopic object. Galileo also first observed the phases of Venus, predicted by Copernicus; as he did also the spots on the sun's disc, unless indeed our countryman Harriot had the advantage of him, in point of time, in this observation. There are others who contest with Galileo for the honour of the important discovery of Jupiter's satellites, as Simon Marius, of Brandebourg, who asserts, that he observed them in 1609; but the truth of this seems to rest on no better foundation than the mere declaration of the author, in his *Nuncius Jovialis, anno* 1609, *detectus,* &c. published in 1614. The spots on the sun were also observed by Fabricius and Scheiner; and it is perhaps doubtful by whom they were first seen. The telescope soon became well known in all parts of Europe; the sun was an object which would naturally claim the regard of astronomers, and it is not unlikely these spots might be observed by many at nearly the same time.

Phases of Venus, and spots on the sun.

Other claims for these discoveries.

Galileo defends the Copernican system.

The system of Copernicus, even before these discoveries, had made considerable progress; and what had now occurred, had rendered the truth of it still more certain. Galileo, therefore, in 1615, had the courage openly to support it; but it caused him a reprimand from the *Holy Inquisition;* he was imprisoned, but afterwards liberated upon certain conditions of silence. Twenty years afterwards, greater advances being made in the progress of this science, he again ventured to assert his opinion, but still with the same effect: he convinced all unprejudiced people; but the holy fathers were not in this class, and he was once more obliged to abjure upon his knees his heretical doctrine. Such are the fruits of ignorance blended with bigotry and superstition.

Huygens discovers Saturn's 6th satellite.
A.D. 1635.

We shall deviate a little in the order of time, in order to bring together those discoveries which have added to the number of the bodies of the solar system. Huygens, therefore, according to this arrangement, is the first to claim our attention. This eminent philosopher, having in 1635 constructed for his own use two excellent telescopes, the one of twelve, and the other of twenty-four feet, discovered a satellite to Saturn, which is now the sixth in the order of distances, and determined the dimensions of its orbit, the duration of its revolution, &c. with an astonishing degree of exactness; but falling into a metaphysical error, concerning the number of the bodies of our system, he sought to find no more, considering that

Discovers Saturn's ring.

this one was all that was wanting to render the whole planetary scheme complete. He was drawn, however, to a minute examination of Saturn itself, and was the first to announce the actual conformation of it. He stated, that this planet was encircled by a flat broad circular ring, detached from it on all sides, but which, according to the position under which it was observed from the earth, would assume the several appearances of a circle, an ellipse, and a right line; he never, however, observed it under its latter form. He discovered that the diameter of the exterior edge of the ring, was to the diameter of the planet as 9 to 4; and that its breadth was equal to the space contained between the globe and its interior circumference; results which, with some slight modifications, have been confirmed by more modern observations.

Cassini discovers four other satellites, A.D. 1684. Herschel two others.

Four other satellites to this planet were discovered by Dominique Cassini: which, in the order of distances, are now the 3d, 4th, 5th, and 7th; the 5th and 7th in 1671, and the 4th and 5th in 1684; two others have since been added by Sir W. Herschel in 1789; making in all seven attendant luminaries to this remote and otherwise solitary planet.

Herschel discovers the Georgium Sidus.

Eight years prior to the above discovery, viz. in 1781, Sir W. Herschel had observed a small star, which after a little attention, he found changed its place; having well ascertained this fact, he communicated a knowledge of the circumstance to M. Laxel, a celebrated astronomer of the academy of St. Petersburgh, who was then in London; the same information was also transmitted to other eminent astronomers, who observed it with great care, and soon afterwards it was announced as a new planet, the most remote in our system, circulating about the sun at the astonishing distance of nearly *eighteen hundred million miles*, and performing its orbicular revolution in about 80 of our years. This new planet was first named by foreign astronomers after its observer, *the Herschel*; but Herschel himself, in imitation of Galileo, dedicated it to his late Majesty, under the name of the *Georgium Sidus*; both these appellations are, however, now nearly become extinct, that of *Uranus* being almost universally adopted. The same indefatigable observer has since discovered six satellites to this planet, which revolve about him under very peculiar circumstances; but the particulars must be reserved for the proper place in the subsequent treatise.

Discovers six of its satellites.

In order to complete this part of our historical sketch, we must now pass to the beginning of the 19th century; the *first day* of which is remarkable for the discovery of another new planet, between the orbits of Mars and Jupiter; this we owe to the observation of Piazzi. This planet has received the name of Ceres. Another new planet was discovered by Dr. Olbers, on the 28th of March, in the same year, 1801, which is called *Pallas*; its distance and periodic revolution being nearly the same as that of the former. A third, having likewise nearly the same mean distance, was discovered by M. Harding, at Lilienthal, near Bremen, September the 4th, 1804; and a fourth by Dr. Olbers, on the 19th of March, 1807, being the second we owe to this indefatigable observer. Of these two, the former has been named *Juno*, and the latter *Vesta*.

Four other newplanets. Ceres, by Piazzi.

Pallas, by Olbers.

Juno, by Harding.

Vesta, by Olbers.

For the elements, and other particulars respecting these new planets, we must refer the reader to the respective articles in the following treatise.

Uniting together these several discoveries, it will appear, that in less than two centuries, there have been added to the known bodies of our system, no less than five planets, and seventeen satellites, about three times as many as were known at the time of the promulgation of our present system by its venerable author Copernicus.

In the paragraphs immediately preceding, we have in some degree disregarded the order of time, for the purpose of bringing under one point of view those discoveries which have increased the number of the bodies in the solar system; we must now, therefore, retrace our steps, in order to glance at some other facts connected with the history of this science; but the bare enumeration of them is all that our limits will allow us to attempt.

In 1603, Bayer formed a catalogue of the stars, which he published under the title of *Uranometria*, a highly important and useful work.

Bay talog the s A.D.

Nov. 7, 1631, Gassendi observed the passage of Mercury over the solar disc, agreeably to the prediction of Kepler; and published his account of it in 1632, in a work entitled *Mercurius in Sole visus*, &c.

Tran Merc A.D.

1638. The transit of Venus was observed over the sun by Horrox, a young English astronomer, which is described in his *Venus in Sole visus*, &c.

Tran Venu A.D.

These are the two first observations of this kind that had ever been made, and much importance was in consequence attached to them, although their great utility in establishing certain astronomical data was not then foreseen.

In 1638, the sciences had to deplore the loss of Hevelius, a celebrated astronomer and senator of Dantzic, to whose indefatigable labours we owe many valuable observations; but our limits will not permit of our entering into particulars.

Heve died. A.D.

We pass now to that period when the Royal Society of London, and the Academy of Sciences at Paris, were first instituted, and the observatories of Paris and Greenwich were erected. The advantage of these institutions to the science of astronomy, is unbounded; instruments of the best kind that could be obtained, were immediately constructed: the members of the two academies communicated with each other; various experiments were proposed and executed; and an impulse was thus given to science in general, and to astronomy in particular, which it would have been in vain to have expected from individual perseverance and talent, however conspicuous.

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The charter of the Royal Society was granted by Charles II. in 1660; that of the Academy of Sciences by Louis XIV. in 1666. In 1667, the observatory of Paris was erected; and the first stone of the observatory of Greenwich, was laid by Flamstead (who was appointed astronomer royal) on the 10th of August, 1675, at the recommendation of Sir Jonas Moore, to whose influence we are indebted even for the institution itself. Flamstead continued to fill this situation in a manner equally honourable to himself and his country, for 43 years; he was succeeded by Dr. Halley, who continued in it for 23 years; Dr. Bradley followed Dr. Halley, and held the office for twenty years: Mr. Bliss only remained two years, and was followed by Dr. Maskelyne, who died in 1811, after holding it for 46 years.

Englis tronom royal.

Never has any appointment been more honourably filled than that of the astronomer royal of England. Of the names which we have above enumerated, four of them will be handed down to the latest posterity; but we shall only have occasion to refer to Halley and Bradley, because we can only notice the more prominent features of the history of this science; and the observations and results of the first and last of those great men, although of infinite advantage to the attainment of accurate knowledge, and would therefore form an important part in a complete history of astronomy, are still not of that striking nature to claim more than a general notice in this place.

Halley.

Dr. Halley, who was equally celebrated as an analyst, a geometer, and astronomer, very early distinguished himself by a geometrical method of determining the apsides, the eccentricities and dimensions of the principal planetary orbits; he afterwards undertook a voyage to the island of St. Helena, for the purpose of forming a catalogue of the stars in the southern hemisphere; he projected the method which was afterwards put in practice for observing the transit of Venus in 1761 and 1769, for determining the parallax of the sun; and predicted the return of a comet in 1759, fixing the period of its revolution at 75 years; this comet, which is the only one whose orbit is known, and which is expected to reappear in 1834, bears his name, as an honourable testimony of the truth of his prediction, and the profundity of his knowledge. Many other valuable works of this author we must necessarily pass over.

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. 1667.

In 1667, a highly important discovery was made by Roemer, a Danish mathematician, at that time resident at Paris. He had long been engaged in making very accurate observations on the motion and eclipses of Jupiter's satellites; in the course of which he noticed that, at certain times, these bodies emerged from the shadow of the planet some minutes later, and at others, as much before the time given by the most accurate tables. By comparing these variations with each other, it appeared, that the satellite emerged too late from the shadow, when the distance between the earth and Jupiter was the greatest, and too soon when that distance was the least; whence, after some conjecture, he hit upon the happy idea, that as the apparent emersion happened too late or too soon, according as the planet was more remote or nearer to us, it must proceed from the time that light employed in passing over the difference in the two distances; in fact, that the propagation of light is not instantaneous; but that it requires a certain time to pass from one point to another; and submitting his ideas to calculation, it appeared, that a luminous ray employs about eleven minutes in describing a distance equal to that of the earth from the sun. This bold idea he communicated to the Academy of Sciences on the 16th of November, 1667: it has since been confirmed with some modification, reducing the time to 7¼ minutes, and has immortalized the name of Roemer.

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Although the system of Copernicus had now gained a complete ascendancy, yet many persons were inclined to imagine, that if it were true, some parallax ought to be observed in the fixed stars: that these bodies should be at such an immense distance, that a base of nearly *two hundred millions of miles*, the diameter of the earth's orbit, should not sensibly alter their positions, was, it must be allowed, a doctrine sufficient to startle those who could not expand their minds to a contemplation of the infinitude of celestial space; and, accordingly, various observations were made with a view of ascertaining whether or not such a parallax had place; and instruments had now arrived at that degree of perfection, that it was thought by many eminent astronomers such an effect ought to be rendered appreciable.

The minuteness and accuracy of these observations, did, indeed, show a small change in the relative position of some stars, but it was, generally speaking, directly contrary to that which ought to result from a parallax. This motion the cause of which was unknown, was denominated *aberration*; and it was this which Dr. Bradley undertook to examine, and endeavoured to reduce to a general law. In the prosecution of this design, he found that certain stars appeared to have, in the course of a year, a sort of vibration, in longitude, without changing their latitude; some varied only in latitude, while others, and that the greater number, appeared to describe in the heavens, in the course of the year, a small ellipse, more or less elongated. This period of a year to which these variations answered, although so different from each other, was a certain indication that they were connected with the annual motion of the earth in its orbit about the sun; and after a time, he fortunately perceived the cause of all the irregularities he had observed. He attributed the apparent aberration of the fixed stars to the combined motion of the earth, and that with which light is propagated.

Dr. Bradley
explains the
cause of it.
A.D. 1736.

Roemer had shown that the velocity of light is about 10,000 times greater than that of the earth in its orbit, therefore, a ray of light issuing from a star, will not carry the impression of this star to the eye, till after the earth has sensibly changed its place; and consequently, when the eye receives the impression, it ought necessarily to refer the object to a different point in the heavens to that in which it is actually placed, or to that in which it would appear, if the earth were at rest. This explanation satisfied every doubt on the subject, and amounted to nearly a mathematical demonstration of the truth of the Copernican system.

Nutation of
the earth's
axis.

We owe to this celebrated astronomer, another discovery no less important than the above, viz. the nutation of the earth's axis. The celestial mechanics of our illustrious Newton had shown, that the unequal attractions of the sun and moon on the different parts of the terrestrial spheroid, ought to produce a variation in the position of its axis as referred to the plane of the ecliptic. Bradley undertook to examine the effect of this motion by means of a long series of delicate observations, made in those positions of the sun and moon most proper to render them manifest. The result of his researches were,—1. That the axis of the earth has a conical motion, by which its extremities describe about the pole of the ecliptic, and contrary to the order of the signs, an entire circle in 25,900 years, or an arc of about 50″ annually. This explains the cause of the precession of the equinoxes. 2dly, That this axis has, with reference to the plane of the ecliptic, a libration, by which it inclines about 18″ in the course of one revolution of the nodes, and is also made contrary to the order of the signs; after which it returns to its first position, inclines again, and so on, in each successive period of nineteen years. Such are the discoveries which render the name of Bradley immortal in the history of this science. We are also highly indebted to him for numerous other important results, to which we shall have occasion to refer in the course of the following treatise.

There still remains one department of practical astronomy, which requires to be briefly noticed in this sketch, viz. the measurement of the terrestrial circumference; which is important, because the terrestrial radius, in all our actual determination, is the only unit of measure to which we can have recourse, in order to reduce the several distances of the heavenly bodies to known measures: but as this is also an important datum in the physical branch of astronomy, we shall defer the consideration of those measurements, till we have to consider it under the latter point of view.

Progress of physical astronomy.

Dr. Hook's ideas of gravitation. A.D. 1668.

We have seen in our account of Kepler, that he had formed some general ideas of the nature of universal gravity; but it was too vague to become the foundation of any mechanical principle. Dr. Hook, an English philosopher of a most extraordinary genius, had made a much nearer approximation to a development of this great law of the universe. In one of his communications to the Royal Society, May 3d, 1668, he expressed himself as follows:—"I will explain a system of the world very different from any yet received. It is founded on the three following positions:

"1. That all the heavenly bodies have not only a gravitation of their parts to their own proper centres, but that they also mutually attract each other within their spheres of action.

"2. That all bodies having a simple motion will continue to move in a straight line, unless continually deflected from it by some extraneous force, causing them to describe a circle, an ellipse, or some other curve.

"3. That this extraction is so much the greater as the bodies are nearer. As to the proportion in which those forces diminish by an increase of distance, I own I have not yet discovered it, although I have made some experiments to this purpose. I leave this to others who have time and knowledge sufficient for the task."

This is a very precise enunciation of a proper philosophical theory. The phenomenon of the change of motion, is considered as the mark and measure of a change of force, and his audience is referred to experience for the nature of this force: he having before exhibited to the society a very neat experiment, contrived to shew the nature of it. A ball, suspended by a long thread from the ceiling, was made to swing round another ball, laid on a table immediately below the point of suspension. When the impulse given to the pendulum was nicely adjusted to its deviation from the perpendicular, it described a perfect circle round the ball on the table; but when the impulse was very great, or very small, it described an ellipse, having the other ball in its centre.

Hook shewed that this was the operation of a deflecting force proportioned to the distance from the other ball; and he added, that although this illustrated the planetary motions in some degree, yet it was not suitable to their case; for the planets describe ellipses, having the sun, not in their centre, but in their focus. Therefore, they are not retained by a force proportional to the distance from the sun.

The exalted genius of Newton can suffer no diminution by the enumeration of the above opinions; for though the idea of such a principle as gravitation was not suggested first by Newton, yet so very obscure were the notions of even the most enlightened philosophers on this subject, that it had never been successfully applied to the explanation of a single astronomical phenomenon.

The important discovery of the law of universal gravitation is so intimately connected with the history of philosophical science, that every circumstance relating to it has been recorded with the greatest care. Dr. Pemberton relates, that Newton, in the year 1666, having retired from Cambridge to the country on account of the plague, was led to meditate on the probable cause of the planetary motions, and upon the nature of that central force which retained them in their orbits; when it occurred to him, that the same force, or some modification of the same force, which caused a heavy body with us to descend to the earth, might likewise retain the moon in her orbit, by causing a constant deflection from her rectilinear path. But before this could be submitted to the test of computation, it was necessary that some hypothesis should be formed relative to the modification of its action with respect to distance; and probably, that which has actually place, namely, that it is reciprocally as the square of the distance, almost immediately suggested itself to his mind, as being observed in all kinds of emanations with which we are acquainted.

When Newton first attempted to verify this conjecture, the requisite data with regard to the distanc of the moon in terrestrial radii, and the measure of th radius itself were but imperfectly known; the resu therefore which he obtained, though nearly, did n accurately agree with observed phenomena; and he consequence abandoned his theory as untenable: remarkable instance of the cool and dispassiona frame of mind which this great philosopher observe even at the moment when he flattered himself with t idea of having discovered one of the most importa secrets of nature.

Sometime afterwards, however, he was induced renew his calculations; as, in the interval, mo correct data had been obtained by the measurement Picard, in France. This attempt succeeded; and he is stated to have been extremely agitated towards the conclusion of his calculation. Whether this were the case or not we cannot attempt to settle: at all events, a moment of greater interest will never be recorded in the annals of science.

Prin of c tion.

Let us briefly illustrate the nature of the calculation to which we have referred. Our author having established as a datum, that the distance of the moon from the earth, was about 60 semi-diameters of the latter, that is, 60 times as far from the earth's centre as a heavy body placed at its surface; and assuming the power of gravity to decrease inversely as the square of the distance, it would follow, that at the moon, this force would only be one 3600th part of what it is at the surface of the earth; and, consequently, the space passed over by a body at this distance, when submitted to the terrestrial gravitation, would be only $\frac{1}{3600}$ of that which is actually described here; or since the in our treatise on physical astronomy, we shall not detain the reader with the nature of the particular solutions and investigations of the authors above alluded to; but shall pass on to the only remaining topic—the figure and magnitude of the earth.

Magnitude and figure of the earth.

We have seen that this problem had been attempted by Eratosthenes, and again by the Arabian mathematician Almamon, but we know very little of their determinations; we may however easily imagine, knowing how great the difficulty of the operation is, even with the most perfect instruments, that their approximations must necessarily have been very remote from the truth; and probably the early attempts of this kind in Europe were not attended with much better success. We shall briefly notice those which seem to be of the greatest importance.

Different measures.

In 1525, M. Fernelius measured a degree of the meridian northward from Paris, which he made 68·7634 English miles. Snellius, professor of mathematics at Leyden, and our countryman, Norwood, measured each of them a meridional degree; the former in Holland in 1620, and the latter in England between London and York, in 1635. Snellius' measure, when reduced to English miles, is 66·91; and that of Norwood 69·545 miles. In 1644, Riccioli also undertook a similar task, and performed it according to three different measures, between mount Parderno and the tower of Modena in Italy, and obtained a mean length of 75·066 English miles to a degree.

Figure of the earth.

These results differed too much from each other to inspire the least confidence in any of them, and consequently nothing could thence be deduced respecting the figure of the earth; nor was this indeed, at that time, suspected to be any other than a perfect sphere, abstracting from the irregularities on its surface. But after the construction of the telescope had received considerable improvement, and when observations had been reduced to greater nicety, it was found that the planet Jupiter was considerably flattened at his poles; and the pendulum experiment of Richer, 1671, had shown that there was a difference in the action of gravity at the equator and in the latitude of Paris; and these two circumstances probably first suggested to Huygens that the earth was not spherical; and its rotatory motion about its axis naturally led him to conclude that it was flattened at its poles; from the combination of its centrifugal force with that of gravity, he calculated that its polar axes were to its equatorial diameter as 578 to 579. But as he regarded all the power of terrestrial attraction to be collected in the centre of the earth only, his solution was obviously defective; and Newton soon after undertook the same determination, on more correct principles, by supposing every particle in the whole mass to have a reciprocal attraction towards all the others; from which he found the figure to be an ellipsoid, having its polar and equatorial diameters to each other at 229 to 230.

Determined by Newton and Huygens.

Determined by practical measurements.

In this state the question remained for many years, till M. Picard undertook the measurement of a degree in France, in 1669, which was afterwards revised by Cassini in 1718; the result of which tended to show, that the earth was not an oblate but a prolate spheroid. Picard obtained for the length of a degree 68·945 English miles, and Cassini 69·119; whereas the latter, being the most southern, ought to have been the shortest. A circumstance so unexpected as this, naturally produced a great curiosity, and a considerable degree of inquiry and controversy between the astronomers and mathematicians of that period; and the French government, at the recommendation of the Academy of Sciences, in 1735, sent out two companies of mathematicians to determine the point in question, by measuring two degrees, one at the equator, and the other in as high a northern latitude as possible. Accordingly, MM. Godin, Bouguer, and Condamine, from France, with Dons Juan and Ulloa, from Spain, proceeded to Peru; while Maupertuis, Clairaut, Camus, La Monnier, &c. accompanied by Celsus, a Swedish astronomer, proceeded to Lapland.

After experiencing many unforeseen difficulties and delays, both parties accomplished the object of their missions, and returned to France; those from Lapland in 1737, and the other division in 1744. The former made the length of their degree, the middle point of which was in latitude 66° 20′, equal to 69·403 English miles; while at the equator the length, as determined by the other party, was found to be 68·724 English miles, taking the mean of three different results, deduced from the same operation. In the interval between the outfitting of the above expeditions, and the publication of their results, the degrees of Picard and Cassini were examined, recomputed, and found now to be, the first, whose middle point was in latitude 49° 22′, 69·121 English miles, and other 69·092 miles, its middle point being 45°.

The results therefore of all these measures confirmed the earth to be an oblate spheroid; but as they were compared together in pairs, they gave very different degrees of ellipticity; nor has all the accuracy that has since been introduced into these operations been sufficient to establish this point. Colonel Mudge in England, has measured an arc extending from the southernmost point in the kingdom to the northernmost of the Shetland islands; and Messrs. Delambre, Mechain, Biot, Arago, &c. have carried another from Dunkirk to Fomentara, one of the Balearic isles. It is perhaps never to be expected that greater accuracy can be introduced into any operations, nor to see greater talents employed in conducting them, than in the cases to which we have last referred; and yet these two measures, compared with each other, in different portions, give very different ellipticities; different parts of the same arcs give very discordant results; even some of those of the English survey, are such as to lead again to the idea of the prolate figure of the earth. What are we then to conclude from the whole of these deductions, but that the irregularities of local attraction, or some other causes which we cannot discover, produce a certain influence or disturbing power, which renders useless the comparisons of small arcs, and that those results only can be depended upon that are drawn from measurements that extend through several degrees of latitude. Adopting this principle, we may state, that the mean length of a degree in the latitude 45° is 68·769 English miles, and that the ellipticity of the earth is between $\frac{1}{307}$ and $\frac{1}{314}$.

One other point still requires to be alluded to before we conclude our history. It was observed by Bouguer, in his operations in Peru, that the high mountains in some parts of that country, very sensibly disturbed the verticality of his plumb-line; that is, the lateral attraction drew the line out of its per-

Number of stars visible to the naked eye.

8. It appears from the preceding summary of the stars in the British catalogue, that the number of them in the entire celestial sphere, including all those of the sixth magnitude, does not much exceed 3000; and it is generally stated, that not more than 1000 are ever visible at one time to the naked eye: but when a telescope is employed, their number appears to increase without any other limit than that of the perfection of the instrument. Sir W. Herschel has observed in the *milky way* above 10,000 stars in the space of one square degree. This luminous track

Milky way.

> "Which nightly, as a circling zone, thou seest
> Powder'd with stars,"

encompasses the heavens, and forms nearly a great circle of the celestial sphere. It traverses the constellations Cassiopeia, Perseus, Auriga, Orion, Gemini, Canis Major, and the Ship, where it appears most brilliant; it then passes through the feet of the Centaur, the Cross, the southern Triangle, and returns towards the north by the Altar, the tail of the Scorpion, and the arc of Sagittarius, where it divides into two branches, passing through Aquila, Sagitta, the Swan, Serpentarius, the head of Cepheus, and returns to Cassiopeia. The ancients had many singular ideas respecting the cause of this phenomenon, but modern astronomers have long attributed it to an immense assemblage of stars too feeble to make distinct impressions; and Herschel has shown, by the observation above referred to, that these conjectures were well founded.

Nebulæ.

Besides the milky way, there are numerous other parts of the heavens, which exhibit an appearance of very nearly the same kind, which are called *Nebulæ*; the most considerable of which is that midway between the two stars in the blade of Orion's sword. This was first observed by Huygens; it contains only seven stars; the other part appearing like a luminous spot upon a dark ground, or like a bright opening into regions beyond.

Proper motion of some stars.

9. We have hitherto spoken of the fixed stars, as preserving actually the same relative situations with regard to each other; this, however, is not strictly true. The delicacy of modern observations has shown that some of these bodies have a progressive motion: Arcturus, for example, has a proper motion, amounting to about two seconds annually; and Dr. Maskelyne found, that out of 36 stars, of which he ascertained the places with great accuracy, 35 of them had a progressive motion.

Mr. Michell and Sir W. Herschel have conjectured that some of the stars revolve round others which are apparently situated near them; and perhaps even all the stars may in reality change their places more or less, although their relative situations, and the direction of their paths may render their motions imperceptible.

Variable stars.

That some of the stars have a periodical change of brightness has been well ascertained from repeated observations; and different hypotheses have been advanced to account for these singular phenomena. New stars have also appeared at certain times, remained stationary like the others, and have afterwards disappeared. Such a temporary star was observed by Hipparchus; and it was this circumstance which suggested to him the idea of forming a catalogue of them, and of delineating their situations. A new star was also discovered in Cassiopeia in 1572, which was so bright as to be seen in the day-time, but it gradually disappeared in 16 months. Another was observed by Kepler in 1604, more brilliant than any other star or planet, and changing perpetually into all the colours of the rainbow, except when it was near the horizon. This star remained visible about a year; and several other cases of the like kind are recorded.

As to the actual magnitude and distance of the stars, we may be said to be almost wholly unacquainted with either; all that we are able to state with certainty is, that their distance is immense. We shall show, in a subsequent chapter, that the distance of the sun from the earth is nearly 100 million miles; and, consequently, the diameter of our orbit is nearly 200 million miles; we therefore view the stars, or any one of them in particular, from two points at different times of the year, which are distant from each other 200 million miles, and yet we are not able to detect any difference in their apparent places. If a change of place, amounting only to one second, actually obtained, there is no doubt that it would be detected by the accuracy of modern observations;* we know, therefore, that an isosceles triangle having the diameter of the earth's orbit for its base, and having its vertex in the nearest fixed star, does not subtend an angle of one second; the nearest star must therefore be distant from us more than 20 *billions of miles*. How much their distance may exceed this, it is impossible for us to say; and much less are we able to offer even the most distant conjecture as to their actual magnitude; judging from analogy, we can only suppose them to be bodies resembling our sun, some of greater, and some of less magnitude; that they shine like the sun by their own light, each forming the centre of its particular system, dispensing light, heat, and animation to thousands of worlds, and to myriads of beings. What a wonderful idea do these reflections give us of the immensity of the universe, and of the power and omniscience of the great Creator and Director of so stupendous a machine!

3. *Of the Solar System.*

10. The system of Copernicus has been already referred to in our historical chapter. We have seen, that, according to this astronomer, all the principal planets revolve about the sun as a general, but not as a common centre; the several planets being supposed by him to describe circular orbits, each however having its particular point of circulation. The Copernican and the modern system are not therefore the same, although these terms are frequently employed synonimously. We shall here confine our description to the modern or true system, without, however, deducing any facts in proof of its accuracy: we shall be content

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* It is proper to observe, that since this was written, Dr. Brinkley, of Trinity College, Dublin, has been led to think, that he has detected a sensible parallax in several stars; but as Mr. Pond, the Astronomer Royal, has not yet been able to verify Dr. Brinkley's observations, we do not feel ourselves justified in stating the existence of a parallax; although, from the confidence we feel in the talents and accuracy of the observer, we have no doubt that his deductions will be hereafter verified by other astronomers.

that the reader receives it at present merely as an hypothesis, or rather as the enunciation of a proposition; and that he suspends his decision, till, in the course of the following chapters we advance such proofs as can leave him no longer in any doubt respecting the constitution of the solar system, nor of the mechanical principles on which its motions depend.

The sun has probably, as well as every body in the universe, a progressive motion in space; but if such a motion has place, he is accompanied in it with the whole system of which he forms the general centre, as well as the source of its light, heat, and motion; we may therefore consider this body at rest, as far as regards the relative motion of the other constituents of our system.

The sun, then, according to the modern hypothesis occupies a fixed centre, about which the several principal planets revolve, in elliptic orbits, one foci of each of which is found in the same point, coinciding very nearly with the solar centre. Several of these planets are again accompanied with attendant luminaries, which observe the same laws in their revolutions about their primaries, as these primaries do with respect to the sun.

11. The orbits of the several planets are not all situated in the same plane, but are variously inclined to each other, and to a fixed plane passing through the sun, called the plane of the ecliptic. Even this plane is not actually fixed, but is subject to a certain annual motion, to which, however, we shall not refer in this description. The several planets of our system, with the exception of the four small ones lately discovered, have their inclinations very small; the particular measures of which will be stated below; and their motions are all made in one direction, viz. from west to east. Several of these bodies are also known to have a rotatory motion on their respective axes, but these axes are differently inclined with respect to the plane of the ecliptic, each, however, always preserving its own direction parallel to itself. This rotatory motion is likewise performed from west to east, at least in all those bodies in which it has been hitherto observed; but some, either from their immense distance, their inconsiderable magnitude, or from their proximity to the sun, have not yet had their diurnal revolution decidedly ascertained.

12. The orbits of the planets we have seen are all elliptical, but these ellipses have very different degrees of eccentricity; that is, the ratio of the major and minor axes vary very considerably in the orbits of the different planets. The orbits of the satellites have also various eccentricities, and are differently inclined to the ecliptic, and to each other; in some of the secondaries this inclination is extremely great, while in all the principal bodies of our system it is very inconsiderable.

These general remarks being premised, we shall proceed concisely to describe the particular circumstances attending the motion of each planet, beginning with the sun, and proceeding orderly with the other bodies, according to their respective distances from him.

13. The sun, which is known to be 882,270 English miles in diameter, performs its diurnal revolution in 25 days 14h. 8′, about an axis, which is inclined 82° 30′ from the plane of the ecliptic.

Mercury.

14. Mercury is the nearest planet to the sun; its mean distance is about 36 millions English miles; the time of its sideral revolution 87 days 23h.; its diameter 3123 miles; the inclination of its orbit to the ecliptic, 7° 0′ 9″; and the ratio of the eccentricity of its orbit to that of its semi major axis ·2055149. The inclination of its axis is unknown, and the time of its diurnal revolution doubtful; it has been stated at 24h. 5′.

Venus.

15. Venus is the second planet in order from the sun; her mean distance being 68 millions of miles, and the time of her sideral revolution 224 days 16 hours. The diameter of this planet is 7702 English miles; the inclination of its orbit 3° 23′; and the ratio of its eccentricity to the semi major axis of its orbit ·0068529. Its diurnal revolution is performed in 23h. 21′, about an axis, which, like that of Mercury, is of unknown inclination.

The Earth.

16. The Earth is the third planet in the system; its distance from the sun is 93 million miles, and the time of its sideral revolution 365 days 6h. 9′; which must not be confounded with the length of the solar year, this being only 365 days 5h. 48′ 48″; the diameter of this planet is 7916 miles. The orbit of the earth being that to which the plane of the other planetary orbits are referred, its inclination is nothing; the eccentricity of its orbit is ·0168531; its diurnal revolution is performed in 23h. 56m. 4″, (which is called a sideral day), about an axis which inclines to the ecliptic in an angle of 66° 32′ 3″, the complement of which, viz. 23 27′ 57″, is the obliquity of the ecliptic as referred to the equator.

Mars.

17. The next primary planet in our system is Mars; its mean distance from the sun is 142 millions of miles, and its sideral revolution is performed in 686 days 23½h.; its diameter is little more than half that of the earth, being only 4398 miles.

Its orbit is inclined to the ecliptic in an angle of 1° 51′ 3″; the eccentricity of its orbit is ·0931340, and its diurnal revolution is performed in 24h. 39m., about an axis, which is inclined to the plane of the ecliptic at an angle of 59° 41′ 49″.

Vesta, Juno, Ceres, Pallas.

18. The next planets in order from the sun, are the four new ones; Vesta, Juno, Ceres, Pallas, of which little is known, except their times of revolution, and the inclination of their orbits; and these may even hereafter be found to stand in need of some corrections: they are at present stated as follows:—

Vesta; mean distance, 225 million miles; time of sideral revolution, 1335 days 4h.; inclination of orbit, 7° 8¾′; eccentricity, ·0932200.

Juno; mean distance, 253 million miles; period of sideral revolution, 1590 days 23h.; inclination of orbit, 21°; eccentricity, ·254944.

Ceres; mean distance, 263 million miles; period of revolution, 1681 days 12h.; inclination of orbit, 10° 37′; eccentricity, ·0783486.

Pallas; mean distance, 265 million miles; period of revolution, 1681 days 17h.; inclination of orbit, 34° 50′; eccentricity, ·245384.

The diameters of these planets (which must, however, be considered as doubtful,) have been given as follow:—Vesta, 238 miles; Juno, 1425 miles; Ceres, 1024 miles; and Pallas, 2099 miles.

Jupiter.

19. Jupiter, the largest of all the planets of our system, is the next after Pallas; its mean distance from the sun is 485 million miles, and its diameter is 91522

miles, more than eleven times that of the earth: the inclination of his orbit to the ecliptic is 1° 18′ 51″, and his sideral revolution 4,332 days 14 hours, the ratio of its eccentricity to the semi major axis ·0481784. The diurnal revolution of Jupiter is performed in 9 hours 55′ 49″, about an axis which is inclined to the ecliptic at an angle of 86° 54½′.

Saturn.

20. Saturn is next to Jupiter, as well in magnitude as in distance; the latter being 890 million miles, and its diameter 76,068 miles; his sideral revolution is performed in 10,758 days, or about thirty of our years, and the eccentricity of his orbit ·0561683. The diurnal revolution of Saturn is made in 10 hours 16 19″, about an axis which is inclined to the ecliptic at an angle of 58° 41′.

Uranus, or Georgium Sidus.

21. Last in the solar system is the Georgium Sidus, or Uranus; whose mean distance is more than double that of Saturn, being no less than 1800 million miles; its diameter is 35,112 miles, and the eccentricity of its orbit ·0446703. Its sideral revolution is performed in about eighty of our years: but its diurnal revolution, and the inclination of its axis, are not at present determined.

22. We have endeavoured to represent, the planetary distances, eccentricities, and proportional magnitude in Plate III., with the exception of the sun, which is there represented nearly as a point, whereas, in comparison with the magnitude we have given to the planets, its diameter ought to have exceeded even that of the orbit of Saturn. The several orbits in the plate are all represented as if situated in one plane; it is necessary therefore for the reader to bear in mind the measures of the several inclinations above indicated.

Characters of the planets.

23. The following are the characters or symbols employed by astronomers for denoting the several planets:—

The Sun	☉	Ceres	⚳
Mercury	☿	Pallas	⚴
Venus	♀	Jupiter	♃
The Earth	⊕	Saturn	♄
Mars	♂	Uranus	♅
Vesta	⚶	The Moon	☽
Juno	⚵		

24. At present we have spoken only of the primary planets, it now remains for us to say a few words with reference to the satellites by which some of them are attended.

The moon.

The first, and that which in this place claims our particular attention, is the moon, which revolves about the earth, as the earth itself does about the sun. Her mean distance from the terrestrial centre is 237,000 miles; her diameter is 2,160 miles, and the eccentricity of her orbit is ·0548553. The period of her orbicular and diurnal revolutions, are exactly alike, being completed in 27 days 7 hours 43′ 4″·7. The inclination of her orbit is 5° 9′, and of her axis 88° 29′ 49″.

Satellites of Jupiter, Saturn, &c.

It is unnecessary in this place to enter into the same minutiæ with respect to the other satellites; it will be sufficient to observe, that Jupiter has four, Saturn seven, and Uranus six, as represented in the plate above referred to.

Saturn's ring.

Besides the seven satellites which accompany Saturn in his dreary path, he is also encompassed by a double ring, by which he is distinguished from all, the other planets of our system. This ring, which is very thin, not exceeding 4,500 miles, is inclined to the plane of the ecliptic at an angle of 31° 19′ 12″, and revolves from west to east in 10 hours 29′ 16″·8; this rotation is performed about an axis perpendicular to the plane of the ring, passing through the centre of the planet.

We shall not enter more particularly in this place into a description of this singularly beautiful telescopic object; as we shall of course have to treat of it at length in a subsequent chapter, in which we shall state its several dimensions as determined from the best observations; it will be sufficient here to observe, that its outside diameter is 204,883 miles, and its inside 146,345, consequently, its mean breadth is about 30,000 miles.

25. We come now to the third class of bodies, which are only visible to us for a short time. The planets perform their revolutions about the sun within the limits of our observation, and at distances from him which vary comparatively very little in the different parts of their orbits; but those to which we now refer, if they all actually revolve about the sun, are, for a very considerable time far beyond the known limits of the solar system. These are called comets: they generally appear attended with a nebulous light, either surrounding them as a coma, or stretched out to a considerable length as a tail, and sometimes they appear to consist of such light only. Their orbits are so eccentric that in the remoter parts of them they are invisible to us, although at other times they approach much nearer to the sun than any of the planets: the comet of 1680, for example, when nearest the sun, was at the distance of only one sixth of the sun's diameter from its surface. Their tails are frequently of great extent, appearing as a faint light directed towards a point always opposite to the sun. It is quite uncertain of what matter they consist, and it is difficult to say which of the conjectures concerning them is the least improbable. Nearly 500 comets are recorded as having been seen at different times, and certain particulars relative to the orbits of about a hundred of them, have been accurately ascertained: but commonly, we have no opportunity of observing a sufficient portion of the cometary path, to determine with accuracy the entire dimensions of the ellipse or other conic section to which the observed part appertains; on which account little can be known of the periods and other circumstances of these wandering bodies. Only one comet has been recognized in its return to our system, which is that of 1759. Dr. Halley, by comparing together the elements of the several comets that had been observed up to his time, conjectured that those recorded to have appeared in 1531, in 1607, and in 1682, were, in fact, one and the same comet, and, consequently, that its return might be expected about the year 1758 or 1759, and its actual appearance in the last of those years, verified the conjecture. Another comet, which appeared in 1770, was suspected to move in an elliptic orbit; and if so, its period ought, by Mr. Lexel's computation (which has been since remade by Burckhardt), to be about 5 years and 7 months; it has never, however, been since observed; but this circumstance must not lead us to discredit either the observations made on it, or the calculations founded on them, for it has

been satisfactorily shown, that supposing all the data correct, this comet must have passed so near to Jupiter, that its orbit would be deranged, and the body rendered in future invisible to us.

We shall not enter further on the subject of comets in this place, except to observe, that whether their orbits be all ellipses, or some of them parabolas or hyperbolas, a very small portion of them fall within the limits of our system; if they are all ellipses, they are of very great eccentricity, and are only for a short period visible in these regions of celestial space. The orbits of two of these bodies are shown in our Plate III.

4. *Phenomena in the heavens, due to the motion of the earth and planets.*

26. We have already in our second section given a general view of the celestial sphere, with an enumeration of the several constellations; we propose now to illustrate a few particular phenomena, and to describe certain circles which astronomers have imagined for the better comprehension of the celestial motions.

We have seen that a person being situated in an open plane, in a star light evening, and watching attentively the motion of the fixed stars, will perceive them rise or emerge above the earth, continue to ascend, till they have attained a certain height, and then descend and disappear in the opposite side of the heavens to that in which they first rose. He will perceive, that accordingly as these stars are nearer the northern or southern points of the heaven, so they will appear visible to him for a greater or less time; and that certain stars very near the north point, never either rise or set, but would be always visible were not their light rendered imperceptible by the more refulgent rays of the sun.

One of the stars in this quarter of the heavens, to which we have already alluded, has scarcely any sensible motion, but, as far as the naked eye can distinguish, retains constantly the same situation; this is called the *pole star*, and those to which we have referred above, that appear to be constantly circulating about it, are, for that reason, called, *circumpolar stars.*

The first impression that this apparent motion of the stars would make on the mind of an uninformed observer would be, that the entire celestial vault was uniformly revolving from east to west, about an imaginary axis, which passes through or near the pole star, in a direction perpendicular to the planes of the circles described by the several stars which are alternately observed to rise and set; and consequently, that this line produced would again meet the celestial sphere in an opposite point, which may thence be denominated the *south pole*. This, as we have observed, would doubtless be the first impression made on an uninformed observer; but at the same time, if he possess the requisite intelligence, it would not be difficult to convince him that the very same appearances would be produced by supposing the earth to be of a globular form, and that it performed a motion of rotation about an axis corresponding with the supposed axis of the heavens, but in an opposite direction, that is, from west to east; numerous instances might be recalled to his recollection in which he had appeared at rest, when he was actually in rapid motion, and when objects were apparently seen to move with great velocity, although they were in an absolute state of rest. This phenomenon must have presented itself to every one who has travelled in a close carriage in a narrow road, when at times it is difficult to be persuaded but that the trees, gates, &c. which we pass are not moving in an opposite direction to that of the vehicle; and the same appearances are observed still more strikingly in the cabin of a ship when sailing with a moderate gale near the land. In fact, every intelligent observer, whether he admits the actual motion of the earth or not, will not for a moment deny, that if it had such a motion as we have supposed, the appearances would be exactly such as he observes in the heavens. This then will be one step towards his conviction, and various others will afterwards suggest themselves to his mind; which, however, we shall not at present insist upon, because from what has been stated, it is obvious, that we may at any rate be allowed to advance such an hypothesis as the diurnal revolution of the earth, without in any respect changing the appearance of any observed phenomenon. We shall therefore proceed upon the supposition of the earth being a sphere or a spheroid of small ellipticity, and that it performs a motion of rotation from west to east in about 24 hours; or rather, as we shall see in a subsequent chapter, in 23 hours 56′ 4″·1.

Apparent motion of the sun.

27. By observing attentively the stars which first appear visible in that part of the heavens where the sun sets, and continuing to observe them for several successive nights, we shall soon perceive that those stars which in the first instance we had observed to set immediately after the sun, are no longer to be seen, but that their places are supplied by certain others, which in their turn will also be lost in the solar beams. If now we observe the heavens in the morning before the rising of the sun, we shall find that those stars which in the first instance we had observed to set just after the sun, and which in the course of a few nights were absorbed by his rays, are now rising before him; the sun therefore has made an apparent motion in the heavens contrary to the general motion of the stars, that is, from west to east; and by following his progress in this manner during the course of a year, we shall find that he has described a complete circle of the heavens, and now rises and sets with the same star as we had observed exactly a year before. This circle which the sun thus appears to describe in the heavens is called the *ecliptic*, it is not directly east and west, but deviates nearly 24° from these points of the heaven, as shown in Plates I. and II., and to his obliquity it is, that we owe the variations in the seasons, and various other phenomena, as we shall show in a future chapter.

This apparent motion of the sun is due to the proper motion of the earth. Fig. 1.

28. This progressive motion of the sun in the heavens, which is only apparent, is due to the actual proper motion of the earth in its orbit about the sun. For let AB (Plate IV., fig. 1) represent two positions of the earth in its orbit ABC about the sun S, and let ♈, ♉, ♊, ♋, &c. represent the ecliptic or the apparent path of the sun. Then when the earth is at A, a spectator will refer that body to that part of the heavens marked ♈; but when the earth is arrived at B, he will then see it in ♊; and being in the mean time insensible of his own motion, the sun will appear to him to have described the arc ♈ ♊,

just the same as if it had actually passed over the arc SS′, and the earth had, during that time, remained quiescent in its first position A. This fact will explain why certain remarkable stars and constellations are seen in the south in different seasons of the year, and at different hours of the night. For the hour depends wholly on the sun; it is noon when the sun is south, and midnight when it is north; the stars directly opposed to him will therefore, by the rotation, appear in the south about midnight; and as the sun, from one day to another, shifts its place in the heavens, so, of necessity, will different stars be opposed to him, and become south at midnight at different seasons of the year.

Proper motion of the planets.

29. Similar phenomena may be observed with regard to the planets; by tracing their motions in the heavens, and comparing them with the stars near to which they appear, they will also be seen to have a motion of their own, but it will be more irregular than that of the sun; for we shall sometimes observe them moving like that body from west to east, then become stationary, maintaining the same position for several nights; then moving in a contrary direction, or from east to west; again become stationary, and again assume their direct motion.

These phenomena, which the ancients spent so much labour and ingenuity to account for, by epicycle upon epicycle, are perfectly consistent with the system as we have described it in our third plate, being due to the proper motions of the earth and planets in their respective orbits. In order to illustrate this, let $v\ v'\ v''$ represent the orbit of the planet Venus; and suppose her to be in the point v when the earth is at A; then it is obvious that a spectator will refer her place in the heavens to ♈, and as her motion is from v towards v', while the earth is moving from A to B, her apparent motion will be direct; or from ♈ towards ♉: if, on the other hand, Venus had been at v'' while the earth was at A, as their motions now are made in the same way, we may suppose Venus to have arrived at v''', while the earth had passed from A to A′; and during this time, it is manifest her place in the heavens will appear to retrograde, or go backward, contrary to the order of the signs: in like manner it will appear, that for a very short time before and after the Earth and Venus attained their positions A and v'', their motions would so agree with each other, that the planet, during this period, would appear stationary. We shall enter more at length upon these phenomena in a subsequent chapter; we have merely referred to the subject here, to show that these appearances are strictly conformable with the constitution which we have supposed of the solar system.

Notwithstanding these irregularities in the apparent motion of the planets, they each, respectively, are observed to describe great circles of the sphere, but more or less inclined to the plane of the ecliptic; they deviate, however, but little from it; all their motions, with the exception of the new planets, being performed in a zone, whose breadth does not exceed 16 degrees, and which we have already spoken of and described in our first and second plates, under the denomination of the zodiac.

Similar observations show that the moon also describes her particular path amongst the stars; but we shall reserve this subject for a subsequent chapter.

5. *Of the Seasons.*

30. It will appear sufficiently obvious from what has been shown in the preceding sections, that the alternations of day and night are attributable generally to the rotation of the earth on its axis; but the difference in the length of the days in different seasons of the year still remains to be illustrated.

We have seen that the two extremities of the terrestrial axis about which the diurnal rotation is performed, are called its poles, as N S, (fig. 2 and 3); and if at a quadrant distance from these we conceive a circle QEQ′ to be described, dividing the earth into two equal hemispheres, that circle is called the equator; the hemisphere towards the north pole N is called the northern hemisphere; and that towards the south pole, the southern. Now if the path of the earth in its orbit, or, which is the same, the apparent motion of the sun in the heavens coincided with the plane of the terrestrial equator, then it is obvious, that, at all times during this revolution, we should have the same equal alternations of day and night, each 12 hours in duration. For example: let NQSQ′ represent the earth; NS, its axis; and QEQ′, its equator: and let the plane of this circle produced pass through the sun S; then it is obvious, that, if we suppose the earth to revolve round the sun at the extremity of the line SE as a radius; and that during this revolution, it performed uniformly its rotatory motion about its axis NS: that line, NS, or the circle of which it is the projection, would terminate the limits of day and night; and the rotatory motion being uniform, every point of the globe, except the two poles, would have an equal succession of light and darkness during the entire revolution. We should then have no spring, no summer, no winter; these changes so pleasing in themselves, and so necessary for the production and re-production of the fruits of the earth would be wanting, and nature would thus be divested of a great portion of its charms.

This, however, is not the case; we have seen that the sun appears to describe an oblique motion in the heavens; it rises higher in the summer than in the winter, and by thus darting upon us more perpendicularly its refulgent beams, produces a greater portion of heat, and describes a larger circuit in the heavens, lengthens out the days, and thus gives time for that heat to become more effective.

The actual motion of the earth is therefore as represented in (fig. 3,) where we still suppose it to describe its annual motion at the extremity of the radius S′E; but such, notwithstanding, that the axis NS preserves its inclination and parallelism, whereby it is always directed to the same point in the heavens. The axis NS being now inclined to the plane of the earth's motion, it is obvious, that, as it revolves on its axis, some parts of the earth will experience perpetual day for a certain portion of the year, while other parts will have to contend with an equal duration of night. In the position of the earth, as shown in the figure, the parts about the north pole will be in continued darkness, and those near the southern pole in perpetual light; while, from the nature of the annual motion, it is clear that it will be exactly enter more particularly on the subject; showing the method of computing the length of the day in any given latitude; and the time of the rising, setting, and southing of the sun on any proposed day, and in any given place.

6. *Of the Phases of the Moon.*

Phases of the moon.

32. The moon, of all the celestial bodies, is that which perhaps attracts most the attention, both of illiterate and scientific observers. The former class are drawn to their observations by the remarkable beauty and serenity of her light, her numerous changes or phases, the inconstancy of her illumination, the enjoyment of her light at certain seasons of the year, when the sun sinks early below the horizon, and the want of it when both luminaries rise and set at nearly the same hour.

The philosophic observer examines her varying phases through his telescope; sees the shadows of the hills on her surface projected to a considerable distance through plains and vallies, rendered by the power of optics nearly as distinct as those of a terrestrial plain at the distance of only a few miles; he examines what he considers to be lunar volcanoes, and marks the progress of the lava from the crater to the vallies below; he looks to determine some indication of eruptions in present action; endeavours to distinguish her seas and continents, to measure the heights of her mountains, and to ascertain the existence or non-existence of a lunar atmosphere.

Again, the practical astronomer is watching carefully her deviating course in the heavens, and is endeavouring to submit her oscillating motion to the general principles of celestial mechanics; and to construct formulæ and tables, for the purpose of predicting her place at any appointed day and hour.

These subjects will each engage our attention in the course of the following chapters of this treatise; but at present we only propose to present to the reader an explanation of her most obvious phenomena, and to show their complete accordance with the motions and constitution of the solar system as described generally in the third chapter of this introduction.

Phases of the moon.

33. The first lunar phenomena to which we shall call the readers' attention, is the continual change of figure, or the phases which she exhibits to a terrestrial spectator. At one time perfectly full or globular, at others half or a quarter illuminated, and at others again exhibiting only a fine arched line, barely perceptible to the naked eye. These appearances are doubtless due to the revolution of the moon in her orbit; and the reflection of her light (which she receives from the sun) towards the earth. The lunar globe is necessarily always one half illuminated as we have shown the earth to be in the last section; and therefore, to a spectator placed in a line between the moon and sun, she would always present a full illuminated circle or hemisphere; but out of that line a greater or less part of the enlightened surface will be perceived, and which will obviously entirely vanish in certain positions. This may be illustrated by means of an ivory ball, as in the experiment described in the last section; for the ball being held before a candle in various positions, will present a greater or less portion of the illuminated hemisphere to the view of the observer, according to his situation with regard to the illuminated axis.

34. The same may be otherwise exhibited by means of our figure 6; where T is the earth, S the sun, and A, B, C, &c. the moon in different parts of her orbit. When the moon is at A, in conjunction with the sun S, her dark hemisphere being entirely turned towards the earth, she will disappear as at *a*, there being no light on that side to render her visible. When she comes to her first octant at B, or has gone an eighth part of her orbit from her conjunction, a quarter of her enlightened side is towards the earth, and she appears horned, as at *b*. When she has gone a quarter of her orbit from between the earth and sun to C, she shows us one half of her enlightened side, as at *c*, and we say she is a quarter old. At D, she is in her second octant; and by showing us more of her enlightened side she appears gibbous, as at *d*. At E, her whole enlightened side is towards the earth; and therefore she appears round, as at *e*; when we say it is full moon. In her third octant at F, part of her dark side being towards the earth, she again appears gibbous, and is on the decrease, as at *f*. At G, we see just one half of her enlightened side; and she appears half decreased, or in her third quarter, as at *g*. At H, we only see a quarter of her enlightened side, being in her fourth octant; where she appears horned, as at *h*. And at A, having completed her course from the sun to the sun again, she disappears; and we say it is new moon. Thus, in going from A to E, the moon seems continually to increase; and in going from E to A, to decrease in the same proportion; having like phases at equal distances from A to E. But as seen from the sun S she is always full.

The moon appears not perfectly round when she is full in the highest or lowest part of her orbit, because we have not a full view of her enlightened side at that time. When full in the highest point of her orbit, a small deficiency appears at her lower edge, and the contrary when full in the lower point of her orbit.

35. The moon, as we have seen, shines by her reflected light; in the same manner, the earth, by throwing back the light it receives from the sun, becomes in its turn a moon to the moon; being full to the inhabitants of the lunar sphere when our moon changes, and *vice versa*. For, when the moon is at A, new to the earth, the whole enlightened side of the earth is turned towards the moon; and when the moon is at *d*, full to the earth, the dark side of the latter is turned towards the former. Hence a new moon answers to a full earth, and a full moon to a new earth. The quarters are also reversed with respect to each other.

36. The position of the moon's cusps, or a right line touching the points of her horns, is very differently inclined to the horizon, at different hours of the same day of age. Sometimes she stands as it were upright on her lower horn, which is then necessarily perpendicular to the horizon; when this happens, she is said to be in her *nonagesimal degree*, which is the highest point of the ecliptic above the horizon; the ecliptic at that time being 90° from each side of the horizon, reckoning from the point where it is then cut by the former circle. But this never happens when the moon is on the meridian, except when she is in the beginning of Cancer or Capricorn.

considerable: regarding only one luminous point of the solar disc, it can only project one ray to every point of the surrounding space, but through the medium of the terrestrial atmosphere, a course of luminous points is collected behind the earth, an object placed in the focus or vertex of this cone, would be more strongly illuminated than by the direct light of the sun; every point of the sun producing a similar effect, the length and extension of the terrestrial shadow are much diminished; and if the atmosphere did not absorb a very great portion of the solar rays, the light reflected from the disc of the moon would be very great; it is, however, so much modified by the circumstances alluded to, as to exhibit only that faint red light above described.

It may be proper to observe, that this faint illumination of the lunar disc at the time of a total eclipse, has been accounted for upon different principles, but the above appears to us the most satisfactory.

Solar eclipse.

Fig. 8.

43. An eclipse of the sun is an occultation of the sun's body, occasioned by the interposition of the moon between the earth and sun. On this account it is by some considered rather as an eclipse of the earth, because the light of the sun is hidden from the earth by the moon, whose shadow involves a part of the terrestrial surface. The cause of a solar eclipse, and the circumstances attending it are represented in fig. 8, where S is the sun, *m* the moon, and CD the earth, *rmso* the moon's conical shadow traversing a part of the earth *C*o*D*, and thus producing an eclipse to all the inhabitants residing in that track, but no where else; excepting that for a large space around it, there is a fainter shade included within all the space *r*CD*s*, which, as in the lunar eclipse, is called the penumbra.

Hence, solar eclipses happen when the moon and sun are in conjunction, whereas the lunar eclipses only take place when they are in opposition; that is, the former happen at the time of the new moon, and the latter at the full.

Different kind.

Total.

Annular.

Partial.

44. Notwithstanding the moon is very considerably less than the sun, yet from its proximity to us, it so happens, that its apparent diameter differs very little from that of the latter body, and even sometimes exceeds it. Suppose an observer situated in a right line which joins the continuation of the sun and moon, he will see the former of these bodies eclipsed, as we have above stated. If the apparent diameter surpasses that of the sun, the eclipse will be total, and the observer will be entirely immerged in the conical shadow which is projected behind the moon: if the diameters are equal, the point of the cone will terminate at the earth's surface, and there will be a momentary total eclipse. If the diameter of the moon be less than that of the sun, the observer will see a zone of the sun surrounding the moon like a ring, and the eclipse will be central and annular. And lastly, if the observer be not exactly in the line joining the centres, the eclipse may be partial; that is, a part of the solar disc may be hid while the remaining part continues perfectly visible. Total eclipses, which are very rare occurrences in any particular place, are remarkable for the darkness which accompanies them, and which they spread over different parts of the surface of the earth, in the same manner as the shadow of a dense cloud, carried along by the wind, sweeps over the mountains and the plains, depriving them for some instants of the light of the sun. This total darkness under the most favourable circumstances may last about five minutes. The smallest apparent diameter of the sun is 31′ 30″; the diameter of the moon at its mean distance 31′ 25″, that is less than that of the sun; consequently there cannot be a total eclipse when the moon is beyond its mean distance. Eclipses of the sun are also modified as to quantity by the height of the moon above the horizon which increases her diameter; other circumstances also contribute to produce certain changes which must be considered in the computations relative to these phenomena, but which it would be useless to detail in this place. From what has been already stated we may draw the following general conclusions.

1. That no solar eclipse is universal; that is, none can be visible to the whole hemisphere to which the sun is risen: the moon's disc being too small and too near the earth to hide the sun from a whole terrestrial hemisphere. Commonly, the moon's dark shadow covers only a spot on the earth's surface, about 180 miles broad, when the sun's distance is greatest, and the moon's least. But her partial shadow or penumbra, may then cover a circular space of 4,900 miles in diameter, within which the sun is more or less eclipsed, as the places are nearer to or farther from the centre of the penumbra. In this case, the axis of the shade passes through the centre of the earth, or the new moon happens exactly in the *node*, and then it is evident that the section of the shadow is circular; but in every other case the conical shadow is cut obliquely by the surface of the earth, and the section will be oval, and very nearly a true ellipsis.

2. Nor does the eclipse appear the same in all parts of the earth, where it is seen; but when in one place it is total, in another it is only partial. Moreover, when the apparent diameter of the moon is less than that of the sun, as happens when the former is in apogee and the latter in perigee, the lunar shadow is then too short to reach the earth's surface; in which case, although the conjunction be central, yet the sun will be to no place totally eclipsed, but to certain observers, a bright rim of light will be seen surrounding the moon while the latter is on the solar disc; and the eclipse is then said to be annular.

3. A solar eclipse does not happen at the same time in all places where it is seen; but appears earlier to the western parts, and later to the eastern; as the motion of the moon, and consequently of her shadow, is from east to west.

4. In most solar eclipses, the moon's disc is covered with a faint light; which is attributed to the reflection of the rays from the illuminated part of the earth.

Lastly. In total eclipses of the sun, the moon's limb is seen surrounded by a pale circle of light, which has been considered as indicative of a lunar atmosphere; others, however, doubt this explanation, and offer different conjectures as to the cause of the phenomenon; but this is not the place for discussing this question.

Having thus given a brief description of the constitution of the solar system, with an illustration of the most remarkable phenomena which it presents to naked vision, as far as they can be illustrated independent of astronomical computation, we shall now conclude our introduction by describing certain astronomical machines, constructed for the purpose of exhibiting, in a simple and popular manner, all the most remarkable celestial movements and phenomena.

8. *Description of astronomical machines.*

45. By astronomical machines is here to be understood any pieces of mechanism constructed for exhibiting the motions and phenomena of the heavenly bodies, being thus distinguished from astronomical instruments, which include all such constructions as are employed for the purpose of measuring altitudes, angles, &c. necessary for astronomical computation. The one, in fact, are merely employed for explanation; the other, for the purpose of research and calculation. Various improvements have been made by different artists in the construction of planetary machines, and that which is now exhibited in the lectures of the Royal Institution, is perhaps the most perfect of its kind; but it would carry us too far to describe this instrument, with all its apparatus, wheel work, &c. Beside, we cannot help observing, that much time, ingenuity, and expence are frequently wasted in these kind of constructions; because, after all, they are only, as we have before observed, explanatory; for which purpose the same degree of accuracy is not required as in instruments employed in astronomical observation. That student who stands in need of the assistance of such machines, will never become a great proficient in astronomy; to pursue this study with effect, a beginner must acquire the habit of constructing his own planetarium in his mind's eye, and of soaring with it into the regions of celestial space; he ought to conceive the orbits of the heavenly bodies in a free non-resisting medium, their nodes, their inclinations, and eccentricities unimpeded by the intervention of brass rings or ebony frames; which have always the effect of giving a stiffness and unnatural representation extremely offensive to the eye of the professed astronomer. We must, however, acknowledge, that to children or mere novices, these machines may be of some assistance, and shall therefore describe one or two of the most simple of them in this place.

Planetarium.

46. The machine exhibited in fig. 9, is a planetarium, constructed by the late Mr. Jones, of Holborn. It represents, in a general manner, by various parts of it, all the principal motions and phenomena of the heavenly bodies.

The sun occupies the centre, with the planets Mercury, Venus, the Earth with its moon, Mars, Jupiter, with his four satellites, Saturn with his seven, and an occasional long arm may be attached for exhibiting the Herschel or Uranus, with his several attendants. To the earth and moon is applied a frame, CD, containing only four wheels and two pinions, which serve to preserve the earth's axis in its proper parallelism in its motion round the sun, and to give the moon her due revolution round the earth at the same time. These wheels are connected with the wheel work in the round box below, and the whole is set in motion by the winch H. The arm M that carries round the moon, points out on the plate C her age and phases for any situation in her orbit, which are engraved upon it. In the same manner, the arm points out her place in the ecliptic B, in signs and degrees, called her geocentric place; that is, as seen from the earth. The moon's orbit is represented by the flat rim A; this orbit is made to incline to any desired angle. The earth of this instrument is usually made of a 3-inch, or 1½-inch globe, papered, &c. for the purpose; and by means of the terminating wire, that goes over it, points out the changes of the seasons, and the different length of days and nights. It may also be made to represent the Ptolemaic system, which places the earth in the centre, and the planets and sun revolving about it. This is done by an auxiliary small sun and earth, which change their places in the instrument; but at the same time it affords a most manifest confutation of it. For it is obviously perceived in this construction, first, that the planets Mercury and Venus being both within the orbit of the sun, cannot at any time be seen to go behind it, whereas, in nature, we see them as often go behind as before the sun in the heavens. It shows that as the planets move in circular orbits about the central earth, they ought at all times to be of the same apparent magnitude; whereas, on the contrary, we observe their apparent magnitude in the heavens to be very variable; and so far different, that Mars, for instance, will sometimes appear nearly as large as Jupiter; while, at others, he will scarcely be distinguishable from a fixed star.

Refutation of Ptolemaic system.

The planetarium, when thus adjusted, shows also that the motions of the planets ought always to be regular and uniform; that they ought always to move in the same direction; whereas, we find them, sometimes direct, at others stationary, and even retrograde; which plainly shows the fallacy of the Ptolemaic hypothesis, at the same time that the modern system is thus clearly represented. Let us, for example, take the earth from the centre, and replacing it by the ball representing the sun, also restoring the earth to its proper situation amongst the planets, and every phenomena will then correspond and agree exactly with celestial observations. For turning the handle H, we shall see the planets Mercury and Venus go both before and behind the sun, or have two conjunctions; we shall perceive also that Mercury can never have more than a certain angular distance 21° from that body, nor Venus a greater than 47°. It will likewise be seen, that the superior planets, particularly Mars, will sometimes be much nearer to the earth than at others; and, consequently, must vary considerably in their apparent magnitude; we shall see that these planets cannot appear from the earth to move with equal velocities; but that this will appear greater when they are nearest, and less as they are more remote; that their apparent motions ought sometimes to be direct, sometimes retrograde, while in particular positions they will seem to be stationary; all which are consistent with the actually observed phenomena.

Fig. 10.

These particulars are shown somewhat more minutely in fig. 10, where a hollow wire, with a slit at top, is placed over the arm of the planet Mercury or Venus, at E. The arm DG represents a ray of light proceeding from the planet at D to the earth, and is put over the centre which carries the earth at F. The machine being then put in motion, the planet D, as seen in the heavens from the earth at F, will undergo the several changes of position as above described; and a similar application may be made to the superior planets.

This apparatus serves also to illustrate the diurnal rotation of the earth on its axis; the cause of the different seasons, the difference in the lengths of the days and nights, &c. For as the earth is placed on an axis inclined to the plane of the ecliptic at an angle of 23½°, we shall have, when the machine is in motion, the most satisfactory illustration of the different inclination of the sun's rays upon the earth. The different quantities which fall on a given space, the unequal quantities of the atmosphere they pass through, and the unequal duration of the sun above the horizon at the same place at different times of the year; which circumstances constitute the primary causes of all that change of seasons and variable lengths of days and nights which we experience.

The globe representing the earth being moveable about an axis, if we draw upon it a circle to denote our own horizon, we may, by means of the terminating wire going over it, very naturally exhibit the cause of the different lengths of the days and nights in our particular latitude, by simply turning the artificial earth with the hand to imitate its diurnal rotation; but in some of the more modern instruments of this kind, this rotatory motion is communicated to the globe by the wheel work of the machine itself.

The eclipses of the sun and moon are still more perfectly shown by this machine than the phenomena to which we have above alluded; for by placing a light in the centre instead of the brass ball, denoting the sun (fig. 11), and turning the handle till the moon comes into a right line, between the centres of the light, or sun, and the earth, the shadow of the moon will fall upon the latter, and all the inhabitants of those parts over which the shadow passes, will see more or less of the eclipse; and on the other side the moon passes through the shadow of the earth, and is by that means eclipsed to the inhabitants of those parts to which the lunar disc is at that time visible.

Fig. 11.

All the phenomena of the satellites of Jupiter, Saturn, &c., might also, with equal facilities, be exhibited by this machine, and are actually so exhibited in some of the larger apparatus, denominated orreries; in the machine we are at present describing, these satellites are only moveable by hand.

Orreries.

Of the Orrery.

47. The term Orrery, to denote such a machine as that we are about to describe, appears rather singular, and is one of those derivations, which, if the history were lost, would involve future etymologists in inexplicable difficulties. The first machine of this kind appears to have been made by the celebrated instrument maker, Graham, by whom it was probably considered only as an improved planetarium: but Rowley, an artist of reputation in his time, copied Graham's machine, and the first of his construction was made for the Earl of Orrery; whence Sir R. Steel, who knew nothing of Graham's original claim, called the instrument after the name of the supposed first purchaser an *Orrery*, which designation it still bears. One of the most usual constructions of this kind is shown in (fig. 12,) which may be briefly described as follows:—

Fig. 12.

The frame of it, which contains the wheel work, &c. and regulates the whole machine, is made of ebony, and about four feet in diameter. Above the frame is a broad ring supported by twelve pillars, which ring represents the plane of the ecliptic. Upon it are two circles divided into degrees with the names and characters of the twelve signs of the zodiac. Near the outside is a circle of months and days, exactly corresponding to the sun's place at noon each day throughout the year. Above the ecliptic stand some of the principal circles of the sphere corresponding with their respective situations in the heavens; viz. *a a* are the two colures, divided into degrees and half degrees; *b* is one half of the equinoctial circle, making an angle of 23½ degrees with the ecliptic. The tropic of Cancer and the arctic circle are each fixed parallel at their proper distances from the equinoctial. On the northern half of the ecliptic is a brass semicircle moveable upon two fixed points in ♈ and ♎.

This semicircle serves as a moveable horizon, to be put to any degree of latitude on the north part of the meridian, and the whole machine may be set to any latitude, without disturbing any of the internal motions, by means of two strong hinges fixed to the bottom frame upon which the instrument moves, and a strong brass arch, having holes at every degree, through which a pin may be passed at any required elevation. These hinges, with the arch, support the whole machine when set to the proposed latitude.

When the Orrery is thus adjusted, set the moveable horizon to any degree upon the meridian, whence may be formed a pretty correct idea of the respective altitude or depressions of the several planets, both primary and secondary. The sun S stands in the centre of the system on a wire making an angle with the ecliptic of about 82°; next in their order follow the planets Mercury, Venus, and the Earth, the axis of the latter being inclined to the plane of the ecliptic, at an angle of 66½°, which is the measure of the inclination of the earth's axis.

Near the bottom of this axis is a dial plate, having an index pointing to the hours of the day, as the Earth revolves; and about the latter is a small ring, supported by two small pillars, representing the orbit of the moon, with divisions answering to the moon's latitude. The motion of this ring represents the motion of the lunar orbit according to that of the nodes; and within it is a small ball with a black cup, or case, by which are exhibited all the phases of this celestial body.

Beyond the orbit of the earth are those of Mars, Jupiter, and Saturn, and in some instruments the Georgium Sidus, or Uranus. Jupiter is attended by his four satellites, and Saturn by his seven satellites and ring.

The machine is put in motion by turning a handle, or winch; and by pushing in, and pulling out a small pin above the handle. When it is in, all the planets, both primary and secondary, will move according to their respective periods. When it is out, the motion of the satellites of Jupiter and Saturn are stopped, while all the rest move without interruption. There is also a brass lamp, having two convex glasses to put in the place of the sun; and also a smaller earth and moon made somewhat in proportion to their distance from each other, and which may be put on or ved at pleasure. The lamp turns round at the time with the earth, and the glasses of it cast a g light upon her. When it is intended to use nachine, the planets must be first placed each in espective position by means of an astronomical meris, and a black patch or wafer may be placed e middle of the sun; against the first degree of Aries); patches may also be placed upon Venus, , and Jupiter. Now turn the handle, one revo n of which corresponds to one diurnal revolution of arth about its axis, and consequently answers to ours upon the dial plate, placed at the foot of the on which the ball is fixed.

ain, when the index has moved over the space hours, Jupiter will have made one revolution on xis, and so of the rest according to their respec eriods of diurnal rotation. By these means the ution of the planets, and their motion round axis, will be represented to the eye, if not ex , yet in nearly their due intervals of time.

We might have entered into the description of ies on other and more correct principles than the e, but the explanation must have been propor lly longer; and we have already observed, that machines are, in our opinion, rather calculated to the ingenuity of their constructors, than to offer dvantages to the student. It is true, that they convey to the uninformed reader some ideas of lanetary motions, but we think it is extremely ble, that the idea thus given, if not actually may, in many cases, be rather injurious than l; and as instruments for computation, the perfect of them are wholly incompetent, we therefore make no apology for not having ex d our description of orreries to a greater length. t has been said, and a reference to the plate, will ite sufficient for showing the general principle eir construction and operation, which, we con , is all that is requisite to be introduced in this

Cometarium.

This machine, which must also be considered rather object of curiosity than utility, shows the motion omet, or very eccentric body, moving round the nd describing equal areas in equal times; and be so adjusted, as to show such a motion for any e of eccentricity. The first projection of it we to Desaguliers.

e dark elliptical groove, *a b c d* &c. (fig. 13,) orbit of the comet Y; which is carried round is groove according to the order of those letters, e wire W fixed to the sun S, and slides on the as it approaches nearer or recedes further from, un; being nearest, or in its perihelion in *a*, and distant in the aphelion *g*. The areas *a* S *b*, *b* S *c*, &c. or the contents of these several trilaterals, are qual; and in every turn of the winch, or handle, e comet Y is carried over one of these spaces; quently, in the same time as it moves from *f* to from *g* to *h*, it will also move from *m* to *a*, or *a* to *b*, and so of the rest, its motion being quick t *a*, and slowest at *g*. Thus the comet's velocity orbit continually decreases from the perihelion e aphelion, and increases in the same proportion g to *a*.

The ecliptic orbit is divided into 12 equal parts or signs, with their respective degrees, as is also the circle *n o p q*, &c., which represents a great circle in the heavens, and to which the comet's motion is referred by a small knob on the point of the wire W. While the comet moves from *f* to *g* in its orbit, it appears to move only about five degrees in this circle, as is shown by the small knob on the end of the wire W; but in as short a time as the comet moves from *m* to *a*, or from *a* to *b*, it appears to describe the large space in the heavens *t n*, or *n o*, either of which spaces contains 120°, or four signs. If the eccentricity of the orbit were greater, the greater also would be the difference in the cometary motion.

The circular orbit ABC, &c. is for showing the equable motion of a body about the sun S, describing equal areas in equal times, with those of a body Y in its elliptic orbit above referred to, but with this difference, that the circular areas ASB, BSC, &c. or the equal arcs AB, BC, &c. are described in the same times as the unequal elliptic arcs *a b*, *b c*, &c.

If we conceive the two bodies Y and R, to move from the points *a*, A, at the same moment of time, and each to go round its respective orbit, and to arrive at the same points again at the same instant, the body Y will be more forward in its orbit than the body R all the way from *a* to *g*, and from A to G: but R will be forwarder than Y through all the other half of the orbit; and the difference is equal to the equation of the body Y in its orbit. At the points *a* A, and *g* G, that is, in the perihelion and aphelion they will be equal; and then the equation vanishes. This shows why the equation of a body moving in an elliptic orbit, is added to the mean or supposed circular motion from the perihelion to the aphelion, and subtracted from the aphelion to the perihelion, in bodies moving round the sun, or from the perigee to the apogee, and from the apogee to the perigee in the moon's motion round the earth.

Fig. 15.

This motion is performed in the following manner by the machine, (fig. 15.) ABC is a wooden bar (in the box containing the wheel-work), above which are the wheels D and E, and below it the elliptic plates FF and GG; each plate being fixed on an axis in one of its foci, at E and K; and the wheel E is fixed on the same axis with the plate FF. These plates have grooves round their edges precisely of equal diameters to one another, and in these grooves is the cat-gut string *g g*, *g g* crossing between the plates at *h*. On H, the axis of the handle or winch N in fig. 13, is an endless screw in fig. 15, working in the wheels D and E, whose numbers of teeth being equal, and equal to the number of lines *a* S, *b* S, *c* S, &c. in fig. 14, they turn round their axis in equal times to one another, as do likewise the elliptic plates. For, the wheels D and E having equal numbers of teeth, the plate FF being fixed on the same axis with the wheel E, and turning the plate GG of equal size by a cat-gut string round them both, they must all go round their axis in as many turns of the handle N as either of the wheels has teeth.

It is easy to see, that the end *h* of the elliptical plate FF being farther from its axis E than the opposite end I is, must describe a circle so much the larger in proportion, and must therefore move through so much more space in the same time; and for that reason the end *h* moves so much faster than the end I, although it goes no sooner round the centre E: at the same time the quick-moving end *h* of the plate FF leads about the short end *h* K of the plate GG with the same velocity; and the slow-moving end I of the plate FF coming half round as to B, must then lead the long end *k* of the plate GG about, with a corresponding slow motion: so that the elliptical plate FF and its axis E move uniformly and equally quick in every part of its revolution; but the elliptical plate GG, together with its axis K, must move very unequally in different parts of its revolution; the difference being always inversely as the distance of any point of the circumference of GG from its axis at K; or in other words, if the distance K *k*, be four, five, or six times as great as the distance K *h*, the point *h* will move in that position, four, five, or six times as fast as the point *k* does when the plate GG has gone half round; and so on for any other eccentricity or difference of the distances K *k*, K *h*. The I on the plate EF, falls in between the two teeth at *k* on the plate GG; by which means the revolution of the latter is adjusted to that of the former, so that they can never vary the one from the other.

On the top of the axis of the equally moving wheel D (fig. 15) is the sun S (fig. 14) which by means of the wire attached to it, carries the ball R round the circle, ABC, &c. with an equable motion, according to the order of the letters; and on the top of the axis K of the unequally moving ellipse GG in (fig. 15) is the sun S (fig. 14) carrying the ball Y unequally round the elliptic groove *a b c d*, &c. which elliptic groove must be exactly equal and similar to the verge of the plate GG, which again is also equal to that of EF.

Eclipsarean.

Eclipsarean

Fig. 16, 17.

50. The eclipsarean is an instrument invented by Mr. Ferguson for exhibiting the time, quantity, duration, and progress of solar eclipses, in all parts of the earth. This machine consists of a terrestrial globe A, (fig. 17) turned by a winch M, round its axis B, inclining 23½°, and carrying an index round the hour circle D; a circular plate E, on which the months and days of the year are inserted, and which supports the globe in such a manner, that when the given day of the month is turned to the annual index G, the axis has the same position with the earth's axis at that time; a crooked wire F, which points to the middle of the earth's enlightened disc, and shows to what place of the earth the sun is vertical at any given time; a penumbra or thin circular plate of brass I, divided into twelve digits by twelve concentric circles and so proportioned to the size of the globe, that its shadow, formed by the sun, or a candle, placed at a convenient distance, with its rays transmitted through a convex lens, to make them fall parallel on the globe, may cover those parts of the globe which the shadow and penumbra of the moon cover on the earth; an upright frame HHHH, on the sides of which are scales of the moon's latitude, with two sliders K and K, fitted to them, by means of which the centre of the penumbra may be always adjusted to the moon's latitude; a solar horizon C, dividing the enlightened from the darkened hemisphere, and showing the places where the general eclipse begins and ends with the rising or setting sun, and a handle M, which turns the globe round its axis by the wheel work, and moves the penumbra across the frames by threads over the pulleys LLL, with a velocity duly proportioned to that of the moon's shadow over the earth as the earth turns round its axis.

51. If the moon's latitude at any conjunction exceeds the number of divisions on the scales, there can be no eclipse; if not, the sun will be eclipsed to some parts of the earth; the appearance of which may be represented by the machine, either with the light of the sun, or of a candle. For this purpose, let the indexes of the sliders KK, point to the moon's latitude, and let the plate E be turned till the day of the given new moon comes to G, and the penumbra be moved till its centre comes to the perpendicular thread in the middle of the frame, which thread represents the axis of the ecliptic; then turn the handle till the meridian of London on the globe comes under the point of the wire F, and turn the hour circle D till 12 at noon comes to its index; also turn the handle till the hour index points to the time of new moon in the circle D, and then screw fast the collar N. Lastly, elevate the machine till the sun shines through the sight holes in the small upright plates OO, on the pedestal, or place a candle before the machine, at the distance of about four yards, so that the shadow of the intersection of the cross thread in the middle of the frame, may fall precisely on that part of the globe to which the wire F points; with a pair of compasses take the distance between the centre of the penumbra and the intersection of the threads, and set the candle higher or lower, according to that distance; and place a large convex lens between the machine and candle, so that the candle may be in the focus of the lens; and thus the machine is rectified for use.

52. Let the candle be turned backward till the penumbra almost touches the side, HF, of the frame, and then turning it forward, the following phenomena may be observed.

1. Where the eastern edge of the shadow of the penumbral plate I, first touches the globe at the solar horizon, those who inhabit the corresponding part of the earth, see the eclipse begin on the uppermost edge of the sun, just at the time of its rising.

2. In that place where the penumbra's centre first touches the globe, the inhabitants have the sun rising upon them centrally eclipsed.

3. When the whole penumbra just falls upon the globe, its western edge at the solar horizon touches, and leaves the place where the eclipse ends at sunrise on his lowermost edge.

4. By continued turning, the cross lines in the centre of the penumbra will go over all those places on the globe where the sun is centrally eclipsed.

5. When the eastern edge of the shadow touches any place of the globe, the eclipse begins there; when the vertical line in the penumbra comes to any place, then is the greatest obscuration at that place; and when the western edge of the penumbra leaves the place, the eclipse ends there, and the times are shown on the hour circle; and from the beginning to the end, the shadows of the concentric penumbral circles show the number of digits eclipsed at all the intermediate times.

6. When the eastern edge of the penumbra leaves

the globe at the solar horizon C, the inhabitants see the sun beginning to be eclipsed on its lowermost edge, at its setting.

7. Where the centre of the penumbra leaves the globe, the inhabitants see the sun centrally eclipsed; and lastly, where the penumbra is wholly departing from the globe, the inhabitants see the eclipse ending on the uppermost part of the sun's edge, at the time of its disappearing in the horizon.

This instrument will likewise serve for exhibiting the time of sun rising and setting; and of morning and evening twilight, as well as the places to which the sun is vertical on any day, by setting the day on the plate E to the index G, turning the handle till the meridian of the place comes under the point of the crooked wire F, and bringing XII on the hour circle D to the index: then if the globe be turned till the place touches the eastern edge of the horizon C, the index shows the time of sun setting; and when the place comes out from below the other edge of C, the index shows the time when evening twilight ends; morning twilight and sun rising are shown in the same manner on the other side of the globe. And the places under the point of the wire F are those to which the sun passes vertically on that day. *Ferguson's Astronomy*, by Brewster, or *Phil. Trans.* vol. xlviii.

53. The celestial and terrestrial globes may also be considered as astronomical machines of the kind we have been describing; but it would too much interrupt the order of our treatise to enter upon a description of these instruments in this place; they will, therefore be described under the proper head in our alphabetical arrangement; and we shall now proceed, having given the foregoing succinct view of the more popular celestial phenomena, to treat the subject under a more scientific point of view, in the following sections.

PART II.

PLANE ASTRONOMY.

§ III. *Containing the principles of astronomical computation.*

1. *Definitions.*

54. Previous to our entering upon this subject, it will be requisite for the reader to render himself familiar with the following definitions. Some of them have been already given, but the convenience of having one decided place of reference will compensate for the few repetitions that occur.

1. A *great circle* of a sphere is any circle QRST (fig. 18) whose plane passes through the centre of the sphere; and a *small circle* is any circle, BHK, whose plane does not pass through the centre. All great circles bisect each other.

2. The *diameter* of a sphere is any line, PE, passing through the centre and terminated on both sides by the circumference; this diameter is said to be the *axis* of that great circle to which it is perpendicular; and the extremity of the axis PE are called the poles of that circle.

Hence it follows, that the pole of a great circle is 90° distant from every point of it upon the sphere; and that the arcs subtending any angles at the centre of a sphere are those of great circles. Consequently, all the triangles formed on the surface of a sphere for the solution of spherical problems must be formed by the arcs of great circles.

Plane Astronomy

Secondaries.

3. *Secondaries* to a great circle are great circles, as PQE, PRE, which pass through its poles, and whose planes are therefore perpendicular to the plane of the latter. Hence every secondary bisects its great circle; a secondary also bisects every small circle that is parallel to the great circle to which it is secondary. Since every secondary passes through the pole of its great circle, and is perpendicular to it; it follows, that if a secondary passes through the poles of two great circles, it is perpendicular to each of them. And conversely, if one circle be perpendicular to two others, it must pass through their poles.

Poles of the celestial and terrestrial sphere.

55. The above definitions belong wholly to the sphere considered abstractedly as a geometrical solid; the following appertain to the sphere considered with reference to astronomy.

Fig. 19.

1. Let pep' of (fig. 19) represent the earth which at present we shall consider as a perfect sphere, and let pp' be the line about which it performs its diurnal rotation; then pp' are called its *poles*, and the line pp' its *axis*. And if we assume the circle PEP′Q to denote the circle of the celestial sphere, and conceive pp' to be produced to the heavens meeting them in PP′, these will be the *poles* of the celestial sphere.

2. The *terrestrial equator* is a great circle $erqs$, of the earth perpendicular to its axis; and if we conceive the plane of this circle to be produced to the sphere of the fixed stars, it will mark out the great circle ERQS, which is called the *celestial equator*.

Hence it follows, that the poles of the terrestrial and celestial spheres are the same as the poles of the respective equators.

Equator.

The equator divides either spheres into two equal portions, called the *northern* and *southern hemispheres*, and the corresponding poles are in like manner denominated the *north* and *south* poles. The northern hemisphere is the part of the earth which lies on the side of the equator which we inhabit, and which in the figure we may assume to be epq.

Latitude terrestrial.

2. The *latitude* of a place on the earth's surface is its angular distance from the equator, measured upon a secondary to it: thus the arc eb, measures the latitude of the point b. Any circle on which we measure the latitude of a place is called a *terrestrial meridian;* and when produced to the heavens a *celestial meridian*.

Meridian.

3. The small circles parallel to the terrestrial equator are called *parallels of latitude*.

4. The secondaries to the celestial equator are called circles of *declination;* and the small circles parallel to the equator on the earth's surface parallels of *declination*. Declination, therefore, in the celestial sphere, corresponds to latitude on the terrestrial sphere, thus the arc eb which measures the latitude of a place on the earth, corresponds to EZ, the declination in the heavens.

Longitude terrestrial

5. The longitude of a place on the earth's surface is an arc of the equator, intercepted between the meridian passing through the place, and another called a first meridian, passing through that place from which you begin to measure; which latter is different in different countries. Most nations account their first

meridian that passing over their capital. The English, for example, take the meridian of London, or rather that of the Royal Observatory, Greenwich; the French, that of Paris, &c.

Horizon.

6. If *b* be supposed to denote any place on the earth, and a tangent plane be supposed to be drawn to that place, and produced to the heavens, meeting them in the points *a c*, the circle *a b c*, which is here projected into a right line, is called the *sensible horizon.*

And the great circle HOR, which is drawn parallel to it, passing through the centre of the earth O, is called the *rational horizon.* It is in the former of those circles, which all the heavenly bodies are observed to rise and set.

Small circles parallel to the horizon are called *almucantars.*

Zenith and nadir.

7. If the radius O *b* of the earth at the place *b* of a spectator be produced both ways to the heavens, that point Z vertical to him is called the *zenith*, and the opposite point N the *nadir*. Consequently, the zenith and nadir are the poles of the rational horizon.

Vertical circles.

8. *Vertical circles* are those secondaries which are perpendicular to the horizon, and which therefore pass through the zenith: it is in these circles the altitude of the heavenly bodies are taken. The celestial meridian of a place is therefore a vertical circle passing through the pole and zenith of that place; as PEHP′ in the figure above referred to.

The two points in the horizon R H, which are cut by the meridian of any place, are called the *north* and *south* points, according as they are towards the north or south poles.

Prime vertical.

9. That vertical circle which cuts the meridian of any place at right angles, dividing it into two equal hemispheres, and which cuts the meridian in the east and west points, is called the *prime vertical*, as ZN, which is projected into the right line ZN.

Azimuth and amplitude.

10. When a body is referred to the horizon by a vertical circle, the distance of that point of the horizon from the north or south points, is called the *azimuth*, and its distance from the east or west points its amplitude.

Ecliptic.

11. The *ecliptic* is that great circle of the heavens which the sun appears to describe in the course of the year.

Obliquity of the ecliptic.

12. The angle which the ecliptic forms with the celestial equator, is called the *obliquity of the ecliptic;* and the two points in which these circles intersect and bisect each other, are called the *equinoctial points.* The times when the sun comes to these points are called the *equinoxes.*

For the signs, order, and characters of the twelve signs of the ecliptic, or zodiac, see our table of constellations, p. 506.

Of these signs, the first six which lie on the northern side of the equator, are called northern signs; viz. Aries, Taurus, Gemini, Cancer, Leo, Virgo: and the other six, Libra, Scorpio, Sagittarius, Capricornus, Aquarius, Pisces, the southern signs.

The equinoctial points correspond to the first points of Aries and Libra.

The six signs, Capricorn, Aquarius, Pisces, Aries, Taurus, and Gemini, are called ascending signs, the sun approaching our's, or the north pole, while it passes through them; the others, Cancer, Leo, &c., are, for a corresponding reason, called descending signs; the sun, while passing through them, being receding from our pole.

14. The *zodiac* is a zone of the celestial sphere, extending 8° on each side of the ecliptic, within which the motion of all the principal planets are performed.

The signs of the ecliptic and of the zodiac are the same.

15. When any of the heavenly bodies appear to move according to the order of the signs, viz. through Aries, Taurus, &c. their motion is said to be *direct*, or *in consequentia;* when contrary to that order, *retrograde*, or *in antecedentia*. (See art. 29)

The real motion of all the planets is according to the order of the signs.

16. We have seen, that the declination of any heavenly body is measured by the arc of a declination circle, or secondary to the equator, and the distance of that secondary on the equator from the first point of Aries, estimated according to the order of the signs, is called its right ascension.

Hence the right ascension of a heavenly body corresponds with the longitude of a terrestrial one, except as to the point whence it is measured.

17. The *latitude* of a heavenly body is measured by a secondary to the ecliptic passing through that body; that is, by the angular distance between it and the ecliptic, as by the arc *m s*, fig. 20: and the *longitude* is measured by the arc of the ecliptic, intercepted between the first point of Aries and the point where the secondary meets the ecliptic, estimated according to the order of the signs, as *o m*, considering *o* as the first point of Aries.

Hence the latitude and longitude of a heavenly body are the same with reference to the ecliptic and its secondaries; as those of a terrestrial body or place with reference to the equator and meridians.

18. The *oblique ascension* is an arc of the equator, intercepted between the first point of Aries, and that point of the equator which rises with any body; and the difference between the right and oblique ascension is called the *ascensional difference.*

19. The tropics are two celestial circles parallel to the equator, and touching the ecliptic; the one at the beginning of Cancer, called the tropic of Cancer; the other at the beginning of Capricorn, called the tropic of Capricorn. The two points where the tropics touch the ecliptic, are called the *solstitial* points.

20. The *arctic* and *antarctic* circles are two parallels of declination; the former about the north, and the latter about the south pole: the distance of each from the pole is equal to the distance of the tropics from the equator.

These two circles, and those of the tropics, when referred to the earth, divides it into five zones; two called the *frigid zones*, which are those towards the two poles; the temperate zones being those between each tropic and its corresponding polar circle; and one torrid zone, including all the space between the two tropics, extending therefore about 23½° on each side of the equator.

21. The *nodes* are the points where the orbit of a planet cuts the plane of the ecliptic; and the nodes of a satellite, are the points where its orbit cuts the plane of the orbit of its primary, or that about which it revolves.

The *ascending* node is that where the body passes from the south to the north side of the ecliptic, and the other is called the *descending* node.

22. The *aphelion* is that point of the orbit of a planet which is farthest from the sun; and the *perihelion* that point where it is nearest.

The *apogee* and *perigee* have the same signification with reference to the earth. The moon, for instance, is said to be in apogee when farthest from us; and in perigee, when nearest.

The above definitions will be sufficient for our present purpose; others, which require a previous knowledge of certain subjects not yet discussed, are reserved for those places in the course of the subsequent sections in which they naturally occur.

2. *Illustration of certain celestial phenomena.*

56. Let us now proceed to show the application of the doctrine of the sphere, to the illustration of certain celestial phenomena, as the rising, setting, southing, &c. of the heavenly bodies.

57. In art. 55. we have defined the sensible and rational horizon; but with reference to the sphere of the fixed stars, these may be considered as coinciding, the angle which the arc Ha (fig. 19) subtends at the earth, becoming then insensible in consequence of the immense distance of these bodies. Now, if we suppose, as we have hitherto done, the earth to revolve daily about its axis, all the heavenly bodies must successively appear to rise and set, or revolve about the pole, in circles, whose planes are perpendicular to the earth's axis, and consequently parallel to each other; and will, to every appearance, be the same as if the spectator were at rest in the centre of a concave sphere, which revolved uniformly about him; or that the stars each revolved in parallel circles on such a sphere. We may therefore consider the earth but as a point with reference to the radius of the sphere of the fixed stars, and leave it out of the consideration in our farther inquiries upon this subject, and only employ the zenith, equator, poles, horizon, &c. of the celestial sphere, and such circles of declinations, as corresponding with the motion of the given bodies.

58. Let then fig. 21. represent the position of the heavens to an observer, whose zenith is Z in north latitude; EQ the equator, PP′ the poles, HOR the rational horizon, PZHP′R the meridian of the spectator.

Draw the great circle ZON perpendicular to the meridian, and passing through the zenith Z, which from our definition will be the prime vertical; and being in the plane of the eye, this being supposed to be perpendicular over the pole of the meridian, will be projected into a right line ZN, as will be shown in our treatise on Projection and Perspective. The same will also be the case with the equator EQ, the horizon HR, and the great circle POP′, supposed to be drawn perpendicular to the meridian; the common intersection of all these circles being in the point O, the pole of the meridian.

Draw the small circles, or parallels of declinations, wH, mt, ae, Rv, yx, which will represent the circles described by any of the heavenly bodies; and as the great circle POP′ bisects the equator, it will bisect all the small circles parallel to it, consequently mt, ae, are bisected in r and c; and we shall have $ac = ce$, and $mt = rt$, each equal to a quadrant, or 90°.

Now if we conceive the figure referred to as exhibiting the eastern hemisphere, the several arcs QE, ae, tm, &c. will represent the paths of bodies placed at those distances from the pole, as they ascend from the meridian under the horizon to the meridian above; and the points b, O, s, will be the places where they rise, or begin to appear above the horizon; and t E e, the points where they attain their greatest or meridian altitude; as ae, QE, mt, are bisected in c, O, r; eb must be greater than ba; QO, equal to OE, and ts less than sm.

Whence it follows, that a body on the same side of the equator as the spectator, will be longer above the horizon than below it; when the body is in the equator it will be as long below as above the horizon; and when it is on the contrary side of the equator to the spectator, it will be longer below the equator than above it; for the arc eb is greater than ba, EO=OQ, and ts less than sm, and the motion with which these arcs are described are uniform.

The bodies describing ae, mt, rise at b and s; and as O is the east point of the horizon, and H and R the north and south points; a body on the same side of the equator as the spectator rises between the east and the north; and a body on the contrary side, between the east and the south, the spectator being supposed in north latitude; and a body in the equator rises in the east at O.

Circles of perpetual apparition.

59. When bodies come to d and r, they are in the prime vertical, or in the east; hence, a body on the same side of the equator as the spectator comes to the east after it has risen; a body in the equator rises in the east, and one on the contrary side of the equator has passed the east before it rises. The body which describes the circle Rv, or any one nearer to P, never sets; and such circles are called circles of *perpetual apparition*, and the stars which describe them *circumpolar stars*. The body which describes the circle wH, just become visible at H, and then instantly descends below the horizon; but those bodies which describe the circles nearer to P′ are never visible.

Such is the apparent diurnal motion of the heavenly bodies, when the spectator is situated any where between the equator and either poles; and this is called an *oblique* sphere; because all bodies rise and set obliquely to the horizon.

In the above deductions we have supposed the figure to represent the eastern hemisphere, and the bodies to ascend through their respective arcs; but it may be equally supposed to denote the western hemisphere, only in this case, these arcs will represent the paths of the body as they descend from their greatest altitude above the horizon to their meridian below the horizon. And hence, it is obvious, that supposing a body not to change its declination, it will be at equal altitudes at equal times before and after it has attained its meridian altitude.

Right sphere.

60. In the preceding article we have supposed the spectator to be in north latitude, or, which is the same, the zenith of the spectator to be between the equator and north pole; and it is obvious, that what we have said would apply equally to a spectator similarly situated in south latitude; but when he is situated either in the equator or in one of the poles,

the considerations become less complicated; in the former case we call it a *right sphere*, and in the latter a parallel sphere. If the spectator be at the equator, then E coincides with Z, and Q with N, as in (fig. 22); consequently also PP′ coincides with HR; and the declination circles *ea*, *tm*, which are always parallel to the equator, are in this case perpendicular to the horizon; and as these circles are always bisected by PP′, they must now be bisected by the horizon HR; hence in this position of the spectator all the heavenly bodies, which change not their declination, will be an equal time above and below the horizon, and will rise perpendicularly to it; whence the denomination of the right sphere.

Fig. 22.

Parallel sphere.

Fig. 23.

On the other hand, if the spectator be situated at the pole, then the sphere will be as represented in fig. 23, that is, we shall have P coincide with Z, and EQ, with HR, or the equator will coincide with the horizon; and all the parallels of declination, *ae*, *mt*, described by the heavenly bodies, will be therefore parallel to the horizon; any body, therefore, which is above the horizon, and which changes not its declination, will remain constantly above the horizon, and at the same altitude; and those which are below the horizon will continue constantly below. Consequently, a spectator at the pole would never see the heavenly bodies rise and set, but would observe them to describe circles in the heavens parallel to the horizon; whence the denomination *parallel sphere*.

Of the change of declination.

61. We have had two or three times occasion to use the words, " those heavenly bodies which change not their declination;" it may be here proper to explain to the reader, that by this we mean the fixed stars only, these being the only celestial body that are not subject to a change of declination; and some of these even are liable to such a change, it is however too inconsiderable to be attended to in this place. But the sun, moon, and planets are constantly changing their declination, in consequence of the proper motion of the earth and themselves; let us, therefore, now bestow a few words in explanation of these cases, particularly as regards the sun.

Apparent motion of the sun.

We have already stated in our introduction (art. 27) that by attentively observing the stars which set and rise with the sun during the course of the year, that he appears to have described a great circle of the celestial sphere, forming with the ecliptic an angle of about 23½°, or more exactly 23° 28′. This circle is called the ecliptic, and is denoted by the line LC, into which it is projected in the three last figures; which circle cuts the equator, as we have seen, in two points called the equinoctional points. The sun, therefore, during one part of the year, is on one side of the equator, and in the other, on the contrary side; and by this means his rising and setting is subject to all that variety which we have noticed in the stars in the two hemispheres, and hence the cause of the different lengths of the days at different times of the year, the succession of seasons, and the several phenomena attending them, as we have already endeavoured to explain in a popular manner in the preceding introduction.

Change of seasons.

Let us now endeavour to illustrate the same a little more particularly by referring again to our fig. 21. Here since in the course of the year, the sun appears to describe the circle of which LC is the projection, it is obvious that he will be sometimes to the north of the equator, as in *q*, at others to the south, as in *p*, and at others in the equator, as at O. In the former case, that is, when he is in *q*, P denoting the north pole, it is obvious, as we have already remarked respecting any body describing the declination circle *ae*, that he will rise between the north and east, attain to the prime vertical, after he has risen, and will be longer above the horizon than below it, which is the case in our latitudes, from about the 21st of March to the 22d of September, these being nearly the times when the sun crosses the equator. On those two days he rises in the east, and is an equal time above and below the horizon to every part of the globe except the two poles; and the days and nights being then equal, these points are called the *equinoctial* points, and the sun itself is said to be in the *equinoxes*. The former of these is called the *vernal*, and the latter the *autumnal* equinox. When the sun is on the south side of the equator, as at *p*, then the same remarks apply as we have already made with respect to any body describing the declination circle, *mt*, that is, he will rise between the south and the east, and will be longer below than above the horizon, and our days will be shorter than our nights, as is the case from the autumnal to the vernal equinox, that is, from about the 22d of September to the 21st of March.

When the spectator is in the equator, then the sphere being right (see fig. 23), the sun will be always an equal time above and below the horizon; and when the sun is also in the equator, he will rise east and describe a great circle corresponding with the prime vertical, and will be vertical over the head of the spectator in the middle of his course: at other times he will rise between the north and the east or the south and the east, according as his declination is north or south. There is, therefore, even in these regions, a change of seasons; but as the sun will dart his vertical beams upon every point of the equator, twice in the course of one revolution, the inhabitants may be said to have two summers and two winters in the course of a year. When the spectator is at the pole, the sphere will be parallel (see fig. 23), and the sun from the vernal to the autumnal equinox, will be constantly visible to the north pole, and perpetually hidden below the horizon during the other half of the year; and the contrary for the south pole. That is, in the former, he will be visible from the time he passes from O to L, and from L to O, and be invisible while he is describing the other half of the ecliptic. At each of the poles, therefore, the days and nights are each half a year in length. It must not, however, be understood here, that the length of the days and nights are each exactly equal to half a year, for we shall see hereafter, that the sun is not so long on the northern as on the southern side of the equator, the cause of which will likewise be illustrated in a subsequent chapter. At present, it will be sufficient to observe, that such is known to be the case from observation.

62. We have seen in our definitions that the earth is considered as divided into five zones, by referring to the earth the two polar circles, and the two tropics. Now from what has been above stated, it is obvious, that to an observer in either hemisphere, in the latitude of 23° 28′, his zenith will coincide with the sun's

place at noon, on that day when it has attained its greatest north declination, if the observer be in north latitude; or south, if he be situated in south latitude; consequently, in either situation, on one day in a year the sun will be vertical to the inhabitants of either tropic, and of course to all places situated between them, except those on the equator, who, as we have seen, have the sun vertical twice.

Beyond the tropics, either to north or south, the sun is never vertical, the zenith of all such places being farther from the equator than the extreme declination of the sun or the obliquity of the ecliptic.

Those who inhabit the two polar circles will have one day and one night of twenty-four hours, or there will be one day in each of those circles when the sun will not set, another on which he will not rise, as will be immediately obvious by referring to the preceding figures.

From these circles to the poles themselves the sun will be for a greater or less time above and below the horizon, till in the actual poles, the nights and days will be half a year each, as we have already stated.

Synopsis of spherical trigonometry.

3. *Synopsis of spherical trigonometry.*

62. As in the course of the following articles, we shall have frequent occasion to refer to the several cases of spherical trigonometry, we conceive that it will be very convenient for the reader to have a general synoptic table of all the principal results and theorems belonging to this doctrine, the investigations of which will be given in our Treatise on Trigonometry, Part I. We collect them here merely for the convenience of reference, and have chosen those only which are most general in their application. They are sufficient for the solution of any spherical problem, although, in certain cases, they may not offer the most expeditious mode of solution; we shall not therefore uniformly adopt the formulæ given in the table, but when a more expeditious form presents itself, we shall avail ourselves of it; in the greater number of cases, however, our solutions will be deduced from the tabula formulæ.

TABLE I. *For the solution of all the cases of right angled spherical triangles.*

Given.	Required.	Value of the Terms required.	Cases in which the terms required are less than 90°.
I. Hypothenuse and one leg.	Angle opposite the given leg.	Its sin $= \dfrac{\text{sin given leg}}{\text{sin hypoth.}}$	If the given leg be less than 90°.
	Angle adjacent to the given leg.	Its cos $= \dfrac{\text{tan given leg}}{\text{tan hypoth}}$	If the things given be of the same affection*.
	Other leg.	Its cos $= \dfrac{\text{cos hypoth.}}{\text{cos given leg}}$	Idem.
II. One leg and its opposite angle.	Hypothenuse.	Its sin $= \dfrac{\text{sin given leg}}{\text{sin given ang.}}$	Ambiguous.
	Other leg.	Its sin $= \dfrac{\text{tan given leg}}{\text{tan given ang.}}$	Idem.
	Other angle.	Its sin $= \dfrac{\text{cos given ang.}}{\text{cos given leg}}$	Idem.
III. One leg, and the adjacent angle.	Hypothenuse.	Its tan $= \dfrac{\text{tan given leg}}{\text{cos given ang.}}$	If the things given be of like affection.
	Other angle.	Its cos $=$ cos giv. leg $\times$ sin giv. ang.	If the given leg be less than 90°.
	Other leg.	Its tan $=$ sin giv. leg tan giv. ang.	If the given angle be less than 90°.
IV. Hypothenuse and one angle.	Adjacent leg.	Its tan $=$ tan hyp. $\times$ cos giv. ang.	If the things given be of like affection.
	Leg opposite to the given angle.	Its sin $=$ sin hyp. $\times$ sin giv. ang.	If the given angle be acute.
	Other angle.	Its tan $= \dfrac{\text{cot giv. angle}}{\text{cos hypothen}}$	If the things given be of like affection.
V. The two legs.	Hypothenuse.	Its cos $=$ rectan. cos given legs	If the given legs be of like affection.
	Either of the angles.	Its tan $= \dfrac{\text{tan opposite leg}}{\text{sin adjacent leg}}$	If the opposite leg be less than 90°.
VI. The two angles.	Hypothenuse.	Its cos $=$ rect. cot given angles	If the angles be of like affection.
	Either of the legs.	Its cos $= \dfrac{\text{cos opposite angles}}{\text{sin adjacent angle}}$	If the opposite be acute.

* Angles or sides are of the same affection when they are both greater or both less than a right angle; and in the third column of our Table we have stated when the result is of the same affection with the things given, and when it is ambiguous.

In working by logarithms the reader must observe, that when the resulting logarithm is the log. of a quotient, 10 must be added to the index; and when it is the log. of a product, 10 must be subtracted from the index. Thus, when the two angles are given,

log. cos hypothenuse = log. cos one angle + log. cos other angle − 10.

log. cos either leg = log. cos opp. angle − log. sin adjac. angle + 10.

Quadrantal triangles.

In a quadrantal triangle if the quadrantal side be called radius, the supplement to the angle opposite to that side be called hypothenuse, the other sides be called angles, and their opposite angles be called legs; then the solution of all the cases will be as in the above table for right angled spherical triangles.

NAPIER's *analogies for right angled spherical triangles.*

Napier's circular parts are, the complements of the two angles that are not right angles, the complement of the hypothenuse and the other two sides; that is, denoting the sides by *a*, *b*, *c*, and the angles by A, B, C, A being the right angle, the parts are 90°−*a*, 90°−B, 90°−C, and *b*, *c*; of these, any one may be the middle part, and the two parts next adjacent, one to either hand (not including the right angle) the adjacent parts; and the other two the opposite parts; then the analogies are

rad × sin middle part = rectangle of the tangents of adjacent parts.

rad × sin middle part = rectangle of the cosines of the opposite parts.

And by making the transformations above explained, these will also apply to quadrantal spherical triangles.

TABLE II. *For the solution of oblique angled spherical triangles.*

Given.	Required.	Values of the quantities required.	
I. Two angles and a side opposite to one of them.	Side opposite to other angle.	By common analogy	Sines of angles, are as sines of opp. sides.
	Third side.	Let fall a perpendicular upon the side contained between the given angles.	Tan 1 seg. of this side = cos adj. angle × tan given side. Sin 2 seg. = $\frac{\text{sin 1 seg.} \times \text{tan ang. adj. given side.}}{\text{tan ang. opp. given side.}}$
	Third angle.	Let fall a per. as before	Cot 1 seg. of this ang. = cos giv. side × tan adj. ang. Sin 2. seg. × $\frac{\text{sin 1 seg.} \times \text{cos ang. opp. given side.}}{\text{cos ang. adj. given side.}}$
II. Two sides and an angle opposite to one of them.	The angle opposite to the other side.	By the common analogy.	Sines of sides are as sines of their opposite angles.
	Angle included between the given sides.	Let fall a perpendicular from the included angle.	Cot 1 seg. ang. req. = tan giv. ang. × cos adj. side. Cos 2 seg. = $\frac{\text{cos 1 seg.} \times \text{tan giv. side adj. giv. ang.}}{\text{tan side opp. given angle.}}$ Tan 1 seg. side req. = cos giv. ang. × tan adj. side.
	Third side.	Let fall a perpendicular as before.	Cos 2 seg. = $\frac{\text{cos 1 seg.} \times \text{cos side opp. giv. ang.}}{\text{cos side adj. given angle.}}$
III. Two sides and the included angle.	An angle opposite to one of the given sides.	Let fall a perpendicular from the third angle.	Tan 1 seg. of div. side = cos giv. ang. × tan side opp. ang. sought. Tan ang. sought = $\frac{\text{tan giv. ang.} \times \text{sin 1 seg.}}{\text{sin 2 seg. of div. side.}}$ Tan 1 seg. of div. side = cos giv. ang. × tan other given side.
	Third side.	Let fall a perpendicular on one of the given sides.	Cos side sought = $\frac{\text{cos side not div.} \times \text{cos 2 seg.}}{\text{cos 1 seg. of side divided.}}$
IV. A side and the two adjacent angles.	A side opposite to one of the given angles.	Let fall a perpendicular on the third side.	Cot 1 seg. of div. ang. = cos giv. side × tan ang. opp. side sought. Tan side sought = $\frac{\text{tan giv. side} \times \text{cos 1 seg. div. ang.}}{\text{cos 2 seg. of divided angle.}}$
	Third angle.	Let fall a perpendicular from one of the given angles.	Cot 1 seg. div. ang. = cos given side × tan other giv. angle. Cos angle sought = $\frac{\text{cos ang. not div.} \times \text{sin 2 seg.}}{\text{sin 1 seg. div. angle.}}$

TABLE II.—*Continued.*

Given.	Required.	Value of the quantities required.
V. The three sides.	An angle by the sine or cosine of its half.	Let a, b, c, be the three sides; A, B, C, the angles; b and c, including the angles sought, and $s = a + b + c$. Then, $\sin \frac{1}{2} A = \sqrt{\frac{\sin. (\frac{1}{2} s - b) \sin (\frac{1}{2} s - c)}{\sin b \sin c.}}$ $\cos \frac{1}{2} A = \sqrt{\frac{\sin \frac{1}{2} s. \sin (\frac{1}{2} s - a)}{\sin b. \sin c.}}$
VI. The three angles.	A side by the sine or cosine of its half.	Let S be the sum of the angles A, B, C, and B and C be adjacent to a, the side required; then, $\sin \frac{1}{2} A = \sqrt{- \frac{\cos \frac{1}{2} S \cos (\frac{1}{2} S - A)}{\sin B. \sin C.}}$ $\cos \frac{1}{2} A = \sqrt{- \frac{\sin (\frac{1}{2} S - B) \sin (\frac{1}{2} S - A)}{\sin B. \sin C.}}$

TABLE III.

For the solution of all the cases of oblique angled spherical triangles, by the analogies of Napier.

Given.	Required.	Value of the terms required.
I. Two angles and the sides opposite to one of them.	Side opposite to the other given angle	By the common analogy; viz. The sines of the angles are as the sines of the opposite sides
	Third side.	tan of its half $\begin{cases} = \frac{\tan \frac{1}{2} \text{ dif. giv. sides} \times \sin \frac{1}{2} \text{ sum opp. ang.}}{\sin \frac{1}{2} \text{ dif. of those angles.}} \\ = \frac{\tan \frac{1}{2} \text{ sum giv. sides} \times \cos \frac{1}{2} \text{ sum opp. ang.}}{\cos \frac{1}{2} \text{ dif. those angles.}} \end{cases}$
	Third angle.	By the common analogy.
II. Two sides and an opposite angle.	Angle opposite to the other known side.	By the common analogy.
	Third angle.	cot of its half $\begin{cases} = \frac{\tan \frac{1}{2} \text{ dif. other two ang.} \times \sin \frac{1}{2} \text{ sum giv. sides}}{\sin \frac{1}{2} \text{ dif. those sides.}} \\ = \frac{\tan \frac{1}{2} \text{ sum other two ang.} \times \cos \frac{1}{2} \text{ sum giv. sides}}{\cos \frac{1}{2} \text{ dif. those sides.}} \end{cases}$
	Third side.	By common analogy.
III. Two sides and the included angle.	The two other angles.	$\begin{cases} \tan \frac{1}{2} \text{ dif.} = \frac{\cot \frac{1}{2} \text{ giv. angle} \times \sin \frac{1}{2} \text{ dif. given sides}}{\sin \frac{1}{2} \text{ sum of those sides.}} \\ \tan \frac{1}{2} \text{ sum.} = \frac{\cot \frac{1}{2} \text{ giv. angle} \times \cos \frac{1}{2} \text{ dif. given sides}}{\cos \frac{1}{2} \text{ sum of those sides.}} \end{cases}$
	Third side.	By the common analogy.
IV. Two angles and the included side.	The other two sides.	$\begin{cases} \tan \frac{1}{2} \text{ dif.} = \frac{\tan \frac{1}{2} \text{ giv. side} \times \sin \frac{1}{2} \text{ dif. giv. angle}}{\sin \frac{1}{2} \text{ sum of those angles}} \\ \tan \frac{1}{2} \text{ sum} = \frac{\tan \frac{1}{2} \text{ giv. side} \times \cos \frac{1}{2} \text{ dif. giv. angle}}{\cos \frac{1}{2} \text{ sum of those angles.}} \end{cases}$
	Third angle.	By common analogy.
V. The three sides.	Either of the angles.	Let fall a perpendicular on the side adjacent to the angle sought; then, $\left\{ \begin{matrix} \tan \frac{1}{2} \text{ sum or } \frac{1}{2} \text{ dif. of the seg-} \\ \text{ments of the base} \end{matrix} \right\} = \frac{\tan \frac{1}{2} \text{ sum} \times \tan \frac{1}{2} \text{ dif. of the sides}}{\tan \frac{1}{2} \text{ base.}}$ Cos angle sought = tan adj. seg. × cot adj. sides.
VI. The three angles.	Either of the sides.	These will be determined, by finding the corresponding angle, by the last case, of a triangle, which has all its parts supplemental to those of the triangles, whose three sides are given.

4. *Problems relative to the determination of the position of the heavenly bodies.*

Rising and setting of the heavenly bodies.

63. It is obvious from what we have stated relative to the position of the different circles of the sphere, that one of the most important data in the solution of astronomical problems is the latitude of the place of the observer, from which the zenith Z in the preceding figures is determined. This may be found as follow.

Problem I.

To find the latitude of any place on the earth's surface.

To determine the latitude of the place.

1. Observe the altitude of the pole above the horizon of any place, and that altitude will be equal to the latitude.

The latitude of any place on the earth is measured by the arc EZ, that is, by the arc subtended between the equator and zenith. But EZ + ZP = 90° and ZP + PR = 90°; whence

$$EZ + ZP = ZP + PR$$

Consequently EZ = PR

That is, the latitude of any place is equal to the elevation of the pole above the horizon of that place.

By observations on the circumpolar stars.

The elevation of the pole above the horizon, may be practically determined by observing the greatest and least altitude of any of the circumpolar stars, and taking half the sum of the two altitudes; the proper corrections being made for refraction, parallax, &c. according to the principles explained in a subsequent chapter.

For let *xy*, fig. 21, represent the circle of declination described by any circumpolar star, then R*x* will be its greatest meridian altitude, and R*y* its least, and it is obvious that

$$RP = \tfrac{1}{2}\,(Ry + Rx)$$

By observations on the sun.

2. The latitude may also be found by observing the altitudes of the sun when he has attained his greatest north and greatest south declination. Half the sum will be the elevation of the equator above the horizon, and the complement of that angle the latitude of the place of the observer.

Referring to the same figure, let *ea* be the declination circle described by the sun when he has the greatest north declination, then *e*H will be his greatest altitude on that day; let *st* in like manner be the declination circle described on the day when he has the greatest south declination; then H*e*, will be its meridian altitude on that day; and since E*e* = E*s*, it is obvious that

$$HE = \tfrac{1}{2}\,(He + Hs)$$

And 90° − HE = EZ the latitude.

64. It is obvious, also, from what is stated above, and referring to our definition of the *obliquity* of the ecliptic, that this angle is measured by half the arc *se*, that

$$\tfrac{1}{2}\,(He - Hs) = \text{the obliquity of the ecliptic}$$

Problem II.

To find the time of the rising, setting, &c. of the heavenly bodies.

To find the time of rising, &c

65. Let the proposed body be the sun, and let us suppose that its declination remains constant during its passage from one meridian to the other, and that a clock is adjusted to go 24 hours during this one apparent revolution of the sun, and moreover, that the clock shows 12 exactly, when the sun is on the meridian; *to find the time of its rising, and its azimuth at that time; the latitude of the place and the declination of the sun being given.*

Referring to fig. 24, and comparing it with what has been stated with reference to fig. 21 (art. 59,) it appears that the sun rises when it comes to *b*; that it is twelve o'clock when the sun is upon the meridian at *e*, and that the whole circle *aea* is described in 24 hours; that is uniformly at the rate of $\frac{360}{24}$° = 15° per hour; to find the time of rising, therefore, we have only to compute the angle ZP*b*, and to convert it into time at the rate of 15° to an hour in time; and to find its azimuth from the north we must compute the angle RP*b*, for which computations we have the following data:

*b*Z = 90°, E*e* = declination; *e*P = P*b* co-declination
EZ = latitude; ZP = co-latitude

Hence in the triangle ZP*b*, we have the three sides given and one of them Z*b* = 90°, to find the angle ZP*b*. This case may therefore be solved by our fifth form in the preceding Table II. for oblique angled triangles, but one of the sides being 90°, it will be more readily solved by means of Napier's analogy; viz.

rad : cot *b*P :: cot ZP : cos ZP*b* = hour angle
or rad : tan.dec :: tan.lat : cos ZP*b* = hour angle

Exam. 1. Let us, for example, suppose the latitude of the place to be 52° 13′ north; the declination 23° 28′, to find the time of sun's rising.

By the above analogy,

rad .... 10·0000000
tan 23° 28′.. 9·6376106
tan 52° 13′.. 10·1105786

Cos 124° 2′ 9·7481892

(taking the supplement of the tabular angle 55° 58′, the angle being obviously greater than 90°.)

According to either solution, therefore, we have the hour angle = 124° 2′,

which converted into time gives

15° : 124° 2′ :: 1h. : 8h. : 19¼′ time from noon

consequently,

h.	m.	″.	
12	0	0	
8	19¼	0	
3	40¾	0	the time of rising

That is in the latitude 52° 13′, on the longest day, or when the sun has 23° 28′ north declination, he will rise at 3h. 40¾m.

To find the azimuth from the north, we have in the same triangle the same data to find the angle PZ*b*, which is the measure of the azimuth sought. This may therefore be determined by means of our form 5, oblique spherical triangles; but more concisely by the following analogy:—

sin ZP : rad. :: cos *b*P : sin PZ*b* = azimuth.

Hence ar. com. cos 52° 13′........ 0·2127683
rad. ....10·0000000
sin 23 28 .... 9·6001181

cos 49° 32′ azim........... 9·8128864

Exam. 2. Let it be required to determine the time of sun rising at the same place, lat. 52° 13′ on the 25th of February, 1818.

The declination on this day by nautical almanack is 9° 11½′ south. Consequently, referring to the same fig. 24, $b' = 99° 11\frac{1}{2}'$. Hence, then, in the triangle ZPb', we have as before, ZP = co. lat. = 37° 47′, Pb' codec. 99° 11½′, Zb' = 90° Whence. as in the former case,

rad.	10·0000000
tan dec. 9° 11½′.........	9·2090197
tan 52° 13′	10·1105786
cos 77° 57′ hour angle..	9·3195983

This converted into time gives 5h. 12m. from noon; which, taken from 12 hours, gives 6h. 48m. for the time sought.

Problem III.

To find the sun's altitude at 6 o'clock, azimuth, &c.

66. Here, since PP′ (fig. 25) bisects *e a*, it is clear that the sun will be at *c*, at 6 o'clock; and we have therefore, in this case, the hour angle ZP*c* = 90° given, and the two sides ZP, P*c*, to find Z*c*, the co-altitude; that is, two sides and the contained angle are given to find the third side. Whence, by our fifth form for right-angled spherical triangles, we have

$$\cos Zc = \frac{\cos ZP \times \cos Pc}{\text{rad.}}$$

Or, which is the same, (adopting the data of the first of the preceding examples,)

rad..................	10·0000000
sin lat. 52° 13′	9·8978103
sin dec. 23° 28′	9·6001181
sin. alt. 18° 21′	9·4979284

67. To find at what time in the day the sun will be east and west; that is, in the prime vertical, and its altitude at that time; taking the latitude of the place 52° 12′ 35″, and the sun's declination 23° 28′.

Here, taking ZP to denote the co-latitude = 37° 47′ 25″ P*b* the co-declination; and the angle *b* ZP being 90°, we have the two sides ZP, P*b*, of the right-angled spherical triangle *b* ZP, and the angle *b* ZP, a right angle, to find the hour angle ZP*b*, and the co-altitude Z*b*.

By preceding Table 1, form 1,

$$\cos Zb = \frac{\cos Pb}{\cos ZP} = \frac{\text{sin dec.}}{\text{sin lat.}} = \text{sin alt.}$$

$$\cos ZPb = \frac{\tan ZP}{\tan Pb} = \frac{\text{cos lat.}}{\text{cos dec.}} = \text{hour angle.}$$

sin dec. 23° 28′	= 9·6001181
sin lat. 52° 12′ 35″	= 9·8977695
sin alt. = 30° 15′ 31″	9·7023486

Again,	cos lat. 52° 12′ 35″ =	9.8995301
	cos dec. 23 28	= 10·3623894
	cos hour angle = 70° 19′ 44″....	9·5371407

Which, converted into time, gives 4h. 41′ 19″ from apparent noon.

Plane Astronomy.

Azimuth and hour.

68. Given the latitude of the place, the sun's declination; and altitude to find the hour and azimuth.

Here, referring to the same figure, we have the co-latitude ZP, the co-altitude Z*b*, and the co-declination P*b* to find the hour angle ZP*b*, and the azimuth PZ*b*. That is, in a spherical triangle, we have three sides to find the angles.

Whence, calling the sum of the three sides *s*, we have, by form 5, of our Table II.,

$$\tfrac{1}{2}\cos ZPb = \sqrt{\left\{\frac{\sin\frac{1}{2}s\ \sin(\frac{1}{2}s - Zb)}{\sin ZP\,.\,\sin Pb}\right\}}$$

Let the latitude be 34° 55′, the declination 22° 22′ 57″ N, the altitude 36° 59′ 39″. Then ZP = 55° 5′, Z*b* = 53° 0′ 21″, P*b* = 67° 37′ 3″. Hence the operation,

P*b* = 67°	37′	3″	sin arith. comp.	0·0340191
ZP = 55	5	0	sin arith. comp.	0·0861939
Z*b* = 53	0	21		
sum 175	42	24		
½*s* 87	51	12	sin =	9·9996942
Z*b* 53	0	21	sin =	9·7569320
½*s*—Z*b*= 34	50	51		2)19·8768392
½ cos ZP*b* = 29°	47′	44″	=	9·9384196

Hence, ZP*b* = 59° 35′ 28″, which, reduced to time, gives 3h. 58′ 22″ the time from apparent noon. In the same manner

$$\tfrac{1}{2}\cos PZb = \sqrt{\left\{\frac{\sin\frac{1}{2}s\,.\,\sin(\frac{1}{2}s - Pb)}{\sin Zb\,.\,\sin ZP}\right\}}$$

The numerical solution of which, we leave as an exercise for the reader.

Meridian line.

69. By means of the azimuth determined as above, it will not be difficult to draw a meridian line, by simply determining its position, so as to make the requisite angle with the direction of the sun at that time. This, however, of course supposes that the altitude of the sun has been properly corrected for refraction and parallax, which being subjects we have not yet touched upon, we merely indicate the principle of the determination, and shall leave our further discussion on it to a subsequent chapter.

Error in altitude.

70. In the preceding examples relative to those questions in which the altitude is supposed to be determined from observation, it is obvious that we not only suppose the requisite corrections to have been applied, but also that the observation is made without any appreciable error; but as such error is easily made, particularly if the instrument be not of the most perfect construction, it may not be amiss to ascertain what effect any such error will produce in the computed time, and the azimuth the sun ought to have, so that an error in altitude shall produce the least error in time; we propose therefore the following problem.

Problem IV.

Given the error in altitude to find the error in time.

Error in time.
Fig. 26.

71. Let EQ, fig. 26, represent the equator, P the pole, *a e* the parallel of declination, in which the sun, or

Astronomy.

any other heavenly body is on the day of observation; and let r be real, and s the apparent place of the body, or $r\,s$ the error in altitude. Draw $m\,s$ parallel to the horizon; from Z, the zenith, draw $Z\,m$, $Z\,r$; and from P the pole, the meridians $P\,m\,p$, $P\,r\,q$, to pass through $m\,r$, and to cut the equator in the points $p\,q$, then m and r will be the real and apparent places of the body, on the parallel of declination; and the arc $m\,r$ on that circle, or the arc $p\,q$ on the equator, will measure the angle $m\,P\,r$, the corresponding error in time. Now the triangle $m\,s\,r$ being of course exceedingly small, it may be regarded as a rectilinear right-angled triangle, right angled at s;
and hence we have

$$s\,r : r\,m :: \sin s\,m\,r : \text{rad.}$$

$$\text{by spherics } m\,r : p\,q :: \cos q\,r : \text{rad.}$$

Multiplying these together, rejecting the like factors, we have

$$s\,r : p\,q :: \sin s\,m\,r \cos q\,r : \text{rad.}^2$$

Whence

$$p\,q = s\,r \times \frac{\text{rad.}^2}{\sin s\,m\,r \cdot \cos q\,r}$$

But $Z\,r\,P = s\,m\,r$, $s\,r\,m$ being the complement of each, and $\sin Z\,r\,P = (\sin s\,m\,r) : \sin PZ :: \sin r\,ZP : \sin P\,r$

Consequently,

$$\sin s\,m\,r \,.\, \sin P\,r = \sin s\,m\,r \,.\, \cos q\,r = \sin PZ \,.\, \sin r\,ZP$$

Whence, substituting for the denominator in the above expression, its equivalent in the last, we have,

$$p\,q = r\,s \times \frac{\text{rad.}^2}{\sin PZ \cdot \sin r\,ZP} = \frac{r\,s}{\cos \text{ lat. } \sin r\,ZP}$$

taking radius = 1.

Hence, since all the quantities in this expression are supposed to be given, except $\sin r\,ZP$, it is obvious, that $p\,q$ will *cæteris paribus* be the least, when $\sin r\,ZP$ is the greatest, or when the azimuth is the greatest; that is, when the body, whose altitude is observed, is in the prime vertical: it is best, therefore, to deduce the time from an observation, when the body is in or near that circle.

As an example in numbers, suppose the latitude to be 52° 12′ 35″; the declination 15° 24′ 25″; and the altitude as corrected, 40°; required the error in time, supposing an error of 1″ in the altitude.
This gives

$$1' \times \frac{1}{\cos \text{ lat. } \sin \text{ azim.}} = 1'.882$$

which answers to 7.528 seconds in time.

Again, supposing the azimuth to be 46° 22′, the latitude 52° 12′, and the error in altitude 1′, we have

$$1' \times \frac{1}{.612 \times .690} = 2'.334 \text{ of a degree} = 9''.336 \text{ in}$$

time.

Time of the sun ascending through an arc equal to its diameter.

72. By means of this solution, we may ascertain the time in which the sun will pass either the horizontal or vertical wire of a telescope: we have seen, for instance, that the time during which the sun will ascend through any small altitude $= r\,s$, or the arc that he will describe when referred to the equator during that time, is

$$p\,q = \frac{r\,s}{\cos \text{ lat. } \sin r\,ZP}$$

in which, substituting d'' the apparent diameter of the sun in seconds for $r\,s$, this becomes

$$\frac{d''}{\cos \text{ lat. } \sin \,.\, \text{azimuth}}$$

consequently,

$$\frac{d''}{15 \,.\, \cos \text{ lat. } \sin \,.\, \text{azim.}} = \text{time in seconds.}$$

73. To find the time in which the sun will pass over the vertical wire, we may take $m\,r$ to denote the apparent diameter of the sun; then, as the seconds in $m\,r$, considered as a small circle, must be increased in proportion as the radius is diminished; (because, when the arc is given, the angle is inversely as the radius,) we have

$\sin P\,r$, or $\cos \,.\, \text{dec.} : \text{rad} :: d''$ the seconds in $m\,r$. considered as a great circle, to the seconds in the same considered as a small circle, which are the seconds in $p\,q$; whence

$$p\,q = \frac{\text{rad.}\, d''}{\cos \text{ dec.}} = d''.\, \text{sec. dec.}$$

and, consequently,

$$\text{the time} = \frac{d'' \times \text{sec. dec.}}{15''}$$

Hence $p\,q = d''$ sec. dec. expresses the sun's diameter in right ascension. If therefore we assume his apparent diameter at 32′ = 1920″, and its declination 20°, its right ascension is

$$1920'' \times \text{sec. } 20^\circ = 34'\ 2''\,.\,88$$

which, divided by 15, will give the same in time.

In the Nautical Almanack a column is given for indicating the time in which the semi-diameter of the sun passes over a vertical wire, by means of which a single observation on either limb, may be referred to the centre.

Problem V.

To find the time when the apparent diurnal motion of a fixed star is perpendicular to the horizon.

74. Let $x\,y$ (fig. 26) be the parallel of declination described by the star; draw the vertical $Z\,h$, touching it at o; then, when the star arrives at o, its apparent motion will be perpendicular to the horizon; the two circles having in the point o, a common tangent. And $Z\,o\,P$ is a right angle, we have

$$\text{rad} : \tan o\,P :: \cos PZ : \cos ZP\,o$$

$$\text{that is, } \quad \text{rad} : \cos \text{dec.} :: \tan \text{lat.} : \cos ZP\,o$$

which, converted into time, gives the time from the star's being on the meridian; the latter therefore being supposed known, the former is readily determined.

Problem VI.

Given the right ascension and declination of a heavenly body, and the obliquity of the ecliptic, to find its latitude and longitude.

75. Let s (fig. 27) be the body, VC the ecliptic, VQ the equator, the angle QVC denoting the given angle of the obliquity of the ecliptic; let s V be joined by the arc of a great circle s V, and let fall the perpendiculars $s\,p$, $s\,r$; then it is clear, from our definitions, that V p will be the right ascension of the body s, $s\,p$

its declination, V r its longitude, and $s\,r$ its latitude: we have therefore given the V p, $s\,p$, and the angle p V r to find V r, and $r\,s$.

Now, first in the right-angled spherical triangle V $p\,s$, we have, by form 5, Table I.

$$\tan s\,\mathrm{V}\,p = \frac{\tan s\,p}{\sin \mathrm{V}\,p}$$

whence the angle s V p becomes known, and consequently also s V r, because

$$s\,\mathrm{V}\,r = s\,\mathrm{V}\,p + p\,\mathrm{V}\,r$$

Again,

$$\cos \mathrm{V}\,s = \cos \mathrm{V}\,p \,.\, \cos p\,s$$

whence V s also becomes known.
Therefore, in the right-angled triangle r V s, we have the hypothenuse V s, and the angle r V s to find the two sides V r, $r\,s$:
which, by our form 4, Table I. are determined as follows:

$$\tan \mathrm{V}\,r = \tan \mathrm{V}\,s \times \cos s\,\mathrm{V}\,r = \tan \text{long.}$$
$$\sin s\,r = \sin \mathrm{V}\,s \times \sin s\,\mathrm{V}\,r = \sin \text{lat.}$$

In like manner, the right ascension and declination of any body may be found, when its latitude and longitude are given; but generally the problem is to determine the latter from the former, these being first ascertained from actual observation: it is thus the tables of the latitudes and longitudes have been computed.

But as both the ecliptic and the equator are subject to a change in their positions, the right ascension, declination, latitude, and longitude of all the fixed stars are constantly varying, and therefore those tables formed for any particular epoch, will not answer correctly after a certain time; the annual variations, however, being computed, their right ascensions, &c. may be determined for any proposed time, as will be hereafter explained.

5. *Of the crepusculum, or twilight.*

76. The crepusculum, or twilight, is that faint light which is perceived before the rising of the sun, and after its setting. It is occasioned by the terrestrial atmosphere, refracting the rays of the sun, and reflecting them amongst its particles.

The depression of the sun below the horizon, at the beginning of the morning and at the end of the evening twilight, has been variously stated at different seasons, and by different authors; for example,

Alhazen observed it to be	19°
Tycho	17°
Rothman	24°
Stevinus	18°
Cassini	15°
Riccioli, at the equinox	16°
summer solstice	21°
winter solstice	17¼°

but we more commonly now in our latitudes assume it 18°; the same both for morning and evening and for all seasons of the year.

Problem I.

Given the latitude of the place and the sun's declination, to find the time when twilight begins.

77. Here S denoting the place of the sun when twilight begins, we have in the triangle ZPS, (fig. 28) ZS = 90° + 18° = 108°, the co-latitude ZP, and the co-declination PS, to find the hour angle ZPS from apparent noon.

Having thus the three sides, the angle required may be found by our 5th form, Table II.; that is, assuming ZP + ZS + PS = s

$$\tfrac{1}{2} \cos \mathrm{ZPS} = \sqrt{\frac{\sin \frac{1}{2} s \,.\, \sin (\frac{1}{2} s - \mathrm{ZS})}{\sin \mathrm{ZP} \sin \mathrm{PS}}}$$

Whence the angle ZPS may be determined, and consequently the time from apparent noon.

Problem II.

Given the latitude of the place to find the time and duration of the shortest twilight.

Time of shortest twilight.

78. Let P fig. 28, denote the pole, Z the zenith of the observer, S the sun, 18° below the horizon ZS = 90° + 18° = 108°, or more generally, = 90° + 2a, where we suppose the twilight to begin when the sun is 2a degrees below the horizon; or that the observer whose zenith is Z, will then see the commencement of the morning twilight.

Now in consequence of the diurnal rotation, the declination circle PS, turning about the axis will bring the sun from S to S′, in the horizon, or which is the same, the zenith Z will approach nearer to S, by describing about P the little circle Z $m\,b$ Q, such that the distances Z m, P m, P b, PQ are all equal.

When, therefore, the zenith shall have descended from Z to any point m, such that mS = 90°, the sun will appear at 90° from the zenith, the day will commence and the twilight terminate. And the arc Zm of the small circle will be the measure of the angle ZPm, and consequently, of the duration of the twilight.

In order to determine this angle, draw the arc ZBm of a great circle, and to its middle B the perpendicular arc PB, then will

$$\sin \tfrac{1}{2}\,\mathrm{ZP}m = \sin \mathrm{ZPB} = \frac{\sin \frac{1}{2}\,\mathrm{Z}m}{\sin \mathrm{PZ}} = \frac{\sin \frac{1}{2}\,\mathrm{Z}m}{\cos \text{lat.}}$$

now in the spherical triangle Z m S, we have

$$\mathrm{S}m + m\mathrm{Z} > \mathrm{ZS}, \text{ or}$$
$$90^\circ + m\mathrm{Z} > 90^\circ + 2a$$

consequently, $m\mathrm{Z} > 2a$ and $\frac{1}{2}\,m\mathrm{Z} > a$

Let $\qquad \frac{1}{2}\,m\mathrm{Z} = a + x$

then,

$$\sin \tfrac{1}{2}\,\mathrm{ZP}m = \frac{\sin (a + x)}{\cos \text{lat.}} = \frac{\sin a \cos x + \sin x \cos a}{\cos \text{lat.}}$$
$$= \frac{\sin a + \cos a \sin x - 2 \sin a \sin^2 \frac{1}{2} x}{\cos \text{lat.}}$$

because, $1 - \cos x = 2 \sin^2 \frac{1}{2} x$ (See Trigonometry)

Now the last expression is equivalent to

$$\sin \tfrac{1}{2}\,\mathrm{ZP}m = \frac{\sin a}{\cos \text{lat.}} + \frac{2 \sin \frac{1}{2} x \,(\cos a + \frac{1}{2} x)}{\cos \text{lat.}}$$

And here $\frac{1}{2} x$ is essentially positive, and a and $\frac{1}{2} x$ small angles, such that $a + \frac{1}{2} x < 90^\circ$. Whence the latter part of the expression will be positive, and we shall have

$$\sin \tfrac{1}{2}\,\mathrm{ZP}m > \frac{\sin a}{\cos \text{lat.}}$$

It is further evident that the twilight will be longer as x is greater, and shorter as x is less, and that it will be the shortest possible when x is zero; because in that case, the second member of the second side of the equation will vanish.

But this is what occurs when the triangle ZmS is reduced to the arc ZS, that is, when the point m falls on b, or when the distance PS is such, that the part Zb, of the vertical ZS, lying within the small circle ZmQ is equal to $2a$, and the exterior part $bS = 90°$. It is also manifest, that if PS increases, the opposite angle PZS will in like manner increase, and that on the contrary that angle will diminish as PS diminishes. In these variations, the point m will approach to or recede from Z, the intercepted part bZ will vary between the limits 0 and $ZQ = 2\ PZ$. Thus the intercepted part may have all values from 0 to $2\ (90° - \text{lat.}) = 180° - 2\ \text{lat.}$ and consequently may have the value $2\ a$: hence in the case where $Zb = r\ a$, and $bS = 90°$, the shortest twilight will obtain; and the semi duration in degrees will be found by means of the equation

$$\sin ZPB = \frac{\sin a}{\cos \text{lat.}} = \sin a \sec \text{lat.}$$

Whence the duration in time is readily determined.

Fig. 29. 79. Other theorems are deducible from the same construction and investigations; for on the arc $Zb = 2a$, and with the complement of ZP of the latitude constitute the isosceles triangle ZPb, (fig. 29) and let fall the perpendicular Pm; then ZPb will be the angle which measures the duration. Prolong Zb, till $bS = 90°$, and draw the arc PS which will be the sun's co-declination for the day of the shortest twilight.

Sun's azimuth. Now,

$$\cos mP = \frac{\cos PZ}{\cos Zm} = \frac{\cos PS}{\cos mS}$$

Wherefore,

$$\cos Zm : \cos mS :: \cos PZ : \cos PS$$

or, $\cos a : \cos (90° + a) : \sin \text{lat.} : \sin \text{dec.}$

or, $\cos a : - \sin a :: \sin \text{lat.} : \sin \text{dec.}$

whence, $\sin \text{dec.} = \dfrac{- \sin a}{\cos a} \sin \text{lat.} = - \tan a . \sin \text{lat.}$

Again, in the right angled triangle ZPm, we have

$$\tan Zm = \cos Z \tan PZ$$

and $\cos Z = \tan Zm \cot PZ = \tan a . \tan \text{lat.}$

Now the angle Z or PZm is the sun's azimuth at the commencement of the twilight, and

$$PZb = PbZ = 180° - PbS$$

But PbS is the sun's azimuth at the instant, when his centre is upon the rational horizon; wherefore *the sun's azimuth at the beginning and ending of the twilight are supplements to each other on the day of the shortest twilight,*

Hence, since $\cos PZS = \tan a . \tan \text{lat.}$

we have $\cos PbS \doteq - \tan a . \tan \text{lat.}$

Hour angle. 80. In like manner ZbS and bPS are the hour angles, at the beginning and end of the twilight; let the former be denoted by P' and the latter by P; then

$$ZPS - bPS = P' - P = ZPb$$

which is the angle that measures the duration of twilight. Hence we have from what has been done above

$$\sin \tfrac{1}{2} (P' - P) = \frac{\sin a}{\sin \text{lat.}}$$

Also,

$$mPS = bPS + mPb = P + \tfrac{1}{2} (P' - P) = \tfrac{1}{2} (P' + P)$$

and

$$\sin mPS = \frac{\sin Sm}{\sin PS} = \frac{\sin (90° + a)}{\cos \text{dec.}} = \frac{\cos a}{\cos \text{dec.}}$$

that is, $\sin \tfrac{1}{2} (P' + P) = \dfrac{\cos a}{\cos \text{dec.}}$

Whence from the two equations,

$$\sin \tfrac{1}{2} (P' - P) = \frac{\cos a}{\cos \text{dec.}}$$

$$\sin \tfrac{1}{2} (P' + P) = \frac{\sin a}{\cos \text{lat.}}$$

the hour angles P' and P are easily determined, the declination itself being given by the equation

$$\sin \text{dec} = - \tan a . \sin \text{lat.}$$

We have also,

$$\cos PSZ = \frac{\sin \text{lat.}}{\cos \text{dec.}}$$

Lastly, let $ST = 2a = Zb$; and draw PT, then $ZT = 90°$, and T is a point in the horizon for the moment when the zenith was in Z. Whence the triangle PZT gives

$$\cos PT = \cos Z \sin PZ \sin ZT + \cos PZ \cos ZT$$
$$= \cos Z \sin PZ = \tan a . \tan \text{lat.} \cos \text{lat.}$$
$$= \tan a . \sin \text{lat.} = \sin \text{dec.}$$

Therefore, $90° - PT = \text{dec.}$, or $PT = 90° - \text{dec.}$

But $PS = 90° + \text{dec.}$

therefore, $PT + PS = 180°$

which is another remarkable property of the shortest twilight.

81. In all the above deductions, the latitude has been left general, or indeterminate, as has also the quantity $2\ a$, which denotes the number of degrees that the sun is below the horizon when the twilight begins. The latitude may therefore be introduced for any given place, as may also $2\ a$; but commonly, we assume $2a = 18°$, that is, the twilight is supposed to commence when the sun is 18° below the horizon.

Example. Required the day on which the twilight is the shortest at Woolwich, the latitude of which is 51° 28′ 40″, in the year 1820, with its duration, and the time of its beginning and end.

First, for the declination we have

$$\sin \text{dec.} = - \tan a \sin \text{lat.}$$

Now $\log \tan a = 9° = 9{\cdot}1997125$

$\log \sin \text{lat. or } 51° 28' \tfrac{2}{3} = 9{\cdot}8934104$

$\log \sin \text{dec.} = - 7° 7' 5'' \quad 9{\cdot}0931229$

The declination, therefore, is 7° 7′ 5″ south, which answers to March 2 and October 11.

Again, for the duration, we have

$$\sin \tfrac{1}{2} (P' - P) = \frac{\sin a}{\cos \text{lat.}}$$

from $\log \sin 9° = 9{\cdot}1943324$

take $\log \cos 51° 28' \tfrac{2}{3} = 9{\cdot}7943613$

$\log \sin 14° 32' 41'' \quad 9{\cdot}3999711$

Whence $P' - P = 29° 5' 22''$

Also $\sin \tfrac{1}{2} (P' + P) = \dfrac{\cos a}{\cos \text{dec.}}$

Hence from $\cos 9° = 9.9946199$

take $\cos 7° 7' 5'' = 9.9966399$

$\sin 95° 31' 19'' \quad 9.9979800$

$$\begin{aligned} \text{whence } P' + P &= 191^\circ\, 2'\, 38'' \\ P' - P &= \ \ 29\ \ 5\ \ 22 \\ \hline & 2)220\ \ 8 \end{aligned}$$

half sum = ... $110 - 4 =$ angle P'
half dif. = ... $80\ 58 - 38 =$ angle P

	h	m	s
The former in time answers to	7	20	16¼
and the latter to	5	23	54¾

The former showing the time when the evening twilight ends, and the latter the time of the sun's setting, or the time of its beginning; consequently their difference, 1h. 56m. 21½s., is the duration.

The above numbers, taken respectively from 12, will leave the time of the beginning and end of the morning twilight.

82. In all the preceding investigations, we have considered only apparent time; that is, we have supposed it to be 12 o'clock when the sun is on the meridian, and that it is exactly 24 hours in passing from one meridian to the same again; but if a clock be adjusted to go thus for one day, that is, if it show exactly 24 hours between the time of the sun being twice successively in the same place, it will not continue to show 12 o'clock every day when the sun comes to the meridian, because the intervals of time from the sun's leaving a meridian to his return to it again, are not always equal. This difference between the sun and a well regulated clock is called the *equation of time*, which will be treated of in a subsequent chapter; at present, we shall not enter farther upon the subject, it is sufficient to apprize the reader that the time as determined in all the preceding problems, is what is called *apparent time*, or the time shown by the sun, and not *mean or true time*, which is that shown by a well regulated clock.

83. Beside this correction for the time, there are also other corrections which must be introduced into the data of the preceding examples, in order to render the results perfectly conformable with observation. Thus we have all along supposed the body to rise as soon as it is found in the rational horizon; but all bodies in the heavens, when in or near the horizon, are elevated 33′ by refraction above their true places; this, therefore, would make them appear when they are actually 33′ below the horizon, or when they are 90° 33′ from the zenith; and all the way from the horizon to the zenith refraction has the effect of elevating the apparent places of the heavenly bodies, but in a less degree as the altitude is greater; till it vanishes in the zenith. The altitudes, therefore, as we have given them, are supposed to have been subjected to these corrections, the method of making which will be explained hereafter. This is one of the principal corrections for the fixed stars; but for the sun or any other of the bodies of our system, a different correction becomes necessary, these being all depressed below their true places by the effect of parallax, as will also be explained in a subsequent chapter; that is, we have confounded the sensible and rational horizon, which is admissible as far as relates to the fixed stars, in consequence of their immense distance; but the angle subtended by them or by the earth's radius, at any of the bodies of our system, is a sensible quantity that must not be neglected in any computations relative to such bodies. The parallax has therefore a tendency to increase the apparent zenith distance of any body in our system, while the refraction tends to diminish it; therefore the actual zenith distance of a body when it first becomes visible to a spectator on the earth, is equal to 90° − hor. parallax + refraction.

84. What has been hitherto done, has been merely to indicate the nature of the calculations after certain observations have been made and corrected; we now propose to explain the instruments used for making these observations, the method of employing them for these purposes, and the nature and quantity of the corrections that are requisite for reducing the results thus obtained, so as to render them proper for the purposes of astronomical computation.

§ IV. *Description and use of the most indispensable astronomical instruments.*

Astronomical instruments. 85. We shall not attempt in this place a general description of astronomical instruments, this will be given under a distinct head in another part of this work; our purpose here is only to describe those whose use is indispensable in the pursuit of this science, and with the principle of whose construction and operation, it is essential that the student be well informed, if he wish to proceed upon any other data than those furnished by the report of other observers.

1. *Of the astronomical telescope.*

Astronomical telescope. 86. We have already in our treatise on Optics described and illustrated the principles of this instrument, we have therefore in this place merely to speak of its application to astronomical observations.

Fig. 30. Let, then, AB (fig. 30) represent the diameter of any heavenly body, as the sun, moon, or planet; *m* its middle point, will send out a luminous pencil of rays, which by means of the refraction of the lenses will be united in the focus of the telescope. The same will be the case with all the points of the disc, and an image of the object GH will be formed, as we have already explained in our treatise on Optics; H being the image of the point B, and G of the point A; that is, the image will be reversed with respect to its actual position. The motion of the object is also reversed, so that if it moves from left to right in the heavens, it will move from right to left in the telescope; but the effect being the same for all bodies, there will result from it no practical inconvenience; it is sufficient that we are apprized of it.

Let us farther observe, that the angle H*e*G = ACB, and if the focal distance *e*F is to the breadth HG in such a ratio that

$$\frac{\frac{1}{2}\text{HG}}{e\text{F}} = \tan \tfrac{1}{2}\,\text{ACB}$$

the object will be shown entire in the telescope; but if the focal distance be *ef* 7 CF, the image *hg*, will be larger than the field of the telescope, and consequently, we shall not then see it entire. On the contrary, if the focal distance be CQ, less than CF, the image *h*′Q*g*′ will not fill the telescope.

Generally, the field of the telescope is found by the formula

$$\tan \tfrac{1}{2}\,\text{field} = \frac{\tfrac{1}{2}\ \text{diam. of the tube}}{\text{focal length}}$$

$$\tan \tfrac{1}{2}\,\text{field} = \frac{\tfrac{1}{2}\ \text{diam. of the tube}}{\text{radius of curvature of the object glass}}$$

Or in case of a diaphragm being placed in the tube, as is commonly practised to prevent the reflection of the oblique rays, then

$$\tan \tfrac{1}{2}\ \text{field} = \frac{\tfrac{1}{2}\ \text{diam. diaphragm}}{\text{focal length}}$$

These results, which are immediately deducible from what has been done in the preceding treatise, are stated here merely for the sake of avoiding frequent references to those articles; with the same view we may state the following relative to the magnifying power of such an instrument as we are here speaking of, viz.

$$\text{magnifying power} = \frac{\text{focal distance obj. glass}}{\text{focal distance eye glass}}$$

By which is to be understood, that the angle under which we view the image is equal to that under which we should see the object if it were brought so many times nearer as is indicated by the above fraction.

Having said thus much with regard to the telescope more commonly employed in astronomical observations, let us offer a few remarks relative to the apparatus attached to it, for rendering those observations more precise than they could be obtained with this instrument in the simple state in which it has been described above.

Use of the cross wires Fig. 31.

87. *Of the reticule or cross wires.*—Let ABDF (fig.31) represent a section of the telescopic tube or of the diaphragm with which it is furnished. On the edges of this ring or circle, are attached with two screws a metallic thread or fine wire DF; this is in a transit instrument called the horizontal wire. Perpendicular to this thread are placed five others; the centre one AB bisecting DF in C, and the other four at equal distances, two on each side of AB.

A star or any other heavenly body passes the field of view of a telescope in different times according to the diameter of the instrument, and the polar distance of the body; therefore, if the telescope were unprovided with some such an apparatus as here described, it would be difficult or even impossible to say precisely the moment when it was in the axis or in the centre of the instrument, and the accuracy of modern science requires this determination to the utmost accuracy: let us see what precision is attainable by means of the cross wires above indicated:

The point C or G being supposed either in the centre of DF, or in the vertical line bisecting DF, at the moment a star passes it, if the diameter of the thread be equal to that of the star, it will be entirely hidden by it, and that moment will be the time of its passage; but more commonly the thread is sufficiently fine, that at the instant of passage, it will bisect the star, an equal portion of the latter being observable on each side of the former; and thus the time of passage might be found to within one-fourth or one-fifth of a second. But it is obvious that we may observe the same with respect to all the other four vertical wires, which being at equal distances, the times of passing each in succession from the first will form an arithmetical progression; and by taking the mean of the five, we shall have the time of the star passing the centre wire still more exactly, and by this means, we may generally depend upon our observation to within one-tenth of a second.

Our figure and the description of it, applies to the case of an instrument fixed in the plane of the meridian, in which case the motion of a heavenly body will be apparently horizontal. In any other case, the star ascends or descends obliquely, and then it is necessary to give to the wire DF, a similar inclination, so that the motion of the star may be parallel to it, as in the line ES, as shown in fig. 32, the proper apparatus being supplied for this purpose. The only difficulty of observation is when the night is very dark, and when we are unable to see the threads except for the moment when the star is bisected by them; which being almost instantaneous, we are not sufficiently prepared for noting the time. In order to obviate this difficulty, the interior of the tube is illuminated by the following apparatus.

Fig.

Meth enlig ing t tube.

88. In the side of the tube of the telescope, and commonly in the axis on which it turns, is made a small hole, directly opposite to which is placed, in the tube, a small mirror, inclined to the axis or sides of the telescope, at an angle of 45°. The light of a small lamp falls on the mirror, and forming with it an angle of 45°, it is reflected at the same angle, and therefore passes in a line parallel to the axis of the instrument, and thus renders the wires sufficiently visible. If the star on which the observation is made be of the 9th or 10th magnitude, we must however be careful to modify the intensity of the illumination, as otherwise, the artificial light will render the natural light of the star imperceptible, and it will be in danger of passing unobserved.

Obser tions o sun or moon.

89. At present we have spoken only of the transit of a fixed star, if it be the sun or moon that we are observing, we must ascertain the time when their respective centres pass the axis of the instrument. For this purpose we note very accurately, the instant when the eastern or western limb comes in contact with each of the five wires in succession, and the sum of the times divided by 5, will be the instant when that limb passed the centre wire. These five observations being made, the other limb of the sun or moon will be just about leaving the first wire, we therefore, in like manner, note these other five instants, the mean of which will give the time when the last limb passed the centre wire, and the mean of the two will be the time of the transit of the centre.

90. Having said thus much with reference to the telescope and the apparatus with which it is supplied, it remains for us to describe the instruments to which it is attached, and the nature of their adjustment; for the accuracy of observations indicated above with reference to the object traversing the axis of the telescope would be to little purpose, unless we could be equally precise in the determination of the direction of that axis, both as regards the horizontal and vertical position of the instrument.

By far the greater number of astronomical observations are made in the plane of the meridian; but some are made at different azimuths, and different instruments are best employed for these purposes; at the same time some are so constructed as to answer in both cases; at present we shall confine our explanation to only one of the most perfect of each sort, but in a future part we shall enter upon the subject more at length, and endeavour to illustrate the advantages and defects of different constructions.

Of astronomical quadrants.

91. These may be either portable or fixed; in the former case they are commonly mounted on a tripod, and may be used for taking altitudes in any azimuth, or be made to follow the body observed in its apparent path; but in the latter case, the instrument is fixed in the plane of the meridian against a substantial wall, and is hence denominated a *mural quadrant*.

2. *Portable astronomical quadrant.*

ble as- nical ant.

92. The great variety of forms under which this instrument has appeared, the numerous methods proposed by different artists for adjusting it to its verticality, &c. would be sufficient of themselves, were we to enter at length upon the subject, to form a considerable volume; we shall, therefore, select one of those esteemed the most perfect, and limit ourselves to the description of that only; which will be amply sufficient for our present purpose.

3:

Fig. 33 represents a portable quadrant, constructed by Ramsden for the observatory of Christ's College, Cambridge. The tripod on which it is mounted has screws of adjustment to set the stem, on which the horizontal motion is performed, perpendicular, which is proved to be so in all directions when the plumb-line bisects both the superior and inferior dots during the whole revolution in a horizontal circle. The visible stem is a brass tube, and through it ascends a solid steel vertical axis, which fitting closely at the upper and lower extremity, has not the least shake, and preserves the position once given to it, so long as the feet screws are unmolested. The telescope is of the achromatic construction, and has the usual apparatus for a slow motion. The telescope lies on a bar that carries the counterpoise, and in which is the centre of its motion. It has a system of wires in the focus of the eye glass, which are adjustable by screws both upwards and sideways, as well as in a circular direction, so that the adjustments for collimation, and for zero in the altitude of the circle, may be thereby effected. The point of suspension of the plumb-line is also adjustable by a proper screw apparatus. At the top of the vertical tube or stem, is a small horizontal circle with a clamping apparatus for slow horizontal motion, by means of which the whole quadrant with its attached telescope turns gradually round in azimuth. When the observation is made in or near the zenith, the plumb-line of this instrument falls in the way of the telescope, and is obliged to be removed. This, however, is supplied by the addition of a spirit level, suspended from an adjustable horizontal brass rod, under the uppermost radical bar of the quadrant, which level not only supplies the place of the plumb-line when taken off, but at all times serves as a check on its adjustment, and when furnished with a graduated scale, may very well be made its substitute. We shall not here attempt to describe the nature and division of the vernier scale, it will be sufficient to observe, that the angles may be ascertained to the 10ths of seconds. We proceed next to the adjustments of the instrument.

To adjust the axis of the pedestal vertical.

Adjustment of the instrument to verticality.

93. This adjustment may be performed either by the plumb-line or by the level. When the plumb-line is used, turn the quadrant in azimuth till its plane, or which is the same thing, till the telescope lies parallel to a line joining any two of its three feet, and turn one of the two screws of the feet till the wire bisects the lower dot, and with the proper screw bring the upper dot to the same wire; then reverse the telescope by turning 180° in azimuth, and if both dots are again bisected, the axis is vertical in the direction that the telescope has pointed; in the next place, turn the telescope the space of a quadrant till it points in the same direction as the third foot of the tripod, and make the wire bisect the lower dot by the screw of this foot, and it will be found to bisect the upper dot also, if the first adjustment was properly made, but if not, repeat the operation till both dots are bisected in all the reversed situations of the telescope, and then the axis will be vertical in every direction.

In making this adjustment by the level alone, the process must be thus; first, the level must be made parallel to the rod which it hangs on, and secondly, this rod must be put perfectly horizontal, and the level will be horizontal also, with the bubble in the middle. In order to make the level parallel to the rod, place it parallel to a line joining two of the feet screws, and bring the bubble to the middle by one of the feet screws in question; then take off and reverse the position of the level, and if the bubble is found in the middle now, the parallelism is perfect; if not, one half of the error must be rectified by the same foot screw, and the other half by the adjusting screws at the end of the rod, by releasing one and screwing up the other. A repetition or two of this process will make the bubble stand in the middle in both of the reversed situations. In the next place, with the level thus parallel to the rod of suspension, turn the quadrant round its axis an entire semicircle as nearly as can be estimated, and if the bubble will now rest in the middle, the rod is level, and being at right angles with the axis of the quadrant's motion, proves that this axis is vertical in every direction; but if the bubble be found to run to one end of the tube, bring it one half way back by the adjusting screws of the rod, releasing one and fixing the other, as the case may require, and the other half by the proper foot screw. A repetition of this process will soon settle the bubble in the middle during a whole revolution in azimuth, and then the adjustment of the axis is perfect, as well as of the rod and level.

Adjustment for collimation.

94. The second adjustment is that by which the line of collimation of the telescope, is made parallel to the horizontal line that passes to the centre of the quadrant to zero on the limb, or quadrantal arc, at the same time that zero on the vernier coincides with zero on the limb. This important adjustment may be made in several ways, some of which are tedious and otherwise objectionable; but we shall confine ourselves to two, which apply one to the vertical, and the other to the horizontal line of the quadrant, which two methods, when duly effected, will not only check each other, but detect the error of

the total arc, if there be any at the same time; which is an acquisition of the utmost importance. First then to adjust by the vertical line, let the axis of the quadrant be made truly perpendicular in all directions by the adjustment we have already described, and fix on a star within a few degrees of the zenith when exactly on the meridian, and measure its altitude by the cross wire in the field of view in the usual way, and note down the result; if these readings prove to be at equal distances from the point 90°, one on the quadrant arc, and the other on the arc of excess beyond 90°, the horizontal wire is truly placed in the eye piece, but if not, half of the difference of the readings must be corrected by the proper screw for raising or depressing the said wire. This may be done by directing the telescope to a distant mark till the cross wire bisects it, then by moving the screw of slow motion of the vernier the half quantity required, and by bringing back the cross wire thus displaced to its original mark again. This operation repeated will place the cross wire in such situation, that zero on the vernier will be in its proper place with respect to the point 90°; or the half difference thus ascertained may remain, without altering the cross wire, as an error of adjustment to be constantly applied with the sign + or —, as the case may be, in all subsequent observations. Again, to adjust by the horizontal line passing through the zero of the quadrantal arc, it will be necessary to have a second telescope turning on pivots in adjustable *y*'s attached to the back of the quadrant, on the same level with the said horizontal line of the quadrant. This telescope may be called the adjusting telescope, and may be also used to watch a distant mark, before and after an altitude is taken, in order to detect any deviation in the position of the vertical axis that may happen during the operation of measuring. Let the adjusting telescope bisect a fine distant mark with its cross wire, and turn the tube of the telescope round one half way on its pivots, as it lies in a horizontal position, and if the wire now bisects the same mark it is truly fixed, if not, look out for a new mark a little higher or lower, as the case may require, and make it cut that in the reversed positions of the cross wire, by means of the proper screw for the purpose; now this adjusting telescope will be adjusted for collimation. In the next place, put zero on the vernier to zero on the limb, and direct the telescope of observation to the distant mark, by which the adjusting telescope had its wire adjusted, and let this mark be bisected by both telescopes, the level and plumb-line at the same time showing that the vertical axis is perpendicular; now turn the quadrant half round its azimuth, and reverse the adjusting telescope so as to view the same distant mark again, and if it be found to bisect it as before, the horizontal line of the quadrant is right, and all the quadrantal arc without error, supposing the telescope of observation to have its adjustment for collimation as fixed by the point 90°, above described; but if this adjustment of the point zero on the limb be first made, half the apparent error must be rectified by the screw at the eye piece by means of reversed positions and marks; and then afterwards the adjustment by a new star near the zenith will detect the error of the whole arc. If, however, no error in the total arc exists, then the adjustment for collimation may be made either from the horizontal or from the vertical measurement, as may be most convenient; one of which is more practicable by day, and the other by night. When this delicate and most essential adjustment is finally settled, the object glass of the telescope should not be disturbed, and therefore it would be adviseable to have its interior surface well cleaned previously. It was taken for granted that the cross wire was perfectly horizontal during the time the preceding adjustment was made, or, which is the same thing, that the parallel wires were perpendicular to the horizon. This is proved in a simple manner thus; direct the telescope to a fine small distant mark, or make the adjustment for vision, if necessary, then if one of the vertical wires will continue to bisect the said mark through the whole field of view while the telescope is elevated or depressed, the wires are right, but if not, they must be made so by the proper screws for that purpose, near the focus of the eye-glas. This preparation ought to precede the last adjustment, and when once made, seldom requires altering, except in case of accidental injury. It has also been assumed in the preceding adjustment, that the maker of the instrument placed the plane of the quadrant parallel to the axis of its motion, and also its line of collimation of the telescope parallel to the said plane. The former may be known to be true thus: if, when the plumb-line is adjusted at its centre of suspension, just to escape touching the limb (which should always be the case) the motion of the quadrant in azimuth will not alter it in this respect, the plane is truly fixed; but if not, the screws that fix the quadrant to its axis must be resorted to for its alteration, which is best done by the maker. When there is no plumb-line, a small spirit level fixed at right angles to the plane of the quadrant will answer the same purpose; for the resting of the bubble during the quadrant's revolution in azimuth, will be a proof that the plane to which it is at right angles is vertical. With respect to the parallel position of the telescope, as this is guided by the vernier sliding on the limb, it is the business of the maker to adjust it properly, which he will best do by a comparison with a good transit instrument of the passages of a high and of a low star in each of the two instruments, but a small deviation of the telescope with respect to parallelism, though to be avoided, if practicable, will not sensibly affect the measurement of altitudes, which is the sole business of this instrument. If, however, this deviation be considerable, the eye end of the telescope must be set nearer to or farther from the limb, as the case may require, by the maker himself. We have been the more minute in describing these adjustments, not only because they are indispensably necessary in making good observations, but because they will apply, one or other of them, by means of the plumb-line or of the spirit level, to all other astronomical quadrants that have a motion in azimuth.

3. *Of the transit instrument.*

Trans strum

95. Transit instruments, as they are now constructed, may be considered either as fixed or portable; the former of which was the original construction, and is still that commonly made use of in permanent observatories, for the purpose of determining in conjunction with a good astronomical clock, the right ascensions of the heavenly bodies; but the latter

some telescopes have the eye-glass and cell, which carries the wires, moveable, while the object-glass is fixed: others have the wires fixed, and the two glasses moveable. In the former case, by pushing in or drawing out the eye piece, adjust the telescope so that the sun or a planet appears perfectly distinct through it; then move the wires nearer to, or farther from, the eye-glass, as may be required, until they also appear perfectly distinct, and the telescope will be adjusted ready for use. In the latter construction, push in, or draw out, the eye piece, till the wires appear perfectly distinct; then alter the object-glass until the sun, or a planet, appears perfectly distinct also, and the telescope will be adjusted ready for use.

As it is of importance to have the telescope adjusted very exactly in this respect, the following method of trying whether it be so or not, may be practised.

The telescope being adjusted to distinct vision for distant objects, when a fixed star is on the meridian, bring the horizontal wire to bisect it very exactly, and the star will run along the wire through the whole extent of the field of the telescope. While the star is thus running along the wire, move your eye a little upward or downward; and if the wires be not exactly in the common focus of the two glasses, the star will appear to quit the wire when the eye is moved. If this be the case, the wires or glasses must be altered until the star will not quit the wire by the motion of the eye; the objects appearing perfectly distinct at the same time.

To bring a transit instrument into the plane of the meridian.

Take the altitude of the sun, noting the times by the watch or clock, and thence find the apparent time, the latitude of the place, and the sun's declination being known. The difference between this time and the mean of the times shown by the watch when the observations were made, will be what the watch is too fast or too slow, for apparent time.

If the watch is too fast, add the difference to 12 hours: but if it be too slow, subtract it from 12 hours, and you will have the time by the watch when the sun will be on the meridian, as near as the going of the watch can be depended upon. Take the time which the sun's semidiameter is in passing the meridian from the Nautical Almanack, and add it to, and subtract it from the time by the watch, when the sun will be on the meridian, and you will have the times when the sun's eastern and western limbs will be on the meridian. A few minutes before the time when the western limb will be on the meridian, let your assistant count the seconds as they pass, by the watch; but instead of calling the 60th second, let him name the minute the watch is then at. While he is doing this, you must bring the sun into the telescope by elevating it to the proper altitude; and turning the whole instrument round on the screw pin U. Having by this means brought the middle wire apparently to the eastward of what appears to be the eastern limb of the sun, (because the sun will appear to move that way in the telescope) tighten the screw U by turning the nut; and when the sun's limb arrives at the middle wire, keep it on it by turning the screw *g*, at the rate the sun moves, till your assistant calls the second by the watch at which you had computed the western limb of the sun would be on the meridian; and the instrument will be nearly in the meridian. Let your assistant count on till the watch arrives at the second, when, according to your calculation, the eastern limb of the sun should be on the meridian; and, if it is not exactly on it, you will have another opportunity of rectifying the instrument by turning the screw *g*.

Having thus brought the instrument into, or very near the meridian, its real situation with respect to the meridian may be verified several ways; of which we shall point out two. If the latitude of the place be considerable, that is, 30 degrees or upward, there are a variety of stars in both hemispheres sufficiently bright, which never set: and consequently, they may be observed with the instrument both above and below the pole.

Let the transits of such a star over the meridian be observed above and below the pole; and it is manifest, that if the time of the first transit above the pole be subtracted from the time of the second transit above the pole (adding 24 hours if necessary), the remainder will be the time by the watch, in which the earth (or the star apparently) makes one diurnal revolution. It is also evident, that if the two intervals between the time of the transit below the pole be equal, the instrument must be exactly in the meridian. If the interval between the first transit above the pole, and the transit below the pole be greater than the interval between the transit below the pole and the second transit above it, the object end of the telescope, when directed toward the elevated pole, lies to the east of the true meridian; but if the latter interval be greatest, the object end of the telescope, when directed towards the elevated pole, lies west of the true meridian.

To correct the error, bring the instrument into the meridian; add 24 hours to the time of the latter transit above the pole, subtract the time of the former from it, and take half the remainder. Take the difference between this and the interval between the transits above and below the pole, and take half this difference. Then, as the time by the watch of an entire revolution is to 24 hours, so is this half difference to the half difference in sideral time. Add to the logarithm of this half difference, the logarithmic tangent of the star's polar distance; and the logarithmic secant of the latitude of the place, the sum, rejecting 20 from the index, will be the logarithm of the number of seconds in time, which expresses the angle made by the instrument and the meridian.

Consider what part this angle makes of the interval, between the wires which are in the focus of the telescope; and turn the instrument on its axis till the telescope points at the horizon. Look out for some tolerably distant object which is cut by one of the wires; and by turning the screw *g*, remove the wire to the east or west of this object (as may be required), such a part of the space between that wire and the next to it, as the angular error which the instrument makes of that interval. You must then proceed to examine the position of the instrument again, either by the same, or some other circumpolar star, and to correct it, if it requires correction, until you get it exactly into the plane of the meridian; and when you have, a mark must be set up in the meridian at as great a distance from the instrument as such observations to actual astronomical determinations.

§ V. *Of sideral and mean solar days, years, &c.*

99. We have seen that the earth performs its revolution on its axis with a motion perfectly uniform; and that the interval between the return of any fixed star to the same meridian, is what is called a sideral day; observing that by the same meridian is here to be understood, the star appearing at the same meridian altitude; for the circumpolar stars, as we have seen, may be observed twice on the same meridian in the course of 24 hours, once in their inferior and once in their superior passage; but they are only once at the same meridian altitude. This interval, then, is the length of the sideral day; but it remains for us now to explain what is to be understood by a mean solar day, or that day which is employed in the common concerns of life.

1. *Of the mean solar day.*

Mean solar day.

100. It will be seen in the following articles, that the interval time between the sun's leaving the first point of Aries to its return again to the same, which is what is denominated a solar year, is performed in about 365¼ solar days; or the sun will have appeared in that interval 365 times on the meridian, and will besides have performed nearly one-fourth of his 366th revolution: hence, if all the solar days were equal, that is, if the sun returned to the meridian of an observer always after the same interval, the increase of his right ascension every day, or the additional angle which the earth (having performed a complete revolution) would have to move through to bring the sun again upon the meridian of the observer, would be equal to $\frac{360^\circ}{365\frac{1}{4}} = 59' \, 8''{\cdot}2$; if, therefore, to the sideral day we add the time which the earth employs in describing 59′ 8″·2, we shall have the length of the mean solar day; that is, the sideral day is to the mean solar day as 360° : 360° 59′ 8″·2.

Consequently, if we call the mean solar day 24 hours, according to common reckoning of time, we shall have

$$360^\circ \, 59' \, 8'' : 360^\circ :: 24\text{h.} : 23\text{h.} \, 56\text{m.} \, 4{\cdot}1''$$

which is the length of a sideral day in mean solar hours; and by reversing the first two terms of the proportion, we shall have the mean solar day expressed in sideral hours.

2. *Sideral year.*

Sideral year.

101. It appears from what is stated above, and we have before made the same remark in a general way, that the sun has a continual motion in the heavens from east to west; and that after a certain period, he will again obtain with respect to the same star, the same relative situation; so that if he were in conjunction with it in the first instance, he will return again to conjunction after this interval, which is therefore called a *sideral year*.

Determined by observation.

In order to determine the duration of this period from observation, take on any day the difference between the sun's right ascension and that of the star, and when the sun returns to the same part of the heavens the next year, compare its right ascension with that of the same star for two days, one when their difference of right ascension is less, and the other when greater than the difference before observed; then it will be obvious that at some instant between these two observations, the right ascension will be exactly the same as it was when first observed.

Now in order to find the precise instant when this happens, let us suppose D to be the difference in right ascension as observed on the two consecutive days; and at the difference between the differences of the sun's and stars' right ascension on the first of these two days, and on the day when the observation was made the day before; and *t* be the exact time between the intervals of the two transits of the sun over the meridian on the two days; then assuming the motion of the sun to be equal in right ascension during this interval, we shall have

$$D : d :: t : \frac{d\,t}{D}$$

the time from the passage of the sun over the meridian on the first day, to the instant when it had the same right ascension, compared with the star, which it had the year before; and the interval between these two times when the difference of right ascension was the same, is the length of the sideral year.

Or if instead of supposing the second observation to have been repeated on the second year, there is an interval of several years between the two observations, and the observed interval of time be divided by the number of years, the length of the year will be had more exactly, any error in the observations being thus rendered less important by being divided into the great number of parts. The best time for these observations is, when the sun is in or near one of the equinoxes or one of the solstices, his motion in right ascension being then exactly or very nearly uniform.

As an example of this kind, we may state the following:—

By MM. P card and La Caille

102. April 1, 1669, at 0h. 3m. 47s. of mean solar time, M. Picard observed the difference between the longitude of the sun and the star Procyon to be 3s. 8° 59′ 36″. And M. La Caille found the difference of longitude between the sun and star to be the same, on April 2, 1745, at 11h. 10′ 45″. The sun, therefore, made 76 complete revolutions with regard to the same fixed star in 76 years 1 day 11h. 6m. 58s.; or in 27,759 days 11h. 6m. 58s.; we have, therefore, by dividing this interval by 76, 365 days 6h 8′ 47″ for the length of the sideral year; more recent observations, however, give 365 days 6h. 9m. 11·5s. for this interval.

3. *Of the tropical year.*

The le s of the pical ye

103. The length of the tropical year is the interval between the sun leaving either equinoctial point to its return again to the same; which it does in a less time than it passes from any fixed star to the same again, the latter we have seen is what is termed the sideral year; and the former being that on which the change of seasons depend is called the *solar* or *tropical* year. To determine the length of this year, we may proceed as follows:

Determi from ob vation.

Observe the meridian altitude of the sun on the day nearest the equinox, and the next year take its meridian altitude again on two successive days, on the one when its altitude is greater, and on the other when it is less, than in the first observation; then it

, that at some intermediate time between
st observations, the sun must have had the
ude or declination as in the first instance.
ind the precise instant, let D be the differ-
itude in the two last observations, t the in-
reen them, which may be here taken as 24
so d the difference between the altitude as
n the first of the two latter days, and that
year before; then, assuming the declination
rm, as it actually is at this time, say, as
24 hours $: \frac{24\,d}{D}$ the time from the first of
tter observations, to the instant when the
was the same as in the preceding year.
therefore, being added to the number of
en the two first observations, will give the
of the solar year.

ain, as in last case, if the two observations
l after an interval of several years, we may
result more nearly approximating to the

e two following observations were made by
l his son, after an interval of 44 years.

0, 1672, Meridian alt. sun's
limb ....41° 43′ 0″
0, 1722 ditto ......41° 27′ 10″
1 ditto ......41° 51′ 0″
0 41 43 0
0 41 27 10
0 : 15 50 :: 24 : 15h. 56m. 39s.

the 20th of March, at 15h. 56m. 39s., the
nation was the same as on the 20th of
72, at noon.

interval between the two first of these ob-
was 44 years, 34 of these were common
55 days each, and 10 of 366 days each,
ll an interval of 16,070 days, and therefore
l between the two periods when the sun's
was the same was, 16,070 days 15h. 56m.
his interval embraces 44 tropical years,

$$\frac{,070\text{d. }15\text{h. }56\text{m. }39\text{s.}}{44}\ 365\text{d. }5\text{h. }49'\,0''\,53'''$$

of the tropical year, as determined from
ts; more recent observations give for the
h of the tropical year, 365 days 5 hours

4. *Precession of the equinoxes.*

ppears from what is stated above, that the
to the equinoxes every year, before it re-
n to the same fixed star, or to the same
e heavens; the equinoctial points must,
have a retrograde motion with respect to
earth, the cause of which it is not for us at
explain; we shall, however, hereafter, see
ue to a regular mechanical effect, viz. the
of the sun and moon upon the earth in con-
f its spheroidal figure. The effect of this
longitude of the stars, which are always
rom the intersection of the equator and ec-
rom the equinoctial point, or first point of
constantly increase, and by comparing the
f the same stars at different times, the mean
motion of the equinoctial points, or the precession of the equinoxes may be determined.

We have observations of this kind from the time of Timocharis and Hipparchus; but we may be allowed to entertain considerable doubt as to accuracy; it will be sufficient to observe, that from a comparison of the best observation, the secular precession, or that which takes place in 100 years, amounts to 1° 23′ 45″ or to 50·34″ annually.

5. *Anomalistic year.*

Anomalistic year.

106. At a certain time of the year, the sun's diameter, if measured instrumentally, would be found to be the least; at which time it is obvious he would be the most distant from the earth or in his apogee, that is, the earth will then be at one extremity of the transverse axis of its orbit, and at that extremity which is furthest from the sun. Now, if at the end of a certain interval, a year from the first observation, a second were made, and the sun were found in precisely the same relative situation with regard to certain fixed stars, when his diameter was least, then it would be obvious that the sun had always the least apparent diameter after the completion of a sideral year; but the astronomical fact is not so; the sun does not return to a point in the heavens where his diameter is least in a sideral year; but in an interval a little exceeding it, and this interval is what astronomers have called the *anomalistic year*, the apogee has therefore a progressive motion, as we have seen the equinoxes have a precession; the quantity of the former, like that of the latter, being found from observation. According to the most recent determination, the increase in longitude of the sun's apogee in 100 years, is 1° 42′ 28″, or 1′ 2·2″ annually; but since the precession, which is a regressive motion, is 50·34″ annually, the annual sideral progression is 62·2″ − 50·34″ = 11·86″. Now, the time of describing 11·86″ added to the length of the sideral year, will compose an anomalistic year; and since the sun near its apogee moves in longitude about 58′ in 24 hours; the time will be about 4m. 50s.; hence the length of the anomalistic year, is equal to

365d. 6h. 9m. 11·4s. + 4m. 50s. = 365d. 6h. 14m. 1·4s.

6. *Obliquity of the ecliptic.*

Obliquity of the ecliptic.

107. The angle contained between the plane of the equator and the ecliptic is what is denominated the obliquity of the ecliptic; which is shown from repeated observations to be variable, like the other quantities we have been just examining. We have already, in the preceding section, indicated the method of determining the measure of this angle by means of the greatest and least meridian altitudes of the sun; it will therefore be sufficient in this place to show the result of a long succession of such observations by different astronomers, which are as follows:—

Determined by different astronomers.

	°	′	″
Eratosthenes, 230 years B.C.	23	51	20
Hipparchus, 140 years B.C..........	23	51	20
Ptolemy, A.D. 140	23	51	10
Pappus, A.D. 390.................	23	30	0
Albatenius, in 880	23	35	40
Arzachel, in 1070	23	34	0
Prophatius, in 1300	23	32	0
Regiomontanus, in 1460	23	30	0

	°	′	″
Waltherus, in 1490	23	29	47
Copernicus, in 1500	23	28	24
Tycho, in 1587	23	29	30
Cassini (the father), in 1656	23	29	2
Cassini (the son), in 1672	23	28	54
Flamstead, in 1690	23	28	48
De la Caille, in 1750	23	28	19
Dr. Bradley, in 1750	23	28	18
Mayer, in 1750	23	28	18
Dr. Maskelyne, in 1769	23	28	8,5
M. de Lalande, in 1768	23	28	0
M. Pond, Ast. Roy. 1816	23	27	50

The observations of Albatenius, an Arabian, are here corrected for refraction. Those of Waltherus, M. De la Caille computed. The obliquity by Tycho is put down as correctly computed from his observations; also the obliquity, as determined by Flamstead, is corrected for the nutation of the earth's axis; these corrections M. de Lalande applied. It is manifest from the above observations, that the obliquity of the ecliptic continually decreases; and the irregularity which here appears in the diminution, we may ascribe to the inaccuracy of the ancient observations, as we know that they are subject to greater errors than the irregularity of this variation. If we compare the first and last observations, they give a diminution of 70″ in 100 years. If we compare the observation of Lalande with that of Tycho, it gives 45″. The same compared with that of Flamstead gives 50″. If we compare that of Dr. Maskelyne with Dr. Bradley's and Mayer's, it gives 50″. The comparisons of Dr. Maskelyne's determination, with that of M. de Lalande, which he took as the mean of several results, gives 50″, as determined from the most accurate observations. This result agrees very well with that deduced from theory; but the observation of Mr. Pond, as compared with those of Bradley, gives 66″ for the variation in the obliquity in 100 years, or 0·40″ annually.

§ VI. *Of the corrections for refraction, parallax, &c.*

1. *Of refraction.*

Refraction.

108. When a ray of light passes out of a vacuum into any medium, or out of a medium into any other of a greater density, it is found to deviate from its regular course, towards a perpendicular to the surface of the medium into which it enters. (See Optics, p. 427) Hence light passing out of a vacuum into the atmosphere, will, where it enters, be bent or deflected towards a radius drawn to the earth's centre, the extreme surface of the atmosphere being supposed spherical and concentric with the centre of the earth; and as in approaching the earth's surface the density of the atmosphere continually increases, the rays of light are constantly entering into a denser medium, and therefore the course of the rays will continually deviate from a right line, and describe a curve; whence, at the surface of the earth, the rays of light enter the eye of a spectator, in a different direction from that in which they would have entered it, if there had been no atmosphere. Consequently, the apparent place of a body from which the light comes must be different from the true place, as shown in fig. 35.

Fig. 35.

Observed by the ancients.

109. Although we here state the fact of the refraction of the atmosphere as a necessary consequence of established laws in Optics, it must not be understood to have been introduced into astronomy as such; for it was observed by the ancients long before they were able to trace its cause to optical principles. Nothing is indeed more easy to be detected in astronomical observation; for by taking the greatest and least altitudes of the circumpolar stars, it will be seen that their apparent north polar distances will be different accordingly as it is taken at the time of their superior or inferior passage; and this variation is observed to be very nearly constant for the same place and the same star, and for all stars that have the same declination; but to vary according to a certain law, in stars that pass at different altitudes; it is also determinable from observation, that refraction does not alter the azimuth of bodies, and the same may also be demonstrated on physical principles, as we have shown in our treatise on Optics. Refraction, therefore, has a tendency only to increase the apparent altitude of a heavenly body, its entire effect being produced in a vertical circle, and its effect is less and less from the horizon to the zenith, where it vanishes.

Both Ptolemy and Alhazen were acquainted with this irregularity, and attributed it to its true cause, but neither of them undertook to determine the quantity of it. Tycho perceiving that the altitude of the equator, as deduced from the two solstices, was not the exact complement of the height of the pole, endeavoured to determine the quantity of refraction due to each zenith distance; he made the horizontal refraction 34″, and supposed it to become insensible at 45° of altitude; in the former, he was not far from the truth, but the latter conclusion was wholly erroneous; it is, in fact, to Dominic Cassini that we are indebted for the first regular hypothesis on the subject of astronomical refraction; but as his solution leads to an expression in which the refraction is made proportional to the tangent of the zenith distance diminished by a quantity which itself depends upon the refraction sought; we shall not insist upon it in this place, but proceed to illustrate the formulæ given by Bradley.

Bradley's formulæ.

110. Various tables of refraction more or less correct had been already formed, when the above celebrated astronomer commenced his observations for the purpose of deducing more exact formulæ than were at that time in existence, and of these tables he availed himself in settling the law of this important astronomical correction.

By means of numerous observations on Polaris and other circumpolar stars, Bradley deduced the apparent zenith distance P of the pole. By observations also on the sun at the equinoxes, when this body had the same zenith distance but opposite right ascensions, he deduced the height of the equator. In this instance, as in the former, it was only the apparent altitude that was obtained on account of the refraction, and therefore greater than the true height, and consequently, the apparent zenith distance was less than the true; whence the sum of the two zenith distances of the poles and the equator (which if true ought to be = 90°) will be less than 90°, by the sum of the two refractions due respectively to the zenith distances for every minute below 3° of altitude; or for every minute more than 87° of zenith distance, the result found as above must be reduced ·00462. By means of these formulæ, Mr. Groombridge has computed a very extensive table of refractions, with the requisite tables of corrections for the different states of barometrical pressure, as well as for thermometrical temperatures, both for the outside and inside of the observatory. We cannot allow ourselves to transcribe this table in the extended form given to it by its author, but the following abridgment of it will, it is presumed, be found highly acceptable to our readers. It will be understood that the correction for the barometer and thermometer will be the sum of the two factors in Table II. and Table III. multiplied into the mean refraction, and the product added or subtracted therefrom, according as the sum of the factors is plus or minus.

Table of mean refractions, computed from the preceding formula.

Zen. Dist.	Refrac.	Zen. Dist.	Refrac.	Zen. Dist.	Refrac.	Zen. Dist.	Refrac.	Zen. Dist.	Refrac.
° ′	′ ″	° ′	′ ″	° ′	′ ″	° ′	′ ″	° ′	′ ″
0 0	0 0·00	22 0	0 23·46	44 0	0 56·03	66 0	2 9·77	87 8	14 46·37
0 30	0 0·51	22 30	0 24·05	44 30	0 57·01	66 30	2 12·85	87 16	15 14·40
1 0	0 1·01	23 0	0 24·65	45 0	0 58·01	67 0	2 16·05	87 24	15 44·00
1 30	0 1·52	23 30	0 25·25	45 30	0 59·03	67 30	2 19·38	87 32	16 15·27
2 0	0 2·03	24 0	0 25·85	46 0	1 0·07	68 0	2 22·85	87 40	16 48·35
2 30	0 2·53	24 30	0 26·46	46 30	1 1·13	68 30	2 26·47	87 48	17 23·37
3 0	0 3·04	25 0	0 27·07	47 0	1 2·20	69 0	2 30·25	87 56	18 0·46
3 30	0 3·55	25 30	0 27·69	47 30	1 3·30	69 30	2 34·21	88 0	18 19·83
4 0	0 4·06	26 0	0 28·32	48 0	1 4·41	70 0	2 38·34	88 6	18 49·98
4 30	0 4·57	26 30	0 28·95	48 30	1 5·55	70 30	2 42·68	88 12	19 21·51
5 0	0 5·08	27 0	0 29·58	49 0	1 6·72	71 0	2 47·23	88 18	19 54·47
5 30	0 5·59	27 30	0 29·80	49 30	1 7·90	71 30	2 52·01	88 24	20 28·89
6 0	0 6·10	28 0	0 30·87	50 0	1 9·11	72 0	2 57·03	88 30	21 5·08
6 30	0 6·62	28 30	0 31·52	50 30	1 10·34	72 30	3 2·33	88 36	21 42·88
7 0	0 7·13	29 0	0 32·18	51 0	1 11·60	73 0	3 7·92	88 42	22 22·47
7 30	0 7·64	29 30	0 32·85	51 30	1 12·89	73 30	3 13·82	88 48	23 3·95
8 0	0 8·16	30 0	0 33·52	52 0	1 14·21	74 0	3 20·07	88 54	23 47·40
8 30	0 8·68	30 30	0 34·20	52 30	1 15·55	74 30	3 26·69	89 0	24 32·94
9 0	0 9·20	31 0	0 34·88	53 0	1 16·93	75 0	3 33·73	89 4	25 4·51
9 30	0 9·72	31 30	0 35·57	53 30	1 18·33	75 30	3 41·22	89 8	25 37·09
10 0	0 10·24	32 0	0 36·27	54 0	1 19·78	76 0	3 49·21	89 12	26 10·71
10 30	0 10·76	32 30	0 36·98	54 30	1 21·25	76 30	3 57·75	89 16	26 45·40
11 0	0 11·29	33 0	0 37·70	55 0	1 22·77	77 0	4 6·89	89 20	27 21·20
11 30	0 11·81	33 30	0 38·42	55 30	1 24·31	77 30	4 16·72	89 24	27 58·14
12 0	0 12·34	34 0	0 39·15	56 0	1 25·91	78 0	4 27·30	89 28	28 36·26
12 30	0 12·87	34 30	0 39·89	56 30	1 27·54	78 30	4 37·72	89 32	29 15·60
13 0	0 13·40	35 0	0 40·64	57 0	1 29·21	79 0	4 51·09	89 36	29 56·19
13 30	0 13·94	35 30	0 41·40	57 30	1 30·93	79 30	5 4·53	89 40	30 38·07
14 0	0 14·48	36 0	0 42·17	58 0	1 32·69	80 0	5 19·18	89 44	31 21·28
14 30	0 15·02	36 30	0 42·95	58 30	1 34·51	80 30	5 35·21	89 48	32 5·85
15 0	0 15·56	37 0	0 43·74	59 0	1 36·38	81 0	5 52·83	89 52	32 51·82
15 30	0 16·10	37 30	0 44·53	59 30	1 38·30	81 30	6 12·26	89 56	33 39·84
16 0	0 16·65	38 0	0 45·34	60 0	1 40·28	82 0	6 33·79	90 0	34 28·13
16 30	0 17·20	38 30	0 46·16	60 30	1 42·32	82 30	6 57·78	90 2	34 53·15
17 0	0 17·75	39 0	0 47·00	61 0	1 44·42	83 0	7 24·63	90 4	35 18·55
17 30	0 18·31	39 30	0 47·84	61 30	1 46·59	83 30	7 54·87	90 6	35 44·34
18 0	0 18·86	40 0	0 48·69	62 0	1 48·83	84 0	8 29·13	90 8	36 10·52
18 30	0 19·43	40 30	0 49·56	62 30	1 51·14	84 30	9 8·18	90 10	36 37·10
19 0	0 19·99	41 0	0 50·44	63 0	1 53·53	85 0	9 53·03	90 12	37 4·08
19 30	0 20·56	41 30	0 51·34	63 30	1 56·00	85 30	10 44·88	90 14	37 31·47
20 0	0 21·13	42 0	0 52·25	64 0	1 58·56	86 0	11 45·57	90 16	37 59·28
20 30	0 21·71	42 30	0 53·17	64 30	2 1·21	86 24	12 41·02	90 18	38 27·50
21 0	0 22·29	43 0	0 54·11	65 0	2 3·96	87 48	13 44·62		
21 30	0 22·87	43 30	0 55·06	65 30	2 6·81	87 0	14 19·80		

Correction to preceding table of refraction.

BAROMETER.				FAHRENHEIT'S THERMOMETER.					
Inches.	Correc. —	Inches.	Correc. +	Degree.	Within. +	Without. +	Degree.	Within. —	Without.
28·60	·0350	29·60	·0000	24·0	·0575	·0420	49·0	·0000	·0080
62	·0342	62	·0007	24·5	·0563	·0410	49·5	·0011	·0090
64	·0335	64	·0014	25·0	·0552	·0400	50·0	·0022	·0100
66	·0328	66	·0020	25·5	·0540	·0390	50·5	·0033	·0110
68	·0321	68	·0027	26·0	·0529	·0380	51·0	·0044	·0120
28·70	·0314	29·70	·0034	26·5	·0517	·0370	51·5	·0055	·0130
72	·0306	72	·0041	27·0	·0506	·0360	52·0	·0066	·0140
74	·0299	74	·0047	27·5	·0494	·0350	52·5	·0077	·0150
76	·0292	76	·0054	28·0	·0483	·0340	53·0	·0088	·0160
78	·0285	78	·0061	28·5	·0471	·0330	53·5	·0099	·0170
28·80	·0278	29·80	·0068	29·0	·0460	·0320	54·0	·0110	·0180
82	·0271	82	·0074	29·5	·0448	·0310	54·5	·0121	·0190
84	·0264	84	·0081	30·0	·0437	·0300	55·0	·0132	·0200
86	·0256	86	·0088	30·5	·0425	·0290	55·5	·0143	·0210
88	·0249	88	·0095	31·0	·0414	·0280	56·0	·0154	·0220
28·90	·0242	29·90	·0101	31·5	·0402	·0270	56·5	·0165	·0230
92	·0235	92	·0108	32·0	·0391	·0260	57·0	·0176	·0240
94	·0228	94	·0115	32·5	·0379	·0250	57·5	·0187	·0250
96	·0221	96	·0122	33·0	·0368	·0240	58·0	·0198	·0260
98	·0214	98	·0128	33·5	·0356	·0230	58·5	·0209	·0270
29·00	·0207	30·00	·0135	34·0	·0345	·0220	59·0	·0220	·0280
02	·0200	02	·0142	34·5	·0333	·0210	59·5	·0231	·0290
04	·0193	04	·0149	35·0	·0322	·0200	60·0	·0242	·0300
06	·0186	06	·0155	35·5	·0310	·0190	60·5	·0253	·0310
08	·0179	08	·0162	36·0	·0299	·0180	61·0	·0264	·0320
29·10	·0172	30·10	·0169	36·5	·0287	·0170	61·5	·0275	·0330
12	·0165	12	·0176	37·0	·0276	·0160	62·0	·0286	·0340
14	·0158	14	·0182	37·5	·0264	·0150	62·5	·0297	·0350
16	·0151	16	·0189	38·0	·0253	·0140	63·0	·0308	·0360
18	·0144	18	·0196	38·5	·0241	·0130	63·5	·0319	·0370
29·20	·0137	30·20	·0203	39·0	·0230	·0120	64·0	·0330	·0380
22	·0130	22	·0210	39·5	·0218	·0110	64·5	·0341	·0390
24	·0123	24	·0216	40·0	·0207	·0100	65·0	·0352	·0400
26	·0116	26	·0223	40·5	·0195	·0090	65·5	·0363	·0410
28	·0109	28	·0230	41·0	·0184	·0080	66·0	·0374	·0420
29·30	·0102	30·30	·0237	41·5	·0172	·0070	66·5	·0385	·0430
32	·0096	32	·0243	42·0	·0161	·0060	67·0	·0396	·0440
34	·0089	34	·0250	42·5	·0149	·0050	67·5	·0407	·0450
36	·0082	36	·0257	43·0	·0138	·0040	68·0	·0418	·0460
38	·0075	38	·0264	43·5	·0126	·0030	68·5	·0429	·0470
29·40	·0068	30·40	·0270	44·0	·0115	·0020	69·0	·0440	·0480
42	·0061	42	·0277	44·5	·0103	·0010	69·5	·0451	·0490
44	·0054	44	·0284	45·0	·0092	—	70·0	·0462	·0500
46	·0048	46	·0291	45·5	·0080	·0010	70·5	·0473	·0510
48	·0041	48	·0297	46·0	·0069	·0020	71·0	·0484	·0520
29·50	·0034	30·50	·0304	46·5	·0057	·0030	71·5	·0495	·0530
52	·0027	52	·0311	47·0	·0046	·0040	72·0	·0506	·0540
54	·0020	54	·0318	47·5	·0034	·0050	72·5	·0517	·0550
56	·0014	56	·0324	48·0	·0023	·0060	73·0	·0528	·0560
58	·0007	58	·0331	48·5	·0011	·0070	73·5	·0539	·0570

Other effects of refraction.

114. What we have hitherto stated, relates principally to corrections requisite to be made in astronomical observations in consequence of the effect of refraction in elevating all the heavenly bodies, but the same principles will also explain some other astronomical phenomena, as for instance, the morning and evening twilight, the oval appearance of the sun in the horizon, the horizontal moon, &c.

Cause of the twilight

115. With respect to the twilight, it arises both from the refraction and reflection of the sun's rays by the atmosphere. It is probable, that the reflection arises principally from the exhalations of different kinds with which the lower beds of the atmosphere are charged; for the twilight lasts till the sun is farther below the horizon in the evening than it is in the morning when it begins, and it is longer in summer than in winter. Now in the former case, the heat of the day has raised the vapours and exhalations; and in the latter, they will be more elevated from the heat of the season; therefore, supposing the reflection to be made by them, the twilight ought to be longer in the evening than in the morning, and longer in the summer than in the winter.

Commonly, it is assumed, that the twilight begins when the sun is 18° below the horizon; but in our investigations relative to the time of shortest twilight (art. 78), we have left the solution general for any number of degrees, and which may therefore be supplied at pleasure.

Oval figure of the sun and moon in the horizon.

116. Another effect of refraction, as we have above observed, is that of giving the sun and moon an oval appearance in consequence of the refraction of the lower limb being greater than the upper; whereby the vertical diameter is diminished. For assuming the diameter of the sun to be 32′, and the lower limb to touch the horizon, then the mean refraction at that limb is 33′; but the altitude of the upper limb being 32′, its refraction is only 28′ 6″, the difference of which is 4′ 54″, the quantity by which the vertical diameter appears shorter than that parallel to the horizon. This, however, will only happen when the body is in or very near the horizon, for when the altitude is any considerable quantity, the refraction of both limbs being then very nearly the same, the apparent disc will not differ sensibly from a circle.

2. *Of parallaxes.*

Paralla .

117. Parallax is a term used by astronomers to denote an arc of the heavens intercepted between the true and apparent place of a star, or other heavenly body, or its place as viewed from the centre and the surface of the earth.

Fig. 37.

Let *s* (fig. 37) represent a star, C the centre of the earth, Z the zenith of a spectator, then the observed zenith distance of *s* is the angle Z A *s*, but its actual zenith distance, as viewed from the centre C, is ZC*s*, and the difference between the angles ZC*s* and ZA*s* is is A*s*C, which is called the angle of parallax.

Now we know from the principles of trigonometry that

$$\sin \mathrm{C}s\mathrm{A} = \sin \mathrm{CA}s \times \frac{\mathrm{CA}}{\mathrm{C}s} = \sin \mathrm{ZA}s \times \frac{\mathrm{CA}}{\mathrm{C}s}$$

whence if CA, C*s* remain the same, the sine of C*s*A, that is, the sine of the parallax varies as the sine of the star's zenith distance. Consequently, the parallax must be greater, the greater the zenith distance; it must therefore be greatest when the body is seen in the horizon, or when the zenith distance is 90°. Let *p* represent the parallax at any zenith distance Z, P the greatest or horizontal parallax, then we shall have

$$\sin p = \frac{\mathrm{CA}}{\mathrm{C}s} \times \sin \mathrm{Z}, \text{ and } \sin \mathrm{P} = \frac{\mathrm{CA}}{\mathrm{C}s} \times \sin 90^\circ = \frac{\mathrm{CA}}{\mathrm{C}s}$$

Consequently, $\sin p = \sin \mathrm{P} \sin \mathrm{Z}$

If therefore the parallax be known for any one zenith distance, it may be determined for any other; and moreover, if CA the earth's radius, and C*s* the distance of the body, were given, the parallax P would become known; and conversely, if the parallax were given, the distance C*s* might also be determined, the radius of the earth being supposed known from geodetic operations. It is, in fact, from knowing the parallax of one or more of the heavenly bodies, that their distances have been determined.

The correction for parallax is one of the most important in practical astronomy, and accordingly, various methods have been proposed by different astronomers for determining it; but of these we shall only specify one or two of the most obvious.

Parall
the mo

118. First, to find the parallax of the moon. Take the meridian altitudes of the moon when it has its greatest north and south latitudes, and correct them for refraction, as explained in the preceding chapter. Then, if there were no parallax, or if the parallax were the same at both altitudes, the difference of the altitudes thus corrected would be equal to the sum of the latitudes, and consequently, what those quantities want of equality will be equal to the difference of the parallax. We have, therefore, by means of these observations, the difference of the parallaxes at these altitudes, and it is required to find them separately. For this purpose, let us denote the two zenith distances, by Z, *z*, the parallaxes by P, *p*: then from what has been stated above we have

$$\sin \mathrm{Z} : \sin z :: \mathrm{P} : p$$

or

$$\sin \mathrm{Z} - \sin z : \sin. z :: \mathrm{P} - p : p$$

whence

$$p = \frac{\sin z\,(\mathrm{P} - p)}{\sin \mathrm{Z} - \sin z}$$

the parallax at the greatest altitude.

We here suppose the moon to be at the same distance from the earth at both observations; when this is not the case, one of the observations must be reduced to what it would have been had the distance been the same.

Determi
by obse
tions.
Fig. 38.

119. Let a body P (fig. 38) be observed from two places, A, B, in the same meridian, then the whole angle APB is the sum of the two parallaxes at those two places. Now we have seen in article 117, that the

$$\text{parallax APC or } p = \mathrm{P} \times \sin \mathrm{PAL}$$

$$\text{parallax PBC or } p' = \mathrm{P} \times \sin \mathrm{PBM}$$

Hence $\mathrm{APB} = p + p' = \mathrm{P} \times (\sin \mathrm{PAL} + \sin \mathrm{PBM})$

Consequently

$$\text{hor. parallax (P)} = \frac{p + p'}{\sin \mathrm{PAL} + \sin \mathrm{PBM}}$$

In order to illustrate this by actual observation, we may state the following example:

October 5, 1751, M. De la Caille, at the Cape of Good Hope, observed Mars to be 1′ 25″·8 below the parallel of λ in Aquarius, and at 25° distance from the zenith. On the same day, at Stockholm, Mars was observed to be 1′ 57″·7 below the parallel of the same star, and at 68° 14′ zenith distance. Hence

$$\begin{array}{ll} \text{from} & 1' \ 57''\!\cdot\!7 \\ \text{take} & 1 \ \ 25\cdot 8 \end{array}$$

$$\text{angle APB} = p + p' = 0 \ \ 31\cdot 9$$

$$\text{Whence P} = \frac{31\cdot 9''}{\sin 25° + \sin 68° \, 14'} = 23\cdot 6$$

The horizontal parallax of Mars being thus determined, we may hence find that for the sun; for we have seen that, generally,

$$\text{P} = \frac{\text{CA}}{\text{C}s} \qquad \text{(see fig. 36)}$$

Consequently, since CA, the radius of the earth, is constant, the parallax will vary reciprocally as the distance of the body; knowing, therefore, the proportional distance of the sun and Mars from the earth at the time of observation; the parallax of the former may be determined when that of the latter is given. This method, however, of determining the solar parallax is not sufficiently accurate for the purposes of modern astronomy.

120. Method of determining the parallax in right ascension and declination.

ax in scen- d tion.).

Let EQ (fig. 39) be the equator, P its pole, Z the zenith, v the true place of the body, and r the apparent place, as depressed by parallax in the vertical circle Zk, and draw the secondaries Pva, Prb, then ab is the parallax in right ascension, and rs in declination.

$$\begin{array}{ll} \text{Now} & vr : vs :: \text{rad} : \sin vrs \text{ or } \mathrm{Z}v\mathrm{P} \\ \text{and} & vs : ab :: \cos va : \text{rad} \end{array}$$

Hence by multiplication, and rejecting the like factors,

$$vr : ab : \cos va : \sin \mathrm{Z}v\mathrm{P}$$

$$\text{therefore} \quad ab = \frac{vr \sin \mathrm{Z}v\mathrm{P}}{\cos va}$$

$$\begin{array}{ll} \text{but} & vr = \text{hor. par. (P)} \times v\mathrm{Z} \\ \text{and} & \sin v\mathrm{Z} : \sin \mathrm{ZP} :: \sin \mathrm{ZP}v : \sin \mathrm{Z}v\mathrm{P} \end{array}$$

$$\text{hence} \quad \sin \mathrm{Z}v\mathrm{P} = \frac{\sin \mathrm{ZP} \times \sin \mathrm{ZP}v}{\sin v\mathrm{Z}}$$

whence by substitution

$$ab = \frac{\mathrm{P} \times \sin \mathrm{ZP} \times \sin \mathrm{ZP}v}{\cos va}$$

Hence it follows, that for the same star, where the *hor. par.* or (P) is given, the parallax in right ascension varies as the sine of the hour angle.

Also,

$$\textit{hor. par.} = \frac{ab \cos va}{\sin \mathrm{ZP} \times \sin \mathrm{ZP}v}$$

In the eastern hemisphere, the apparent place b lies on the equator to the east of a its true place, and therefore the right ascension is diminished by parallax; but in the western hemisphere b lies to the west of a, and therefore the right ascension is increased. Hence, if the right ascension be taken before and after the meridian, the whole change of parallax in right ascension between the two observations, is the sum (s) of the two parts before and after the meridian, we have therefore

$$s = \frac{vr}{\cos va} \times \mathrm{S}$$

S denoting the sum of the sines of the two hour angles, and

$$\textit{hor. par.} \ (\mathrm{P}) = \frac{s \cos va}{\sin \mathrm{ZP} \times \mathrm{S}}$$

On the meridian there is no parallax in right ascension.

Illustration of this method.

In order to apply this rule, observe the right ascension of the planet when it passes the meridian, compared with that of a fixed star, at which time there is no parallax in right ascension; about six hours after, take the difference of their right ascensions again, and observe how much the difference (d) between the apparent right ascension of the planet and fixed star has changed in that time. Next observe the right ascension of the planet for three or four days when it passes the meridian in order to get its true motion in right ascension. Then if its motion in right ascension in the above interval of time, between the taking of the right ascensions of the fixed star and planet on and off the meridian, be equal to d, the planet has no parallax in right ascension; but if it be not equal to d, the difference is the parallax in right ascension, and hence, on the above principles, the horizontal parallax will be known. Or one observation may be made before the planet comes to the meridian and another after, by which a greater difference will be obtained.

By example.

121. In order to illustrate this method by an example, let the following be taken:

On August 15, 1719, Mars was very near a star of the 5th magnitude in the eastern shoulder in Aquarius: and at 9h. 18m. in the evening, Mars followed the star in 10′ 17″; and on the 16th, at 4h. 21m. in the morning, it followed it in 10′ 1″, therefore in that interval, the apparent right ascension of Mars had increased 16″ in time.

But according to observations made in the meridian for several days after, it appeared that Mars approached the star only 14″ in that time, from its proper motion; therefore 2″ in time or 30″ in motion was the effect of parallax in the interval of the observations.

$$\begin{array}{ll} \text{Now the declination of Mars was} & 15° \\ \text{the co-latitude} & 41° \ 10' \\ \text{the two hour angles} & \begin{cases} 49° \ 15' \\ 56° \ 39' \end{cases} \end{array}$$

Consequently, the horizontal parallax

$$\mathrm{P} = \frac{30'' \times \cos 15°}{\sin 41° \, 10' \, (\sin 49° \, 15' + \sin 56° \, 39')} = 27\tfrac{1}{2}$$

At the time of these observations, the distance of the earth from Mars was to its distance from the sun as 37 : 100; whence the sun's horizontal parallax is found to be 10·17″.

Effect of parallax in latitude and longitude

122. Besides the effect of parallax in right ascension and declination, it is manifest that the latitude and longitude of the moon and planets must also be effected by it, and as the determination of this, in respect to the moon, is, in many cases, particularly in solar eclipses, of great importance, we shall proceed to show how it may be computed, supposing the latitude of the place, the time, and consequently the sun's right ascension, the moon's true latitude and longitude, and her horizontal parallax to be given.

Fig. 40.

Let HZR (fig. 40) be the meridian, ♈EQ the equator, p its pole; ♈C the ecliptic, P its pole; ♈ the first point of Aries, HQR the horizon, Z the zenith, ZL a secondary to the horizon, passing through the true place r, and apparent place t, of the moon; draw Pt, Pr, which produce to s, drawing the small is in her apogee and in conjunction) is 53′ 52″ in the latitude of Paris: now as the parallax varies inversely as the distance, the parallax at the mean distance of the moon is 57′ 24″ viz. an harmonical mean between the two former.

But Delambre recalculated the parallax from the same observations from which Mayer calculated it, and found a slight disagreement in the two results. He made the equatorial parallax 57′ 11″·4; Lalande made it 57′ 5″ at the equator, 56′ 53″·2 at the pole, and 57′ 1″ for the mean radius of the earth; upon a supposition that the ratio of the equatorial and polar axes was as 300 : 299.

Assuming then the mean parallax to be 57′ 1″, we have, referring to fig. 36,

AC : mean radius r :: D, dist. of moon : sin 57′ 1″

hence D = 60·3 rad = 60·3 × 3964 = 239029 miles the mean distance of the moon from the earth.—*Vince's Astronomy*.

The preceding methods by which the parallaxes of the moon and of Mars have been determined are not sufficiently exact for us to employ them in determining that of the sun. And since this is in astronomy a most important element, and requires the most exact determination, it has, as we have before remarked, engaged the most anxious attention of philosophers, and no one has rendered in this respect a more essential service to the science than Dr. Maskelyne, our late worthy astronomer royal; but the method which he employed, and which was first pointed out by the celebrated astronomer Dr. Halley, cannot be with propriety illustrated in this place, because it supposes the planetary motions to be determined to the utmost accuracy: in a subsequent chapter we shall, however, enter at some length upon the explanation of this method, at present it will be sufficient to state that it depends upon the transit of either of the two inferior planets, but particularly that of Venus over the sun's disc; and that it has been thus found to be 8″·75 according to Maskelyne, but 8″·81 according to Laplace; whereas we have seen (art. 121) that as deduced from observations on Mars, it was found to be 10″·17.

From what has been stated it appears, that the parallaxes of the planets answer two important purposes, for we hence may determine their actual distances from the earth, and moreover, without a knowledge of the quantity of this important datum, we should be unable to correct our observations, and much uncertainty would consequently attend all our deductions. The parallax of the sun being very small, its mean horizontal parallax may be considered as constant, viz. 8″·75, and consequently, the parallax for any altitude is readily determined by means of the formula

$$p = \mathrm{P} \times \text{sin zen. dist., or}$$
$$p = 8''{\cdot}75 \times \text{sin zen. dist.}$$

But for the moon, the parallax being considerable, we ought, in delicate observations, to compute it for her actual distance, and accordingly, in the *Nautical Almanack*, we find the lunar parallax, as well as her semi diameter, stated for 12 o'clock, both at noon and midnight, for every day in the year.

125. Let us now propose an example to show the method of introducing the corrections for parallax and refraction, agreeably to what has been taught in the preceding articles.

Observation.

	°	′	″	
Alt. ⊙ upper limb....	62	44	11	
Lower limb....	62	15	41	
2)	124	59	52	
App. alt. ⊙ centre ..	62	29	56	
App. zen. dist.	27	30	4	sin = 4617
+ Refraction........	0	0	29	
	27	30	33	
(8″¾ × ·4617) parallax	0	0	4	
Corr. alt. ⊙'s centre..	27	30	29	
Latitude of the place..	48	50	14	
⊙'s declination......	21	29	45	

3. *Of the correction for aberration.*

Aberration.

126. In our historical chapter, we have given some account of this important astronomical discovery; it remains for us in this place to enter a little more at length into an illustration of the principles, and to describe the method of computing and applying the requisite corrections.

The situation of any object in the heavens is determined by the position of the axis of the telescope, attached to the instrument with which we measure; for such a position is given to the telescope, that the rays of light from the object may descend down the axis, and in that situation, the index shows the angular distance required; if, therefore, the observer were in a state of absolute rest, while he was making his observation, the direction of his telescope would coincide with that of the object, as it would also, although he were in motion, if light was instantaneously transmitted from the luminous body to the eye. But if, as is actually the case, the motion of light is progressive, while the observer is also carried forward in space, then, except in the particular instance in which both motions take place in the same line, a different direction must necessarily be given to the axis of the telescope; and consequently, the place measured in the heavens will be different from the true place.

Illustrated. Fig. 41.

127. This may be illustrated in a general manner as follows. Let S′ (fig. 41) be a fixed star, VF the direction of the earth's motion, S′F the direction of a particle of light, entering the axis ac of the telescope, at a, and moving through aF, while the earth moves from c to F; and if the telescope be kept parallel to itself, the light will descend in the axis. For let the axis nm, Fw, continue parallel to ac; and if each motion be considered as uniform, that of the spectator occasioned by the earth's rotation being disregarded on account of its being too small to produce any sensible effect, the spaces described in the same time will preserve the same proportion; but cF and Fa being described in the same time, and as we have

$$c\mathrm{F} : \mathrm{F}a :: cn : av$$

it follows, that cn and av will be described in the same time; therefore, when the telescope is in the situation nm, the particle of light will be at v in the telescope; and the case being the same at every moment of its descent, the place measured by the telescope at F is s', and the angle $S'F's'$, is the aberration or the difference between the true place of the star and the place measured by the instrument.

Hence it appears, that if we take

FS : Ft :: vel. of light : vel. of the earth,

join St and complete the parallelogram Ft Ss, the aberration will be equal to FSt; S will be the true place of the star, and s the place measured by the instrument, and this latter is the same with the apparent place of the object as it would be seen by the naked eye. This will appear as follows.

The place shown by the telescope the same as shown to the naked eye.

128. If a ray of light fall upon the eye in motion, its relative motion with respect to the eye, will be the same as if equal motions were impressed in the same direction upon each, at the instant of contact; for equal motions in the same direction impressed upon two bodies will not effect their relative motions, and therefore the effect one upon the other will not be altered. Let then VF be a tangent to the earth's orbit at F, and therefore represent the direction of the earth's motion at that instant, and S′ a star: join S′F and produce it to G; take

FG : Fn :: vel. of light : vel. of the earth,

and complete the parallelogram FGHn; join also FH: then since FG, and Fn represent the motions of light and of the earth, we shall have on the principle of the composition of motion explained (art. 24) Mechanics, FH, for the corresponding resulting motion; that is, the object will appear in the direction of the diagonal FH, and GFH or its equal $S'Fs'$ or FSt will be the aberration; consequently, the apparent place to the naked eye is the same as that determined by the telescope.

Ratio of vel. of light to the vel. of earth.

129. Now we have by trigonometry,

sin FSt : sin FtS :: Ft : FS :: vel. of earth : the vel. of light : whence

$$\sin \mathrm{FS}t = \sin \mathrm{F}t\mathrm{S} \times \frac{\mathrm{F}t}{\mathrm{F}s}$$

$$\text{or} \qquad \sin \text{of aberration} = \sin \mathrm{F}t\mathrm{S} \times \frac{\text{vel. of earth}}{\text{vel. of light}}$$

therefore considering the ratio of the velocity of light and of the earth as constant, the sine of aberration or the aberration itself, will vary as the sine of FtS, and is therefore greatest when that angle is a right angle, and it will be zero when the angle FtS vanishes, that is, when the motions of the earth and of light are made in the same right line.

By observations it has been determined, that the greatest effect of aberration is 20″, and as this corresponds to the case of FtS = 90°, or sin FtS = 1 we have

$$\sin 20'' = 1 \times \frac{\text{vel. of earth}}{\text{vel. of light}}$$

vel. of earth : vel. of light as sin 20″ : rad. :: 1 : 10314

Compared with other observations.

130. This result is obtained from observation, and is independent of any deductions drawn from the actual velocity of the luminous rays; it only shows that if light moved with the velocity indicated, that such phenomena ought to have place, and that it may therefore be employed as an illustration of them. But the actual velocity of light has been otherwise determined; let us see, therefore, how nearly the results in the two cases agree with each other.

For this purpose, call the radius of the earth's orbit r, and we shall have 3·1416 × 2r for the circumference which is described by the earth in 365¼ days, or 31557600 seconds; consequently, $\frac{3{\cdot}1416 \times 2r}{31557600}$ will be the velocity per second, and by the above determination,

$$\frac{10314 \times 3{\cdot}1416 \times 2r}{31557600} = \text{the velocity of light per second.}$$

$$\text{Hence as } \frac{10314 \times 3{\cdot}1416 \times 2r}{31557600} : 2r : 1 \text{ sec.}$$

$$\frac{31557600}{10314 \times 31416} = 97''4 \text{ or } 16\tfrac{1}{4} \text{ minutes, the time}$$

light employs in traversing a space equal to the diameter of the earth's orbit, which agrees very nearly with the results previously deduced by astronomers from observations on the eclipses of Jupiter's satellites.

Con as a of t eart volu

The coincidence of these deductions, founded upon observations wholly independent of each other is generally considered as furnishing one of the most satisfactory proofs of the truths of the Copernican or modern system of astronomy.

Dir of tion

131. The aberration $S's'$ lies from the true place of a star in a direction parallel to that of the earth's motion, and towards the same part; and its effect is therefore produced at one time wholly in declination, at another wholly in right ascension, and at others, it will effect both these quantities, but in a greater or less degree, according to circumstances.

Fig.

In order to illustrate this, let E, E′, E″, E‴, (fig. 42) represent four positions of the earth when the sun is in the signs ♈, ♑, ♎, and ♋, and let tT, t'T′, t''T″, &c. be tangents to the earth's orbit at those points; s, s' the same star seen on the meridian of any place. Now in the position E, corresponding to the vernal equinox, the sun is in the equator; therefore, if Pmp be a meridian passing through the poles of the equator and ecliptic, Pmp will coincide with the solstitial colure, and a line drawn from S to E will be perpendicular to the plane of the meridian Pmpn: therefore, if the plane of the ecliptic EE′E″, be conceived to lie in the plane of the paper, that of Pmpn must be perpendicular to it; and SE will be also perpendicular to the tangent tT, which tangent, therefore, is in the plane of the meridian Pmpn.

Now the earth being supposed to rotate on its axis in the direction nEm, and SE being perpendicular to the plane mPpn, the position as shown in the figure corresponds to that of 6 o'clock in the evening; and supposing the star s, to be on the meridian at this instant, it is obvious that it will be situated in a plane corresponding with the line of the earth's motion; the aberration will therefore produce its effect wholly in that plane, and will tend to elevate the star s, from s to s', that is, it will increase its declination, or diminish its zenith distance; and it is obvious that a directly contrary effect will be produced in the opposite position E″ corresponding to the autumnal equinox; for the direction of light being then in the line SE″, while that of the earth is from E″ towards t'', and in the plane of sE″t'' the aberration will be produced wholly in the same plane, and will tend to depress the star; that is, to diminish its declination; or to increase its zenith distance.

quadrature before opposition, and greatest in quadrature after. From the mean of a great number of observations, Dr. Bradley determined the value of the greatest aberration to be 20″ as we have already stated (art. 129.)

Deductions.

134. It follows from what has been shown above,

1. That the greatest *aberration in latitude* is equal to 20″ multiplied by the sine of the star's latitude, and that the aberration in latitude *for any time* is equal to 20″ multiplied by the star's latitude and by the sine of the elongation found for the same time; and that it is subtractive before opposition and additive after it.

2. The *greatest aberration in longitude* is equal to 20″ divided by the cosine of the latitude; and the aberration for *any time* equal to 20″ multiplied by the cosine of the elongation, and divided by the cosine of the latitude; it will be subtractive in the first and last quadrant of the argument, or of the difference between the longitude of the sun and star, and additive in the second and third.

EXAMPLES.

1. Find the greatest aberration of γ *Ursa Minoris*, whose latitude is 75° 13′.

Here sin 75° 13′ is ·9669
Mult. by 20″

19″·34 the *greatest aberration in lat.*

And $\frac{20''}{\cos 75^\circ\ 13'} = \frac{20''}{\cdot 2551} = 78''\cdot 4$ the *greatest aberration in longitude.*

2. Required the aberration in latitude and longitude of the same star when the earth is 30° from syzygies.

$\sin 30^\circ = m = \cdot 5$; hence

$20'' \times (\sin 75^\circ\ 13') + \cdot 5 = 9''\cdot 67$ aberration in latitude; and since $\cos 30^\circ = \cdot 866$

$\frac{20'' + \cdot 866}{\cdot 2551} = 67''89$ the aberration in longitude.

In the case of the sun, we have always $\sin sSc = o$ and $\cos = 1$, also cos lat. $= 1$; and, consequently, $20'' \times \sin sSc = o$, or there is no aberration in latitude, and the aberration in longitude is constant and equal to 20″. This quantity 20″ of aberration of the sun answers to its mean motion in $8'\ 7''\frac{1}{2}$; and is therefore the time in which light moves from the sun to the earth, at its mean distance, agreeably to what we have already stated (art. 130.) Hence the sun always appears 20″ behind its true place.

The following table is intended to expedite the calculation in the preceding cases.

The argument for the longitude is *long sun — long star.* The argument for the latitude is *long sun — long star — 3 signs.*

TABLE.—*continued.*

Degree.	0 VI. − +	I VII. − +	II. VIII. − +	Degree.
0	20″00	17·32″	10·0″	30
1	20·00	17·14	9·70	29
2	19·99	16·96	9·39	28
3	19·97	16·77	9· 8	27
4	19·95	16·58	8·77	26
5	19·92	16·58	8·45	25
6	19·89	16·18	8·13	24
7	19·85	15·97	7·81	23
8	19·81	15·76	7·49	22
9	19·75	15·54	7·17	21
10	19·70	15·32	6·84	20
11	19·63	15· 9	6·51	19
12	19·56	14·86	6·18	18
13	19·49	14·63	5·85	17
14	19·41	14·39	5·51	16
15	19·32	14·14	5·18	15
16	19·23	13·89	4·84	14
17	19·13	13·64	4·50	13
18	19· 2	13·38	4·16	12
19	18·91	13·12	3·81	11
20	18·80	12·86	3·47	10
21	18·67	12·59	3·12	9
22	18·54	12·21	2·78	8
23	18·41	12· 4	2·44	7
24	18·27	11·76	2· 9	6
25	18·13	11·47	1·74	5
26	17·98	11·18	1·40	4
27	17·82	10·89	1·50	3
28	17·66	10·60	0·70	2
29	17·49	10·30	0·35	1
30	17·32	10· 0	0· 0	0

135. For the aberration *in longitude*, multiply the corresponding quantities in the table, by the secant of the star's latitude.

For the aberration *in latitude* multiply the quantities taken from the table by the sine of the star's latitude.

EXAMPLE.

1. Let the longitude of the sun be 7s 5° 18′, the longitude of the star 5s. 11° 14′, and its latitude 31° 10′; to find its aberration in latitude and longitude.

long. ⊙ 7s. 5° 18′
long. ✱ 5 18 14′

1 17 4 correspond in tab. to 13″·62
see 31° 10′........ 1″·69

aberration in long. — 15″·92 product

For the latitude.

s ° ′
long. ⊙ − long. ✱ = 1 17 4
− 3 signs 3 0 0

10 17 4 cor. t a . − 14″·65
sin 31° 10 0·5175

aberration in latitude — 7·58 prod.

To find the aberration in declination and right ascension.

136. Let AEL (fig. 44) represent the equator, *p* its pole, ACL the ecliptic, P its pole, S the true place of a star, *s* the apparent place in the ellipse; draw the great circles PS*a*, P*sb*, *p*S*w*, *p*S*v*, and S*t*, S*v* per-

Use of the Tables

For the aberration in right ascension.

Use of the tables.

138. Enter Table I, with the argument A − S, and Table II. with A + S. Then the sum of the two corresponding numbers, multiplied by the sec. of D will be the aberration in *right ascension.*

For the aberration in declination.

Enter Table I. with the argument A − S + 3 signs, and Table II. with A + S + 3 signs, and the sum of the two corresponding numbers multiplied by the sine of D, will be the *first part* of the aberration in declination.

Enter Table III., with the arguments S + D and S − D, by which will be found the other two parts of the aberration in declination, and the sum of the three will give the whole of the *aberration in declination.*

If the declination of the star be south, add 6 signs to the arguments S + D and S − D.

Example.

Illustrated by an example.

Required the aberration of *a* Aquilæ, Feb. 15, 1819, at 12 o'clock in the evening.

Here by the tables, A = 9s 25° 29′ D = 8° 24′ 25″
S = 10 26 2

A − S = 10s 29° 27′ Tab. I. . . −16″·5
A + S = 8 21 31 Tab. II. . − 0·15

− 16·6
Sec. dec. 8° 24′ 25″ = 1·0108

Aberration in *right ascen.* 16·66 Product
A − S + 3 signs 1s 29° 27′ Tab. I. −16″·68
A + S + 3 signs 11 21 31 Tab. III. + 0.82

− 15·86
Sin dec. 8° 24′ 25″ ·1461

S = 10 26° 2′ 9″ − 2·317

S + D = 11 4° 26′ 34″ Tab. III. − 3·595
S − D = 10 17 37 44 Tab. III. − 2·945

Aberration in declination − 8·857

If the declination had been south, the two latter arguments would have been S + D + 6 signs, and S − D + 6 signs, as stated above.

4: *Of nutation.*

Nutation.

139. We have in the preceding articles endeavoured to illustrate the principles of the most important astronomical corrections, and have shown the method of computation; there however still remains for explanation the doctrine of nutation, but as the complete developement of the principles upon which this theory rests, involves considerations of a physical nature which we have not hitherto examined, we must in this place content ourselves with a very general view of the subject, leaving the more minute particulars for our treatise on physical astronomy. By nutation is to be understood a kind of trepidation or tremulous motion of the earth's axis, whereby its inclination to the plane of the ecliptic is not always the same; but vibrates within certain limits, never, however, exceeding a few seconds; the period of variation is also limited to a certain number of revolutions of the earth in its orbit. This inequality in the terrestrial motion was, like that described in the foregoing chapter, discovered by Dr. Bradley, to whom we owe likewise a just explanation of the cause of it, and a near approximation of its effects.

140. We have observed above, that it is impossible in this place to give more than a very general explanation of this doctrine; it will be perhaps sufficient to observe here, that the first cause of nutation is due to the mutual gravitation of matter, and to the laws which it is known to observe; viz. that it is directly as the mass, and inversely as the square of the distance. If the orbit of the earth were a circle, and the terrestrial globe a perfect sphere, the attraction of the sun would have no other effect than to keep it in its orbit, and would cause no irregularity in the position of its axis; but neither of these conditions takes place, the earth is not a perfect sphere, nor is its orbit a circle: when the position of the earth is such, that the plane of its equator passes through the centre of the sun, the attractive power of the latter body will still have no other tendency than that of drawing the earth towards it, and the parallelism of its axis will not be disturbed: this happens in the equinoxes. But as the earth recedes from these points, the sun deviates so much the more from the plane of the equator, and the latter, in consequence of its protuberance is more powerfully attracted than the rest of the globe, which causes some alteration in its position, that is, in the inclination of its axis to the plane of the ecliptic; and at that part of the orbit, which is described between the vernal and autumnal equinox, is less than that passed over between the latter and the former; it follows that the irregularity caused by the sun, during its passage through the northern signs, is not entirely compensated by that which takes place during the other part of the revolution; and consequently, that the parallelism of the terrestrial axis, and its inclination to the ecliptic will be a little changed.

141. Again, the same effect which the sun produces upon the earth by its attraction, or at least an analogous effect, is also produced by the moon, which is more powerful in proportion as it is more distant from the equator; and therefore when its nodes concur with the equinoctial points, the power which causes the irregularity in the position of the terrestrial axis acts with the greatest force; and the revolution of the moon's nodes being performed in about eighteen years, the nodes will twice in this period concur with the equinoctial points; and consequently, twice in the same period, or once every nine years, the earth's axis will be more influenced than at any other time; and during this interval, the pole of the earth will describe an ellipse in the heavens, whose transverse axis is 19″·2, and conjugate axis 15″, which correspond with the ratio between the cosine of the obliquity and the cosine of twice the obliquity of the ecliptic; or to the ratio of cos 23° 28′ and cos 46° 56′.

142. Let π (fig. 45) be the pole of the ecliptic, and P the mean place of the pole of the equator, ADC a circle whose radius is equal to the semi transverse axis of the ellipse C*d*A described by the pole as above stated, A the true pole of the equator when the ascending node of the moon's orbit is at ♈, and let A be supposed to move contrary to the order of the signs. Take

Epochs of the mean longitude of the moon's ascending node.

Years.	S.	D.	M.	Years.	S.	D.	M.	Years.	S.	D.	M.	Years	S.	D.	M.
1752	8	1	37	1785	10	23	20	1817	2	4	27	1849	5	15	31
1753	7	12	17	1786	10	4	1	1818	1	15	7	1850	4	26	11
1754	6	22	57	1787	9	14	41	1819	0	25	48	1851	4	6	51
1755	6	3	37	1788	8	25	18	1820	0	6	25	1852	3	17	28
1756	5	14	14	1789	8	5	58	1821	11	17	5	1853	2	28	8
1757	4	24	55	1790	7	16	39	1822	10	27	45	1854	2	8	48
1758	4	5	35	1791	6	27	19	1823	10	8	26	1855	1	19	29
1759	3	16	15	1792	6	7	56	1824	9	19	3	1856	1	0	6
1760	2	26	52	1793	5	18	36	1825	8	29	43	1857	0	10	46
1761	2	7	33	1794	4	29	17	1826	8	10	23	1858	11	21	26
1762	1	18	13	1795	4	9	57	1827	7	21	4	1859	11	2	7
1763	0	28	53	1796	3	20	34	1828	7	1	41	1860	10	12	44
1764	0	9	30	1797	3	1	14	1829	6	12	21	1861	9	23	24
1765	11	20	11	1798	2	11	55	1830	6	23	1	1862	9	4	4
1766	11	0	51	1799	1	22	35	1831	5	3	42	1863	8	14	45
1767	10	11	31	1800	1	3	16	1832	4	14	19	1864	7	25	22
1768	9	22	8	1801	0	13	55	1833	3	24	59	1865	7	6	2
1769	9	2	49	1802	11	24	36	1834	3	5	39	1866	6	16	42
1770	8	13	29	1803	11	5	16	1835	2	16	20	1867	5	27	23
1771	7	24	9	1804	10	15	53	1836	1	26	57	1868	5	8	0
1772	7	4	46	1805	9	26	33	1837	1	7	37	1869	4	18	40
1773	6	15	27	1806	9	7	14	1838	0	18	17	1870	3	29	20
1774	5	26	7	1807	8	17	54	1839	11	28	58	1871	3	10	1
1775	5	6	47	1808	7	28	31	1840	11	9	35	1872	2	20	38
1776	4	17	24	1809	7	9	11	1841	10	20	15	1873	2	1	18
1777	3	28	5	1810	6	19	52	1842	10	0	55	1874	1	11	58
1778	3	8	45	1811	6	0	32	1843	9	11	35	1875	0	22	39
1779	2	19	25	1812	5	11	9	1844	8	22	13	1876	0	3	16
1780	2	0	2	1813	4	21	49	1845	8	2	53	1877	11	13	56
1781	1	10	42	1814	4	2	30	1846	7	13	33	1878	10	24	36
1782	0	21	23	1815	3	13	10	1847	6	24	13	1879	10	5	17
1783	0	2	3	1816	2	23	47	1848	6	4	51	1880	9	15	54
1784	11	12	40												

Table *of nutation.*

(148) Argument the mean longitude of the node.

Degrees.	0s.		VI	Deg.	Deg.	I.		VII	Deg.	Deg.	II.		VIII.	Deg.
	log.*b*—	B —	C — +			log.*b*—	B —	C — +			log.*b*—	B —	C — +	
0	0·9944	0° 0′	0″·00	30	0	0·9588	6° 45′	8″·27	30	0	0·8960	7° 48	14″·35	30
1	0·9944	0 15	0·29	29	1	0·9571	6 54	8·52	29	1	0·8939	7 40	14·47	29
2	0·9943	0 31	0·58	28	2	0·9554	7 3	8·77	28	2	0·8917	7 32	14·61	28
3	0·9942	0 46	0·87	27	3	0·9536	7 12	9·01	27	3	0·8896	7 23	14·74	27
4	0·9940	1 1	1·15	26	4	0·9518	7 20	9·25	26	4	0·8875	7 14	14·87	26
5	0·9937	1 16	1·44	25	5	0·9500	7 28	9·49	25	5	0·8854	7 4	14·99	25
6	0·9834	1 32	1·75	24	6	0·9481	7 36	9·72	24	6	0·8834	6 53	15·11	24
7	0·9830	1 47	2·02	23	7	0·9462	7 43	9·96	23	7	0·8814	6 42	15·23	23
8	0·9825	2 2	2·03	22	8	0·9442	7 49	10·19	22	8	0·8795	6 29	15·34	22
9	0·9821	2 17	2·59	21	9	0·9422	7 55	10·41	21	9	0·8776	6 17	15·45	21
10	0·9815	2 31	2·87	20	10	0·9402	8 1	10·63	20	10	0·8758	6 3	15·55	20
11	0·9809	2 46	3·16	19	11	0·9382	8 6	10·85	19	11	0·8740	5 49	15·64	19
12	0·9802	3 1	3·44	18	12	0·9361	8 10	11·07	18	12	0·8723	5 35	15·73	18
13	0·9795	3 15	3·72	17	13	0·9340	8 14	11·28	17	13	0·8707	5 20	15·82	17
14	0·9787	3 29	4·00	16	14	0·9318	8 17	11·49	16	14	0·8692	5 4	15·90	16
15	0·9779	3 45	4·28	15	15	0·9297	8 20	11·70	15	15	0·8677	4 48	15·98	15
16	0·9770	3 57	4·56	14	16	0·9275	8 23	11·90	14	16	0·8663	4 31	16·05	14
17	0·9760	4 11	4·84	13	17	0·9253	8 24	12·10	13	17	0·8659	4 14	16·12	13
18	0·9750	4 24	5·11	12	18	0·9231	8 25	12·30	12	18	0·8637	3 56	16·18	12
19	0·9739	4 37	5·39	11	19	0·9208	8 25	12·49	11	19	0·8625	3 38	16·24	11
20	0·9728	4 50	5·66	10	20	0·9186	8 25	12·67	10	20	0·8615	3 20	16·29	10
21	0·9716	5 3	5·93	9	21	0·9163	8 24	12·86	9	21	0·8605	3 1	16·34	9
22	0·9704	5 16	6·20	8	22	0·9140	8 23	13·04	8	22	0·8596	2 41	16·38	8
23	0·9691	5 28	6·46	7	23	0·9118	8 21	13·21	7	23	0·8588	2 22	16·42	7
24	0·9678	5 40	6·73	6	24	0·9095	8 18	13·38	6	24	0·8582	2 2	16·45	6
25	0·9664	5 51	6·99	5	25	0·9072	8 15	13·55	5	25	0·8576	1 42	16·48	5
26	0·9650	6 3	7·25	4	26	0·9050	8 11	13·72	4	26	0·8571	1 22	16·50	4
27	0·9635	6 14	7·51	3	27	0·9027	8 6	13·83	3	27	0·8568	1 2	16·52	3
28	0·9620	6 24	7·77	2	28	0·9005	8 1	14·03	2	28	0·8565	0 41	16·53	2
29	0·9604	6 35	8·02	1	29	0·8983	7 55	14·18	1	29	0·8563	0 21	16·54	1
30	0·9588	6 45	8·27	0	30	0·8960	7 48	14·33	0	30	0·8563	0 0	16·54	0
	log *b*	+ B	— + C			log. *b*	+ B	— + C			log. *b*	+ B	— + C	
	V.		XI.			IV.		X.			III.		IX.	

hence AMC = AED; take away the area ACD, which is common to both, and the area MCD = EDC, but MCD = $\frac{1}{2}$ DM × CD; and EDNC = $\frac{1}{2}$ ET × CD, therefore ET = DM.

Now if t be given, the arc AM will be given; for as the body in the circle moves uniformly,

$$p : t :: 360° : \text{AD}$$

Thus the mean anomaly may be found for any given period, the time of the planets passing its aphelion being supposed known; and therefore, if we can find ET or DM, we shall know the angle DCA called the *eccentric* anomaly; and hence, also, we shall be able to find the angle AEP the true anomaly. With a view to determine the eccentric anomaly, Woodhouse gives the following easy approximation.

Approximate determination of eccentric anomaly.

148. (1.) Draw MO parallel to DC; find half the sum of the angles at the base of the triangle ECM; which may be done by means of the formula

$$\tan \tfrac{1}{2} \{\text{CEM} - \text{CME}\} = \tan \tfrac{1}{2} \{\text{CEM} + \text{CME}\} \times \frac{1-e}{1+e}$$

(2.) Find CEM by the addition of $\frac{1}{2}$ (CEM − CME,) and $\frac{1}{2}$ (CEM + CME), and use this angle as an approximate value of the eccentric anomaly DCA; from which, however, it really differs by the angle EMO.

Use this approximate value of ∠ DCA = ∠ ECT, in computing ET, which is equal to the arc DM; for since $t = \frac{\text{P}}{\text{area} \odot} \times \text{DEA}$, and the body being supposed to revolve in the circle ADM,

$$t = \frac{\text{P}}{\text{area} \odot} \times \text{ACM}$$

therefore area DEA = area ACM.
Or the area DEC + area ACD = area DCM + area ACD, consequently, area DEC = area DCM.
Hence expressing their values,

$$\frac{\text{ET} \times \text{DC}}{2} = \frac{\text{DM} \times \text{DC}}{2}, \text{ or ET} = \text{DM}$$

Having then computed ET = DM, find the sine of the resulting arc DM, which sine = OT; the difference then of the arc and sine (ET − OT) gives EO.

(3.) Use EO in computing the angle EMO, the real difference between the eccentric anomaly DCA, and the angle MEC: add the computed angle EMO to MEC in order to obtain ∠ DCA: which result, however, is not the exact value of DCA, since EMO has been computed only approximatively; that is, by a process which commenced by assuming ∠ MEC, for the value of ∠ DCA.

(4.) For the purpose of finding the eccentric anomaly, the above is the entire description of the process, which, if greater accuracy be required, must be repeated; that is, from the last found value of ∠ DCA = ∠ ECT, ET, EO, and ∠ EMO must be again computed.

(5.) A sufficiently correct value of the eccentric anomaly being found, investigate the true anomaly as follows.

True anomaly determined.

149. Let v be the other focus, and make AC = 1; then $\text{EP}^2 - \text{P}v^2 = v\text{E}^2 + 2v\text{E} \times v\text{N} =$

$$(v\text{E} + 2v\text{N}) \times v\text{E} = (2\text{C}v + 2v\text{N}) \times 2\text{EC} = 2\text{CN} \times 2\text{EC}$$

Hence (EP + Pv) : 2CN :: 2EC : (EP − Pv), or

$$2 : 2\text{CN} :: 2\text{EC} : \{\text{EP} - (2 - \text{EP})\}$$

$$1 : \text{CN} :: \text{EC} : \text{EP} - 1$$

Therefore, $\text{EP} = 1 + \overline{\text{CE} \times \text{CN}} = 1 + \text{CE} \times \cos \text{ACD}$
Now by trigonometry,

$$\frac{1 - \cos \text{AEP}}{1 + \cos \text{AEP}} = \tan^2 \tfrac{1}{2} \text{AEP}$$

And EP : rad. :: EN : cos AEP
or 1 + EC cos ACD : 1 :: CE + CN : cos AEP

$$\text{whence } \cos \text{AEP} = \frac{\text{CE} + \cos \text{ACD}}{1 + \text{CE} \cos \text{ACD}}$$

$$\text{Hence } \tan^2 \tfrac{1}{2} \text{AEP} = \frac{1 - \cos \text{AEP}}{1 + \cos \text{AEP}} =$$

$$\frac{1 + \text{CE} \cos \text{ACD} - \text{CE} - \cos \text{ACD}}{1 + \text{CE} \cos \text{ACD} + \text{CE} + \cos \text{ACD}}$$

$$= \frac{1 - \text{CE} + \cos \text{ACD}\,(\text{CE} - 1)}{1 + \text{CE} + \cos \text{ACD}\,(\text{CE} + 1)}$$

$$= \frac{\text{EB} - \cos \text{ACD} \times \text{EB}}{\text{EA} + \cos \text{ACD} \times \text{EA}}$$

$$= \frac{1 - \cos \text{ACD}}{1 + \cos \text{ACD}} \times \frac{\text{EB}}{\text{EA}} = \tan \tfrac{1}{2} \text{ACD} \times \frac{\text{EB}}{\text{EA}}$$

Hence $\sqrt{\text{EA}} : \sqrt{\text{EB}} :: \tan \tfrac{1}{2} \text{ACD} : \tan \tfrac{1}{2} \text{AEP}$
consequently, we get AEP the *true anomaly*.

This may be more concisely expressed as follows,

$$\tan \tfrac{1}{2} v = \sqrt{\frac{1-e}{1+e}} \times \tan \tfrac{1}{2} u$$

Where v is the true anomaly, u the eccentric anomaly, and e the eccentricity of the orbit.

150. The following examples will illustrate the above approximation.

Examples.

1. The eccentricity of the earth's orbit being assumed at ·01691, and the mean anomaly 30°, it is required to find the eccentric and true anomalies.

(1) log. tan 15 . . . . 9·4280525
log (1−e) or log ·98309 . . . . −1·9925933
arith. com. (1+e) or of 1·01691 . . . . −1·9927218

log tan $\frac{1}{2}$ (CEM−CME) = 14° 31′ 22″ 9·4133676
(2.) $\frac{1}{2}$ (CEM − CME) = 14° 31′ 22″
$\frac{1}{2}$ (CEM + CME) = 15 0 0

CEM = 29 31 22 1st approx. of DCA

(3) log sin 29° 31′ 22″ . . . . 9·6926438
log ·01691 . . . . −2·2281436
+ log (arc = rad) . . . . 5·3144251

log DM in seconds = 1718 7 . . . . . . 3·2352125

DM = 28′ 38″·7, and its sine expressed as seconds differs from the arc DM, by less than half a second.

(4.) The operation prescribed under this number (art. 149) is unnecessary in this case, the correction for the angle MEC being so small, that the first eccentric value may be considered as correct.

(5.) log tan $\frac{1}{2}$ u, or log tan 14° 45′ 41″ . . . . = 9·4207651
$\frac{1}{2}$ log (1−e) or of . . . . ·98309 = − 1·9962966
A.C. $\frac{1}{2}$ log (1+e) or of 1·01691 = 1·9963609

log tan $\frac{1}{2}$ v = 14° 31′ 28″ . . . . 9·4134226
Mult. 2

29 2 56 true anomaly

Mean anomaly = 30° 0′ 0″
true anomaly = 29 2 56

equation of centre 0 57 4

2. As a second example, take the following from Woodhouse's Astronomy, p. 197. The eccentricity of the orbit, that of Pallas, for example, being 0·259, the mean anomaly 45°. To find the eccentric and true anomalies,

(1) log. tan 22° 30′............9·6172243
log. tan ·741..............—1·8698182
arith. com. 1·259..........9·8999743

log. tan $\frac{1}{2}$ (CEM — CME) = 13° 42′ 3″·3 } 9·3870168

(2) $\frac{1}{2}$ (CEM—CME) = 13° 42′ 3″·3
$\frac{1}{2}$ (CEM + CME) = 22 30

CEM = 36 12 3·3 1st approx. ∠CDA
CME = 8 47 56·7

(3) log. sin 36° 12′ 3″·3.........9·7713071
log. ·259 ....—1·4132998
log. (arc = rad.) ....5·3144251

log. MD in sec. = 31552·4 — 4·4990320

therefore DM = 31552″·4 = 8° 45′ 52″.4
log. sin...............9·1829067
log. (arc. — rad.) ....5·3144251

log. 31429 ....4·4973318
therefore, since DM = 31552·4
and sin DM = 31429

EO = 123·4

(4) log. ·259.............—1·4132998
log. sin 45° ....9·8494850

9·2627848
log. sin 8° 47′ 56″·7....9·1845968

·0781880
log. (arc. = rad.)......5·3144251

5·3926131

log. rad. ....10·0000000
log. 123·4.............2·0913152

12·0913152
log. (arc = rad.) + log. EM — 5·3926131

log. sin EMO ....6·6987021
therefore EMO = 1′ 43″·1

Hence, since CDA = 36° 12′ 3″·3
and EMO = 0 1 43 ·1

corrected val. CDA or *eccentric anom.* } = 36 13 46 ·4
log. tan 18° 6′ 53″·2 ....9·5147282
$\frac{1}{2}$ log. (1 — ·259)...........—1·9349091
A.C $\frac{1}{2}$ log. (1 + ·259) ....—1·9499871

log. tan $\frac{1}{2}$ v = 14° 5′ 19″ ....9·3996244
and v = 28° 10′ 38″ the *true anomaly.*

To find the distance of a planet, the anomaly being given.

Plane Astronomy.

Distance determined.

151. The eccentricity of an orbit, and either the true or eccentric anomaly being given, the corresponding distance of the planet from the sun may be readily determined, by means of what is denominated the polar equation of the ellipse: viz. calling a the semi-transverse axis, e the eccentricity, v the true anomaly, and ρ the distance sought; we shall have, (See art. 58 DYNAMICS)

$$\rho = \frac{a\,(1 - e^2)}{1 - e \cos v}$$

and hence, again,

$$\rho = a\,(1 + e \cos u)$$

where u is the eccentric anomaly.

One example will be a sufficient illustration of the latter formula; and we will take the conditions proposed in the last example to determine the corresponding distance of that planet from the sun.

We there found the eccentric anomaly, or the value of u to be 36° 13′ 46″·4, and the eccentricity was assumed ·259. Hence, we have

log. cos u = 36° 13′ 36″·4 ....9·9066881
log. e = ·259 ....—1·4132998

log. ·20892 ....—1·3199879

Hence 1 + ·20892 = 1·20892 = ρ the distance sought.

We might determine precisely the same with formula 1, using the true anomaly, v, but the operation is obviously more laborious.

Of the greatest equation of the centre.

Greatest equation of the centre.

152. We have seen (art. 147) that the equation of the centre is the difference between the mean and the true anomalies; the greatest equation of the centre then is obviously as its name imports, the greatest difference between the mean and true anomalies; and this must necessarily happen at that time when the planet is moving with its mean angular velocity. For if we conceive a body to move uniformly in a circle about E as a centre, with the mean angular velocity of the planet, so that, departing from BA, the line of the apsides, it shall arrive at it again in the same time as the planet: then it is plain, that at the commencement of the motion, P moving, we will suppose, with its least angular velocity, describes about E a less angle than the fictitious body does in a given time, a day for example: the second day it also describes a less angle than the supposititious body, although a greater one than on the preceding; and so on for the third, fourth, &c days. The angular distances of the two bodies from the aphelion will therefore differ more and more from each other, till, by the still increasing angular velocity of P, it at length attains to and surpasses the uniform angular velocity of the fictitious body; after which the difference between the two angles will decrease, till at length they coincide again in the perihelion. Consequently, at that time, when the angular velocities of the two bodies are equal, the difference of the two angles, or the angle of the centre, will be the greatest.

Radius vector.

153. The point in the elliptic orbit at which this takes place is easily determined; for it must be that where the circle in which the body moves with the mean angular velocity, intersects the elliptic orbit. Let x be the radius of that circle; then, since the

Whence from 3s 8° 51′ 1″·5
take 19 21

leaves 3 8 31 40·5

the longitude of the apogee, viz. 8° 31′ 40″·5 past the summer solstice.

Results.

159. At the time of Hipparchus, 140 B. C., the sun was in apogee in 5° 30′ of Gemini; consequently, from that time to 1744, a space of 1884 years, it advanced 33°, which divided by 1884 gives

$$\frac{33^\circ}{1884} = 1'\,3'' \text{ for the annual progression.}$$

More recent observations state the longitude of the perigee to be as follows, viz.

1750	9s	8°	37′	28″
1800	9	9	29	3
1810	9	9	39	22

The first difference, divided by 50, gives

$$\frac{51'\,35''}{50} = 61'\,9''$$

which is now generally assumed for the annual motion of the perigee, and consequently, for the apogee.

The progressive motion of the apogee or perigee being determined, the position of either, or of the axis of the solar ellipse for any other time may be found by a simple proportion. Thus, if the time were required, when the line of the apsides was perpendicular to the line of the equinoxes, that is, the time when the longitude of the perigee was 9s. or 270°, we should find

$$\frac{8^\circ\,37'\,28''}{61{\cdot}9} = \text{about 500 years}$$

and consequently, the year sought 1250 of the Christian era.

The epoch when the axis coincided with the line of the apsides will be found in like manner by dividing 3s 8° 37′ 28″ by 61·9, which answers to about 5720 years, or about 4000 years before the birth of Christ, at which time chronologists have fixed the creation of the world.

Remarks.

160. It may not be amiss to observe, that there is one apparent objection to the latter method of determining the longitude of the apogee; viz. that the length of the anomalistic year, which depends upon the progression in question is supposed to be already known to some degree of accuracy; this, however, as we have frequently had occasion to remark, is a defect in most astronomical determinations; for in this more than in any other science, we are obliged to proceed step by step, from approximation to approximation, till we arrive at the conclusion sought, which is, however, still only a more accurate approximation than the former. In the present instance, if we suppose only the tropical revolution to be known, we must employ it as we have done the anomalistic period, and thus obtain a first value, and then using this again, we should soon arrive at the conclusion as above given.

6. *Of the equation of time.*

Equation of time.

161. We have already (art. 82) had occasion to explain the difference between solar and mean time. We have seen that the sun returns from one meridian to the same again after unequal intervals, at different seasons of the year, at one time preceding and at another subsequent to the noon or 12 o'clock, as shown by a well regulated chronometer properly adjusted to mean time, and this difference is what is denominated the equation of time. It remains for us now to enter a little more minutely into this doctrine, and to show how the difference in question may be computed.

First, then, we may observe, that if we imagine two meridians to be drawn through the two extremities of the arc of the ecliptic described by the sun in one day, the arc of the equator which they intercept will measure the difference between the solar and the sideral day; for when the complete revolution of the celestial sphere is terminated, this small arc of the equator must traverse the meridian before the centre of the sun can arrive there; and as from two distinct causes, which are explained below, this arc is not always of the same length, the solar days must necessarily, as we have already stated, be also unequal.

In the first place, the motion of the sun being in the ecliptic, it is obvious, that if he described in this circle, equal arcs every day, these arcs when referred to the equator would be unequal, and consequently, the solar days also; but we have seen that the sun's motion in the ecliptic is not uniform, being greatest at the time he is in or near his perigee, and least in apogee; this, therefore, is another cause of the unequal lengths of the arcs of the equator intercepted between the two meridians supposed to pass through the extremities of the arc of the ecliptic described by the sun in a day. Thus the inequality in the solar days arises from two distinct causes, the obliquity of the ecliptic and the unequal motion of the earth in its orbit.

Mean s
day defi

162. The mean solar day to which we have referred is marked by no obvious phenomenon, but the noon of which may be conceived to be shown by a fictitious sun or star moving in the equator with the mean motion of the real sun in right ascension, viz. with such an uniform motion, that supposing both to depart from the equinox at the same instant, they may both arrive at it again together, after one complete revolution, and as the length of the tropical year, or the time in which the sun passes through 360° in right ascension is 365d. 5h. 48m. 48s., we shall have

$$\frac{360^\circ}{365\text{d. }5\text{h. }48\text{m. }48\text{s.}} = 59'\,8''{\cdot}3$$

for its mean daily motion in right ascension, and therefore for the mean motion of the fictitious sun in the equator; and hence the length of the mean day will be the time during which any point on the earth revolves through 360° 59′ 8″·3, and it is this interval which answers to 24 hours by a clock well regulated to mean time, and the difference between the noon, or 12 o'clock, shown by such a time piece, and that of the real noon when the sun is on the meridian, is the equation of time, which is sometimes, as we have before stated, additive, at others subtractive, and four times in the year zero.

Equation of time illustrated.

163. In order to a more complete illustration of this phenomenon, let us imagine two suns S″ and S‴ to set out at the same time from the equinox, both moving with an uniform motion, the one S″ in the ecliptic, the other S‴ in the equator, and let the motion of each be referred to the latter plane. Now on this supposition, S‴ will first advance upon the me-

Table *of the equation of time.*

Computed for 1810, *with the secular variation.*

Deg.	Sign 0.	Diff.	Secular Variation. ±	Diff.	Sign I.	Diff.	Secular Variation. +	Diff.	Sign II.	Diff.	Secular Variation. +	Diff.
	′ ″	″	″		′ ″	″	″		′ ″	″	″	
0	+6 59·2	18·8	2·97	25	−1 31·2	12·5	4·31	24	−3 43·4	4·5	10·83	18
1	6 40·4	18·9	2·72	24	1 43·7	12·0	4·55	24	3 38·9	5·0	11·01	17
2	6 21·5	18·9	2·48	24	1 55·7	11·4	4·79	23	3 33·9	5·5	11·18	18
3	6 2·6	18·8	2·24	24	2 7·1	10·8	5·02	24	3 28·4	6·1	11·36	17
4	5 43·8	18·9	2·00	25	2 17·9	10·4	5·26	23	3 22·3	6·6	11·53	16
5	5 24·9	19·0	1·75	24	2 28·3	9·8	5·49	24	3 15·7	6·9	11·69	16
6	5 5·9	18·8	1·51	24	2 38·1	9·3	5·73	23	3 8·8	7·7	11·85	16
7	4 4·1	18·8	1·27	25	2 47·4	8·8	5·96	23	3 1·1	8·0	12·01	15
8	4 28·3	18·7	1·02	24	2 56·2	8·3	6·19	23	2 53·1	8·4	12·16	15
9	4 9·6	18·6	0·78	24	3 4·5	7·6	6·42	23	2 44·7	8·8	12·31	14
10	3 51·0	18·4	0·54	25	3 12·1	7·1	6·65	23	2 35·9	9·2	12·45	14
11	3 32·6	18·5	0·29	24	3 19·2	6·6	6·88	23	2 26·7	9·8	12·59	14
12	3 14·1	18·3	0·05	25	3 25·8	5·9	7·11	22	2 16·9	10·0	12·73	13
13	2 55·8	18·0	0·20	24	3 31·7	5·4	7·33	23	2 6·9	10·3	12·86	13
14	2 37·8	17·9	0·44	24	3 37·1	4·9	7·56	22	1 56·6	10·8	12·99	12
15	2 19·9	17·6	0·68	25	3 42·0	4·2	7·78	22	1 45·8	11·2	13·11	12
16	2 2·3	17·3	0·93	24	3 46·2	3·4	8·00	22	1 34·6	11·4	13·23	11
17	1 45·0	17·1	1·17	24	3 49·6	3·0	8·22	21	1 23·2	11·6	13·34	11
18	1 27·9	16·9	1·41	25	3 52·6	2·5	8·43	22	1 11·6	11·9	13·45	10
19	1 11·0	16·7	1·66	24	3 55·1	1·9	8·65	20	0 59·7	12·1	13·55	9
20	0 54·3	16·4	1·90	24	3 57·0	1·3	8·85	21	0 47·6	12·3	13·64	9
21	0 37·9	16·0	2·14	25	3 58·3	0·6	9·06	21	0 35·3	12·5	13·73	9
22	0 21·9	15·5	2·39	24	3 58·9	0·1	9·27	21	0 22·8	12·7	13·82	8
23	+0 6·4	15·1	2·63	24	3 59·0	0·5	9·48	20	−0 10·1	12·8	13·90	8
24	−0 8·7	14·7	2·87	24	3 58·5	1·1	9·68	20	+0 2·7	12·9	13·98	7
25	0 23·4	14·3	3·11	24	3 57·4	1·7	9·88	19	0 15·6	13·1	14·05	6
26	0 37·7	13·9	3·35	24	3 55·7	2·2	10·07	20	0 28·7	13·1	14·11	6
27	0 51·6	13·5	3·59	24	3 53·5	2·8	10·27	20	0 41·8	13·2	14·17	5
28	1 5·1	13·2	3·83	24	3 50·7	3·4	10·47	18	0 55·0	13·1	14·22	5
29	1 18·3	12·9	4·07	24	3 47·3	3·9	10·65	18	1 8·1	13·0	14·27	4
30	1 31·2		4·31		3 43·4		10·83		1 21·1		13·31	

TABLE *of the equation of time.*

Computed for 1810, *with the secular variation.*

Deg.	Signs VI.	Diff.	Secular Variation. ±	Diff.	Signs VII.	Diff.	Secular Variation. −	Diff.	Signs VIII.	Diff.	Secular Variation. −	Diff.
	′ ″	″	″		′ ″	″	″		′ ″	″	″	
0	− 6 56·3	21·2	3·54	18	− 15 17·8	8·8	2·56	23	− 13 54·9	16·0	9·64	24
1	7 17·5	21·0	3·36	19	15 26·6	8·1	2·79	23	13 38·9	16·8	9·88	23
2	7 38·5	20·8	3·17	18	15 34·7	7·4	3·02	23	13 22·1	17·5	10·11	22
3	7 59·3	20·7	2·99	19	15 42·1	6·7	3·25	23	13 4·6	18·2	10·33	22
4	8 20·0	20·6	2·80	19	15 48·8	6·0	3·48	23	12 46·4	19·1	10·55	22
5	8 40·0	20·4	2·61	19	15 54·8	5·2	3·71	23	12 27·3	19·9	10·77	21
6	9 1·0	20·1	2·42	19	16 0·0	4·4	3·94	24	12 7·4	20·6	10·98	21
7	9 21·1	19·9	2·23	19	16 4·4	3·7	4·18	23	12 46·8	21·3	11·19	21
8	9 41·0	19·7	2·04	19	16 8·1	2·9	4·41	24	11 25·5	22·0	11·40	21
9	10 0·7	19·4	1·85	19	16 11·0	2·0	4·64	24	11 3·5	22·7	11·61	20
10	10 20·1	19·2	1·66	20	16 13·0	1·2	4·89	24	10 40·8	23·4	11·81	20
11	10 39·3	18·8	1·46	20	16 14·2	0·5	5·13	24	10 17·4	24·0	12·01	19
12	10 58·1	18·4	1·26	20	16 14·7	0·5	5·37	24	9 53·4	24·5	12·20	19
13	11 16·5	18·0	1·06	20	16 14·2	1·3	5·61	24	9 28·9	25·2	12·39	19
14	11 34·5	17·7	0·86	20	16 12·9	2·1	5·81	24	9 3·7	25·8	12·58	19
15	11 52·2	17·3	0·66	20	16 10·8	3·0	6·09	24	8 37·9	26·4	12·77	18
16	12 9·5	16·8	0·46	21	16 7·8	3·8	6·33	24	8 11·5	26·9	12·95	17
17	12 26·3	16·4	0·25	21	16 4·0	4·7	6·57	24	7 44·6	27·3	13·12	17
18	12 42·7	16·0	0·04	20	15 59·3	5·6	6·81	24	7 17·5	27·6	13·29	16
19	12 58·7	15·5	0·16	21	15 53·7	6·4	7·05	24	6 49·7	28·1	13·45	16
20	13 14·2	14·9	0·37	21	15 47·3	7·3	7·29	24	6 21·6	28·5	13·61	15
21	13 29·1	14·4	0·58	22	15 40·0	8·3	7·53	24	5 53·1	28·8	13·76	15
22	13 43·5	13·9	0·80	21	15 31·7	9·1	7·77	24	5 24·3	29·2	13·91	14
23	13 57·4	13·4	1·00	22	15 22·6	10·0	8·01	24	4 55·1	29·5	14·05	14
24	14 10·8	12·8	1·23	22	15 12·6	10·9	8·25	24	4 25·6	29·7	14·19	13
25	14 23·6	12·1	1·45	22	15 1·7	11·7	8·49	23	3 55·9	30·0	14·32	12
26	14 35·7	11·5	1·67	22	14 50·2	12·5	8·72	24	3 25·9	30·1	14·44	12
27	14 47·2	10·8	1·89	22	14 37·5	13·4	8·96	23	2 55·8	30·3	14·56	11
28	14 58·0	10·2	2·11	22	14 24·1	14·2	9·19	23	2 25·5	30·5	14·67	11
29	15 8·2	9·6	2·33	23	14 9·9	15·0	9·42	22	1 55·0	30·3	14·78	10
30	15 17·8		2·56		13 54·9		9·64		1 24·7		14·88	

Plane Astronomy.

Table *of the equation of time.*

Computed for 1810, *with the secular variation.*

Deg.	Signs IX.	Diff.	Secular Variation. —	Diff.	Signs X.	Diff.	Secular Variation. —	Diff.	Signs XI.	Diff.	Secular Variation. —	Diff.
	′ ″	″	″		′ ″	″	″		′ ″	″	″	
0	− 1 24·7		14·88		+11 36·3		14·89		+14 6·9		9·96	
		30·5		10		17·0		10		7·0		22
1	0 54·2		14·98		11 53·3		14·79		13 59·9		9·74	
		30·5		9		16·2		11		7·8		22
2	− 0 23·7		15·07		12 9·5		14·68		13 52·1		9·52	
		30·5		8		15·3		11		8·5		22
3	+ 0 6·8		15·15		12 24·8		14·57		13 43·6		9·30	
		30·4		8		14·5		11		9·1		22
4	0 37·2		15·23		12 39·3		14·46		13 34·5		9·08	
		30·3		7		13·8		12		9·6		22
5	1 7·5		15·30		12 53·1		14·34		13 24·9		8·86	
		30·1		6		13·0		13		10·2		23
6	1 37·6		15·36		13 6·1		14·21		13 14·7		8·63	
		29·9		5		12·1		13		10·7		22
7	2 7·5		15·41		13 18·2		14·08		13 4·0		8·41	
		29·7		5		11·2		14		11·6		23
8	2 37·2		15·46		13 29·4		13·94		12 52·4		8·18	
		29·5		5		10·4		14		12·0		22
9	3 6·5		15·51		13 39·8		13·80		12 40·4		7·96	
		29·3		4		9·5		15		12·5		23
10	3 36·0		15·55		13 49·3		13·65		12 27·9		7·73	
		28·9		3		8·7		15		13·0		24
11	4 4·9		15·58		13 58·0		13·50		12 14·9		7·49	
		28·6		2		7·9		15		13·4		23
12	4 33·5		15·60		14 5·9		13·35		12 1·5		7·26	
		28·3		1		7·0		16		14·0		23
13	5 1·8		15·61		14 12·9		13·19		11 47·5		7·03	
		27·8		1		6·1		16		14·4		24
14	5 29·6		15·62		14 19·0		13·03		11 33·1		6·79	
		27·3		0		5·2		17		14·8		24
15	5 56·9		15·62		14 24·2		12·86		11 18·3		6·55	
		26·8		0		4·4		17		15·1		23
16	6 23·7		15·62		14 28·6		12·69		11 3·2		6·32	
		26·3		1		3·7		17		15·6		23
17	6 50·0		15·61		14 32·3		12·52		10 47·6		6·09	
		25·9		2		2·8		18		16·0		24
18	7 15·9		15·59		14 35·1		12·34		10 31·6		5·85	
		25·3		2		2·0		18		16·3		24
19	7 41·0		15·57		14 37·1		12·16		10 15·3		5·61	
		24·6		3		1·1		19		16·7		24
20	8 5·8		15·54		14 38·2		11·97		9 58·6		5·37	
		24·1		4		0·3		19		17·0		24
21	8 29·9		15·50		14 38·5		11·78		9 41·6		5·13	
		23·4		5		0·6		19		17·2		23
22	8 53·3		15·45		14 37·9		11·59		9 24·4		4·90	
		22·8		4		1·3		19		17·4		24
23	9 16·1		15·41		14 36·6		11·40		9 7·0		4·66	
		22·1		6		2·0		20		17·7		24
24	9 38·2		15·35		14 34·6		11·20		8 49·3		4·42	
		21·5		6		2·7		20		17·9		24
25	9 59·7		15·29		14 37·9		11·00		8 31·4		4·18	
		20·9		7		3·6		21		18·1		24
26	10 20·6		15·22		14 28·3		10·79		8 13·3		3·94	
		20·2		8		4·4		20		18·3		24
27	10 40·8		15·14		14 23·9		10·59		7 55·0		3·74	
		19·3		8		5·0		21		18·5		25
28	11 0·1		15·06		14 18·9		10·38		7 36·5		3·45	
		18·5		8		5·6		21		18·6		24
29	11 18·6		14·98		14 13·3		10·17		7 17·9		3·21	
		17·7		9		6·4		21		18·7		24
30	+11 36·3		14·89		+14 6·9		9·96		+ 6 59·2		2·97	

§ VIII. *Of the phenomena and theory of the planetary motions.*

1. *Phases of the planets, &c.*

Phenomena of the planets.

167. Having in the preceding sections treated of the theory of the earth's motion about the sun, as well as examined some of the circumstances attending the planetary orbits generally, it remains for us now, first, to explain some of the more obvious phenomena which these bodies exhibit; and then to illustrate the principles upon which all the circumstances of their motion are computed. The hypothesis of the motion of the earth in its orbit, combined with its rotation on its axis, have been found sufficient to illustrate all the observed phenomena of the sun and stars; let us now see how the phenomena of the planets can be reconciled with the same hypothesis, when we add to it the motions of those bodies in their respective orbits. The circumstances to which we here allude, are the phases of the inferior, and the stationary positions and retrograde and direct motions of the whole; to which we have already slightly referred in our introductory chapter.

Of the phases of the inferior planets.

Phases of Venus.

168. As the same remarks will obviously apply to both our inferior planets, we shall select Venus for the subject of our explanation, her phases being far more conspicuous than those of Mercury.

Venus is seldom seen but for a certain time before sun rising, or after he is set; though sometimes her lustre is sufficient to render her visible while that body is yet above the horizon. When seen in the east, she is called the *morning* star; and when in the west, the *evening* star.

The motion of this planet sometimes direct, at others retrograde.

In the latter case, if observed for several successive nights, she will be found to vary her distance from the sun; first increasing it to about 45°, after which she approaches the sun again, and is finally lost in the more refulgent rays of this great luminary. If now we watch in the east for a few days, we shall perceive Venus rising before the sun, increasing her distance from him as before, till, after having attained her maximum in this direction, she again approaches him, and is again lost in the lustre of his beams.

These appearances, which continually recur, shows decidedly that Venus does not revolve about the earth; for, if this were the case, she must sometimes be seen in opposition to the sun; but it does not prove with the same force that her revolution is made about the sun; this, however, is otherwise rendered incontrovertible.

Venus sometimes a crescent, full, &c.

To the naked sight the disc of this planet, like that of all the others, appears circular, and always nearly of the same magnitude; but the telescope and its micrometer, both prove this to be a deception or optical illusion. For when thus viewed as an evening star, at her greatest elongation from the sun, she assumes the form of a crescent, like the moon in its first quarter, the horns being opposite to the sun. As she approaches the sun the horns diminish; and after her reappearance in the morning, we see her again assume the crescent form, but her horns pointing now in a contrary direction. Moreover, by means of the micrometer, it is found that her apparent diameter is greatest when she is least illuminated; that is, when she is ceasing to be an evening star; and least when most illuminated, or when ceasing to be a morning star; the ratio of her diameter being in these two cases as 166 to 26; that is, 2′ 46″ and 26″

Deducti from obs vations.

From these facts, deduced from observation, it appears, *first*, that Venus never attains an angle of elongation that far exceeds 45°; *secondly*, that she is at various distances from the earth; and other observations, not referred to above, have shown that she sometimes passes like a black spot over the solar disc: *thirdly*, it appears, that Venus, when an evening star, and separating from the sun, moves from west to east, or according to the order of the signs; but when she returns, her motion is necessarily in a contrary direction, or contrary to the order of signs; and the same phenomena have place in the opposite direction, after this planet first becomes a morning star.

These appearances are what it is requisite for us to show to be the necessary consequences of Venus revolving in an orbit interior to that of the earth, and of her being an opaque body enlightened by the sun.

The abo consister with the received hypothe Fig. 50.

169. First, that the phases of Venus correspond very correctly with such an hypothesis, will appear by referring to (fig. 50), in which S represents the sun, KVN the orbit of Venus, and EE′ part of the orbit of the earth. Let W, N, V, be three different positions of Venus in her orbit with respect to the earth at E; then, since that hemisphere of Venus which is perpendicular to the sun is the enlightened hemisphere, and that perpendicular to the earth is the visible one to the inhabitants of this planet, it is clear that all the phenomena above described, ought to obtain, as represented in the figure; viz. Venus will appear as a crescent at W, half illuminated at N, NE being a tangent to her orbit, gibbous at V, and full at K.

The same appearances precisely obtain with respect to Mercury; but being small, at a greater distance from the earth, and much nearer to the sun, they are not so easily observed as in the case of Venus.

It is also equally obvious, without our entering into a minute detail of the particulars, that, according to the supposition we have made, Venus ought to appear to move from K towards M according to the order of the signs; and from M to N contrary to that order; while about the points M and N her motion will be imperceptible, and she will appear stationary.

As we had here a view only to the general illustration, we have not considered the motion of the earth, but in any computation relative to these phenomena, it is obvious that this circumstance must be introduced, as we shall see in a subsequent article.

Stations, &c. of th superior planets.

170. None of the superior planets, with the exception of Mars, offer any phases such as those we have been explaining above; their distance from the earth being such as to render these phenomena imperceptible; but Mars not being so far distant, does sometimes exhibit a gibbous appearance, but is never horned, as we have shown to be the case with Venus and Mercury. We have seen, that neither Venus nor Mercury are ever seen in opposition to the sun, from which we have concluded, that they do not either revolve about the earth as a centre, nor include the orbit of the earth in their revolution about the sun; but Mars, and all the other superior planets, being found sometimes in opposition, and at others in conjunction, we are equally certain, that they do either revolve about the

earth, or enclose its orbit in their's. If they revolved about the earth as a centre, then their motions ought to be always direct; whereas, it is found from observation, that they are sometimes direct, at others stationary, and others retrograde; which is strictly what ought to take place, supposing them to revolve about the sun in circular or elliptic orbits superior to that of the earth. Their phases also, although not variable, are consistent with the same hypothesis; namely, that they revolve about the sun; that they are dark opaque bodies, which shine only by the borrowed light of the great central luminary.

Problem I.

To find the position of a planet when stationary.

171. To solve this problem with the utmost accuracy, the distances and relative motions of the earth and planets ought to be first ascertained, but the principle of computation may be explained independent of these data, and the subject having an intimate relation with the preceding parts of this chapter, we shall make no apology for introducing the question here, although we may not be able to give the most correct numerical results.

Let S (fig. 51) be the sun, E the earth, P the cotemporary position of a superior planet, and MN the sphere of the fixed stars, to which all motions are referred. Let EE′, PQ, be two indefinitely small arcs described in the same time; and let EP, E′Q produced, meet at L; then it is manifest, that while the earth moves from E to E′, the planet will appear stationary at L; and, in consequence of the immense distance of the fixed stars EPL, E′QL may be considered as parallel. Draw the other lines as in the figure: and then since EP, E′Q are considered as parallel, we shall have

$$\angle \mathrm{QE'S} - \angle \mathrm{PES} = \angle \mathrm{P}w\mathrm{S} - \angle \mathrm{PES} = \mathrm{ESE'}$$

and $\angle \mathrm{SP}w - \angle \mathrm{SQE'} = \angle \mathrm{S}v\mathrm{E'} - \mathrm{SQE'} = \mathrm{PSQ}$

Hence the cotemporary variations of the angles at E and P, are as the angles ESE′, PSQ; or, since the angular velocities are inversely as the periodic times, or as the $\frac{3}{2}$d power of the distances, we shall find those angles to each other as $\mathrm{SP}^{\frac{3}{2}} : \mathrm{SE}^{\frac{3}{2}}$, or denoting the ratio of the distances by $a : 1$, they will be as $a^{\frac{3}{2}} : 1$.

Now the sines of the angles of a triangle being in the ratio of their opposite sides, we shall have

$$\sin \mathrm{E} : \sin \mathrm{P} : \mathrm{SP} : \mathrm{SE} : a : 1$$

and the variation of angles, whose sines are constant being as their tangents; we shall have also,

$$\sin \mathrm{E} : \sin \mathrm{P} : a : 1$$

$$\frac{\sin \mathrm{E}}{\cos \mathrm{E}} (= \tan \mathrm{E}) : \frac{\sin \mathrm{P}}{\cos \mathrm{P}} (= \tan \mathrm{P}) :: a^{\frac{3}{2}} : 1$$

or $$\frac{\sin \mathrm{E}}{\sqrt{(1-\sin^2 \mathrm{E})}} : \frac{\sin \mathrm{P}}{\sqrt{(1-\sin^2 \mathrm{P})}} : a^{\frac{3}{2}} : 1$$

whence $$\sin^2 \mathrm{E} = \frac{a^3 - a^2}{a^3 - 1} = \frac{a^2}{a^2 + a + 1}$$

and $$\sin \mathrm{E} = \frac{a}{\sqrt{(a^2 + a + 1)}}$$

the planet's elongation from the sun when stationary.

By taking P to represent the earth, and E an inferior planet, the same results will apply to the case of Venus or Mercury.

Problem II.

To find the time when a planet is stationary; the time of opposition, if a superior planet, or of its inferior conjunction, if an inferior one, being given.

172. Let m and n be the daily motions of the earth and planet; and v, the angle PSE when the planet is stationary, then $m - n$, or $n - m$ is the daily variation of the angle at the sun between the earth and planet, according as it is a superior or inferior planet; hence,

$$\left.\begin{matrix} m - n \\ \text{or } n - m \end{matrix}\right\} : v :: \overset{\text{day}}{1} : \frac{v}{m \backsim n}$$

the time from opposition or conjunction to the stationary points both before and after.

Hence the planet must be stationary every *synodic* revolution.

Remarks.

173 It is to be observed, that the above solutions depend upon the supposition of the orbits being circular, which we know is not precisely the case; but as the subject is altogether one rather of curiosity than utility, we have not thought it necessary to enter upon it at greater length. It will be sufficient to observe, that in the case of elliptic orbits, the stationary positions will depend upon the position of the apsides, and the places of the bodies at the time. A near approximation however may be obtained by computing, on the supposition of circular orbits, the time when the planet would be stationary, and finding for several days about that time the geocentric place of the planet, so as to obtain two results, such, that on one day the motion of the planet be direct, and on the other retrograde, within which interval it must have been stationary; and the point of time when this happens may be determined by interpolation.

Superior planets retrograde in opposition.

174. It is obvious from what has now been stated, that a superior planet will be retrograde in opposition, and an inferior in its inferior conjunction; and that the latter would appear to retrograde whether the earth were in motion or at rest, but that the apparent retrogradation of a superior planet depends wholly upon the motion of the earth; this will, however, perhaps be rendered more evident by referring to fig. 52, in which we shall suppose E to represent the earth; P a superior planet, in opposition; then, as the velocities are as the inverse square roots of the radii of the orbits, the superior planet moves slowest; hence, if EF, PQ, be two indefinitely small cotemporary arcs, PQ is less than EF; and, on account of the immense distance of the sphere of the fixed stars, FQ must cut EP in some point x between P and m; consequently, the planet appears retrograde from m to n; whereas, if the earth were at rest, then the motion of P must necessarily appear throughout its whole circle to move in the same direction.

Fig. 52.

175. Again, if P be the earth, and E an inferior planet in its inferior conjunction, it will appear retrograde from v to w; which being due to the more rapid motion of the planet, it is clear that the same phenomena would occur, whether P were at rest or moving with a velocity less than that of the planet. The periods and positions of retrogradations would however be obviously changed in the former case.

PROBLEM III.

To delineate the phase of a planet in any given position.

Delineation of the phases.

176. The principle of this construction will appear from what has been stated in the preceding part of this section, where we have observed that the enlightened hemisphere will be that whose plane or base is perpendicular to a line drawn from the sun, and the hemisphere of vision that whose plane is perpendicular to a line drawn from the earth, each being directed to the centre of the planet. If, therefore, in fig. 53, we take S to denote the sun, E the earth, V any planet, aVb will be the plane of the illuminated hemisphere, or the plane of illumination, and cVd perpendicular to EV the plane of vision; draw av, perpendicularly to cd then ca is the breadth of the illuminated part which is projected into cv, the versed sine of cVa, or SVZ, SVc being the complement to both. Now the circle terminating, the illuminated part of the planet being seen obliquely, appears to be an ellipse, therefore if $cmdn$ represent the projected hemisphere of the planet next the earth, mn, cd, being two diameters perpendicular to each other, and we take cv = the versed sine of SVZ. and describe the ellipse mvn, then mcn vm will represent the visible enlightened part, as it appears at the earth; and from the property of the ellipse this varies as cv. Hence

Fig. 53.

The visible enlightened part
Is to the whole disc
As the versed sine of SVZ
Is to the diameter.

From this construction it appears, that the more remote a planet is from the sun with respect to the earth, the nearer will the illuminated part as seen from the earth approach to a circle; in Mars a gibbous form may sometimes be perceived, but in the more distant bodies the angle SVZ never differs enough from two right angles to give them the same appearance, that is, they must necessarily always appear full.

2. On the reduction of observations made on the planets from the earth, to the corresponding observations that would be made from the sun.

Determination of the heliocentric longitude of a planet.

177. The earth being in constant motion in its orbit, and being thus very differently situated with regard to the other bodies of our system, as well in respect to distance as position, it is actually necessary for the purpose of comparing different observations with each other, to refer them all to one common point of observation; and it is obvious that no point can be better suited for this purpose than the sun, the common centre of all their motions. The theory of the earth's motion being determined, the dimensions of its orbit may be taken as the base of the geometric operations necessary for the reduction in question, of which we have had occasion to speak in describing the laws of Kepler; we shall now illustrate the same a little more at length.

Fig. 54.

Let E (fig. 54) be the earth's place in its orbit, S the sun, and P any planet, P′ its projected place in the ecliptic, SEP′ the observed angle, measuring the difference between the longitude of the sun and that of the planet seen from the earth. After an entire revolution of the planet, it will return to the same point P; let then the earth be at E′, and the angle P′E′S be observed. Then by the theory of the earth's motion, the angle ESE′, and the lines SE, E′S are known, therefore the angles SEE′, SE′E, and the line EE′ may be also determined.

Again, in the triangle EME′, there is known EE′ and the angles P′EE′, P′E′E, therefore the sides P′E, P′E′ are likewise known or may be found. Lastly, in the triangle P′SE, the sides P′E, ES and the included angle P′ES are given to find SP′, the sun's distance from the planet at the point P′ of its orbit, as also the angle ESP′, whence the difference between ♈SE, the longitude of the earth, and ♈SP′, the longitude of the planet, become also determined.

178. To find the latitude seen from the sun, the following proportion may be used,

As the sine of the angle P′ES : sine of ESP′ : : the tangent of the latitude of the planet, observed from the point E, to the tangent of its heliocentric latitude, that is, to its latitude as observed from the sun. To find the true distance of the planet from the sun, that is, the line SP, say, as the cosine of the planet's heliocentric latitude is to radius, so is the SP′ to the distance sought.

We may otherwise determine the longitude, latitude, and distance by means of the triangle SEP. By either of these means the distance of a planet, its longitude and latitude may be found for various points of its orbit, but as will be seen in the following problem, three such points being found, the orbit itself may be described.

PROBLEM.

To describe an ellipse through three given points, that is, having given three distances, and the three corresponding heliocentric longitudes, to find the orbit.

179. In the preceding part of this section, we have shown the method by which the distance and heliocentric longitude of a planet may be determined from observation, the dimensions of the earth's orbit being supposed known; but the distances thus determined apply only to those particular points of the orbit in which the planet is at the times of observation, and we are still at a loss to determine the nature and elements of its entire orbit; the present problem then is intended to obviate this difficulty, and to show that, from three points being given, the orbit itself may be ascertained. This may be effected either geometrically or analytically, but we prefer the latter.

Fig. 55

Let SP, SP′, SP″, fig. 55, be three given distances, and ♈SP, ♈SP′, ♈SP″, three given longitudes, and AB the line of the apsides of the planet's orbit. Let the angles PSP′, PSP″, which are known, be represented by m, n, and the angle ASP which is unknown by x; let the three distances SP, SP′ SP″ be denoted by ρ ρ' ρ''; the semi transverse Ac by c, and the eccentricity by e; then the three unknown quantities which are to be determined are, a, e, and the angle x, and the three equations from which their values are to be deduced are,

$$\rho = \frac{a\,(1 - e^2)}{1 + e\cos x} \qquad (1)$$

$$\rho = \frac{a\,(1-e^2)}{1 + e\cos\,(m + x)} \qquad (2)$$

$$\rho'' = \frac{a\,(1 - e^2)}{1 + e\cos\,(n + x)} \qquad (3)$$

See art. 58. Dynamics.

From these three we deduce the two following, involving only e and x; viz.

$$\left.\begin{array}{l} \rho\,(1 + e\cos x) = \rho'\,\{1 + e\cos\,(m + x)\} \\ \rho\,(1 + e\cos x) = \rho''\,\{1 + e\cos\,(n + x)\} \end{array}\right\}$$

or which are still the same,

$$e = \frac{\rho'' - \rho}{\rho\cos x - \rho''\cos\,(n + x)} \qquad (4)$$

$$e = \frac{\rho' - \rho}{\rho\cos x - \rho'\cos\,(m + x)} \qquad (5)$$

Let $\rho'' - \rho = r''$ and $\rho' - \rho = r'$; then

$r'\,\rho\cos x - r'\,\rho''\cos\,(n + x) = r''\,\rho\cos x - r''\,\rho'\cos\,(m + x)$

or

$\cos x\,(r'\rho - r''\rho) = r'\rho''\cos\,(n + x) - r''\rho'\cos\,(m + x)$

substituting for $\left.\begin{array}{l}\cos\,(n + x) = \cos n\cos x - \sin n\sin x \\ \cos\,(m + x) = \cos m\cos x - \sin m\sin x\end{array}\right\}$

and dividing by $\cos x$, we have,

$r'\rho - r''\rho = r'\rho''\,(\cos n - \sin n\tan x) - r''\rho'\,(\cos m - \sin m\tan x)$

Hence $(r''\rho'\sin m - r'\rho''\,\sin m)\,\tan x = r'\rho - r''\rho - (r'\rho''\cos n - r''\rho'\cos m)$

Therefore, finally,

$$\tan x = \frac{r'\rho - r''\rho - (r'\rho''\cos n - r''\rho'\cos m)}{r''\rho'\sin m - r'\rho''\sin m}$$

The angle x being thus determined, the three angular distances from the perihelion become known; also x being given, the eccentricity e may be computed from either equation 4 or 5, and hence again a, the semi transverse axis from equation 1, 2, or 3.

180. We have thus the means of determining very accurately all the preceding elements of a planet's orbit, but it will be perceived, that it rests upon one important condition, namely, that the planet has been observed from the same point of its orbit after one, or some number of complete revolutions; for it is on this condition that the heliocentric longitudes are determined. The problem, therefore, will still not directly apply to any newly discovered body, as for example, Uranus or the Georgian planet, whose period of revolution being about 80 years, we must have waited till this period had passed by, before its heliocentric longitudes could have been computed on the principles explained in (art. 179). But as such a delay is wholly incompatible with the anxiety and impatience of astronomers, recourse has been had to a method of trial and error, and by assuming a distance in the first instance, and comparing the results as determined according to Kepler's laws, with those deduced from observations, and making such successive corrections as the case requires, astronomers have found the method of approximating to the elements of a planet's orbit, from three geocentric longitudes only, the time between these observations being of course also given. It was by these means that the elements of the orbit of Uranus were determined by Lalande to a very considerable degree of accuracy, within one year of the first discovery of that planet by Sir W. Herschel.

Plane Astronomy.

Of the methods of determining the elements of a planetary orbit.

Elements of a planet's orbit.

181. We have in the preceding articles explained the methods of determining certain of the elements of a planet's orbit; but there are others equally necessary to be known, in order to complete the theory of the planetary motions.

The quantities known under the denomination of elements, are the seven following.

1. The longitude of the ascending node.
2. The inclination of the orbit to the plane of the ecliptic.
3. The mean motion of the planet about the sun.
4. The mean distance from the sun.
5. The eccentricity of the orbit.
6. The longitude of the aphelion.
7. The epoch of the planet being in its aphelion

1. *Determination of the place of the nodes.*

Longitude of the node.

182. The *nodes* of a planet's orbit are the two points where the orbit intersects the plane of the ecliptic, and the line joining those points is called the line of the nodes. That node at which the planet passes from the southern to the northern side of the ecliptic is called the *ascending* node, and is marked ☊, the other is called the *descending* node, and is marked ☋.

Since the node is the intersection of the orbit with the ecliptic, and the latitude of a planet being always estimated from that plane, it is obvious that if by a comparison of observations made on a planet when near the node, and when its latitude is decreasing, we compute according to the principles explained in several preceding articles the exact time when its latitude was zero, that time will be the exact time of the node. Whence again, as explained in the last section, find the corresponding heliocentric longitude of the planet, and the result will be the longitude of the planet when in the ecliptic or the *longitude of the node.*

Second method.

Second method. Fig. 56.

183. Let N, (fig. 56,) be the place of the node, nMm a portion of the ecliptic, aNb, a portion of the planet's orbit, $am\,bn$, two heliocentric latitudes, reduced from their geocentric latitudes; then by the analogies of Napier.

$$\text{rad.}\sin Nm = \cot N\tan am \qquad (1)$$

$$\text{rad.}\sin Nn = \cot N\tan bn \qquad (2)$$

Whence

$$\frac{\sin Nm}{\sin Nn}\ \text{or}\ \frac{\sin\,(mn - Nn)}{\sin Nn} = \frac{\tan am}{\tan bn}$$

and by reducing

$$\sin nm \times \cot Nn - \cos mn = \frac{\tan am}{\tan bn}$$

Whence $\tan Nn = \dfrac{\sin nm\tan bn}{\cos mn\tan bn + \tan an}$

where bn, am, the two reduced latitudes are known; also nm, the difference of the longitudes in the interval of the observations; and hence Nn, or the difference of the longitudes n and N is also known; and the longitude of n being given, that of N, the node, is determined.

2. *To find the inclination of the orbit.*

Inclination of the orbit.

184. This is immediately deducible from the preceding investigation; for as Nn is determined, and bn known, we have from equation (2)

$$\cot N = r \times \frac{\sin Nn}{\tan bn} \text{ or } \tan N = r \times \frac{\tan bn}{\sin Nn}$$

3. *To determine the periodic time of a planet.*

Periodic times

185. This may be effected in different ways; viz. we may determine by means of (art. 179 et seq.) the semi axis of its orbit, and hence compute its periodic time by Kepler's laws; or we may observe the planet in its node, and again, when it returns hither; and the interval elapsed will be the period nearly but not exactly, by reason of the retrogradation of the nodes and some other minor circumstances.

Another method, and the most accurate of any, is to observe the planet, if a superior one when in opposition, and if an inferior when in conjunction, and to observe the same again after one or some number of complete revolutions, whence the time may be found very correctly, and the tables of its motion corrected. The time of opposition, it will be perceived, is selected for this determination, in consequence of the heliocentric and geocentric longitudes being then the same; whereby that reduction necessary in all other cases is avoided.

Periodic times by opposition.

186. Therefore when the planet is approaching towards opposition, observe the mean time that it passes the meridian, its right ascension and meridian altitude, whence by means of the corrections already explained, find its declination on those days; and, hence again, compute its latitude and longitude; make similar observations on the sun, and determine its longitude in like manner. By comparing these results with each other, it will be seen that the difference of longitude from being more than 180°, has become less than 180°, consequently, the opposition must have taken place in that interval, that is, between the two observations that give these opposite results.

In order now to find the exact instant, compute the sun's longitude at the moment when the planet was on the meridian, for that observation that gives the difference in longitude too little; and let $180° - d$ denote the difference of longitude between the sun and planet at this instant; let r represent the sun's daily increase in longitude at this time, and r' the daily motion in that of the planet; then since when a planet is in opposition its motion is retrograde, the sun and planet will approach each other at the rate of $r + r'$, per day; and we shall have

$$r + r' : d : 1 \text{ day } \frac{d}{r+r'}$$

the time prior to the latter of the two observations, when the difference of longitude was exactly 180°; which time, therefore, subtracted from the time of the said observation will be the time of opposition required.

If the above be, as we have supposed, repeated for several days, and the mean of the several results be taken, the time will be had more accurately, and still more so if the observations be repeated after a number of complete revolutions; as by this means the errors will be so much the more divided, and rendered of less consequence. If the time of opposition, as determined above, be compared with the time given in the tables, the difference will be the error of the tables, and may be employed as a means of correcting them.

187. Mr. Vince in his Astronomy, gives the following example of this kind of determination. On October 14, 1763, M. Lalande observed the difference between the right ascension of β Aries and Saturn, (which latter passed the meridian at 12h. 17m. 17s. apparent time) to be 8° 5′ 7″, the star passing first. Now the apparent right ascension of the star at that time was 25° 24′ 33″·6, hence the apparent right ascension of Saturn was 1s. 3° 29′ 40″·6 at 12h. 17m. 17s. apparent time, or 12h. 1′ 37″ mean time. On the same day, he found from observation of the meridian altitude of Saturn, that its declination was 10° 35′ 20″ N. Hence the longitude was found to be 1s. 4° 50′ 56″, and latitude 2° 43′ 25″ S. At the same time the sun's longitude was found by calculation to be 7s. 1° 19′ 22″, which subtracted from 1s. 4° 50′ 56″, gives 6s. 3° 31′ 34″; hence Saturn was 3° 31′ 34″ beyond opposition, but being retrograde must afterwards come into opposition. Now, from the observations made on several days at that time, Saturn's longitude was found to decrease 4′ 50″ in 24 hours, and by computation, the sun's longitude increased 59′ 59″, in the same time, therefore $r + r'$, or the sum of the two, was 64′ 49″: hence

64′ 49″ : 3° 31′ 24″ : 24 : 78h. 20m. 20s.

which added to Oct. 24, 12h. 1m. 37″ gives 27d. 18h. 21′ 57″ for the time of opposition.

To find Saturn's longitude at this time, say

24h. : 78h. 20m. 20s. 4′ 50″ : 15′ 47″

which latter is the retrograde motion of Saturn in 78h. 20m. 20s., and this subtracted from 1s 4° 50′ 56″ leaves 1s. 4° 35′ 9″, the longitude of Saturn at the time of opposition; and by a similar process, the longitude of the sun for the same time was found to be 7s. 4° 35′ 9″.

In a manner nearly analogous, we may find the latitude of Saturn at the instant of opposition; viz. by observing the daily variation in latitude. In the present case between the time of observation and opposition, the latitude had increased 6″, and consequently, the latitude was 2° 43′ 31″ S at the time of opposition.

188. The most ancient observation of this kind on record is one by Hipparchus; from which it appears that

228 B.C. Mar. 2, 1h. Saturn's longitude in opposition was .... } 98° 23′ 0″
lat. N 2° 50′

1714, A.D. Feb. 26, 8h. 15m. his longitude in opposition was .... } 97° 56′ 46″
lat. N 2° 3′

Differen. of longitude in 1943y. 105d. 7h. 15m. = 26′ 14″

Now we find the time of describing this 26′ 14″ to be 13d. 14h. Adding this to the former interval, we have 1943y. 118d. 21h. 15m. for the interval between these two oppositions of Saturn, and the period of this planet is known to be about 29½ years, this time must therefore contain 66 complete periods of revolution, and hence by division we have

$$\frac{1943\text{y. } 118\text{d. } 21\text{h. } 15\text{m.}}{66} = 29\text{y. } 162\text{d. } 4\text{h. } 27\text{m.}$$

for the exact time of one revolution.

ral re- tion of rn.

189. It will be observed, that the longitudes in both these cases is reckoned from the first point of Aries, and in consequence of the annual precession of this point of 50″·1, the entire precession due to the above interval of 29y. 162d. 4h. 27m. will be 24′ 35″. The period then determined for Saturn belongs to his tropical revolution, and is shorter than his sideral by the time that he employs in describing 24′ 35″, which is about 12d. 7h.; hence his sideral revolution will be 29y. 174d. 11h. 27m. His mean annual motion is hence found to be 12° 13′ 35″ 14‴ and his daily motion 2′·0097.

Exactly the same process may be observed with all the superior planets.

iod for nferior ets.

190. With respect to the inferior planets, we may proceed in the same manner when circumstances will admit of our making the requisite observations, but it is obvious that this will only happen when the elongation of the planet from the sun, near conjunction, is sufficient to render the former visible. The most accurate method of observing the time of an inferior conjunction of either Mercury or Venus is from observations made on them in their transits over the sun's disc.

3. *To find the semi axis major or mean distance.*

e mean nce.

191. Let d denote the mean distance of the earth, and t the time of its sideral revolution, D the required mean distance of the planet, and T its period of revolution, as determined in the last article. Then by Kepler's law

$$t^2 : T^2 :: d^3 : D^3 = d^3 \times \frac{T^2}{t^2}$$

whence $$D = d \times \left(\frac{T}{t}\right)^{\frac{2}{3}} \text{ the distance sought.}$$

4. *To determine the eccentricity of the orbit.*

tri-

192. This may be done by means of our general equation.

$$\rho = \frac{a\,(1 - e^2)}{1 + e \cos x}$$

the angle x, and radius vector ρ being determined as in arts. 179 and 180, and a as in the preceding article; but more conveniently from the equation (4) art. 179. viz.

$$e = \frac{\rho'' - \rho}{\rho \cos x - \rho'' \cos\,(n + x)}$$

in which all the quantities on the second side of the equation are supposed to have been determined.

5. *Of the aphelion of a planet's orbit.*

itude helion.

193. This is deducible immediately from article 179, where we have.

$$\tan x = \frac{r'\rho - r''\rho - (r'\rho'' \cos m - r''\rho' \cos n}{r''\rho \sin m - r'\rho'' \sin m}$$

for the quantity x is the angular distance of the aphelion from the observed place of the planet, the longitude of which is known.

nd me-

194. Another method of determining the same element, is by means of the greatest equation of the centre, for example, let the planet be observed when near A (fig. 48); if it should be exactly there, it will be known by the mean angle proportional to the time from M to A, minus the difference of the observed longitudes at M and A, being equal to the greatest equation of the centre. But suppose it to be less by some quantity e, then the planet is at some point S; make a second observation, and now let the difference of the true and mean motions be greater than the greatest equation, by another quantity e'; then the planet is at T, past the aphelion. Now the longitudes of S and T are known; consequently the difference, SET is known, and the intervals between the two observations; then, the points S and T being supposed very near to A, we shall have

$$e + e' : \angle \text{SET} :: e : \angle \text{SEA}$$

Add, therefore, the ∠ SEA to ∠ SEM, and ∠ MEA or the angular distance of M from the aphelion will be determined.

6. *To determine the epoch at which a planet is in the aphelion.*

Epoch determined.

195. This may be readily deduced from the preceding observations; for since the interval of time t, between the two observations at S and T are known, and the angle SEA has been determined, we have

$$e + e' : e :: t : \frac{e\,t}{e + e'}$$

the time from S to A: which time thus determined added to the time of observation at S, gives the time at which the planet was in its aphelion.

Table of the elements of the planets.

196. We shall here conclude this division of our treatise, by giving the following table of the elements of the planets, according to the most recent determination of Laplace, reduced where necessary to English measures.

Table of the elements of the planetary orbits.

Sideral periods of the planets.

	Days
Mercury	87·96258
Venus	224·700824
The Earth	364·256384
Mars	686·979619
Vesta	1335·205
Juno	1590·998
Ceres	1681·539
Pallas	1681·709
Jupiter	4332·596308
Saturn	10758·969840
Uranus, or the Georgian Planet	30688·712687

Mean distances, or semi axes major of the orbits.

	Days
Mercury	0·387098
Venus	0·723332
The Earth*........	1·000000
Mars	1·523694
Vesta	2·373000

---

* The distance of the earth from the sun is computed to be 93,726,900 English miles.

Juno	2·667163
Ceres	2·767406
Pallas	2·767592
Jupiter	5·202791
Saturn............	9·538770
Uranus, or the Georgian Planet......	19·18305

Ratio of the eccentricities to the semi axes major at the beginning of 1801, *with the secular variation of the ratio. The sign — indicates a diminution.*

	Ratio of eccentricity.	Secular variation.
Mercury......	0·205514	0·000003867
Venus........	0·006853	0·000062711
The Earth	0·016853	0·000041632
Mars	0·093134	0·000090176
Juno	0·254944	not ascertained
Vesta	0·093220	
Ceres	0·078349	
Pallas........	0·245384	
Jupiter	0·048178	0·000159350
Saturn	0·056168	0·000312402
Uranus	0·046670	0·000025072

Mean longitudes at the beginning of 1801, *reckoned from the mean equinox at the epoch of the mean noon, Jan.* 1*st*, 1801, *Greenwich.*

	°	′	″
Mercury	166	0	48·2.
Venus	11	33	16·1
The Earth	100	39	10·0
Mars	64	22	57·5
Vesta	267	31	49·0
Juno	290	37	16·0
Ceres	264	51	34·0
Pallas	252	43	32·0
Jupiter	112	15	7·0
Saturn............	135	21	32·0
Uranus	177	47	38·0

Mean longitudes of the perihelia, for the same epoch as above, with the sideral and secular variations.

	Mean lon. per.			Sec. var.	
	°	′	″	′	″
Mercury........	74	21	46	9	43·5
Venus..........	128	37	08	4	28
The Earth	99	30	5	19	39
Mars	332	24	24	26	22
Vesta	249	43	0	not known	
Juno	53	18	41		
Ceres	146	39	39		
Pallas	121	14	1		
Jupiter	11	8	35	11	4
Saturn..........	89	8	58	32	17
Uranus	167	21	42	4	0

Inclination of the orbits to the ecliptic at the beginning of 1801, *with the secular variations.*

	Inclination.			Sec. var.
	°	′	″	″
Mercury	7	0	1	19·8
Venus	3	23	32	4·5
The Earth	0	0	0	
Mars	1	51	3·6	1·5
Vesta	7	8	46	not known
Juno	13	3	28	
Ceres	10	37	34	
Pallas	34	37	7·6	
Jupiter	1	18	51	23
Saturn........	2	29	34·8	15·5
Uranns	0	46	26	3·7

Longitudes of the ascending nodes on the ecliptic at the beginning of 1801, *with the sideral and secular motions.*

	Lon. of ascen. node.			Sec. var.	
	°	′	″	′	″
Mercury	45	57	31	13·2	
Venus	74	52	38·6	31	10
The Earth...	0	0	0		
Mars	48	14	38	38	48
Vesta	103	0	6	not known	
Juno	171	6	37		
Ceres	80	55	2		
Pallas	172	32	35		
Jupiter	98	25	34	26	17
Saturn	111	55	46	37	54
Uranus	72	51	14	59	57

§ IX. *On the computation of eclipses.*

197. We have already given a popular illustration of the phenomena of eclipses; and have shown that they arise from the interposition of the earth between the moon and sun, or of the moon between the earth and sun. If the lunar orbit were in the plane of the ecliptic, we must necessarily have an eclipse at every conjunction and opposition of the sun and moon, and these phenomena would therefore, in this case, be governed by the phases of the moon; but the duration of the obscuration, and the other circumstances attending the eclipse, would still depend upon the relative distances and apparent magnitudes of these bodies with respect to each other.

When the moon is at her mean distance, the apparent diameters of the earth and sun as seen from the moon's centre are 1° 55′ 8″ and 31′ 59″·08. But the angles subtended by the sun and moon at the extremity of the earth's shadow in a lunar eclipse, are the same, about 64′; consequently, the moon must fall within that shadow at a very considerable distance from its vertex; and by computation it will be seen, that the length of the shadow is actually about equal to four times the distance of the moon from the the earth. This has reference, however, only to the mean distance, but the eccentricity of the lunar orbit being only ·0548553, the ratio between the length of the shadow and the distance of the moon is never materially changed. For example, the greatest distance of the moon from the earth never exceeds 64 terrestrial radii; and the length of the shadow when

the moon is in perigee is	212·896 rad.
mean distance	216·531
in apogee	220·231

Consequently, as we have observed above, if the lunar orbit, were wholly in the plane of the ecliptic, we must necessarily have an eclipse of the moon at every opposition.

The case, however, is somewhat different in a solar eclipse; for here the shadow of the moon in no instance passes far beyond the earth's centre, and in some it does not even reach the surface; but still the eclipse must, on the supposition we have made, actually take place with every conjunction of the sun and moon: this phenomenon depending, not upon the shadow of the moon, but upon the actual interposition of that body between the sun and earth.

The moon, however, revolves in an orbit which inclines to the plane of the ecliptic at an angle of 5° 9′; and, consequently, she may be in opposition, that is, the difference of longitude of the sun and moon may be 180°, and yet the latter body be either above or below the terrestrial shadow, in which case no eclipse will take place; and the same of course applies to the moon being in conjunction, or having the same longitude as the sun. It is not necessary, however, that the moon be actually in her node at the time of opposition, but only that she be within certain limits of that position; for the shadow of the earth subtending a larger angle at its vertex than the moon does, an eclipse may happen although the moon be not in her node at the time of opposition; and the same again, except that the limits are more narrow, at the time of conjunction. Let us, therefore, first examine what are called the *ecliptic limits,* that is to say, the least degree of proximity of the moon that will render an eclipse possible.

198. For this purpose, let SO (fig. 57) denote the radius of the sun, TE that of the earth, and L the centre of the moon at the moment of opposition: the three centres being in the same right line. Let OENC represent a solar ray, being a tangent to the sun at O, and to the earth at E: SO and TE will be perpendicular to the radius OC; and CS will be the axis of the conical shadow projected by the earth, and which, therefore, will be dark between the earth and the vertex C.

Let L u K be a portion of the lunar orbit; then its tangent LN will represent the semi diameter of a section of the cone, drawn perpendicularly to the axis in the region of the moon, and this section will be a circle; u will be the point where the edge of the lunar disc enters into the conical shadow; the centre of the moon being in K, or K u being supposed equal to a semi diameter of the moon at that time, as seen from the earth at T. Let now TE, the terrestrial radius = 1, then

$$\text{TS} = \text{dist. of sun from the earth} = \frac{1}{\sin p}$$

$$\text{TL} = \text{dist. of moon from the earth} = \frac{1}{\sin \text{P}}$$

$$\text{TC} = \text{axis of conical shadow} = \frac{1}{\sin \text{C}}$$

where p denotes the parallax of the sun, P that of the moon, and C the angle of the cone, or C = TCE.

$$\text{Now LC} = \text{TC} - \text{TL} = \frac{1}{\sin \text{C}} - \frac{1}{\sin \text{P}}, \text{ or}$$

$$\text{LC} = \frac{\sin \text{P} - \sin \text{C}}{\sin \text{C} \sin \text{P}} = \frac{2 \sin \frac{1}{2} (\text{P}-\text{C}) \cos \frac{1}{2} (\text{P}+\text{C})}{\sin \text{C} \sin \text{P}}$$

Let the apparent semi diameter of the sun, as seen from the earth be called δ, then

$$\text{SO} = \text{TS} \sin \delta = \frac{\sin \delta}{\sin p}$$

$$\text{and TC} = \frac{\text{SC TE}}{\text{SO}} = \frac{\text{SC}}{\text{SO}} = \frac{\text{ST} + \text{TC}}{\text{SO}}$$

Or, substituting for these quantities their respective values, we have

$$\frac{1}{\sin \text{C}} = \frac{\dfrac{1}{\sin p} + \dfrac{1}{\sin \text{C}}}{\left(\dfrac{\sin \delta}{\sin p}\right)} = \frac{1 + \dfrac{\sin p}{\sin \text{C}}}{\sin \delta}$$

Whence,

$$\frac{\sin \delta}{\sin \text{C}} = 1 + \frac{\sin p}{\sin \text{C}} = \frac{\sin \text{C} + \sin p}{\sin \text{C}}$$

Therefore,

or,

$$\sin \delta = \sin \text{C} + \sin p$$
$$\sin \text{C} = \sin \delta - \sin p$$

Now, $\sin \text{C} = 2 \sin \frac{1}{2} \text{C} \cos \frac{1}{2} \text{C}$

and $\sin \delta - \sin p = 2 \sin \frac{1}{2} (\delta - p) \cos \frac{1}{2} (\delta + p)$

Wherefore,

$$2 \sin \tfrac{1}{2}\text{C} \cos \tfrac{1}{2} \text{C} = 2 \sin \tfrac{1}{2} (\delta - p) \cos \tfrac{1}{2} (\delta + p)$$

In this expression, if $(\delta - p)$ and $(\delta + p)$ were equal to each other, we should have, accurately,

$$2 \sin \tfrac{1}{2} \text{C} = 2 \sin \tfrac{1}{2} (\delta - p)$$

or

$$\text{C} = \delta - p$$

which may, in fact, be taken as an approximate equation; for since δ never surpasses 16′ 18″, nor p 7 9″, we have the greatest value of $\frac{1}{2} (\delta + p) = 8' \, 14''$ and of $\frac{1}{2} (\delta - p) = 8' \, 5''$; consequently, the cos $\frac{1}{2}$ $\frac{1}{2} (\delta - p) = \cos \frac{1}{2} (\delta + p)$ *very nearly.* Hence the angle subtended by the side and axis of the conical shadow, is equal to the sun's apparent semi diameter minus the sun's parallax, or,

$$\text{C} = \delta - p$$

Now $\text{CTE} = 90° - \text{C} = 90° + p - \delta = b\text{TE} = b\text{E}$
$2 b\text{E} = 180° + 2p - 2 \delta = \text{E} b \text{F} = 180° - 2 (\delta - p)$

E b F is the obscure part of the earth's surface, which is less than a hemisphere by a zone whose altitude is C, or $\delta - p$; and the enlightened part is for the like reason greater than a hemisphere by the same quantity.

The angle $\text{LT}u = \text{T}u\text{E} - \text{C} = \text{P} - \text{C} = \text{P} - \delta + p = \text{P} + p - \delta$

Hence the sum of the two parallaxes, minus the sun's semidiameter, will be the angular distance of the moon from the axis of the conical shadow at the moment when the edge of the lunar disc enters into the shadow. To this angle, adding the moon's semidiameter (d), we shall have the angular distance of the axis of the cone from the moon's centre for the same instant. The two expressions therefore are

$$\text{P} + p - \delta, \text{ and}$$
$$\text{P} + p - \delta + d,$$

and they are those generally adopted; although, as we have seen, they are in fact only approximative, but very closely so.

199. If we conceive the moon to be placed between the earth and the sun in L′, L′n' being a part of its orbit, n' will be the point at which the moon enters into the luminous cone, and we shall have

$$\text{L}'\text{T}n' = \text{T}n'\text{E} + \text{C} = \text{P} + \delta - p = \text{P} - p + \delta.$$

Whence the difference of the parallaxes, plus the semidiameter of the sun, will be the angular distance of the centre of the moon from the axis of the cone, at the moment when the centre of the moon enters into the luminous cone; and to this expression, adding the semidiameter (d) of the moon, we shall have the angular distance corresponding to the moment when the edge of the lunar disc enters the cone.

The formulæ therefore for solar eclipses are

$$P - p + \delta$$
$$P - p + \delta + d.$$

We have thus the general expressions for the limits beyond which no eclipse of either luminary can take place.

Limits of total eclipses.

200. Within these limits an eclipse will happen, but they must be still further restricted in the case of the sun, in order that the eclipse may be total. With respect to the moon, the eclipse is necessarily total, provided the latitude of this luminary be such that it enters completely within the shadow. But this condition only is not sufficient in the case of a solar eclipse; for although the moon may be completely interposed between the earth and sun, yet its shadow may not arrive at the earth; and in this case the eclipse will not be total, but annular or partial.

Draw KVO, a tangent to the moon and sun; then, if KL′ be equal to TL′, the eclipse is necessarily total wherever it is central; and the duration of it will be greater or less according to the situation of the observer on the terrestrial surface. The same will also happen if KL′ be greater than TL′; but the obscuration will continue longer. If KL′ be less than L′T, but not less than L′a, the eclipse will still be total to those observers who are nearer the moon than the vertex of the cone, and total for an instant only to those who are precisely at the same distance as the vertex: to others the eclipse will be partial or annular. If KL′ be less than a L′, then the eclipse cannot be total to any observer on the terrestrial surface.

Now it may be shown, that

$$TL' - KL' = \frac{\dfrac{1}{\sin P} - \dfrac{\sin d}{\sin P \sin \delta}}{1 - \dfrac{\sin \delta \sin P}{\sin \delta \sin P}}$$

Hence assuming $d = 16'\ 28''$, $\delta = 16'$, $P = 60'\ 20''$ $p - 8''' \cdot 6$; we have, expressing the above in numbers,

$$TL' - KL' = \frac{\dfrac{1}{\sin P}\left(1 - \dfrac{\sin d}{\sin \delta}\right)}{1 - \cdot 002445107}$$
$$= \frac{1}{\sin P}(0 \cdot 00245110)\left(1 - \frac{\sin d}{\sin \delta}\right)$$
$$= \frac{1}{\sin P} - \frac{1 \cdot 2736100}{\sin \delta} + \frac{0 \cdot 0024511}{\sin P}$$

Consequently, when this expression is equal to zero, we shall have

$$\sin P = 3 \cdot 663792 \sin \delta$$

and when it is equal to unity, that is, to T a,

$$\sin P' = \frac{1 \cdot 002508336}{1 + \dfrac{0 \cdot 279996}{\sin \delta}}$$

and it is only between these limits, in the value of sin P, that the eclipse can be total.

201. By means of these two expressions, we may compute the several values of P corresponding to the different apparent diameters of the sun that must have place to produce a total or annular eclipse. For example, in the following table, the semidiameter of the sun being such as stated in the first column, the lunar parallax (P) must not be less than that stated in the second column, in order that the eclipse be necessarily total: if the parallax be less than in the second column, but greater than in the third, it may be total to certain observers; but if the parallax be less than that given in the third column, it can be only annular or partial.

Limits of total and annular eclipses.

δ		P		P′		P−P′
′	″	′	″	′	″	″
15	45	57	42	56	45	57
15	50	58	01	57	3	58
15	55	58	19	57	21	58
16	00	58	37	57	39	58
16	05	58	36	57	56	60
16	10	59	14	58	14	60
16	15	59	32	58	32	60
16	20	59	51	58	50	61

1. *Eclipses of the moon.*

202. Having in the preceding articles explained the principles by which the ecliptic limits may be determined, we proceed now to the computation; beginning with those of the moon.

Let NZ (fig. 58) be the ecliptic, O the point of it opposite the sun, or the nadir of the sun; the position of which will always be known by adding 180° to the sun's longitude. This point O will therefore be in the axis of the cone, and be the centre of the circular section cut by a plane perpendicular to the axis. Let

OE = OG = P + p − d, be the semidiameter of the shadow.

Conceive NAV to represent the orbit of the moon inclined about 5° 9′ to the ecliptic; then OA will be the latitude of the moon at the moment of conjunction. We shall suppose the point O immoveable during the time of the eclipse, although in fact it advances along the ecliptic about 2′½ in an hour, but the moon's motion is about 32′½ in an hour; we may therefore consider the point O constant, and conceive the moon to advance at the rate of about 30′, this being called its relative velocity or motion in the ecliptic.

While the moon advances from N to A (fig. 59), her latitude, which is nothing at N, will be OA when she has arrived at A, or is in conjunction. Fig. 59

Let m M denote the motion of the moon in longitude, and $m\lambda$ her motion in latitude, and m S the motion of the sun in longitude.

Then $\dfrac{m\lambda}{m M}$ = tan of inclination of the true orbit = tan N;

$\dfrac{m\lambda}{m M - m S}$ = tan of inclination of relative orbit = tan I;

for while the moon passes from N to A by a combination of the motions $m\,M$, $m\,\lambda$, the nadir of the sun will have passed from S to O; so that the moon and the nadir of the sun are found at the same instant in the same circle of latitude AO.

Now in the right angled triangle ONA, we have

$$\tan N = \frac{OA}{NO}$$

but by placing the nadir immoveable in O, it is necessary to suppose that the moon left the point I, such that $NI = SO$; whence IO will be the relative motion, which we have denoted by $m\,M - m\,S$: the motion in latitude will be actually OA; and the triangle OAI gives

$$\tan I = \frac{AO}{IO} = \frac{m\,\lambda}{m\,M - m\,S}$$

as stated above.

The motions being supposed uniform during the eclipse, if we assume $m\,M$, $m\,\lambda$, $m\,S$, to be the hourly motions, and n any fraction of an hour; then $n\,m\,M$, $n\,m\,\lambda$, $n\,m\,S$, will be the actual motions in the time n, and we shall still have

$$\frac{n\,m\,\lambda}{n\,m\,M - n\,m\,S} = \frac{m\,\lambda}{m\,M - m\,S} = \tan I.$$

Hence the relative motion of the moon to the sun, will always have the same inclination to the ecliptic; and if we suppose the nadir of the sun immoveable, the relative orbit of the moon will be IA, of which the inclination is I

These suppositions are made for the sake of simplifying the calculation; and they are so nearly true for the interval we have employed, that they produce no sensible error in the calculation: but if more accuracy should be thought desirable in any case, we may employ the common methods, for calculating the distances of the centres; or we may conceive the period of duration of the eclipse divided into a certain number of equal parts, for each of which the angle I of inclination may be computed. All the motions to which we have referred are the true motions as seen from the centre of the earth; and no notice is taken of the parallax; for in order to know whether the moon actually enters into the shadow, and loses the light which she receives from the sun, it is not necessary to know the point in the heavens to which the moon is referred, but the point where she actually is, before we can ascertain whether she is in the light or in the shadow.

203. These preliminaries being established, and supposing the time of opposition to have been determined by tables, or otherwise to within half an hour of the truth, we commence the calculation of the eclipse as follows:

Calculate first,

$$\tan I = \frac{m\,\lambda}{m\,M - m\,S}$$

$$\text{make } ♂ = 180 + S$$

then

$$(m\,M - m\,S) : 1 \text{ hour} = (3600'') : ♂ - M : t = \frac{(♂ - M)\,3600''}{m\,M - m\,S}$$

So that T being the time given by the first calculation, the time of opposition will be

$$T + t = T + \frac{(♂ - M)\,3600''}{m\,M - m\,S}$$

In which t will be negative if $M > ♂$.

Again,

$$3600'' : m\,\lambda :: t : \frac{t\,m\,\lambda}{3600''}$$

Hence the latitude at the opposition will be

$$\lambda + \frac{t\,m\,\lambda}{3600''} = \lambda + \left(\frac{♂ - M}{m\,M - m\,S}\right) m\,\lambda = l,$$

where λ is supposed to be the latitude by computation.

Now $l \cos I$, will be the shortest distance; for, letting fall the perpendicular $m\,O$, fig. 58, on the relative orbit NA, the point m will be the nearest to the centre O of the shadow; $O\,m$ will be the shortest distance from the centre of the moon to the point O, and

$$O\,m = OA \cos m\,OA = l \cos I$$

and it is evident that $m\,OA = 90° - m\,ON = 90° - I$; also, $m\,A$ will be the path of the moon in her orbit between the time of the shortest distance in m, and the time of the conjunction in A, or

$$m\,A = OA \sin m\,OA = l \sin I = O\,m \tan I.$$

Let fall the perpendicular $m\,m'$ on the ecliptic, and we shall have

$$m'\,O = m\,O \cos m\,O\,m' = l \cos I \sin m\,OA = l \sin I \cos I$$

$$(m\,M - m\,S) : 3600'' :: m'\,O : \text{time in } m'\,O = \frac{l \sin I \cos I\; 3600''}{m\,M - m\,S};$$

Or,

$$\text{time in } m\;O = \frac{\dfrac{l \sin I\; 3600''}{m\,M - m\,S}}{\cos I} = \frac{\dfrac{m\,A \cdot 3600''}{m\,M - m\,S}}{\cos I}$$

Thus, in order to convert the arc $m'\,A$ into time, that is to say, to determine the time which the moon employs to describe any arc of its relative orbit, it is sufficient to multiply this arc by the constant quantity

$$\frac{\dfrac{3600''}{m\,M - m\,S}}{\cos I} = \frac{3600'' \cos I}{m\,M - m\,S}$$

and reciprocally, to know the arc of the orbit when the time is given, we have only to divide by the above quantity, or multiply by its reciprocal,

$$\frac{m\,M - m\,S}{3600'' \cos I}$$

This is denominated the horary motion in the relative orbit.

204. From the centre O, (fig. 58) and with the radius $P + p - \delta = OE$, describe the circle ERG, which will be the circle of the shadow, or the section of the dark cone. With the radius $OL = OV = P + p - \delta + d$ mark the points L and V on the relative orbit; and from the points L and V, on the relative orbit, and with the radius $LE = LV = d$, describe two circles, which will represent the disc of the moon; and which two circles will be tangents in E and in G to the circle of the shadow. The points L and V will therefore be those of the beginning and end of the eclipse: the right angled triangles $m\,OL$, $m\,OV$, will have two sides equal each to each; and hence it follows, that

$$mL = mV = \surd(OL^2 - Om^2) = \surd(OV^2 - Om^2)$$
$$= \surd\{(OL - Om)(OL + Om)\}$$
$$= \surd\{(P + p - \delta + d - l\cos I)(P + p - \delta + d + l\cos I)\}$$

whence the value of mL or mV becomes known.

Duration of the eclipse. 205. It follows from this, that the point m is equally distant from the beginning in L, and from the ending in V; it is therefore the middle of the eclipse, and the time at m is called the time of the middle of the eclipse; the time in Lm is the half duration; wherefore the

$$\text{half duration} = \frac{mL \cdot 3600'' \cdot \cos I}{mM - mS}$$

middle of the eclipse − ½ duration = time of commencement,

middle of the eclipse + ½ duration = time of the end.

Hence we may make

$$\sin OLm = \sin u = \frac{Om}{OL} = \frac{l\cos I}{P + p - \delta + d}$$

and

$$mL = OL\cos u = (P + p - \delta + d)\cos u.$$

Quantity of the eclipse. As the moon advances from L towards m, its distance from the centre O will diminish, because we have always for any point L′,

$$L'O^2 = L'm^2 + Om^2;$$

Now the distance cannot diminish without a part of the disc entering into the shadow, and being eclipsed; and the part eclipsed will be

LO − L′O = $(P + p - \delta + d)$ − the actual dist. of centres.

It is obvious, supposing the centre to be in L′, that the semidiameter drawn from L′ towards O will be in the shadow; and that the part RL of another semidiameter will be also in the shadow. The eclipsed part therefore will be

$$d + RL' = d + OR' - L'O = d + P + p - \delta - L'O$$
$$= P + p - \delta + d - L'O.$$

The least value of L′m is Om, and therefore the greatest quantity of the eclipse is found by substituting for L′m, Om, or its value l cos I, in which case it becomes,

$$P + p - \delta + d - l\cos I.$$

If the part eclipsed be equal to the lunar disc = 2 d, we shall have,

$$2d = P + p - \delta + d - L'O$$
$$\text{or } L'O = P + p - \delta + d - 2d = (P + p) - (\delta + d)$$
$$= \text{sum of the parallaxes} - \tfrac{1}{2}\text{ sum of diameters.}$$

Fig. 60. Projection of an eclipse. 206. Take L′O = $(P + p) - (\delta + d)$, (fig. 60,) and describe about the two points L′, L′ two circles touching the circle of the shadow OR in the points R, R′, and we shall have

$$mL' = \surd\{(OL' - Om)(OL' + Om)\}$$
$$= \surd\{(P + p - \delta - d\, l\cos I)(P + p - \delta + d + l\cos I)\}$$

and the half duration of the total eclipse will be

$$\tfrac{1}{2}\text{ duration} = \frac{mL'\, 3600\cos I}{mM - mS}$$

In order to make these calculations of the half duration, the radius of the shadow is commonly assumed equal to $\frac{61}{60}(P + p - d)$; because it has been found that the observed duration always exceeds the computed; a circumstance which is attributed to the atmosphere of the earth, viz. its intercepting the light of the sun, and thereby producing an effect equivalent to an increase of $\frac{1}{60}$ in the diameter of the earth.

It is customary to express the quantity of the eclipse in digits, or $\frac{1}{12}$ths of the lunar diameter, under which form our equation (art. 205.) becomes

$$\text{eclipse} = \frac{12}{2d}(P + p - \delta + d - l\cos I)$$

which, when $l = o$, or when the eclipse happens in the node, becomes.

$$\text{eclipse} = (P + p) + (d - \delta) = (P + d) + (p - \delta)$$
$$= \frac{60{\cdot}20}{16{\cdot}28}d + d - (\delta - p) = \frac{76{\cdot}48}{16{\cdot}28}d - (\delta - p)$$

which exceeds the quantity 2 d very considerably. Whence we learn that a lunar eclipse may be more than total; that is, the number of digits may exceed twelve.

207. Again let us suppose

$$2d = \text{eclipse} = P + p + d - \delta - l\cos I$$

and it will follow, that

$$l\cos I = P + p - (\delta + d), \text{ and}$$
$$l = \frac{P + p - (d + \delta)}{\cos I}.$$

In this case the eclipse will be total, but only for an instant; if l be less than this quantity, the eclipse will be total with a greater or less duration: but if l be greater, the eclipse can only be partial.

The least value of P + p is 53′, and in this case $d + \delta = 31'$ at the greatest: hence

$$\frac{P + p - (d - \delta)}{\cos I} = \frac{53' - 31'}{\cos I} = \frac{22'}{\cos I}$$ at the minimum; consequently, if the latitude do not exceed 22′ we may conclude that the eclipse is necessarily total.

When we have measured the quantity of the eclipse

$$E = P + p - \delta + d - l\cos I,$$

we shall have

$$P = E - p + \delta - d + l\cos I$$

and we thus know the parallax of the moon, provided we know that of the sun, the semidiameters and the latitude; but practically this method, though employed by the ancients is by no means conclusive, on account of the difficulty in measuring the eclipse in consequence of the penumbra.

The figure of the shadow is always circular, which demonstrates the spherical figure of the earth; but as the arc which terminates the shadow is never more than a small part of the circumference, it is impossible from it to decide whether it be accurately a circle or an ellipse of small eccentricity.

208. As the penumbra, which encompasses the pure shadow, renders the beginning and end of the eclipse difficult to observe with certainty, astronomers prefer noting carefully the moment when the edge of the shadow arrives and it passes the divers spots which are always observed on the disc of the moon. Thus the same eclipse may present many different observations, which we may compare with others made in a different place, in order to ascertain the difference of longitude between the meridians; for these phenomena happen precisely at the same instant of time to all observers; and the time shown at these different places

will furnish of course the means of determining the difference of longitude between them.

The ancients, who had only the eclipses of the moon for finding the longitude, might easily be deceived as much as 2°, or even more, in their results; for two degrees in measure answers only to eight minutes in time; and we may readily imagine, that with the means they possessed, two observers might very easily be in error each four minutes in opposite directions, and thus cause a difference in the results to the extent we have stated, and even considerably beyond it.

Graphical construction of an eclipse.

209. Let AX (fig. 61) be a line representing the ecliptic, and divided into six equal parts by six perpendiculars, which will be so many parallels of latitude. Take each interval, AA', $A'A''$, &c. for the hourly relative motion in the ecliptic, and divide each of these intervals into 60 parts, which will be minutes in time; place the corresponding numbers at every five minutes, and mark with zero each intersection with a circle of latitude. This figure, once constructed and divided, will serve for all kinds of eclipses and occultations.

Let A''' be the place of the moon for the instant given by calculation; then, on the first interval, take

$$AB = m\,M - m\,S = \begin{cases}\text{relative motion of moon in}\\ \text{the ecliptic.}\end{cases}$$

If, for example, the relative horary motion be 32′, we must make $AB = 32'$, and the perpendicular $BD = 60'$; and the lines BD, AB, will have to each other the ratio of the times to the angular motions. Through the points A and D draw the indefinite right line ADH. This being done, take $A\,c$ equal to the latitude for the time shown by the calculation; and the perpendicular $c\,d$ will be the latitude in parts of the figure: apply $c\,d$ from A''' to a''', and a''' will be the place of the moon at the time given by the calculation.

Take $A\,a$ equal to the latitude an hour before the time by calculation, and apply the perpendicular $a\,b$, from A'' to a'', and the latter will be a second point in the relative orbit; draw the indefinite line $a''\,a'''$, which will be the relative orbit.

Take Ae equal to the difference of longitude between the moon and the sun at the time by calculation, and ef will be the distance of the conjunction in the figure; apply then ef from A''' to O, and O will be the place of the conjunction, and the centre of the shadow. If the conjunction is past the time by calculation, OA''' must be applied backwards on the ecliptic; but otherwise it must be taken forward towards X; and the divisions for the time will show how many minutes the point O precedes or follows the time by calculation: and we shall thus have the instant of opposition.

Draw the perpendicular $O\,m$, which will be the shortest distance: let fall the perpendicular $m\,n$, and the point n will show how much the middle exceeds the time by calculation, or how much it follows the time from A''; that is to say, the time in A''' diminished by an hour.

Take $AG' = P + p + d - \delta$; make $OC = G'H'$, and with the radius OC mark the points C and F from the beginning and the end: and the perpendiculars, $C\,c$, $F\!f$, will give the instant of these two phases.

Again, take $AG'' = P + p - d - \delta$, and make $OI = G''H''$; and with this as a radius, mark the points I and E, which will be those of immersion and emersion; viz. from the commencement and end of the total eclipse. The perpendiculars $I\,i$ and $E\,e$ will give these two instants.

Take $AG = P + p - \delta$, and with the radius $OG = GH$ describe from the point O the circle GM, which will be the section of the shadow.

With the radius CG describe circles about the points C, I, m, E, F, and we shall have all the phases of the eclipse.

Produce Om to M; $(d + m\,M)$ will be the quantity of the eclipse. If $O\,m$ is less than $P + p - \delta - d$, the eclipse will be total; if $O\,m = OM$, the eclipse will be six digits; if $O\,m$ be greater than $P + p - \delta$, the eclipse will be still less; and if $O\,m = P + p - \delta + d = OC$, the eclipse will not take place, being in this case only a simple contact.

We have stated, that the figure above described, when once constructed, will serve for all eclipses; it is therefore advisable, in order to preserve it, not to draw actually the several perpendiculars $C\,c$, $I\,i$, mn, &c., but to employ a square for simply getting the points of intersection, which is all that is required. See Delambre's *Abrégé d'Astronomie.*

2. *Eclipses of the sun.*

Eclipses of the sun.

210. At the instant when the centre of the moon enters into the luminous cone which extends from the sun to the earth, the angular distance of the moon from the axis of the cone is $P - p + \delta$; at the instant when the edge of the moon enters into the cone, the distance is $P - p + \delta + d$. But the edge of the moon cannot penetrate into the cone without obstructing the light of a part of the sun from an observer at E (fig. 57,) who will see the edge of the solar disc in the horizon, his zenith being in the prolongation of the radius TE; but an observer at E′ will see the centre of the sun in the line $E'rS$, and the inferior limb in the line $E\,o\,O'$, and consequently at a certain distance from the edge of the moon which enters at n'; the eclipse therefore will not have commenced to the observer at E′.

Let us conceive the plane $L'r\,o$ to be a tangent to the orbit $V'n'L'$ of the moon. This may be called the *plane of projection*; $o\,r$ will be the projection of the semidiameter of the sun; the point r the projection of the centre, and o that of the edge. The observers placed in different parts of the enlightened hemisphere, of which the section is $E\,a\,F$, will all see the centre of the sun answering to different points on the plane of projection.

Fig. 62.

The section of the cone by this plane will be circular, the radius of the circle seen from the centre of the earth, subtends an angle equal to $P - p + \delta$. Let therefore O (fig. 62) be the centre of the sun, considered as immoveable, and with the radius $OE = P - p + \delta$ describe the circle ED, which will represent the section of the cone. Let OA be the latitude of the moon in conjunction, NLAV the relative orbit; and the perpendicular $O\,m$ will be the shortest distance of the centres. Let $OL = P - p + \delta + d$, and mark the points L, V, which will indicate the first and the last instant of the eclipse; the point E will be that at which the first edge of the moon enters into the cone; and the point S that at which the other edge leaves it; therefore from the points L and V, and with the radius LE = VS = d, describe two circles, which will represent the lunar disc; the points E and S will be the points of contact of these two circles with the circle of the section of the cone EDS.

We shall thus have, as in the eclipses of the moon,

$$\tan \mathrm{I} = \frac{m\lambda}{m\mathrm{M} - m\mathrm{S}}; \quad \mathrm{O}m = l \cos \mathrm{I}; \ \mathrm{A}m = l \sin \mathrm{I};$$

$$\text{time in A}m = \frac{l \sin \mathrm{I} \cos \mathrm{I}.\ 3600''}{m\mathrm{M} - m\mathrm{S}}$$

$$m\mathrm{L} = \surd\{(\mathrm{OL} + \mathrm{OM})\ (\mathrm{OL} - \mathrm{O}m)\}$$

$$\tfrac{1}{2} \text{ duration} = \frac{m\mathrm{L} \cdot 3600'' \cdot \cos \mathrm{I}}{m\mathrm{M} - m\mathrm{S}}$$

211. Instead of the trigonometrical calculation, we may employ the graphic construction, (art. 209) and we shall thus have all the general circumstances of the eclipse; viz. we shall have the instant when the shadow of the moon begins to abstract the light of the sun from certain points on the earth, and that, when the eclipse is entirely terminated; but the most important and the most difficult to determine, is the precise places on the earth where the eclipse will be visible; and to find rules for calculations of the eclipse for a given place; for these circumstances vary with the position of the observer relative to the moon and sun, which changes every instant by the diurnal motion.

In all these calculations it is necessary to take into account the parallax, which is different for every place on the earth, and of which the effect changes every instant; in short, it is involved in so much intricacy, notwithstanding the ingenious application which has been made of the plane of projection, that M. Delambre has preferred employing the direct solution, and in which we propose to follow him in the remaining part of this article.

Effect of parallax illustrated. Fig. 63.

212. Let T (fig. 63) represent the centre of the earth, O a point on its surface, of which Z is the zenith. Let S be the sun, whose zenith distance will be ZTS for the centre of the earth, and ZOS to the observer at O; let the moon be in the same vertical at the distance ZOL; then the observer will see the centre of the moon on the centre of the sun; and if we suppose the apparent diameters of both bodies to be the same, as they are nearly, the edges of the two discs will coincide, and the sun will be totally eclipsed to the observer at O, while to an observer supposed at the centre, T, the entire disc, will be luminous, as it will consequently to all points on the surface comprised between the radii aT, bT, and to observers situated towards D and E, the distance of the two bodies will be still greater; we see therefore how very considerable is the effect due to the parallax.

Fig. 64.

Let HR (fig. 64) be the horizon, and HMPR the meridian of any place P; PS the circle of declination of the sun S; PL, that of the moon L; La, the semi-diameter of the lunar disc; Sb, that of the solar disc; at the centre of the earth, the distance of the edges of the moon and of the sun will be the arc $b\,a$, and therefore there will be no eclipse to an observer supposed at the centre; but if the arc $a\,b$ does not exceed a few minutes of a degree, the parallax, which has a tendency to diminish this distance, may bring the disc of the moon within the luminous cone, and produce an eclipse more or less considerable.

If it be desired to have the entire effect of the parallax, in the place of referring the moon to the zenith V, of the place P, and to the vertical VL, which is oblique to the distance of the centres SL, let us place it at Z on this distance SL, prolonged to M and Z; then the parallax will carry the edge a of the earth directly towards S. Make aZ = 90°, La will descend towards S by all the horizontal parallax; $a\,b$ will be diminished by a quantity equal to it; and if therefore $a\,b$ is equal to this parallax, the distance of the edges of the sun and moon is reduced to zero, supposing no parallax of the sun, but this last is p, and therefore the actual diminution of the arc $a\,b$, will be $\mathrm{P} - p$; consequently, when $a\,b = \mathrm{P} - p$, we shall have a simple contact at the place, whose zenith is Z, and which obviously happens when

$$\mathrm{SL} = \mathrm{P} - p + \delta + d$$

which is precisely the radius we have employed for determining the commencement and the end of the general eclipse. (art. 210.)

213. Let us conceive the arc ZP to be drawn, then PZ will be the meridian of the place, where the simple contact will be observed; PZ will be the distance from the pole, or the co-latitude of it; HMP is the meridian of the given place P, and consequently ZMP is the difference of the meridians. Now we know the time at P, for which we have calculated the true distance LS; and we have therefore the horary angle MPS. Hence, if we can find ZPS, we shall have ZPM = ZPS − MPS; but in the spherical triangle ZPS, we know

$$\mathrm{PS} = 90° - \text{dec. of sun} = 90° - \mathrm{D}$$
$$\mathrm{SZ} = \mathrm{S}b + 90° = 90° + \tfrac{1}{2}\text{diam.} \odot = 90° + \delta;$$

and we may calculate the angle PSL; we shall therefore have

$$\cos \mathrm{PZ} = \sin \text{lat.} = \cos \mathrm{S} \sin \mathrm{PS} \sin \mathrm{ZS} + \cos \mathrm{PS} \cos \mathrm{ZS}$$
$$= \cos \mathrm{S} \cos \mathrm{D} \cos \delta - \sin \mathrm{D} \sin \delta$$

$$\cot.\ \mathrm{ZPS} = \frac{\cot \mathrm{ZS} \sin \mathrm{PS}}{\sin \mathrm{S}} - \cos \mathrm{PS} \cot.\ \mathrm{S};$$

ZPS will therefore be the horary angle of the place, which is greater than the same angle corresponding to P, and the sun therefore more distant from the former place than from the latter: here we not only know the difference of the meridians, but we shall know also whether it be to the east or west, and the place required is therefore correctly known; that is,

$$\text{lon. sought} = \text{lon. P} + (\mathrm{ZPS} - \mathrm{MPS})$$
$$= \text{lon. P} - \text{horary ang. of P} + \mathrm{ZPS},$$

and this longitude will be west of P, supposing S to be in the east part of the heavens, and the hour at P to be in the morning.

If, on the contrary, we suppose the figure to represent the western part of the heavens, the time at P will be afternoon, and the east longitude of the place will be

$$= \text{lon. of P} + \mathrm{MPS}$$
$$= \text{lon. of P} + \mathrm{ZPS} - \mathrm{MPS}$$
$$= \text{lon. of P east} - \text{hor. ang. of P.}$$
$$+ \text{ the hor. angle of the place.}$$

The point thus determined will be the first that has simple contact.

214. The instant after that at which the simple contact has place, the true distance LS will be diminished, and $a\,b$ will be less than $\mathrm{P} - p$.

$$\sin ZV = \frac{LV}{P-p}; \; ZL = ZV - LV$$

Then, in the triangle ZPL, we shall have ZL, PL, and the contained angle, to calculate PZ, ZPL, ZPS; and we have still the latitude and longitude by the same formulæ.

In making use of the oblique parallax, we may take for our data one of the angles, with the sides SL and SV, for example LSV, and hence find LV, ZV, and ZL; and for the rest, the calculation is the same as the above. When $SL \angle (P-p) + (\delta + d)$, we have two points V, V′ two equal and similar triangles SLV, SLV′; two zeniths, Z, Z′, which have a simple contact in the horizon: the triangles SLV, at first very oblique, have the angle at L very acute, the angles V or V′ very obtuse; L will open more and more, and V will close; the angle LSV will increase from 0 to 180°, and we shall then have, (fig. 65,)

$$P - p = SL + (\delta + d), \text{ or}$$
$$SL = (P-p) - (\delta + d)$$

at this moment the points V and V′ are in one, we shall have only one zenith, one point of contact in the horizon; the curve formed by all these zeniths which give the contacts in the horizon, will finish as it begun, in one point only; it will form a sort of oval, of which we may have the breadth at each instant; for this breadth is the arc ZZ′, which separates the two zeniths, where the contact is simultaneous: now $ZL = 90° - LV = Z'L = 90° - LV'$; therefore

$$\sin \tfrac{1}{2} ZZ' = \sin ZL \sin SLV =$$
$$\sin \{90° - (P-p)\} \sin SLV =$$
$$\cos (P-p) \sin SLV.$$

The breadth ZZ′ can never become zero, except when sin SLV vanishes; therefore the two branches do not cross each other, and they touch each other only in the extreme points.

If the shortest distance $= (P-p) - (\delta + d)$, it will be at the middle of the general eclipse that the angle LSV becomes equal to 180°; and in the other half of the eclipse we shall see return in the inverse order, the same angles S, L, V, and the same triangles SLV, but not the same triangles LPZ, because the angles PLS, and the sides PL will have changed.

If the shortest distance $S m \angle (P-p) - (\delta + d)$, it is before the middle of the eclipse that the angle LSV will be 180°, and the contacts at the horizon will cease, to commence again after the middle, when we shall have $SL = (P-p) - (\delta + d)$, by which interruption we see that the curve of the contacts will close, and form an oval of an unequal breadth.

While $SL \angle (P-p) - (\delta + d)$, the contacts at the horizon are impossible; and in order to have the simple contact, we must employ the parallax in altitude.

At the conjunction in right ascension, (fig. 67,) if $SL = (P-p) - (\delta + d)$, the zenith Z is on SP; so that $SZ = 90° - \delta$; but if $SL \; 7 \; (P-p) - (\delta + d)$, we shall have two zeniths Z and Z′, which will correspond to places having contacts at the horizon; $LZ = LZ' = 90° - (P-p)$: these arcs will be commonly greater than LP, and the distance ZZ′ of the zeniths will be found by the formula,

$$\sin \tfrac{1}{2} ZZ' = \cos (P-p) \sin SLV.$$

The difference of longitude is the sum of the two horary angles after the conjunction. The point V, which gave a zenith more westerly, will give one more easterly, the two branches will intersect each other in the interval, and the curve will resemble the figure of 8. We might extend these remarks to a greater length; but as the circumstances attending the different forms of the curve under particular data, are rather matters of curiosity than utility, we shall proceed to other inquiries.

217. In order to calculate by this method all the circumstances of an eclipse, we are supposed to have found very exactly by the tables of the sun and moon, and for a time not too distant from the middle of the eclipse, or the conjunction, the longitude of the sun, his horary motion, his parallax and semi-diameter; the longitude and latitude of the moon, its horary motions, the parallax, and its diurnal change, and semi-diameter.

From these first data we compute, hour by hour, the longitudes and the latitudes, from which we draw the right ascensions and declinations both of the moon and sun.

218. Let us take as an example the annular eclipse of 1764, the most remarkable which has been observed in Europe for many years, supposing that we have found the following quantities:

True time.	Æ ☉	Declination ☉ = D	Æ ☾	Declination ☾ = D′	P − p	d	Motion in Æ = mot. Æ	D′ − D
8	11° 5′ 24″	4° 46′ 28″	9° 49′ 14″	4° 49′ 44″	54° 2′ 5″	14° 47′ 5″	1° 16′ 10″	3′ 16″
9	7 40	47 26	10 15 34	5 3 52	2 0	47 3	0 52 6	16 26
10	9 56	48 24	10 41 55	5 18 0	1 5	47 2	0 28 1	29 36
11	12 13	49 22	11 8 18	5 32 8	1 0	47 1	0 3 55	42 46
0	14 29	50 20	11 34 40	5 46 16	0 5	46 9	+0 20 11	55 56
1	16 46	51 18	12 1 4	6 0 24	54 0 0	14 46 8	+0 44 18	69 6

All these quantities increase with sufficient regularity, to enable us to extend the table to every ten minutes of time, by means of simple proportional parts. We may perform this operation on the declination of the sun and moon, for the differences of right ascension and parallax. And as to the diameters, we may suppose throughout $d = 14' 47''$, $\delta = 15' 57''$, and $\delta + d = \sigma = 30' 44''$. We may, moreover, suppose the relative motion in right ascension to be $P = 24' 6''$ on the equator; it will then be $(24' 6'') \cos D'$ in the region of the moon, and the relative motion in declination $= 13' 10''$.

The calculation of the relative orbit with regard to the equator, is a little less simple than when referred to the ecliptic, because the sun has a motion in declination as well as the moon. Astronomers, however, are satisfied with multiplying the relative motion in right ascension by the sine of the polar distance of the moon, in order to obtain the motion in the region of the moon, which motion will be $(24' 6'') \cos 5° 25' = 23' 59''\cdot5$; the tangent of the inclination of the relative orbit will be

$$\tan I = \frac{\text{mot. } D'}{\text{mot. Æ} \cos D} = \frac{13' 10''}{23' 59''\cdot5} = 28° 45' 26''$$

and the shortest distance $S m = 43' 54''\cdot6 \cos I = 39' 22''\cdot3$.

This method is sufficiently exact, taking, as we do, the declination of the moon corresponding to the middle of the interval.

We may determine the conjunction in right ascension by means of the following table A, where we see that at 11 hours the difference of right ascension was $3' 55''$, therefore

$$24''\cdot6 : 1\text{h.} :: 3' 55'' : 9' 46'';$$

that is, the conjunction will be at 11h. 9′ 46″, and subtracting 47′ 20″·4, we shall have for the middle, 10h. 22′ 26″.

At 11 hours, the difference of declination is $42' 46''$, and it increases by $13' 10''$ per hour · hence

$$1\text{h.} : 13' 10'' :: 9' 46'' . 2' 8''\cdot6,$$

and the declination in conjunction will be $44' 54''\cdot6$, s in the preceding approximation.

219. In order to calculate the angles S and L, and the distance of the centres $E = SL$, we may employ the following formulæ:

$$\tan\tfrac{1}{2}(L - S) = \frac{\sin\frac{1}{2}(D' - D)\cot\frac{1}{2}P}{\cos\frac{1}{2}(D' + D)} = \frac{\frac{1}{2}(D' - D)}{\frac{1}{2}\text{mot. Æ} \cos\frac{1}{2}(D' + D)};$$

$$\tan\tfrac{1}{2}(L + S) = \frac{\cos\frac{1}{2}(D' - D)\cot\frac{1}{2}P}{\sin\frac{1}{2}(D' + D)};$$

$$L = \tfrac{1}{2}(L + S) + \tfrac{1}{2}(L - S)$$

$$S = \tfrac{1}{2}(L + S) - \tfrac{1}{2}(L - S),$$

$$\sin E = \frac{\cos D \sin \text{mot. Æ}}{\sin L} = \frac{\cos D' \sin \text{mot. Æ}}{\sin S}.$$

We may observe, that the angles at S are always diminishing till the conjunction, when they pass through zero, change their positions, and then increase to the end. The angle at L, on the contrary, increases till the conjunction, when it becomes 180°, and then diminishes to the end.

$$\left.\begin{array}{l} P - p = 54' \; 3'' \\ \delta - d = 30' \, 44'' \end{array}\right\} (P - p) + (\delta - d) = 84' 47''$$

the distance of the centres for the commencement. Now we see by the table A, that this difference happens a little before 7h. 40′, at which time, $E = 83' 56''$, with a diminution of $4' 3''$ for 10 minutes; therefore

$$243'' : 10' \text{ or } 243'' : 600'' :: 51 : 126'' \text{ or } 2' 6'';$$

hence the commencement of the eclipse will take place at 7h. 37′ 54″.

The angle S' changes $1° 23' 20''$ in 10 minutes; it therefore changes $25'$ in $2' 6''$, it will consequently be $91° 6' 33''$ for 7h. 37′ 54″.

The declination of the sun changes 9″ every 10′, or $0''\cdot9$ in one minute; it will therefore be $4° 46' 7''$.

With these formulæ we find the horary angle H, viz.

$$\sin H = \cos D \cos S \sin ZS + \sin D \cos ZS$$

$$= \cos 4° 46' 7'' \cos 91° 6' 33'' = \sin \text{lat.} = 1° 6' 19'' \text{ S}$$

$$\cot. ZPS = \frac{\cos D \cot ZS'}{\sin S} - \sin D \cot. S$$

$$= -\sin 4° 46' 7'' \cot. 91° 6' 33'' = 89° 55' 17''$$

$$\text{ang. at Paris } (-4\text{h. } 22' 6'') \qquad = 65 \;\; 31 \;\; 30$$

longitude west of Paris.............. 24 23 47
Hour at the place, or dist. of merid. .. 5h 59 41·8

By making $ZS = 90°$ we have put the centre of the sun in the horizon, but we might, without sensibly lengthening the calculation, bring its upper limb in the horizon by making $ZS = 90° 15' 57''$, and by making $ZS = 90° 30' 44''$, we should bring the centre of the moon to the horizon.

There is no other calculation to make for this instant; this is the only place where the contact is seen, and it is seen in the horizon. This place being near the equator, we may employ the equatorial parallax, which is the greatest of any:

at 7h. 40′, the distance is only 83′ 56″
$\delta + d = \sigma = 30 \;\; 44$

The distance of the edges...... 53 12

It will be here sufficient, with a parallax of 53′ 12″ to work a contact in the vertical; the horizontal parallax produces an eclipse of 51″ in the northern limb of the sun.

220. In order to find the contact, we make

$$\sin N = \frac{53' 12''}{54 \cdot 3}, \quad ZS = N + \delta;$$

and afterwards with D and S for 7h. 40′ and the formulæ,

$$\sin H = \cos D \cos S \sin ZS + \sin D \cos ZS$$

$$\cot ZPS = \frac{\cos D \cot ZS}{\sin S} - \sin D \cot. S$$

we shall have the latitude and the horary angle, the hour and the longitude, as given in table B.

In this first calculation, the angle S may be acute or obtuse, according to the position of the moon with respect to the north pole, from which we reckon the angle PSL; but the rules for the signs of the trigonometrical lines are sufficient guides in this respect. At 13h. 10′. $E \; 7 \; (P - p) + \delta + d$, we ascertain by simple proportion when $E = P - + \delta + d$, as for the commencement, and by an operation, in all respects similar, we find the last point which will have a simple contact in the horizon.

221. For the following operation, that is, for forming table C, we have

$$ZS = 90° + E - (P - p) - d$$

$$\text{thus, for 7h. 40m. } 90° + E = 91° 23' 56''$$

$$-(P - p) = -54 \;\; 3$$

$$-d = -14 \;\; 47$$

$$ZS = 90 \;\; 15 \;\; 06$$

D and S are found as in the preceding operation.

The quantity of the eclipse is

$$(P-p)-(\delta+d)-E=p'+\sigma-E$$

which will augment as E diminishes; but it can never surpass $2d=29'\,34''$, it is to this quantity that we must reduce it if it be too great. Thus at 9h.

$p'+\sigma=84'\,46''$, $E=54'\,27''$ $p'+\sigma-E=30'\,19''$.

This quantity is less than 2δ, and greater than $2d$, we must therefore reduce it to $29'\,34''$. The moon has not then reached the southern limb of the sun, but it has passed the northern one, the eclipse is therefore annular.

At 9h. 10′ $p'+\sigma=84'\,46$, $E=51'\,25''$, $p'+\sigma-E$
$=33'\,21''$,

which is greater than 2δ or $31'\,54''$ by $1'\,27''$; the edge of the moon is therefore beyond the sun, and the eclipse is $2d-1'\,27''=29'\,34''-1'\,27''=28'\,7''$.

$$\begin{array}{lrr} \text{This found, by making} \;\ldots\; p'= & 54' & 2'' \\ E= & 51 & 25 \\ \hline p'-E= & 2 & 37 \\ \sigma= & 30 & 44 \\ \hline \end{array}$$

$$E-(p'-\sigma)=E+\sigma-p'=\sigma\,(p-E)=28\quad 7$$

This rule will serve till the eclipse surpasses $2d$, without being so great as 2δ, then we have to reduce it to $2d$, thus

$$\begin{array}{lrr} \text{at 11h. 40'}\ldots\ldots E= & 52' & 56'' \\ \sigma= & 30 & 44 \\ \hline E+\sigma= & 83 & 40 \\ p'= & 54 & 1 \\ \hline E+\sigma-p'= & 29 & 39\ \text{⁊}\ 2d \\ \text{the eclipse will be}\ldots\ldots & 29 & 34=2d. \end{array}$$

When the eclipse surpasses 2δ, we must have recourse to the formula

$$p'+\sigma-E=\sigma+(p'-E)=\sigma-(E-p'),$$

and the eclipse will again be northern, as in the commencement. The eclipse will be central for a point on the earth, when we have $E=P-p=54'\,2''$, which happens between the times 9h. and 9h. 10′. We may find this instant as we found the first contact. But from 9h. 10′ to 11h. 40′ inclusively, the distance being less than the horizontal parallax, we must use a parallax in altitude. We then search the distance from the zenith by the formula

$$\sin N=\frac{E}{P-p};\ (N-p\sin N)=ZS;$$

but we may, without sensible error, make $ZS=N$; then we shall have

$$\sin H=\cos D\cos S\sin ZS+\sin D\cos ZS,$$

$$\cot.\,ZPS=\frac{\cos D}{\sin S},\ \cot.\,ZS-\sin D\cot S,$$

where it is only necessary to pay attention to the signs. It is in this way that table D is formed, which shows all the points where the eclipse is central.

222. The places determined by the three first operations for the same instant are in the same vertical, and in the same great circle of the globe: therefore, in joining them by an arc of a great circle, we shall have on this arc all the places where the eclipse will be seen of divers magnitudes, from simple contact to where it is central, (when this happens,) and in all other cases to the *maximum* shown in table C.

223. After having made the calculations of the effects of the direct parallax, we must pass to the oblique parallax contained in tables E and F, where we find the contacts at the horizon.

We must at first calculate the angle $SLV=L'$ by the formula

$$\sin\tfrac{1}{2}L'=\frac{(p'+\sigma-E)\,(-p'+\sigma+E)}{2\,p'E};$$

from which we draw

$$PLV=PLS-L'=(L-L'),\ \text{and}$$
$$PLV'=PLS+L'=(L+L')$$

$PLZ=180^\circ-(L-L')$, $ZL=Z'L=90^\circ-(P-p)$,
$PLZ'=180^\circ-(L+L')$

$$\sin H=\cos PLZ\sin PL\sin ZL+\cos PL\cos ZL$$
$$=-\sin(L-L')\cos D'\cos(P-p)+\sin D'\sin(P-p)$$

$$\sin H'=-\sin(L+L')\cos D'\cos(P-p)+\sin D'\sin(P-p)$$

$$\cot.\,ZPL=-\frac{\cos D'\tan(P-p)}{\sin(L-L')}-\sin D'\cot(L-L')$$

$$\cot.\,Z'PL=-\frac{\cos D\tan(P-p)}{\sin(L+L')}-\sin D'\cot(L+L')$$

$ZPS=ZPL+LPS=ZPL+\text{mot. Æ}$,

$Z'PS=Z'PL+\text{mot. Æ}$.

These will be the horary angles of the two places; whence their longitudes may be deduced as above. These formulæ require no modification from the commencement to the conjunction in right ascension; but in order that the triangle LSV may be possible, it is necessary that

$$E\angle(P-p)+(\delta+d)\ \text{and}\ \text{⁊}\,(P-p)-(\delta+d).$$

If the shortest distance $Sm=\text{mot. }D\cos I=(P-p)-(\delta+d)$ the points V and V′ are united in one on the arc LS, and there will be but one point of contact in the horizon, and this in the prolongation of mS.

If $Sm\angle(P-p)-(\delta+d)$, the curve of contacts at the horizon of the first half will have no point common with the other half; and the two will be separated by an interval of greater or less extent.

When we have again $E=(P-p)-(\delta+d)$ the curve of contacts, recommences at one point. When $E\ \text{⁊}\ (P-p)-(\delta+d)$, there are two points, till $E=(P-p)+(\delta+d)$, then the curve is shut by a point only, which is the last of the eclipse.

For this second curve we have, as for the first, the same formulæ for L′, for PLV and PLV′, for PLZ and PLZ′, for H and H′, and for ZPL and Z′PL; but $ZPS=ZPL-\text{mot. Æ}$, and $Z'PS=Z'PL-\text{mot. Æ}$, because mot. Æ has a change of sign; by this means ZPS and Z′PS may be negative, if ZPL and Z′PL are very small: in this case, the contact at the horizon will be east, and it will be west when ZPS becomes positive: the same may be said of the triangle Z′PS.

The curve of contacts which embraces the pole will have a node. The two branches will cross each other, as we may perceive by putting the results of the calculation into table E and F; the former will contain the quantities given by the formula (L—L′), and the latter by (L + L′); but at 10h. 20′ the longitude and the latitude follow in more regular order, by putting the (L + L′) in table F, and the (L—L′)

Central Eclipse.

Table D.

Time at given place.		Longitude.		Latitude. north.		Time at the place where the eclipse is central.	
h.	m.	°	′	°	′	h.	′
9	10	30	42W	21	42	—4	53
9	20	23	36	24	55	—4	14
9	30	19	14	27	58	—3	47
9	40	15	47	30	58	—3	23
9	50	13	2	33	55	—3	2
10	0	10	15	36	57	—2	41
10	10	7	47	40	2	—2	21
10	20	5	22	43	12	—2	1
10	30	2	49	46	29	—1	41
10	40	0	1E	49	58	—1	20
10	50	4	55	53	21	—0	52
11	0	7	21	57	8	—0	32
11	10	12	13	61	15	—0	1
11	20	20	38	65	40	+0	42
11	30	34	2	70	24	+1	46
11	40	67	10	75	20	+4	9

The longitudes here are from the given place, Paris; and the negative time is the time during the morning, counted from the next noon. Table B gives the minimum, and table C the maximum in the same vertical. Table D shows the places where the eclipse is central. By the maximum, however, we must not understand the maximum absolutely, but the greatest immersion of the moon in the vertical of the sun. If the centre of the moon is above that of the sun, the eclipse is north; and it is south in the contrary case. The places situated on the same great circle have all an eclipse of greater or less magnitude, as they are nearer or more distant from the point where it is seen central; the distance between the first and second place is the greater axis or length of the figure of the shadow. The places marked in Tables B and F, by their distance, give the breadth of the shadow.

Contact in the horizon.

Table E.

Time, given place.		Longitude.		Latitude. north.		Time at place sought.	
h.	m.	°	′	°	′	h	′
7	40	24	57W	5	21	—6	4
7	50	29	12W	15	13	—6	7
8	0	32	14W	21	48	—6	9
8	10	35	10W	26	57	—6	11
8	20	38	10W	32	14	—6	13
8	30	40	59W	35	26	—6	14
8	40	44	2W	40	6	—6	16
8	50	47	27W	46	37	—6	20
9	0	50	34W	50	18	—6	22
9	10	54	1W	54	55	—6	26
9	20	57	42W	59	27	—6	31
9	30	61	54W	64	15	—6	38
9	40	66	58W	69	18	—6	48
9	50	73	56W	74	33	—7	6
10	0	86	19W	80	1	—7	45
10	10	120	44W	84	46	—9	53
10	20	165	4E	84	10	+9	20
10	30	135	38E	78	47	+7	35
10	40	124	13E	72	24	+6	57
10	50	127	36E	66	2	+6	40
11	0	112	32E	60	17	+6	30
11	10	108	56E	54	57	+6	26
11	20	105	27E	50	18	+6	22
11	30	102	18E	46	24	+6	18
11	40	99	20E	43	16	+6	17
11	50	96	32E	40	45	+6	14
12	0	93	53E	39	8	+6	13
0	10	91	20E	38	9	+6	12
0	20	68	55E	37	49	+6	12
0	30	86	33E	38	15	+6	12
0	40	84	11E	38	53	+6	12
0	50	82	20E	41	51	+6	14
1	0	80	42E	46	15	+6	17

Contact at the horizon.

TABLE F.

Hours, given place.	Longitude.	Latitude.	Time at place sought.
h. m.	° ′	° ′	h. ′
7 40	24 56W	6 50S	—6 0
7 50	26 44W	13 39	—5 59
8 0	28 52W	16 49	—5 55
8 10	31 9W	18 9	—5 54
8 20	33 3W	19 20	—5 52
8 30	36 1W	17 53	—5 51
8 40	38 29W	17 22	—5 54
8 50	40 55W	17 20	—5 54
9 0	43 33W	15 14	—5 54
9 10	46 14W	12 32	—5 55
9 20	49 1W	8 51	—5 56
9 30	51 51W	4 34	—5 57
9 40	54 46W	0 41N	—5 59
9 50	57 41W	6 8	—6 1
10 0	60 15W	12 20	—6 3
10 10	63 47W	18 53	—6 5
10 20	67 14W	25 31	—6 9
10 30	70 7W	32 9	—6 10
10 40	73 23W	38 24	—6 14
10 50	76 44W	44 15	—6 17
11 0	80 17W	49 58	—6 21
11 10	83 59W	55 13	—6 26
11 20	87 57W	60 10	—6 22
11 30	91 28W	64 48	—6 40
11 40	97 14W	69 2	—6 49
11 50	103 47W	73 36	—7 5
12 0	112 50W	77 49	—7 31
0 10	127 6W	81 42	—8 22
0 20	161 44W	84 40	—10 27
0 30	143 46E	84 16	+10 5
0 40	114 46E	81 2	+8 9
0 50	97 38E	75 9	+7 21
1 0	88 14E	67 50	+6 53

3. *Eclipse of the sun for a particular place.*

225. In determining the several circumstances attending an eclipse of the sun in a particular place, we have no occasion for that preparation, which has been explained in the preceding articles: we have a ready means of ascertaining whether the eclipse will or will not be visible; for, if the difference of declination at the instant of the mean conjunction be less than the sum $(P - p) + (\delta + d)$, and the moon nearer the pole than the sun (or than the star when the question is an occultation), it is nearly certain there will be an eclipse, at least if the conjunction does not take place considerably out of the meridian; and in all cases it will be sufficient to compute two triangles to decide the question.

When we have determined that the eclipse will happen, we must search for the time of the middle, known within half an hour or an hour, the longitudes of the two bodies, their latitudes, horary motions, parallaxes, and diameters. For two instants differing by an hour, we must calculate the right ascension of the midheaven M, the nonagesimal degree n, the height h of the pole, of the ecliptic, and the parallax in longitude and latitude, by the formulæ of art. 202, et seq.; substituting there the relative instead of the absolute parallax; the parallax in latitude and longitude being applied according to their signs, and to the places of the moon. We must then compare the position of the moon with that of the body eclipsed, and thence deduce the apparent difference of latitude and longitude for the two instants in question. It is necessary to make these calculations with great precision, but we may neglect the seconds in the subsidiary arcs n and h, which is the principle advantage of the nonagesimal method.

Preparations for observation.

226. This being done, and our object being to prepare ourselves for observation, we may proceed according to the principles of the graphical construction illustrated in art. 209; that is, on the two perpendiculars $A''a''$, $A'''a'''$, (fig. 61,) mark the two apparent latitudes, if the eclipse is of the sun, and the two apparent differences of latitude, if it be an occultation of a star or planet; and through the summits of the two latitudes draw a line, which will be the apparent orbit.

The interval between the two perpendiculars will be the horary motion; it will correspond to 60′ in time, answering to the relative motion of the moon to the sun, or more generally to the relative motion multiplied by the cosine of the latitude of the body eclipsed.

Find on the line AX the place of the apparent conjunction, as we found the centre of the shadow in the eclipse of the moon. Let fall from this point the perpendicular on the apparent orbit to obtain the shortest distance. With the sum of the diameters, mark the points for the beginning and end on the relative orbit; and by means of a square, mark on AX, the points which answer to the beginning and end.

If the eclipse be central or total mark the points of immersion and emersion, with an opening of the compasses equal to the difference of the semidiameters. If it be a planet which is eclipsed, these two phases will differ very little from the two first; and if it be a star, the points of contact will be confounded with those of immersion and emersion.

From the point m, as a centre, describe the lunar disc; and from the point O, as a centre, describe the disc of the body eclipsed, and we shall have the figure of the greatest eclipse.

Draw from the centre O, to the place of commencement C, the radius OC; and through the same point O raise a perpendicular to the ecliptic, which will be the circle of latitude; with the declination of the body calculate the angle of position, placing the circle of declination to the east, if the tangent of the angle of position be positive.

For the time of the beginning compute the angle of

Astronomy. the vertical of the sun with the circle of declination, that is, POZ; form the angle POZ on the figure, and OZ will be the vertical.

Now the eclipse ought to commence with the contact at G on the line OC. Place before you the figure of the eclipse in such a manner, that OZ may be a vertical line; and we shall see the position of the point G, at which the eclipse will commence; and thus we shall have the place of the first impression of the lunar disc on the body eclipsed. If it be a planet or a star, of which the disc is very small, through the point C, the centre of the moon, draw the parallel Z′C, which will indicate the point where the eclipse ought to begin.

This is indeed useless for the commencement, because the star or the planet will be seen by the side of the moon, but it is indispensable for the end, in order to know precisely the place of emersion, the planet till then being invisible behind the moon; and it is therefore of importance, in order to ascertain well the instant of emersion, and particularly if it takes place at the enlightened part of the disc. We must perform then, for the emersion of the star, an operation similar to that which we have employed for the beginning of the eclipse of the sun.

If however it is Jupiter that is eclipsed, we need not use the above proceeding, for some one or more of his satellites will commonly emerge before him, and indicate the place of his emersion: and if not, the disc of this planet is such, that if we miss the emersion of his first edge we may get that of the second.

227. In the occultations of the stars and planets, no other considerations than those of immersion and emersion are required; but in eclipses of the sun, we may continually measure the lines of the horns. This line is shortened by refraction; and, in order to compute the quantity, we must know the inclination to the horizon, which we may take at any time from the figure; and the line thus corrected, is a common chord to the lunar and solar discs, from which we may obtain the distance of the centres. The observation is however attended with difficulty, and is scarcely worth attention.

228. An observation more easy to make, and more useful in its results, is of the greatest quantity of the eclipse. Towards the middle of the eclipse, the line of the horns is of a constant magnitude, at least to observation, for one or two minutes, which is plenty of time for observing it very exactly.

Fig. 69. Let then $a\,b$ (fig. 69), the distance of the horns, be the greatest of all those measured during the eclipse, and corrected for refraction; then

$$\frac{\frac{1}{2}\,a\,b}{\mathrm{S}\,a}=\frac{a\,b}{2\,\delta}=\sin a\,\mathrm{SL},$$

$$\frac{\frac{1}{2}\,a\,b}{\mathrm{L}\,a}=\frac{a\,b}{2\,d}=\sin a\,\mathrm{LS};$$

$\delta\cos a\,\mathrm{SL}+d\cos a\,\mathrm{LS}=$ shortest distance $=(\lambda-p)\cos\mathrm{I}$; thus

$$\frac{\delta\cos a\,\mathrm{SL}+d\cos a\,\mathrm{LS}}{\cos\mathrm{I}}=(\lambda-p)=\text{app. lat.}$$

Calculate now the parallax of latitude p for the middle, then

$$(\lambda-p)+p=\lambda,$$

and we shall have the latitude in conjunction. SL sin I will be the difference of apparent longitudes on the ecliptic; SL sin I—parallax long. will be the true difference of longitude at the instant of the greatest eclipse; we shall thus have the time of the true conjunction by the true relative motions if we have well determined the time of the middle of the eclipse, and by calculating for this instant the true latitude by the tables, we shall know the error of the tables in latitude. It is true that the motion in latitude is only 6″ in one minute of time; and, therefore if we deceive ourselves by a minute as to the time of the middle, we shall not know nearer than within 6″ the errors of the tables in latitude.

If amongst the line of chords measured there are found many that are equal two and two, these will give the time of the middle. We shall also have the time of the middle very nearly by the observations on the beginning and end; and we shall have thus very nearly the errors of the tables.

If the eclipse be annular, the time of the two interior contacts, denominated the formation and breaking of the ring, will give us the difference of the diameters; for the interval of time between the two observations will give us the motion of the centre of the moon on its relative orbit. Let m be the half of this motion, ϵ the shortest distance, then we shall have

$$(\delta-d)^2=m^2+\epsilon^2;$$

also, at the instant of the interior contact, the breadth of the ring in the part opposite the contact will be the difference of the two diameters.

If the eclipse be total without duration for more than an instant, the apparent diameters are equal; and this instant will be that of the shortest distance $=(\lambda-p)\cos\mathrm{I}=0$; whence $\lambda=p$. Calculate therefore for this instant the parallax in latitude, which will be the true latitude, and the parallax in longitude will be the true difference of longitude for this instant; and the augmentation of the diameter will be the difference of the true diameters. We shall have therefore the correction of the three elements, the longitude, the latitude, and the difference of the diameters.

If the sun remains for some time in the shadow, we shall have the apparent relative motion; and if we have deduced the shortest distance, we shall have as above, the difference of the diameters. These observations, however, are very rare; and after all, perhaps they will not admit of great precision.

§ X. *Of Comets.*

229. We have already, in our introductory chapter, made a few observations on this class of celestial bodies, which are occasionally seen in the heavens with ill defined and faint discs, and usually accompanied with a *coma*, or stream of faint light, in the direction of a line drawn from the sun through the comet.

These bodies, as far as regards their change of place in the heavens amongst the fixed stars, resemble the moon and planets; but they differ from them in almost every other respect; they have never, (with one or two exceptions,) been known to return; that is, they do not appear to have definite elliptic orbits: the inclination of their orbits also are inclined to the ecliptic at a great variety of angles, from 0° to 90°; their motions are, some retrograde and some direct:

and the eccentricities of their orbits are so great, that it has been found that they may be very nearly represented by parabolas.

What are called the elements of a comet's orbit, are less than those of a planet's in number, being only five. It is impossible, from the observations made during one appearance of a comet, to compute the major axis of its orbit, and its period with any accuracy: what astronomers endeavour to compute, and what they do with difficulty compute, are, the perihelion distance, its place or longitude, the epoch of that longitude, the longitude of the ascending node, and the inclination of the orbit.

The elements of the orbits of planets are capable of being determined by observations made in the meridian; by longitudes and latitudes computed from right ascensions and declinations; but comets require observations of a different kind, by the rotation of the earth they are brought on the meridian, but generally from their proximity to the sun while they are visible, this does not take place in the night, and it is only then that they can be observed, on account of the faintness of their light. They must therefore be observed out of the meridian; and in that position the difference between the right ascension and declinations, and those of a known contiguous star must be determined.

It is difficult to make these latter observations with accuracy, by reason of the doubtful and ill defined disc of the comet; and a small error in the observations will materially affect the elements of the orbit.

The rigorous solution of the problem of the elements of a comet's orbit requires three observations only. But then the solution is attended with so many difficulties, that in this, as in other like cases, astronomers have sought, by the indirect method of trial and conjecture, to avoid them. If, what generally happens, more than three observations are obtained, the redundant ones are employed in correcting and confirming previous results.

The periodic time, as we have observed, cannot be determined from observations during one appearance of the comet. If known, it can only be so by recognising the comet during its second appearance; and the only mode of recognition is by the identity of the elements of its orbit, with those of the orbit of a comet already observed. If the perihelion distance, the position of the perihelion and of the nodes, and the inclination of the orbit, are the same, or nearly so, we may presume, with considerable probability, that the comet we are observing has been previously in the vicinity of the sun; and that, after moving round by the aphelion of its oval orbit, it has again returned towards its perihelion.

Comets not having been formerly observed with great accuracy, it so happens that, till lately, the period of one alone, that of the comet observed in 1759, 1682, 1607, and 1531, is known to any degree of certainty. Its period being about 76 years: assuming the earth's mean distance to be unity, the perihelion distance of this comet was 0·58, and the major axis of the orbit 35·9. The inequalities which are noted in its period are supposed to arise from the influence of some disturbing forces. This comet was foretold by Dr. Halley; and its appearance in the former of the above years verified the conjecture, and immortalized the name of this celebrated astronomer.

It would carry us far beyond our due limits to enter minutely into all the particulars necessary for illustrating the various methods of computations adopted by astronomers in their inquiries concerning the orbits of comets; and it does not belong to this part of our treatise to discuss the various hypotheses which have been advanced by different philosophers respecting the nature of those bodies, their tails, comas, &c.; we shall therefore conclude this section by giving in the form of a table, the several particulars of one hundred comets, whose elements have been determined with some degree of accuracy; whereby we should be able to ascertain their identity should any of them reappear.

It must not, however, be understood that this table is intended to include all, or even the principal number of the comets belonging to our system; this number being very considerable. From the accounts of the ancient and the more accurate observations of modern astronomers, it has been ascertained that more than 500 comets have been seen since the commencement of our æra. When the attention of astronomers was called to this subject by the expectation of the return of the comet of 1759, no less than seven were observed within the course of as many years; and from 1780 to 1800, about 30 comets were seen, and the elements of their orbits computed. These circumstances therefore, and the probability that most of the comets of small apparent magnitudes were overlooked by the ancients, and moreover that even in the present day many may pass unnoticed in the southern hemisphere, it is reasonable to conclude that the number of these bodies is very considerable, and far beyond any estimation that can be made from the observations we now possess.

Table of the elements of the orbits of 105 *comets.*

Number.	Passage through the perihelion in mean time, Greenwich.		Longitude of perihelion of the orbit of the comet.	Perihelion distance, that of the earth being 1.	Longitude of the ascending node.	Inclination of the orbit.	Motion.	Authorities.
	Anno A.C.	h. m. ″	s ° ′ ″					
1	539 Oct. 20,	15 0 0	10 13 30 0	0·3412	1ˢ 28° or 70 28°	10° + or −	D	Burckhardt.
	Anno Domini, Old Stile.							
2	837 March		9 19 3 0	0·5800	6ˢ 26° 33 0	10° or 12	R	Pingré.
3	1097 Sept. 21,	21 36 0	11 2 30 0	0·7385	6 27 30 0	73° 30′ 0″	D	Burckhardt.
4	1231 Jan. 30,	7 12 40	4 14 48 0	0·9478	0 13 30 0	6° 5′ 0	D	Pingré.
5	1264 July 6,	8 0 0	9 21 0 0	0·445	5 19 0 0	36 30 0	D	Dunthorn.
	—— July 17,	6 0 40	9 5 45 0	0·41081	5 28 45 0	30 25 0		Pingré.
6	1299 March 31,	7 28 40	0 3 20 0	0·3179	3 17 8 0	68 57 0	R	Pingré.
7	1301 Oct.		9s. or 10s.	0·457	0 15° + or −	70° + or −	R	Pingré.
8	1337 June 2,	6 25 0	1 7 59 0	0·40666	2ˢ 24 21 0	32 11 0	R	Halley.
	June 1,	0 30 40	0 20 0 0	0·6445	2 6 22 0	32 11 0		Pingré.
9	1351 Nov. 26,	12 0 0	2 9 0 0	1·0000			D	Burckhardt. [years.
10	1456 June 8,	22 0 0	10 1 0 0	0·5855	1 18 30 0	17 56 0	R	Pingré, period 75½
11	1472 Feb. 28,	22 23 0	1 15 33 30	0·54273	9 11 46 20	5 20 0	R	Halley.
	1531 Aug. 24,	21 18 30	10 1 39 0	0·56700	1 19 25 0	17 56 0	D	Halley.
12	1532 Oct. 19,	22 12 0	3 21 7 0	0·50910	2 20 27 0	32 36 0	D	Halley.
13	1533 June 16,	19 30 0	4 27 16 0	0·20280	4 5 44 0	35 49 0	R	Douwes.
14	1556 April 21,	20 3 0	9 8 50 0	0·46390	5 25 42 0	32 6 30	D	Halley.
15	1577 Oct. 26,	18 45 0	4 9 22 0	0·18342	0 25 52 0	74 32 45	R	Halley.
16	1580 Nov. 28,	13 44 40	3 19 11 5	0·59553	0 19 7 37	64 51 50	D	Pingré.
		15 0 0	3 19 5 50	0·59628	0 18 57 20	64 40 0	D	Halley.
17	1582 May 7,		8ˢ 5° or 9ˢ 11°	0·23 or 0·04	7ˢ 5° or 21°	59 or 61°	R	Pingré.
18	1585 Sept. 27,	19 20 0	0 8 51 0	1·09358	1ˢ 7° 42′ 30″	6° 4′ 0′	D	Halley.
19	1590 Jan. 29,	3 45 0	7 6 54 30	0·57661	5 15 30 40	29 40 40	R	Halley.
20	1593 July 8,	13 38 40	5 26 19 0	0·08911	5 14 15 0	87 58 0	D	De la Caille.
21	1596 July 31,	19 55 0	7 18 16 0	0·51293	10 12 12 30	55 12 0	R	Halley.
	29,	15 33 40	7 28 30 50	0·549415	10 15 36 50	52 9 45	R	Pingré.
*10	1607 Oct. 16,	3 50 0	10 2 16 0	0·58680	1 20 21 0	17 2 0	R	Halley.
22	1618 Aug. 7,	3 2 40	10 18 20 0	0·51298	9 23 25 0	21 28 0	D	Pingré.
23	1618 Oct. 29,	12 23 0	0 2 14 0	0·37975	2 16 1 0	37 34 0	D	Halley.
24	1652 Nov. 2,	15 40 0	0 28 18 40	0·84750	2 28 10 0	79 28 0	D	Halley.
25	1661 Jan. 16,	23 41 0	3 25 58 40	0·44851	2 22 30 30	32 35 50	D	Halley.
26	1664 Nov. 24,	11 52 0	4 10 41 25	1·025755	2 21 13 55	21 18 40	R	Halley.
27	1665 April 14,	5 15 30	2 11 54 30	0·10649	7 18 2 0	76 5 0	R	Halley.
28	1672 Feb. 20,	8 37 0	1 16 59 30	0·69739	9 27 30 30	83 22 10	D	Halley.
29	1677 April 26,	0 37 30	4 17 37 5	0·28059	7 26 49 10	79 3 15	R	Halley.
30	1678 Aug. 16,	14 3 0	10 27 46 0	1·23801	5 11 40 0	3 4 20	D	Douwes.
31	1680 Dec. 8,	0 1 2	8 22 40 10	0·006030	9 1 57 13	61 22 55	D	Pingré.
	7,	0 4 0	8 22 33 0		9 1 53 0	61 20 20		Halley.
	7,	23 9 0	8 22 44 25	0·006170	9 2 2 0	61 6 48		Halley.
	7,	20 38 39	8 23 26 48	0·006565	9 2 59 9	58 39 50		Euler.
	8,	0 4 0	8 27 43 0	0·005920	9 1 53 0	61 20 20		Newton.
*10	1682 Sept. 4,	7 39 0	10 2 52 50	0·58328	1 21 16 30	17 56 0	R	Halley.
		21 31 0	10 1 36 0	0·58250	1 20 48 0	17 42 0		Halley.
32	1683 July 2,	3 50 0	2 25 29 30	0·56020	5 23 23 0	83 11 0	R	Halley.
33	1684 May 29,	10 16 0	7 28 52 0	0·96015	8 28 15 0	65 48 40	D	Halley.
34	1686 Sept. 6,	14 33 0	2 17 0 30	0·32500	11 20 34 40	31 21 40	D	Halley.
35	1689 Nov. 21,	14 55 40	8 23 44 45	0·016889	10 23 45 20	69 17 0	R	Pingré.

TABLE—*continued.*

Number.	Passage through the perihelion in mean time, Greenwich.	Longitude of perihelion of the orbit of the comet.	Perihelion distance, that of the earth being 1.	Longitude of the ascending node.	Inclination of the orbit.	Motion.	Authorities.
	h m ″	s ° ′ ″					
36	1698 Oct. 8, 16 57 0	9 0 51 15	0·69129	8 27 44 15	11 46 0	R	Halley.
37	1699 Jan. 3, 8 22 19	7 2 31 6	0·74400	10 21 45 35	69 20 0	R	De la Caille.
38	1702 March 2, 14 12 19	4 18 41 3	0·64590	6 9 25 15	4 30 0	D	De la Caille.
39	1706 Jan. 19, 4 22 39	2 12 29 10	0·42580	0 13 11 40	55 14 10	D	De la Caille.
	4 56 4	2 12 36 25	0·426865	0 13 11 23	55 14 5	D	Struyck.
40	1707 Nov. 30, 23 29 39	2 19 54 56	0·8597	1 22 46 35	88 36 0	D	De la Caille.
	23 43 6	2 19 58 9	0·85904	1 22 50 29	88 37 40	D	Struyck.
41	1718 Jan. 3, 23 38 39	4 1 30 0	1·02650	4 8 43 0	30 20 0	R	De la Caille.
	4, 1 14 55	4 1 26 36	1·02565	4 7 55 20	31 12 53		Douwes.
42	1723 Sept. 16, 16 10 0	1 12 15 20	0·998651	0 14 16 0	49 59 0	R	Bradley.
43	1729 June 14, 11 6 40	10 22 40 0	4·26140	10 10 32 37	76 58 4	D	De la Caille.
	12, 6 36 2	10 22 16 53	4·0698	10 10 35 15	77 1 58		Douwes.
44	1737 Jan. 19, 8 20 0	10 25 55 0	0·22282	7 16 22 0	18 20 45	D	Bradley.
45	1739 June 6, 9 59 40	3 12 38 40	0·67358	0 27 15 14	55 42 44	R	De la Caille.
46	1742 Jan. 28, 4 38 40	7 7 35 13	0·76568	6 5 38 29	66 59 14	R	De la Caille.
	4 20 50	7 7 33 44	0·765555	6 5 34 45	67 4 11		Struyck.
	27, 4 14 39	7 10 49 23	0·7521	6 9 32 7	61 43 44		Euler.
47	1742 Dec. 30, 20 25 40	3 2 41 45	0·83501	2 8 21 15	2 19 33	D	De la Caille.
	21 15 16	3 2 58 4	0·838115	2 8 10 48	2 15 50		Struyck.
48	1743 Sept. 9, 21 16 18	8 6 33 52	0·52157	0 5 16 25	45 48 21	R	Klinkenberg.
49	1744 Feb. 19, 8 27 0	6 17 12 55	0·22206	1 15 45 20	47 8 36	D	Betts.
50	1747 Feb. 20, 7 10 40	9 7 2 0	2·19851	4 27 18 50	79 6 20	R	De la Caille.
	17, 11 44 38	9 10 5 41	2·29388	4 26 58 27	77 56 55		Chezeaux.
51	1748 April 17, 19 25 0	7 5 0 50	0·84067	7 22 52 16	85 26 57	R	Maraldi.
52	1748 June 7, 1 24 15 New Style.	9 6 9 24	0·65525	1 4 39 43	56 59 3	D	Struyck.
53	1757 Oct. 21, 9 46 40	4 2 49 0	0·33800	0 4 4 0	12 48 0	D	Pingré.
54	1758 June 11, 3 17 40	8 27 38 0	0·21535	7 20 50 0	68 19 0	D	Pingré.
*10	1759 Mar. 12, 13 31 40	10 3 16 0	0·58349	1 23 49 0	17 39 0	R	De la Caille.
	13 50 4	10 3 8 10	0·58490	1 23 45 35	17 40 14		Lalande.
	13 48 16	10 3 16 20	0·58360	1 23 49 21	17 35 20		Maraldi.
55	1759 Nov. 27, 0 2 37	1 23 34 19	0·80139	4 19 39 41	79 6 38	D	Pingré.
56	1759 Dec. 16, 21 3 40	4 18 24 35	0·96599	2 19 50 45	4 51 32	R	De la Caille.
57	1762 May 28, 15 17 40	3 15 15 0	1·01240	11 19 20 0	84 45 0	D	Lalande.
	28, 6 51 29	3 14 29 46	1·00985	11 19 2 22	85 3 2		Struyck.
	29, 0 18 28	3 15 22 23	1·01415	11 18 55 31	85 22 21		Maraldi.
58	1763 Nov. 1, 19 43 18	2 24 51 54	0·49876	11 26 23 26	72 40 40	D	Pingré.
	20 58 14	2 25 1 6	0·49820	11 26 27 0	72 28 0		Burckhardt.
59	1764 Feb. 12, 13 42 16	0 15 14 52	0·55522	4 0 4 33	52 53 31	R	Pingré.
60	1766 Feb. 17, 8 40 40	4 23 15 25	0·50533	8 4 10 50	40 50 20	R	Pingré.
61	1766 April 22, 20 46 20	8 2 17 53	0·33274	2 14 22 50	11 8 4	D	Pingré.
62	1769 Oct. 7, 12 20 40	4 24 5 54	0·12376	5 25 0 43	40 37 32	D	Lalande.
	13 36 53	4 24 11 7	0·12272	5 25 6 33	40 48 49		Prosperin.
	14 56 39	4 24 16 0	0·12265	5 25 3 0	40 50 0		Euler.
	15 28 16	4 24 10 51	0·1227	5 25 4 41	40 49 33		Lexell.
	12 34 9	4 24 11 8	0·12328	5 25 2 24	40 48 29		Sir H. Englefield.
	15 42 2	4 24 15 53	0·12275	5 25 6 4	40 46 42		Pingré.
	12 44 38	4 24 11 32	0·12327	5 25 3 40	40 47 56		Legendre.
63	1770 Aug. 14, 0 4 4	11 26 26 13	0·676893	4 12 17 3	1 34 30	D	Pingré.
	13, 12 55 40	11 26 16 26	0·674381	4 12 0 0	1 33 40		Lexell.
	13, 12 37 35	11 26 15 0	0·67435	4 11 54 50	1 34 31		Burckhardt.

TABLE—*continued.*

Number.	Passage through the perihelion in mean time, Greenwich.	Longitude of perihelion of the orbit of the comet.	Perihelion distance, that of the earth being 1.	Longitude of the ascending node.	Inclination of the orbit.	Motion.	Authorities.
	h m ″	s ° ′ ″					
64	1770 Nov. 22, 5 38 40	6 28 22 44	0·52824	3 18 42 10	31 25 55	R	Pingré.
65	1771 April 18, 22 5 7	3 13 28 13	0·90576	0 27 51 0	11 15 20	D	Pingré.
	19, 5 1 21	3 14 2 54	0·90340	0 27 50 27	11 16 0		Burckhardt.
66	1772 Feb. 18, 20 41 15	3 18 6 22	1·01815	8 12 43 5	18 59 40	D	Lalande.
67	1773 Sept. 5, 11 9 25	2 15 35 43	1·13390	4 1 15 37	61 25 21	D	Pingré.
	14 33 48	2 15 10 58	0·10120	4 1 5 30	61 14 7		Burckhardt.
68	1774 Aug. 15, 10 46 15	10 17 22 4	1·4286	6 0 49 48	83 0 25	D	Mechain.
69	1779 Jan. 4, 2 2 40	2 27 13 11	0·71312	0 25 5 51	32 24 0	D	Mechain.
	2 15 10	2 27 13 40	0·7132	0 25 3 57	32 25 30		Chev. D'Angos.
70	1780 Sept. 30, 18 3 30	8 6 21 18	0·09925	4 4 9 19	53 48 5	R	Mechain.
71	1781 July 7, 4 32 0	7 29 11 25	0·77586	2 23 0 38	81 43 26	D	Mechain.
72	1781 Nov. 29, 12 32 26	0 16 3 28	0·96101	2 17 22 52	27 13 8	R	Mechain.
	12 33 25	0 16 3 7	0·96099	2 17 22 55	27 12 4		Legendre.
73	1783 Nov. 15, 5 44 3	1 15 24 46	1·5653	1 24 13 50	53 9 9	D	Mechain.
74	1784 Jan. 21, 4 47 40	2 20 44 24	0·70786	1 26 49 21	51 9 12	R	Mechain.
75	1784 April 9, 21 7 26	10 28 54 57	0·65053	2 26 52 9	47 55 8	R	Chev. D'Angos.
76	1785 Jan. 27, 7 48 44	3 19 51 56	1·143398	8 24 12 15	70 14 12	D	Mechain.
77	1785 April 8, 8 58 52	9 27 29 33	0·427300	2 4 33 36	87 31 54	R	Mechain.
78	1786 July 7, 21 50 52	5 9 25 36	0·41010	6 14 22 40	50 54 28	D	Mechain.
79	1787 May 10, 19 48 40	0 7 44 9	0·34891	3 16 51 36	48 15 51	R	P. de Saron.
80	1788 Nov. 10, 7 25 40	3 9 8 27	1·06301	5 7 10 38	12 28 20	R	Mechain.
81	1788 Nov. 20, 9 4 25	0 23 12 22	0·766911	11 21 42 15	64 52 32	D	Mechain.
82	1790 Jan. 15, 5 5 39	2 0 14 32	0·7581	5 26 11 46	31 54 15	R	De Saron.
83	1790 Jan. 28, 7 36 13	3 21 44 37	1·063286	8 27 8 37	56 58 13	D	Mechain.
84	1790 May 21, 5 46 54	9 3 43 27	0·79796	1 3 11 2	63 52 27	R	Lalande.
85	1792 Jan. 13, 13 35 9	1 6 29 42	1·293	6 10 46 15	39 46 55	D	
86	1795 Dec. 14, 23 17 53	5 13 37 0	0·227	11 29 11 0	24 17 0	D	Zach.
	15, 15 6 18	5 7 37 0	0·258	11 13 23 0	20 3 0		Bouvard.
	14, 18 43 0	5 15 34 0	0·215	0 1 7 0	24 42 0		Prosperin.
87	1796 April 2, 19 47 51	6 12 44 0	1·578	0 17 2 0	64 55 0	R	Olbers.
88	1797 July 9, 2 44 31	1 19 34 48	0·52545	10 29 16 35	50 35 50	R	Bouvard.
89	1797 April 4, 11 32 21	3 14 59 0	0·48476	4 2 9 0	43 52 16	D	Burckhardt.
90	1798 Dec. 31, 12 58 57	1 4 29 48	0·77968	8 9 30 44	42 23 25		Burckhardt.
91	1799 Sept. 7, 5 49 48	0 3 39 12	0·839865	3 9 31 59	50 55 37	R	Mechain.
	5 34 4	0 3 39 10	0·840178	3 9 27 19	50 57 30		Von Zach.
	4 24 39	0 3 36 0		3 9 34 0	50 52 30		Burckhardt.
92	1799 Dec. 25, 21 30 49	6 10 20 12	0·625810	10 26 49 11	77 1 38	R	Mechain.
93	1801 Aug. 8, 13 22 39	6 3 49 0	0·2617	1 14 28 0	21 20 0	R	Burckhardt.
94	1803 Sept. 9, 20 33 54	11 2 8 0	1·0942	10 10 17 0	57 0 0	D	Mechain.
95	1804 Feb. 13, 13 31 7	4 28 44 51	1·07117	5 26 47 58	56 28 40	D	Gauss.
96	1805 Nov. 18, 0 15 39	4 29 0 28	0·37567	11 15 6 51	15 58 12	D	Legendre.
97	1805 Dec. 31, 6 1 36	3 19 23 40	0·89159	8 10 35 24	16 25 25	D	Legendre.
	1806 Jan. 2, 10 35 10	3 19 30 2	0·911786	8 11 28 23	12 43 10	D	Legendre.
	1805 Dec. 31, 6 19 27	3 19 21 55	0·89203	8 10 34 42	16 30 35	D	Bessel.
	31, 18 39 36	3 19 23 13	0·89682	8 10 48 5	15 36 10	D	Bessel.
	31, 6 51 8	3 19 23 39	0·89181	8 10 33 34	16 31 10	R	Legendre.
	31, 6 20 59	3 17 21 50	0·89193	8 10 33 35	16 30 32	D	Gauss.
	31, 8 41 18	3 19 28 54	0·890347	8 10 31 34	16 35 9	R	Bouvard.
98	1806 Dec. 28, 22 2 20	3 4 4 30	1·08193	10 22 18 38	35 4 5	D	Bessel.

TABLE—*continued.*

Number.	Passage through the perihelion in mean time, Greenwich.	Longitude of perihelion of the orbit of the comet.	Perihelion distance, that of the earth being 1.	Longitude of the ascending node.	Inclination of the orbit.	Motion.	Authorities.
	h m ″	s ° ′ ″					
	1806 Dec. 28, 22 30 43	3 7 2 3	1·081571	10 22 19 15	35 2 50	D	Burckhardt.
99	1807 Sept. 18, 18 56 0	9 0 56 53	0·647491	8 26 39 40	63 14 1	D	Bouvard.
	18, 19 6 0	9 1 0 13	0·64788	8 26 38 31	63 12 36	D	Gauss.
	18, 5 31 12	9 2 24 0	0·65320	8 26 44 0	62 57 0	D	Beck Colkoen.
	18, 19 49 0	9 0 53 38	0·647617	8 26 29 25	63 16 15	D	Brojelmann.
	18, 20 7 5	9 1 6 8	0·64846	8 26 40 52	63 13 7	D	Oriani.
	18, 15 15 51	9 0 51 5	0·646267	8 26 42 12	63 12 51	D	Ferrari.
	18, 17 38 50	9 0 45 1	0·646213	8 26 39 9	63 12 30	D	Lemaur.
	18, 11 44 26	9 0 5 55	0·646212	8 26 40 26	63 27 14	D	Friesneker.
	18, 15 52 50	9 0 39 22	0·648123	8 26 36 57	63 17 56	D	Santini.
	18, 19 27 55	9 0 59 55	0·649620	8 26 25 3	63 9 57	D	Bowditch.
	18, 20 55 32	9 1 6 53	0·648769	8 26 33 4	63 11 18	D	De Montfort.
	18, 19 49 0	9 0 53 38	0·647617	8 26 29 25	63 16 15	D	Brojelmann.
	18, 19 3 0	9 0 58 22	0·64746	8 26 40 46	63 13 2	D	Cacciatore.
	18, 19 51 7	9 1 6 8	0·64896	8 26 36 52	63 14 28	D	Bessel.
	18, 17 41 43	9 1 53 51	0·64587	8 26 48 9	63 10 11	D	Bessel.
100	1808 July 12, 4 10 49	8 12 38 50	0·60786	0 24 11 15	39 18 59	D	Bessel.
101	1810 Oct. 5, 19 54 11	2 3 9 10	0·96914	10 8 53 4	62 46 17	D	Bessel.
102	1811 Sept. 21, 18 30 0	3 8 34 0	1·76770	4 13 12 0	65 9 0	R	Burckhardt.
	15, 10 0 0	3 18 12 30	1·23370	4 19 0 0	71 50 0	R	Burckhardt.
	12, 9 54 21	2 14 48 14	1·035679	4 20 20 25	73 9 40	R	Bessel.
	12, 5 21 15	2 15 17 34	1·04006	4 20 24 13	75 17 34	R	Gauss.
	12, 4 51 3	2 15 17 34	1·040064	4 20 24 13	73 7 16	R	Gauss.
	12, 3 18 9	2 13 40 45	1·01045	4 20 10 13	72 59 55	R	Olbers.
	Ellipse 10, 0 21 9	2 13 14 35	0·98069	4 21 5 59	73 48 2	R	Gauss.
	Ellipse 12, 6 57 30	2 14 29 40	1·02716	4 20 16 56	72 59 10	R	Flaugergnes.
	Ellipse 12, 5 32 19	2 15 1 9	1·03541	4 20 24 30	73 2 36	R	Bessel.
	12, 7 40 13	2 14 56 0	1·03483	4 20 19 50	72 3 3	R	Bouvard.
	12, 6 0 23	2 15 1 44	1·03539	4 20 21 58	73 4 31	R	Nicolaï.
	12, 9 19 1	2 14 51 58	1·03618	4 20 23 18	73 3 44	R	Piazzi.
	12, 9 15 18	2 14 54 24	1·03620	4 20 23 46	73 3 23	D	Calandrelli
103	1811 Nov. 9, 5 52 47	1 16 10 50	1·58533	3 2 57 1	31 29 28	D	Werner.
	11, 4 46 1	1 17 32 10	1·58892	3 2 53 9	31 32 53	D	Werner.
	11, 13 9 14	1 17 47 0	1·58963	3 2 55 1	31 31 52	D	Werner.
	11, 2 45 9	1 17 29 30	1·58791	3 2 56 13	31 29 14	D	Zach.
104	1812 Sept. 14, 14 17 16	3 2 40 29	0·78212	8 13 36 25	74 1 32	D	Werner.
	14, 20 35 55	3 1 54 45	0·78830	8 13 43 25	73 53 51	D	Werner.
	15, 0 0 0	3 2 54 38	0·77835	8 13 18 50	74 20 30	D	Nicollet.
105	1813 March 5, 16 46 0	2 6 52 30	0·67451	2 17 27 30	27 33 30	R	Werner.

Catalogue of three hundred and ninety stars;

WITH THEIR RIGHT ASCENSION IN TIME AND DEGREES; THEIR DECLINATION; ANNUAL PRECESSION AND PROPER MOTION, BOTH IN RIGHT ASCENSION AND DECLINATION; NUMBERED ACCORDING TO THE CATALOGUES OF FLAMSTEAD AND PIAZZI.

Numbers according to		Names of the stars.	Magnitude.	Right ascension.				Declination.		
Piazzi.	Flamstead.			In time.	In degrees.	Annual precession in AR	Annual proper motion in AR	In degrees.	Annual precession.	Annual proper motion.
				h m ″	° ′ ″	″	″	° ′ ″	″	″
9	88	γ Pegasi	2·3	0 2 57	0 44 15·9	+46·076	−0·03	+14 4 16·6	+20·044	−0·09
42	8	ι Ceti	4	0 9 14	2 18 30·6	45·870	+0·06	− 9 55 58·5	20·030	+0·04
69	..	α Phœnicis	2	0 16 21	4 5 30·9	44·659	−0·24	−43 23 35·8	19·994	−0·01
99	15	κ Cassiopeæ....	4	0 21 44	5 26 0·2	49·555	....	+61 49 32·3	19·955	+0·02
123	17	ζ Cassiopeæ....	4	0 25 54	6 28 30·7	48·989	+0·10	+52 47 39·4	19·918	0·00
134	30	ε Andromedæ ..	4	0 28 1	7 0 12·0	47·323	−0·10	+28 13 25·5	19·896	−0·20
136	31	δ Andromedæ ..	3	0 28 39	7 9 57·3	47·441	+0·35	+29 45 54·2	19·889	−0·09
139	18	α Cassiopeæ....	3	0 29 14	7 18 35·7	49·700	−0·05	+55 26 17·6	19·883	−0·07
159	16	β Ceti	2·3	0 33 32	8 23 11·0	45·000	+0·21	−19 5 11·0	19·831	+0·07
182	34	ζ Andromedæ ..	4	0 36 46	9 11 28·0	47·382	−0·10	+23 10 36·8	19·788	−0·10
185	24	η Cassiopeæ....	4	0 37 5	9 16 21·0	50·938	+1·78	+56 45 2·3	19·783	−0·72
194	35	ν Andromedæ ..	4	0 38 49	9 42 29·4	48·846	+0·80	+39 59 11·8	19·759	−0·10
226	27	γ Cassiopeæ....	3	0 44 45	11 11 7·6	52·648	0·00	+59 37 51·4	19·664	+0·04
232	37	μ Andromedæ ..	4	0 45 51	11 25 29·7	49·049	+0·20	+37 24 41·0	19·648	+0·40
263	1	Polaris........	2·3	0 52 25	13 6 19·5	193·942	....	+88 14 24·3	19·524	0·00
264	71	ε Piscium......	4	0 52 34	13 8 37·8	46·555	−0·15	+ 6 48 37·5	19·521	+0·06
300	31	η Ceti	3·4	0 58 32	14 38 0·0	45·006	+0·28	−11 14 42·8	19·396	−0·10
301	43	β Andromedæ ..	2	0 58 34	14 38 33·7	45·501	+0·35	+34 33 23·5	19·395	−0·10
62	37	δ Cassiopeæ....	3	1 12 51	18 12 48·8	56·517	+0·64	+59 11 26·2	19·041	−0·16
67	45	θ Ceti	3	1 14 1	18 30 25·8	44·979	−0·19	− 9 13 9·5	19·009	−0·17
98	99	η Piscium......	4	1 20 48	20 12 2·4	47·777	+0·07	+14 18 37·3	18·811	−0·06
124	51	r 2Andromedæ	3·4	1 25 47	21 26 48·6	54·041	....	+47 36 33·4	18·658	....
163	52	τ Ceti	3·4	1 34 47	23 41 39·0	43·550	−1·86	−16 59 40·5	18·356	+0·84
184	45	ε Cassiopeæ....	3·4	1 40 10	25 2 24·6	62·433	−0·05	+62 40 36·8	18·161	−0·15
192	55	ζ Ceti	3	1 41 35	25 23 51·0	44·289	−0·16	+11 19 42·5	18·108	+0·11
193	2	α Trianguli	3·4	1 41 42	25 25 43·5	50·704	+0·04	−18 35 52·3	18·104	−0·36
202	6	β Arietis	3	1 43 36	25 54 12·6	49·168	+0·14	+19 49 29·0	18·032	−0·23
215	50	f Cassiopeæ....	4·5	1 46 39	26 39 40·5	72·806	....	+71 26 37·3	17·914	....
236	57	γ Andromedæ ..	3·4	1 51 41	27 55 11·5	54·276	+0·26	+41 21 45·5	17·712	−0·11
238	113	α Piscium......	5	1 51 42	27 55 39·4	46·305	−0·06	+ 1 47 32·8	17·711	−0·09
253	13	α Arietis	3	1 55 55	28 58 54·0	50·036	+0·20	+22 30 36·5	17·536	−0·20
260	4	β Trianguli	4	1 57 41	29 25 21·0	52·662	+0·33	+34 2 4·6	17·460	−0·07
144	82	δ Ceti	4	2 29 14	37 18 39·0	45·896	+0·07	− 0 32 31·0	15·994	−0· 5
149	83	ε Ceti	4·5	2 29 53	37 28 27·4	43·257	+0·15	−12 43 41·0	15·909	−0·44
150	13	θ Persei	4	2 30 37	37 39 12·0	59·790	+0·67	+48 22 20·7	15·871	−0·10
157	35	Arietis........	4	2 31 45	37 56 10·8	52·249	+0·15	+26 50 50·0	15·810	−0·05
161	86	γ Ceti	3	2 32 57	38 14 14·4	49·528	−0·35	+ 2 23 6·5	15·745	−0·20
167	87	μ Ceti	4	2 34 8	38 32 10·5	48·048	+0·06	+ 9 15 44·0	15·702	+0·20
170	89	π Ceti	4	2 34 36	38 39 7·2	42·724	−0·03	−14 42 41·0	15·655	+0·11
178	39	Arietis........	4	2 36 2	39 0 31·5	52·836	+0·23	+28 24 26·0	15·574	−0·13
186	41	Arietis........	3	2 38 15	39 33 40·5	52·357	+0·20	+26 25 38·0	15·454	−0·13
219	3	η Eridani......	3	2 46 39	41 39 59·7	43·733	+0·16	− 9 42 4·0	14·975	−0·30

CATALOGUE—*continued.*

Numbers according to		Names of the stars.	Magnitude.	Right ascension.				Declination.		
Piazzi.	Flamstead.			In time.	In degrees.	Annual precession in Æ	Annual proper motion in Æ	In degrees.	Annual precession.	Annual proper motion.
				h m ″	° ′ ″	″	″	° ′ ″	″	″
234	23	γ Persei	3·4	2 50 24	42 35 56·1	+63·829	−0·15	+52 42 39·0	+14·756	−0·05
238	..	θ Eridani præc.	2·3	2 50 41	42 40 15·0	34·135	−0·10	−41 6 44·5	14·739	0·00
244	92	α Ceti	2	2 51 49	42 57 34·3	46·798	−0·08	+ 3 17 48·8	14·670	−0·15
246	25	ρ Persei	3·4	2 52 24	43 6 5·4	56·734	+0·26	+38 3 16·4	14·636	−0·11
249	11	Eridani	4	2 53 34	43 23 39·9	39·760	....	−24 24 54·4	14·566	....
253	..	Persei	4	2 54 42	43 40 36·6	61·844	....	+48 50 10·0	14·498	−0·00
13	12	Eridani	3·4	3 3 24	45 53 44·4	37·774	+0·64	−29 46 56·0	13·951	0·82
22	13	ζ Eridani	4	3 6 7	46 31 53·7	43·558	−0·11	− 9 34 14·5	13·791	−0·02
41	33	α Persei	2·3	3 10 7	47 31 42·4	63·103	−0·21	+49 8 13·0	13·535	−0·02
43	16	Eridani	3·4	3 10 37	47 39 21·9	39·875	....	+22 29 39·0	13·502	....
47	..	e Eridani	4	3 11 56	47 59 13·5	31·711	+4·30	−43 50 34·0	13·417	+0·83
51	..	Camel. 2 Hevel.	4	3 12 59	48 14 51·0	71·125	....	+59 13 40·0	13·349	....
55	1	ο Tauri........	4	3 14 4	48 30 58·2	48·206	−0·24	+ 8 16 56·1	13·278	−0·13
63	2	ξ Tauri........	4	3 16 20	49 5 10·8	48·418	−0·08	+ 9 1 33·5	13·128	0·00
89	18	ε Eridani	4	3 23 30	50 52 43·9	43·229	....	−10 8 37·0	12·648	....
95	19	Eridani	4	3 24 57	51 14 24·0	39·597	....	−22 18 42·5	12·687	....
106	39	δ Persei	3·4	3 38 45	52 11 12·6	63·073	+0·04	+49 8 2·0	12·290	−0·10
123	..	Persei 31 Hevel.	4	3 31 48	52 57 12·4	55·870	....	+31 38 33·3	12·077	....
134	23	δ Eridani	3·4	3 33 40	53 25 9·3	43·063	−0·23	−10 26 56·2	11·947	+0·60
152	25	η Tauri........	3	3 35 37	53 54 16·3	53·046	−0·04	+20 28 31·0	11·810	−0·09
183	..	f Eridani seq. ..	4	3 41 13	55 18 18·7	33·024	....	−38 14 17·0	11·410	....
185	44	ζ Persei	3·4	3 41 35	55 23 50·4	56·035	−0·14	+31 16 37·5	11·383	0·00
196	45	ε Persei	3·4	3 44 29	56 7 7·3	59·722	....	+39 25 5·8	11·175	....
210	34	γ Eridani	2·3	3 48 42	57 10 33·6	41·784	+0·12	−14 5 12·0	10·866	−0·11
218	35	λ Tauri........	4	3 49 36	57 24 10·3	49·575	+0·16	+11 54 54·3	10·799	0·00
39	54	γ Tauri........	3·4	4 8 25	62 6 22·8	50·802	+0·14	+15 7 57·4	9·378	−0·09
50	4	i Eridani	3·4	4 10 19	62 34 54·6	33·876	....	−34 17 40·5	9·230	....
57	61	δ 1 Tauri	4	4 11 25	62 51 13·2	51·486	+0·10	+17 3 43·6	9·146	−0·04
87	74	ε Tauri........	4	4 16 57	64 14 17·1	52·131	+0·08	+18 43 27·5	8·713	−0·13
125	87	Aldebaran	1	4 24 27	66 6 50·4	51·299	+0·04	+16 5 42·0	8·117	−0·21
133	48	υ Eridani	4	4 26 20	66 34 59·4	44·799	−0·02	− 3 46 19·6	7·967	−0·05
144	52	υ 2 Eridani	3	4 27 46	66 56 43·0	34·936	+0·05	−30 58 50·0	7·850	0·00
150	53	Eridani	4	4 29 1	67 15 22·9	41·160	−0·32	−14 42 15·1	7·750	−0·13
166	54	Eridani	4	4 31 42	67 55 28·2	39·227	+0·12	−20 3 52·2	7·534	−0·06
201	1	Orionis........	4	4 38 59	69 44 54·3	48·187	+0·54	+ 6 35 59·5	6·939	−0·07
213	3	Orionis........	4	4 40 34	70 8 29·5	47·744	+0·22	+ 5 15 6·2	6·809	+0·44
232	8	ζ Orionis	4	4 43 50	70 57 31·8	46·707	....	+ 2 6 7·0	6·540	....
235	3	ι Aurigæ	4	4 43 59	70 59 46·8	58·242	0·00	+32 50 6·0	6·527	+0·02
256	7	ε Aurigæ	4	4 47 39	71 54 37·5	64·101	....	+43 30 39·5	6·224	....
262	8	ζ Aurigæ	4	4 48 32	72 7 51·0	62·460	....	+40 46 3·1	6·151	....
283	10	η Aurigæ	4	4 52 31	73 7 39·6	62·656	+0·10	+40 56 51·5	5·818	−012
303	2	ε Leporis	3·4	4 56 59	74 14 54·0	37·961	....	−22 38 55·0	5·451	....
312	67	β Eridani	3	4 58 1	74 30 20·8	44·200	−0·30	− 5 21 22·0	5·355	−0·11
323	69	λ Eridani	4	4 59 34	74 53 40·5	42·939	....	− 9 1 16·5	5·224	....
6	13	Capella	1	5 1 56	75 29 0·6	65·951	+0·12	+45 46 37·5	5·025	−0·44
18	19	Rigel	1	5 4 55	76 13 57·4	43·121	−0·05	− 8 26 36·4	4·770	−0·02
40	20	τ Orionis	4	5 7 54	76 58 30·0	43·589	....	− 7 4 18·0	4·518	....
72	112	β Tauri........	2	5 13 39	76 24 51·9	56·642	−0·03	+28 25 25·5	4·026	0·17

CATALOGUE—*continued.*

Numbers according to		Names of the stars.	Magnitude.	Right ascension.				Declination.		Annual proper motion.
Piazzi.	Flamstead.			In time.	In degrees.	Annual precession in AR	Annual proper motion in AR	In degrees.	Annual precession.	
				h m ″	° ′ ″	″	″	° ′ ″	″	″
80	24	γ Orionis	2	5 14 24	78 36 8·2	+48·131	−0·17	+ 6 9 21·3	+ 3·961	—0·03
81	28	η Orionis	4	5 14 25	78 36 24·0	45·112	....	− 2 35 33·9	3·960	....
113	9	β Leporis......	4	5 19 41	79 55 11·8	38·464	−0·03	−20 55 42·5	3·508	—0·07
126	34	δ Orionis	2	5 21 47	80 26 53·7	45·853	−0·12	− 0 27 32·7	3·326	+ 0·05
130	36	υ Orionis	4	5 22 15	80 33 57·7	43·422	....	− 7 27 38·0	3·286	....
139	11	α Leporis......	3	5 23 54	80 58 39·7	39·588	....	−17 56 33·3	3·149	....
140	..	ε Columbæ	4	5 24 6	81 1 43·5	31·812	....	−35 37 39·0	3·125	....
141	39	λ Orionis	4	5 24 7	81 1 54·0	49·427	....	+ 9 47 17·2	3·125	....
151	44	ι Orionis	3·4	5 25 39	81 24 52·0	43·910	....	− 6 3 5·6	2·998	....
152	123	ζ Tauri........	3·4	5 25 41	81 25 22·8	53·623	−0·03	+ 1 0 25·0	2·996	—0·04
160	46	ε Orionis	2·3	5 26 4	81 31 2·1	45·547	−0·14	− 1 20 29·4	2·957	—0·06
172	48	σ Orionis	4	5 28 42	82 10 37·5	45·066	−0·21	− 2 43 36·0	2·728	—0·03
188	50	ζ Orionis	3	5 30 40	82 40 4·0	45·296	−0·12	− 2 3 36·6	2·558	—0·06
196	..	α Columbæ	2	5 32 24	83 6 7·2	32·492	−0·20	−34 11 21·0	2·407	—0·10
219	13	γ Leporis......	4	5 36 7	84 1 53·1	37·744	−0·42	−22 31 24·0	2·084	—0·40
234	53	κ Orionis	3	5 38 16	84 34 4·9	42·582	−0·07	− 9 45 4·0	1·808	0·00
262	33	δ Aurigæ	3·4	5 43 4	85 45 57·0	73·780	....	+54 14 57·1	1·480	—0·42
267	..	β Columbæ	3	5 43 54	85 58 44·7	31·561	....	−35 51 4·0	1·406	....
268	58	α Orionis	1	5 44 20	86 5 12·5	48·594	−0·03	+ 7 21 25·0	1·368	+ 0·03
269	34	β Aurigæ	2	5 44 51	86 12 52·9	65·950	−0·03	+44 54 35·3	1·323	+ 0·06
277	37	θ Aurigæ	4	5 46 5	86 31 14·5	61·190	+0·11	+37 11 0·0	1·217	—0·04
281	16	η Leporis......	4	5 47 17	86 49 26·7	40·942	....	−14 12 50·5	1·111	....
297	..	γ Columbæ	4	5 50 26	87 36 43·5	31·823	....	−35 18 47·0	0·835	....
18	44	κ Aurigæ	4	6 2 37	90 39 22·9	57·378	−0·30	+29 33 22·3	− 0·230	—0·35
74	13	μ Gemin.	3	6 10 51	92 42 40·9	54·347	+0·13	+22 36 8·5	0·882	—0·16
81	1	ζ Can. maj.	3	6 12 38	93 9 34·5	34·465	−0·20	−29 59 4·0	1·105	—0·25
92	2	β Can. maj.	2	6 13 53	93 28 23·1	39·561	−0·04	−17 52 5·0	1·214	—0·06
95	3	λ Can. maj.	4	6 14 48	93 42 3·0	32·850	−0·34	−33 20 38·5	1·294	—0·03
169	24	γ Gemin.	3	6 26 9	96 32 16·9	51·932	+0·05	+16 33 24·6	2·282	—0·08
204	27	ε Gemin.	3	6 31 37	97 54 16·3	55·403	−0·03	+25 18 54·0	2·757	0·00
205	..	υ Argo navis ..	3	6 31 38	97 54 40·5	27·041	−0·14	−43 1 41·0	2·759	—0·20
217	31	ξ 2 Gemin.	4	6 34 4	98 30 55·2	50·624	....	+13 5 56·5	2·968	....
227	9	Sirius	1	6 36 20	99 4 59·2	40·166	−0·51	−16 27 6·2	3·165	— 1·14
248	34	θ Gemin.	4	6 39 36	99 53 52·5	59·425	−0·16	+34 11 12·0	3·446	....
259	13	κ 2 Can. maj. ..	4	6 42 22	100 35 34·5	33·563	−0·10	−32 17 9·5	3·685	+ 0·04
279	16	ο 1 Canis	4	6 45 49	101 27 30·7	37·298	−0·10	−23 56 37·0	3·892	0·00
304	21	ε Canis........	2	6 50 45	102 41 28·9	35·301	−0·05	−28 42 30·3	4·404	+ 0·07
312	43	ζ Gemin.	4	6 52 14	103 3 33·9	53·449	0·00	+20 51 10·0	4·530	—0·05
320	22	σ Canis maj.....	3·4	6 53 45	103 26 18·1	35·794	−0·04	−27 39 27·4	4·659	0·00
328	24	ο 2 Canis	4	6 54 40	103 49 5·4	37·522	−0·01	−23 33 2·0	4·737	+ 0·02
325	23	γ Canis........	4	6 54 42	103 40 36·3	40·666	0·00	−15 20 51·0	4·740	—0·04
57	45	δ Gemin.	3·4	7 8 9	107 2 27·6	53·787	−0·06	+22 20 14·3	5·875	—0·05
68	..	π Argo	3	7 10 5	107 31 1·5	31·738	....	−36 44 46·7	6·033	....
90	60	ι Gemin.	4	7 13 17	108 19 18·0	56·208	−0·27	+28 10 56·4	6·301	+ 0·08
104	31	η Canis min.....	2·3	7 16 10	109 2 42·6	35·542	−0·12	−28 55 18·0	6·544	0·00
106	3	β Canis min.....	3	7 16 17	109 4 24·4	48·904	−0·04	+ 8 40 52·0	6·550	+ 0·04
127	66	Castor præc....	3·4	7 21 49	110 27 7·2	57·892	−0·16	+32 18 45·0	7·006	—0·10
128		seq.	3	7 21 49	110 27 18·0					

CATALOGUE—*continued.*

Numbers according to		Names of the stars.	Magnitude.	Right ascension.				Declination.		
Piazzi.	Flamstead.			In time.	In degrees.	Annual precession in AR	Annual proper motion in AR	In degrees.	Annual precession.	Annual proper motion.
				h m ″	° ′ ″	″	″	° ′ ″	″	″
136	..	σ Argo	4	7 22 53	110 43 10·5	+28·587	−0·43	−42 54 10 5	− 7·092	+0·30
168	10	Procyon	2	7 28 49	112 12 21·7	47·873	−0·71	+ 5 43 38·5	7·576	−0·98
184	77	κ Gemin.	4	7 33 21	113 5 15·0	54·556	−0·16	+24 51 52·7	7·861	−0·02
191	78	β Gemin.	2	7 33 3	113 15 45·6	56·005	−0·72	+28 29 46·8	7·917	−0·11
214	..	C Argo præc. ..	4	7 33 8	114 31 1·5	32·021	....	−37 29 27·0	8·318	....
230	7	ξ Navis........	4	7 40 53	115 13 15·0	37·798	−0·07	−24 22 0·0	8·542	0·00
306	..	ζ Argo	2·3	7 56 33	119 8 19·5	31·605	....	−39 26 46·8	9·761	....
320	15	Navis	3·4	7 59 1	119 45 20·8	38·358	−0·21	−23 44 8·7	9·949	+0·09
28	17	β Cancri	4	8 5 39	121 24 49·6	48·964	−0·13	+ 9 47 30·0	10·448	−0·02
57	1	ο Ursæ maj.....	4	8 13 32	123 22 45·0	76·675	....	+61 22 14·0	11·029	....
114	4	δ Hydræ	4	8 27 3	126 45 49·5	47·811	....	+ 6 23 31·7	11·998	....
164	11	ε Hydræ	4	8 36 10	129 2 34·5	47·962	....	+ 7 8 34·0	12·622	....
210	16	ζ Hydræ	4	8 44 49	131 12 13·8	47·783	+0·14	+ 6 41 54·1	13·205	−0·48
212	9	ι Ursæ maj.....	3·4	8 45 27	131 21 37·5	63·208	−1·05	+48 48 57·5	13·246	−0·32
222	65	α 2 Cancri	4	8 47 32	131 52 59·4	49·354	....	+12 37 22·0	13·383	....
1	..	λ Argo	2·3	9 0 38	135 9 45·7	33·001	−0·04	−42 37 48·0	14·215	+0·06
29	38	Lyncis	4	9 6 21	136 35 16·0	56·634	+0·28	+37 38 22·8	14·562	−0·20
82	23	h Ursæ........	4	9 15 35	138 53 52·5	72·941	−0·09	+63 55 30·0	15·105	−0·04
89	30	α Hydræ	2	9 17 45	139 26 20·2	44·226	−0·15	− 7 47 54·5	15·229	−0·05
98	25	θ Ursæ........	3	9 19 23	139 50 52·8	62·902	−0·80	+52 34 45·4	15·322	−0·60
151	14	ο Leonis	4	9 30 27	142 36 53·7	48·331	−0·22	+10 47 43·4	15·928	−0·04
164	17	ε Leonis	3	9 34 28	143 37 2·2	51·477	−0·03	+24 41 18·0	16·138	+0·02
194	24	μ Leonis	4	9 41 21	145 20 21·3	51·806	−0·39	+26 56 31·4	16·489	−0·07
245	30	η Leonis	3·4	9 56 24	149 6 7·5	49·303	−0·05	+17 43 55·0	17·201	−0·02
251	32	Regulus	1	9 57 42	149 25 33·4	48·354	−0·28	+12 56 22·0	17·259	−0·01
20	53	λ Ursæ maj.....	3·4	10 4 58	151 14 38·4	55·294	0·00	+43 54 24·5	17·573	−0·05
29	..	q Argo	4	10 6 21	151 35 25·0	37·601	....	−41 8 0·0	17·632	....
38	41	γ Leonis	2	10 8 55	152 13 50·7	49·568	+0·35	+20 50 52·2	17·737	−0·20
45	34	μ Ursæ........	3	10 10 21	152 35 22·3	54·468	−0·29	+42 30 0·0	17·795	+0·03
74	42	μ Hydræ	4	10 16 25	154 6 18·0	43·530	−0·21	−15 49 9·2	18·033	−0·06
192	47	ρ Leonis	4	10 22 16	155 34 0·9	47·513	−0·14	+10 19 54·5	18·250	0·00
121	37	Leonis min.....	4	10 27 25	156 51 24·0	51·130	−0·03	+33 0 39·3	18·432	+0·01
167	4	ν Hydræ	4	10 39 46	159 56 26·7	44·150	+0·06	−15 8 59·8	18·830	+0·20
207	48	β Ursæ........	2	10 49 40	162 25 9·0	55·498	+0·12	+57 27 4·5	19·109	+0·06
209	7	α Crateris......	4	10 50 2	162 30 33·0	44·142	−0·59	−17 14 11·2	19·118	+0·06
217	50	α Ursæ........	2	10 51 15	162 48 52·2	57·549	−0·24	+62 49 38·4	19·151	0·00
253	52	ψ Ursæ	3·4	10 58 21	164 35 30·0	51·447	−0·08	+45 34 52·8	19·325	−0·07
6	11	β Crateris......	3·4	11 1 50	165 27 34·5	44·005	+0·05	−21 44 9·0	19·404	−0·11
10	68	δ Leonis	3	11 3 27	165 51 43·5	47·952	+0·18	+21 37 4·0	19·438	−0·11
13	70	θ Leonis	3	11 3 43	165 55 54·1	47·457	−0·16	+16 31 15·0	19·445	−0·06
28	53	ξ Ursæ maj.....	4	11 7 28	166 52 10·0	48·930	−0·52	+32 39 7·5	19·522	−0·64
29	54	ν Ursæ maj.....	4	11 7 39	166 54 36·1	49·095	−0·03	+34 11 2·0	19·525	+0·05
38	12	δ Hydræ	3·4	11 9 21	167 20 15·0	44·941	−0·19	−13 41 48·6	19·558	−0·06
42	77	σ Leonis	4	11 10 49	167 42 14·4	46·545	−0·16	+ 7 7 24·5	19·586	−0·05
54	78	ι Leonis	4	11 13 29	168 22 18·0	46·843	−0·00	+11 37 47·5	19·634	−0·03
62	15	γ Hydræ........	4	11 14 54	168 43 29·5	44·844	−0·29	−16 35 12·5	19·658	+0·02
76	84	τ Leonis	4	11 17 39	169 24 42·1	46·266	−0·03	+ 3 57 24·5	19·704	−0·02
86	1	λ Draconis	3·4	11 19 22	169 50 27·0	55·960	−0·15	+70 25 59·7	19·731	−0·09

CATALOGUE—*continued.*

Numbers according to		Names of the stars.	Magnitude.	Right ascension.				Declination.		
Piazzi.	Flamstead			In time.	In degrees.	Annual precession in AR	Annual proper motion in AR	In degrees.	Annual precession.	Annual proper motion.
				h m "	° ′ ″	″	″	° ′ ″	″	″
103	19	ξ Crateris......	4	11 23 11	170 47 46·5	+44·104	−0·42	−30 45 8·5	−19·788	− 0·11
114	21	θ Hydræ	4	11 26 32	171 38 6·3	45·565	−0·14	− 8 41 47·0	19·832	+ 0·05
150	27	ζ Crateris......	4	11 34 38	173 39 35·1	45·325	−0·06	−17 14 21·5	19·923	− 0·09
152	63	χ Ursæ maj.....	4	11 35 25	173 51 22·5	48·470	−0.36	+48 53 16·6	19·931	− 0·07
159	93	Leonis........	4	11 37 39	174 24 45·9	46·774	....	+21 19 49·0	19·950	....
163	94	β Leonis	3	11 38 50	174 42 42·0	46·530	−0·53	+15 41 24·7	19·961	− 0·08
166	5	β Virginis	3·4	11 40 16	175 4 7·8	46·098	+0·76	+ 2 53 30·0	19·972	− 0·30
172	28	β Hydræ	4	11 42 50	175 42 35·5	45·045	−0·20	−32 47 44·0	19·990	− 0·13
174	64	γ Ursæ maj....	2	11 43 14	175 48 37·2	48·088	+0·06	+54 48 23·0	19·992	− 0·03
241	1	α Corvi	4	11 58 7	179 31 50·1	45·940	+0·06	−23 36 44·7	20·045	+ 0·16
248	2	ε Corvi	4	11 59 51	179 57 52·5	46·007	....	−21 30 25·0	20·045	− 0·09
22	69	δ Ursæ maj.....	3	12 5 27	181 21 46·0	45·244	−0·06	+58 8 40·2	20·140	− 0·08
24	4	γ Corvi........	3	12 5 32	181 23 3·3	46·154	−0·32	−16 25 47·0	20·040	+ 0·06
44	15	η Virginis	3·4	12 9 40	182 25 10·2	46·005	−0·05	+ 0 26 47·0	20·028	− 0·04
92	..	μ Centauri	4	12 17 47	184 26 53·4	47·222	....	−37 55 46·0	19·985	....
101	7	δ Corvi........	3·4	12 19 32	184 52 59·4	46·481	+0·07	−15 23 58·6	19·973	− 0·20
123	9	β Corvi........	2·3	12 23 54	185 58 35·1	46·867	....	−22 17 19·5	19·937	....
129	5	κ Draconis	3	12 24 52	186 12 49·0	39·748	−0·50	+70 53 32·7	19·928	− 0·18
157	29	γ 1 Virginis....	3	12 31 32	187 52 57·0	46·028	−0·72	− 0 20 59·0	19·856	+ 0·10
220	77	ε Ursæ maj.....	3	12 45 11	191 17 43·2	39·954	+0·24	+57 2 52·5	19·657	− 0·10
223	43	δ Virginis	3·4	12 45 31	191 22 57·9	45·701	−0·65	+ 4 29 16·2	19·651	− 0·02
226	12	Can. venatic ..	2·3	12 46 38	191 39 42·3	42·683	−0·34	+39 24 5·0	19·632	+ 0·04
249	47	ε Virginis	3·4	12 52 13	193 3 17·4	45·046	−0·37	+12 2 16·7	19·528	+ 0·07
273	41	Coma Beren ..	4	12 57 34	194 23 29·2	43·283	....	+28 42 7·0	19·417	....
2	42	Coma Beren ..	4	13 0 15	195 3 46·0	44·259	−0·45	+18 35 27·7	19·357	+ 0·15
45	2	γ Hydræ	3·4	13 8 4	197 1 9·0	48·395	+0·14	−22 6 41·0	19·168	− 0·10
53	..	ι Centauri	3	13 9 24	197 21 2·5	50·299	....	−35 39 8·0	19·142	....
75	67	Spica	1	13 14 40	198 40 6·3	47·156	−0·09	−10 6 44·0	18·991	− 0·03
78	79	ζ Ursæ maj. pr.	3	13 15 51	198 57 27·0	36·366	−0·08	+55 58 25·8	18·958	− 0·01
85	80	G Ursæ maj. ..	4	13 17 10	199 17 46·5	36·178	+0·13	+56 2 1·0	18·919	− 0·18
99	..	D Centauri	4	13 19 30	199 52 33·0	51·407	....	−38 22 5·5	18·851	....
128	79	ζ Virginis	4	13 24 30	201 7 41·1	45·957	−0·49	+ 0 25 55·6	18·699	+ 0·08
197	..	ν Centauri	4	13 37 34	204 23 34·5	53·124	....	−40 40 1·0	18·257	....
198	..	μ Centauri	4	13 37 38	204 24 32·1	53·332	....	−41 28 11·5	18·254	....
209	84	η Ursæ maj.....	3	13 39 39	204 54 33·7	35·836	−0·50	+50 18 59·2	18·181	0·00
210	5	υ Bootis	4	13 39 49	204 57 24·6	43·458	....	+16 47 46·2	18·174	....
240	8	η Bootis	3	13 45 9	206 17 22·5	42·884	−0·10	+19 24 22·0	17·972	− 0·40
293	5	θ Centauri	2	13 54 58	208 41 31·8	52·865	−0·63	−35 22 41·0	17·576	− 0·40
312	11	α Draconis	3·4	13 58 59	209 44 36·6	24·353	−0·15	+65 20 7·7	17·405	+ 0·09
14	98	κ Virginis......	4	14 2 14	210 33 40·8	47·687	−0·18	− 9 20 8·0	17·261	+ 0·13
28	99	ι Virginis......	4	14 5 32	211 23 7·8	46·923	−0·01	− 5 2 20·0	17·113	− 0·47
32	16	Arcturus	1	14 6 32	211 38 6·6	42·137	−1·17	+20 13 48·3	17·067	− 0·96
37	100	λ Virginis	4	14 8 18	212 4 36·7	48·360	−0·08	−12 26 33·0	16·986	+ 0·07
41	19	λ Bootis	4	14 8 47	212 11 31·0	34·553	−0·55	+47 0 44·0	16·964	+ 0·27
42	21	ι Bootis	4	14 9 4	212 16 4·8	32·167	−0·34	+52 17 41·0	16·950	− 0·03
92	23	θ Bootis	3	14 18 23	214 35 41·4	31·028	−0·80	+52 46 48·0	16·501	− 0·54
109	..	η Centauri	3	14 22 52	215 43 4·0	56·280	....	−41 16 0·0	16·275	....
112	25	ρ Bootis	4	14 23 12	215 48 7·0	38·894	−0·07	+31 15 20·0	16·258	+ 0·09

CATALOGUE—*continued.*

Numbers according to		Names of the stars.	Magnitude.	Right ascension.				Declination.		
Piazzi.	Flamstead.			In time.	In degrees.	Annual precession in Æ	Annual proper motion in Æ	In degrees.	Annual precession.	Annual proper motion.
				h m ″	° ′ ″	″	″	° ′ ″	″	″
136	5	A Ursæ min. . . .	4	14 28 9	217 2 4·5	− 4·609	. . . .	+76 35 6·5	−16·006	. . . .
147	29	π Bootis	4	14 31 20	217 49 52·2	+42·186	. . . .	+17 17 0·4	15·833	. . . .
152	30	ζ Bootis	3·4	14 31 36	217 53 59·4	42·805	0·00	+14 35 40·0	15·818	+0·03
174	109	Virginis	4	14 36 8	219 2 10·0	45·409	. . . .	+ 2 44 40·4	15·570	. . . .
175	26	ε Bootis	3	14 36 5	219 3 43·2	39·316	+0·23	+27 55 28·5	15·565	−0·04
187	9	α 2 Libræ.	3	14 39 50	219 57 34·0	49·510	−0·20	−15 12 4·0	15·365	−0·08
197	37	ξ Bootis	3	14 42 10	220 32 28·0	41·285	−0·23	+19 56 17·6	15·234	−0·18
211	..	β Lupi	3	14 45 30	221 22 28·5	58·074	. . . .	−42 18 58·0	15·042	−0·21
216	7	χ Centauri	3	14 46 13	221 33 13·0	57·688	. . . .	−41 17 27·0	15·001	. . . .
240	7	β Ursæ min.	3	14 51 28	222 51 40·0	− 4·779	−0·30	+74 58 20·4	14·694	−0·18
251	20	Libræ	3·4	14 52 23	223 5 55·5	+52·249	−0·14	−24 29 6·5	14·636	+0·08
259	42	β Bootis	3·4	14 54 25	223 36 8·1	33·915	−0·20	+41 11 11·0	14·517	−0·02
26	27	β Libræ	2·3	15 6 15	226 33 55·0	48·222	−0·30	− 8 38 4·7	13·782	+0·07
29	49	δ Bootis	3·4	15 7 26	226 51 32·4	36·119	+0·10	+34 4 8·0	13·707	−0·07
31	..	δ Lupi	4	15 8 18	227 4 31·8	58·288	. . . .	−39 54 36·0	13·652	. . . .
34	..	φ 1 Lupi	4	15 9— 9	227 17 24·0	56·517	. . . .	−35 31 27·0	13·597	. . . .
73	51	μ Bootis	4	15 16 56	229 13 59·1	34·113	−0·30	+38 5 11·2	13·090	+0·16
86	3	β Coronæ bor. . .	4	15 19 35	239 53 45·0	37·229	. . . .	+29 48 13·0	12·913	+0·15
92	12	ι Draconis	3	15 20 30	230 7 25·0	29·716	. . . .	+59 40 14·0	12·852	+0·15
95	13	γ 2 Ursæ min. . .	4	15 21 9	230 17 10·8	− 3·031	. . . .	+72 32 44·0	12·808	. . . .
98	..	γ Lupi	4	15 21 51	230 28 3·0	59·206	. . . .	−40 28 49·0	12·760	. . . .
106	37	Libræ	4	15 23 15	230 48 57·0	48·575	+0·33	− 9 22 5·5	12·606	−0·30
111	38	γ Libræ	4	15 24 21	231 5 25·5	+49·933	+0·14	−14 6 38·0	12·590	+0·03
117	13	δ Serpentis	3	15 25 15	231 18 48·0	42·908	−0·06	+11 13 2·0	12·530	+0·07
121	5	α Coronæ bor. . .	2	15 26 15	231 33 17·7	37·875	−0·10	+27 23 48·0	12·463	−0·10
152	7	ζ Coronæ bor. . .	4	15 31 51	232 57 39·0	33·824	. . . .	+37 17 37·0	12·075	−0·40
163	24	α Serpentis	2·3	15 34 25	233 36 22·2	44·012	−0·10	+ 7 3 53·7	11·894	+0·05
174	5	λ Lupi	4	15 38 17	234 34 19·5	56·621	. . . .	−33 0 16·2	11·620	. . . .
178	34	μ Serpentis	4	15 39 11	234 47 56·1	46·814	−0·04	− 2 48 24·1	11·555	0·00
182	35	κ Serpentis	4	15 39 44	234 56 5·4	40·436	−0·06	+18 46 8·0	11·516	−0·10
187	37	ε Serpentis	3·4	15 40 51	235 12 48·6	44·545	+0·12	+ 5 5 24·4	11·437	−0·01
206	6	π Scorpii	3·4	15 46 46	236 41 42·6	54·012	+0·10	−25 31 30·8	11·005	−0·20
217	..	η Lupi	4	15 46 54	236 43 42·0	59·015	. . . .	−37 48 35·0	10·997	. . . .
219	41	γ Serpentis	3·4	15 47 13	236 48 20·1	41·099	+0·35	+16 19 27·8	10·985	−1·31
225	7	δ Scorpii	3	15 48 32	237 7 59·4	52·827	0·00	−22 2 20·5	10·879	−0·10
248	..	θ Lupi	4	15 53 29	238 22 26·4	58·523	. . . .	−36 14 37·0	10·511	. . . .
251	8	β Scorp. præc. . .	2	15 53 50	238 27 27·6	51·976	+0·03	−19 14 42·0	10·487	−0·10
277	13	θ Draconis	3·4	15 58 10	239 32 19·0	18·136	. . . .	+56 6 12·6	10·162	+0·30
4	14	ν Scorpionis. . . .	4	16 0 23	240 5 54·3	51·971	−0·06	−18 55 42·3	9·993	−0·08
21	1	δ Ophiuchi	3·4	10 3 52	240 58 7·5	46·981	−0·15	− 3 10 3·0	9·728	−0·08
41	2	ε Ophiuchi	3·4	16 7 45	241 56 15·6	47·308	+0·15	− 4 11 33·5	9·430	+0·03
50	20	σ Scorpii	4	16 9 3	242 15 49·8	54·322	−0·06	−25 5 55·8	9·329	−0·04
66	20	γ Herculis	3·4	16 13 6	243 16 31·0	39·624	−0·04	+19 37 59·5	9·015	+0·09
73	22	τ Herculis	4	16 13 44	243 26 2·1	26·923	. . . .	+46 47 39·0	8·965	. . . .
84	21	Antares	1	16 17 10	244 17 32·2	54·810	−0·05	−25 58 26·0	8·695	−0·10
100	10	λ Ophiuchi	4	16 20 50	245 12 28·5	45·238	−0·06	+ 2 26 1·3	8·406	−0·06
102	14	η Draconis	3·4	16 21 18	245 19 27·0	11·797	. . . .	+61 58 11·0	8·369	−0·05
103	27	β Herculis	2·3	16 21 37	245 24 20·7	38·671	−0·22	+21 56 6·5	8·343	−0·04

CATALOGUE—*continued.*

Numbers according to		Names of the stars.	Magnitude.	Right ascension.				Declination.		
Piazzi.	Flamstead.			In time.	In degrees.	Annual precession in AR	Annual proper motion in AR	In degrees.	Annual precession.	Annual proper motion.
				h m ″	° ′ ″	″	″	° ′ ″	″	″
113	23	τ Scorpii	3·4	16 23 27	245 51 50·4	+55·651	−0·04	−27 47 10·5	− 8·197	− 0·17
123	13	ζ Ophiuchi	3	16 26 9	246 32 22·7	49·303	+0·16	−10 8 56·3	7·980	+ 0·10
132	35	σ Herculis	4	16 27 40	246 54 52·0	28·901	−0·30	+42 51 25·5	7·860	+ 0·03
165	40	ζ Herculis	3	16 33 45	248 36 10·5	34·374	−0·70	+31 58 26·5	7·368	+ 0·47
173	44	η Herculis	3	16 36 3	249 0 34·0	30·687	−0·24	+39 18 38·5	7·181	− 0·09
184	26	ε Scorpii	3	16 37 14	249 18 32·2	58·620	−0·65	−33 54 53·0	7·083	− 0·18
189	1	μ 1 Scorpii	4	16 38 21	249 35 16·0	60·525	....	−37 41 15·0	6·991	....
193	2	μ 2 Scorpii	3·4	16 38 29	249 42 17·7	60·521	....	−37 39 35·4	6·953	....
233	29	ι Ophiuchi	4	16 44 33	251 8 16·0	42·489	....	+10 30 25·0	6·481	0·00
252	27	κ Ophiuchi	3·4	16 48 12	252 3 7·0	42·752	−0·30	+ 9 41 51·2	6·177	+ 0·08
272	58	ε Herculis	3·4	16 52 38	253 9 34·5	34·378	−0·18	+31 13 48·4	5·807	+ 0·05
302	..	η Scorpii	4	16 57 51	254 27 50·4	63·994	....	−42 57 27·0	5·369	....
306	35	η Ophiuchi	2·3	16 58 55	254 43 48·6	51·362	−0·03	−15 27 46·0	5·279	+ 0·09
4	21	μ Draconis	4	17 1 12	255 18 0·0	18·587	−0·20	+54 44 19·2	5·087	+ 0·16
29	64	α Herculis	3·4	17 5 32	256 22 57·1	40·926	−0·11	+14 37 47·7	4·720	+ 0·12
35	65	δ Herculis	3·4	17 6 49	256 42 13·2	36·879	−0·20	+25 5 7·4	4·610	− 0·14
36	22	ε Ursæ min. ..	4	17 6 58	256 43 55·0	−99·116	−0·82	+82 20 35·4	4·601	+ 0·01
39	67	π Herculis	3·4	17 8 5	257 1 16·5	+31·268	....	+37 2 34·4	4·502	....
42	22	ζ Draconis	3	17 8 14	257 3 28·0	2·211	−0·32	+65 57 42·3	4·499	+ 0·04
53	42	θ Ophiuchi	3	17 9 44	257 26 5·4	55·045	−0·08	+24 47 4·0	4·361	− 0·07
105	75	ρ Herculis	4	17 16 48	259 11 44·1	30·990	....	+37 20 21·3	3·758	....
106	34	υ Scorpii	3·4	17 17 10	259 17 46·0	60·918	+0·06	−37 7 7·4	3·723	0·00
121	35	λ Scorpii	3	17 20 2	260 0 39·6	60·856	−0·08	−36 56 26·6	3·477	− 0·02
153	55	α Ophiuchi	2	17 25 39	261 24 48·6	41·538	+0·09	+12 43 3·0	2·993	+ 0·18
155	23	β Draconis	2	17 25 55	261 28 45·6	20·217	−0·27	+52 27 18·7	2·971	0·00
157	55	ξ Serpentis	4	17 26 8	261 32 7·0	51·420	−0·17	−15 15 29·5	2·951	− 0·15
174	..	κ Scorpii	3	17 28 40	262 9 58·8	62·041	....	−38 54 34·0	2·732	....
209	60	β Ophiuchi	3	17 33 36	263 23 55·5	44·387	−0·17	+ 4 39 48·1	2·304	+ 0·22
211	85	ι Herculis	4	17 33 49	263 27 12·0	25·302	....	+46 7 11·5	2·285	− 0·02
229	..	γ Telescop.	4	17 36 15	264 3 46·5	61·015	....	−36 57 42·0	2·073	....
239	62	γ Ophiuchi	4	17 37 52	264 28 1·5	45·037	+0·02	+ 2 47 43·0	1·933	− 0·10
244	86	μ Herculis	4	17 38 38	264 39 28·5	35·468	−0·29	+27 50 48·8	1·866	− 0·84
303	64	ν Ophiuchi	4	17 48 1	267 0 16·0	49·446	....	− 9 44 3·5	1·048	0·00
309	91	θ Herculis	4	17 49 24	267 20 54·0	30·765	−0·17	+37 17 8·1	0·987	+ 0·06
314	92	ξ Herculis	4	17 50 0	267 29 55·5	34·782	....	+29 16 46·7	0·875	0·00
316	32	ξ Draconis	3.4	17 50 5	267 31 1·0	15·280	....	+56 54 29·0	0·868	+ 0·30
322	67	ο Ophiuchi	4	17 50 38	267 39 26·1	44·978	....	+ 2 57 16·0	0·819	....
335	33	γ Draconis	2	17 51 58	267 59 26·4	20·810	−0·31	+51 31 4·5	0·703	− 0·07
343	10	γ 2 Sagittarii ..	4	17 52 57	268 14 30·0	−57·770	−0·04	+30 24 35·5	− 0·615	− 0·15
344	95	Herculis	4	17 53 1	268 15 16·0	+38·075	0·00	+21 36 26·8	− 0·611	+ 0·05
374	72	s 2 Ophiuchi ..	4	17 57 52	269 28 1·0	42·640	....	+ 9 32 52·3	− 0·186	....
388	103	o Herculis	4	17 59 45	269 56 8·1	35·016	....	+28 44 41·6	− 0·023	....
7	13	μ 1 Sagittarii ..	4	18 1 48	270 27 3·1	53·745	−0·08	−21 5 45·7	+ 0·158	− 0·09
17	..	β Telescop.	4	18 4 5	271 1 28·5	61·007	−0·19	−36 48 14·7	0·179	− 0·07
32	19	δ Sagittarii	3·4	18 8 11	272 2 50·4	57·529	+0·10	−29 53 50·5	0·716	− 0·10
46	20	ε Sagittarii	3	18 10 54	272 43 27·0	59·753	−0·15	−34 27 40·7	0·953	− 0·08
48	58	η Serpentis	4	18 10 58	272 44 28·0	47·039	−0·67	− 2 56 16·5	0·959	− 0·68
66	22	λ Sagittarii	3·4	18 15 37	273 54 24·3	55·558	−0·11	−25 31 1·0	1·366	− 0·25

CATALOGUE—*continued.*

Numbers according to		Names of the stars.	Magnitude.	Right ascension.				Declination.		
Piazzi.	Flamstead			In time.	In degrees.	Annual precession in AR	Annual proper motion in AR	In degrees.	Annual precession.	Annual proper motion.
				h m ″	° ′ ″	″	″	° ′ ″	″	″
162	22	β Aquarii......	3·4	21 21 1	320 15 17·7	+47·459	−0·06	− 6 26 33·0	+15·414	−0·15
198	8	β Cephei	4	21 26 0	321 30 14·2	12·310	−0·17	+69 41 2·8	15·688	−0·03
223	40	γ Capricorni ..	3	21 28 59	322 14 51·0	49·894	+0·15	−17 33 26·2	15·850	−0·10
260	8	ε Pegasi	3	21 34 21	323 35 25·0	44·134	+0·11	+ 8 57 55·3	16·133	0·00
269	10	κ Pegasi	4	21 35 35	323 53 57·0	40·571	....	+24 43 57·3	16·196	....
276	49	δ Capricorni....	3	21 35 59	323 59 46·5	49·620	+0·24	−17 1 36·2	16·217	−0·26
368	..	γ Gruis.........	4	21 41 46	325 26 11·5	54·990	0·00	−38 17 47·2	16·509	−0·03
387	34	α Aquarii	3	21 55 30	328 52 36·0	46·244	−0·12	− 1 17 6·1	17·160	−0·05
402	24	ι Pegasi	4	21 57 42	329 25 39·0	41·392	+0·33	+24 22 28·0	17·259	+0·06
1	26	θ Pegasi	4	22 0 6	330 1 39·0	45·096	+0·26	+ 5 13 12·8	17·365	+0·11
6	29	π Pegasi	4	22 1 7	330 16 48·6	39·753	....	+32 12 7·7	17·409	....
26	21	ζ Cephei	4	22 3 56	330 58 52·0	30·911	−0·14	+57 13 7·3	17·525	−0·12
72	48	γ Aquarii	3·4	22 11 19	332 49 48·3	46·398	−0·11	− 2 23 20·4	17·834	+0·05
95	3	Lacertæ	4	22 15 43	333 55 40·0	35·032	....	+51 13 50·6	18·006	....
111	55	ζ Aquarii	4	22 18 31	334 37 56·1	46·167	−0·10	− 1 2 17·6	18·113	−0·14
123	17	β Piscis austr. ..	4	22 20 6	335 1 31·0	51·586	+0·03	−33 21 56·7	18·166	−0·01
141	7	Lacertæ	4	22 23 4	335 46 8·7	36·461	−0·05	+49 15 29·0	18·280	0·00
151	62	η Aquarii	4	22 25 4	336 16 7·5	46·172	−0·16	− 1 8 33·8	18·351	−0·13
175	18	ε Piscis austr. ..	4	22 29 33	337 23 30·0	50·123	+0·05	−28 4 52·5	18·522	+0·02
189	42	ζ Pegasi	3·4	22 31 29	337 52 21·7	44·708	+0·08	+ 9 47 32·0	18·570	−0·06
205	44	η Pegasi	3	22 33 38	338 24 36·7	41·893	−0·03	+29 10 46·5	18·689	−0·12
231	48	μ Pegasi	4	22 40 21	340 5 22·2	43·036	....	+23 32 57·1	18·847	....
235	73	λ Aquarii	4	22 42 10	340 32 34·5	47·026	−0·15	− 8 38 22·6	18·901	−0·04
238	32	ι Cephei	4	22 42 33	340 38 49·5	31·667	−0·24	+65 9 2·9	18·913	−0·11
245	76	δ Aquarii	3·4	22 44 1	341 0 19·0	47·991	−0·17	−16 52 47·7	18·954	−0·13
253	24	Fomalhaut	1	22 46 34	341 38 32·1	49·757	+0·33	−30 40 41·2	19·026	−0·26
284	1	ο Andromedæ ..	4	22 52 45	343 11 7·5	40·915	−0·07	+41 15 14·0	19·189	−0·03
288	53	β Pegasi	2	22 54 5	343 31 25·0	43·114	+0·24	+27 0 5·2	19·222	+0·20
290	54	Markab	1	22 54 48	343 42 5·4	44·595	+0·02	+14 7 57·1	19·240	−0·07
313	88	c 2 Aquarii	4	22 58 46	344 41 28·5	48·177	....	−22 25 14·6	19·384	....
138	16	λ Andromedæ ..	4	23 37 49	351 57 11·1	43·168	+0·27	+45 12 34·9	19·848	−0·42
155	35	γ Cephei	3	23 31 15	352 48 38·2	35·549	−0·24	+76 30 59·0	19·888	−0·11
246	28	ω Piscium.......	4	23 49 2	357 15 43·8	45·915	+0·06	+ 5 45 23·0	20·023	−0·18
264	2	g Ceti	4	23 53 29	358 22 16·5	46·201	+0·24	−18 26 54·5	20·037	−0·04
281	21	α Andromedæ ..	1	23 58 4	359 31 6·6	45·922	+0·14	+27 59 9·0	20·045	−0·21
283	11	β Cassiopeæ....	2·3	23 58 35	359 38 43·8	45·813	+0·82	+58 2 45·5	20·045	−0·25

NAUTICAL ASTRONOMY.

: situation of a ship at sea is ascertained by aining its latitude and longitude. For this pur- wo methods are employed, the one called the l reckoning" depending on the course and dis- sailed; the other founded upon astronomical ations, and which may be termed *nautical my*; this last will form the subject of the ing article.

dead reckoning proceeds upon the assumption he ship's course, (that is the angle which her iakes with the meridian,) and the distance she re known, and could these data be accurately ed, the latitude and longitude might be readily by trigonometrical calculation; but a variety cumstances generally operate to render this sible. The difficulty of steering within some s of the intended course, uncertainty respecting y, errors arising from the log, the heave of the d the influence of currents, contribute more or render the dead reckoning, at best, but a rude imation to the truth.

ust not, however, be understood that the dead ing is without its value; on the contrary, when ied with astronomical observations, it is of con- le utility in detecting the existence and velocity ents, and is indispensably necessary to fill up ort intervals which may occur in unfavourable r between celestial observations. But the too l practice of relying exclusively upon it, cannot ficiently deprecated; and numerous instances be adduced of the fatal consequences of this e in the loss of vessels, from errors of such tude, that they might have been detected by ost superficial knowledge of nautical astro- and the aid of even a good common watch.

Preliminary Definitions.

t circles are those which divide the sphere into jual parts, or hemispheres, and consequently he same diameter and centre with each other, th the sphere.

eat circle is called a *secondary* to another when ie is perpendicular to that of the first.

pole of a great circle is that point which is where distant 90° from its circumference.

er circles are those which divide the sphere into iequal parts.

ry circle is supposed to be divided into 360 called degrees; each degree into 60 parts, minutes; each minute into 60 seconds, and second into 60 thirds, or into decimals of a

distance of any two points in the heavens is expressed by the number of degrees, &c. comprised between them on the arc of a great circle.

A spherical triangle is formed by the intersection of three great circles.

The horizon of any spot on the globe is an imaginary plane, touching the earth's surface at that spot. This is frequently called the *sensible horizon*. A plumb line is perpendicular to this place.

The rational horizon is an imaginary plane, passing through the centre of the earth parallel to the sensible horizon. It is, therefore, a great circle dividing the sphere into two equal parts.

If we observe the fixed stars, we may remark that some of them appear to rise, attain a certain height, decline, and then set. Others describe concentric circles in the heavens about a certain point. This point in the northern hemisphere is called the *north pole*, and in the southern, the *south pole*.

An imaginary line passing through the centre of the earth, in the direction of these points, is called *the axis of the earth*: about which it makes its diurnal revolution, causing the apparent motions of the fixed stars before described.

The zenith is that point which is every where 90° distant from the horizon. The zenith is, therefore, the pole of the horizon.

The nadir is 180° distant from the zenith.

Vertical, or *azimuth circles*, are secondaries to the horizon, and pass through the zenith and nadir.

The *prime vertical* intersects the meridian at right angles. It is sometimes called the six o'clock line.

Cardinal points are the four points where the meridian and the prime vertical intersect the horizon. Those of the meridian, north and south; those of the prime vertical, east and west.

The altitude of an object is its distance from the horizon measured on a vertical circle.

The zenith distance, or complement of altitude is the distance from the object to the zenith; and is, therefore, what the altitude wants of 90°.

The meridian of any spot is that vertical circle which passes through the poles as well as the zenith.

The meridian altitude is the greatest altitude the object can attain; and is its altitude when upon the meridian, and not under the elevated pole.

The azimuth is an arc of the horizon, comprised between the meridian (either from the north or south) and a vertical circle passing through the object.

The amplitude is an arc of the horizon between the east or west point, and the object when rising or setting.

The equator is a great circle at right angles to the axis of the earth, and dividing the globe or heavens into two hemispheres. It intersects the horizon in the east and west points.

Hour circles are secondaries to the equator, and pass through the poles. When named with reference to the earth they are called meridians.

The declination of a celestial object is its distance from the equator, measured upon a secondary, and is either north or south, according to the position of the object with respect to the equator.

Polar distance is the distance of the object from that pole which is elevated above the horizon.

The latitude of any spot upon the globe is its distance north or south of the equator, measured upon a meridian, and is equal to the altitude of the pole.

Complement of latitude is what the latitude wants of 90°, and is equal to the angle which the equator forms with the horizon.

Parallels of latitude and *parallels of declination* are lesser circles, parallel to the equator.

The longitude of any spot on the globe is an arc of the equator, comprised between the meridian of that spot and the *first meridian*: or the meridian from which the longitude is counted; and is said to be east or west, according as the place is to the east or west of the first meridian.

The ecliptic is that great circle which the sun appears to describe in the heavens among the fixed stars, by reason of the motion of the earth in its orbit.

The obliquity of the ecliptic is the angle which it forms by its intersection with the equator; the axis of the earth not being perpendicular to the plane of the ecliptic but inclined about 23½°.

The equinoctial points are those points where the equator intersects the ecliptic.

The ecliptic is divided into twelve parts, called *signs*; reckoning from the vernal equinoctial point towards the west, each sign into 30°, &c.

The *longitude* of a *celestial* body is its distance upon a secondary, north or south of the ecliptic.

Right ascension is an arc of the equator, intercepted (reckoning according to the order of the signs) between the vernal equinoctial point, or the first point of aries, and a secondary to the equator, passing through the object. This, supposing the equator to be divided into twenty-four hours, may be expressed either in time or space.

The tropics are two lesser circles, parallel to the equator, and are the limits of the path of the sun, north and south; they are consequently distant about 23½° from the equator. The northern tropic is called the tropic of cancer; the southern, the tropic of capricorn.

The zodiac is a space extending about 8° on either side of the ecliptic.

OF TIME.

In order to render this subject as intelligible as possible, imagine the earth to be at rest, and the fixed stars and sun to move round it. We may then observe, that the interval between any two successive returns of the same fixed star to the meridian is invariable. This is called a *sidereal day*, and is divided into twenty-four hours, &c.; to show which we will suppose a clock to be constructed.

We may also remark that the sun takes a longer time than a star to complete its circle round the earth, but this time will be found to vary. The *mean* interval is called a *mean* solar day, and is divided by a clock or chronometer into twenty-four hours. Such are the clocks and watches in common use, which are intended to shew *mean solar time.*

Whilst the solar clock passes through twenty-four hours, the sidereal clock will have gone through $24^{h}\ 3^{m}\ 56^{s}{\cdot}55$. Such, therefore, is the ratio of mean solar time to sidereal time. As $24^{h}\ 3^{m}\ 56^{s}{\cdot}55$; $3^{m}\ 56^{s}{\cdot}55$; $24^{h}\ 3^{m}\ 55^{s}{\cdot}91$; the quantity to be subtracted from twenty-four hours of sidereal time to convert it into mean solar time; and so in proportion for any other period.

It has been remarked that the motion of the sun is not equable, but that it takes sometimes more and sometimes less than a mean solar day to complete its revolution. When the sun is upon the meridian we say it is *apparent* noon, and all time deduced immediately from the sun, is called *apparent time*, and is that portion of an *actual* revolution which the sun has passed through since it left the meridian.

In order to convert apparent into mean solar time, the *equation of time* must be applied to it; the equation of time being the accumulated errors of the sun, (if the expression may be allowed,) from mean solar time. The result will be the time which should be indicated by a solar clock or chronometer.

To convert degrees into time and the contrary.

As every circle is divided into 360°, and also where time is concerned, into twenty-four hours, the proportion for converting degrees, &c. into time, will be the same in all cases, that is, 15° to one hour; but, it must be carefully noted, whether the circle or arc to be converted into time is that described by the motion of a star, the actual motion of the sun, or the motion of the sun supposed to move equably. In the first case, the result will be sidereal time, in the second, apparent time, and in the last, mean solar time.

To convert degrees, &c. into time and the contrary, a table is generally used, which may be found in various authors; but the following method is more expeditious, and it is always desirable to be as independent of extraneous aid as possible.

As 15° degrees give one hour.
1° degree will give 4^{m} minutes of time.
1′ minute . . . . . . 4^{s} seconds, and
1″ second . . . . . . 4^{ths} thirds.

If then the seconds of space be multiplied by four, the result will be *thirds* of time. In like manner the minutes will produce *seconds*, and the degrees *minutes* of time.

Let it be required to convert 57° 49′ 54″ *into time.* The operation will stand thus :

$$\begin{array}{r} 57^{\circ}\ 49'\ 54'' \\ 4 \\ \hline 3^{h}\ 51^{m}\ 19^{s}{\cdot}36. \end{array}$$

To convert time into degrees, &c. the following method will be found the most convenient. Multiply the time by ten, and add to this half the product; the sum will be the corresponding degrees, &c.

Let it be required to convert $3^{h}\ 51^{m}\ 19^{s}{\cdot}36^{th}$ into degrees, &c.

h m s th
3 51 19 36
10

38 33 16 0
One half 19 16 38 0

57° 49′ 54″ 0‴ corresponding space.

Having the apparent time at the ship, and the longitude, to find the apparent time at Greenwich.

As all the calculations in the *Nautical Almanac*, with the exception of the eclipses of Jupiter's satellites, are made for the instant of apparent noon at Greenwich; it becomes necessary when any of the data contained in the *Almanac* are required for a given time at the ship, to find in the first place the corresponding time at Greenwich. This may be done in the following manner:

Turn the longitude of the ship into time, and add it to the given time if the longitude be west, but subtract it from the given time if the longitude be ast. The result will be the apparent time at Greenwich.

In this operation, if the result of the addition hould exceed 24 hours, the excess will be the time rom the noon of the following day. And in performng the subtraction, 24 hours must be added, if necesary, to the given time at the ship, and the remainder will be the Greenwich time from the noon of the preceding day. This must be carefully recollected.

EXAMPLES.

What is the apparent time at Greenwich, when the sun is upon the meridian in 76° of west longitude.

76° 0′ ship's longitude.
4

5^h 4^m 0^s longitude west in time.
0 0 0 apparent time at the ship.

5 4 0 apparent time at Greenwich.

Given the ship's longitude on the 15th of January, 5° 13′ west, and the apparent time 6^h 5^m, required the parent time at Greenwich.

105° 13′ longitude west.
4

h 0^m 52^s longitude west in time.
5 0 apparent time at the ship.

5 52 apparent time the 15th Jan. at Greenwich.

Given the ship's longitude on the 13th August, 96° 7′ st, and the apparent time 5^h 7^m 30^s, required the appant time at Greenwich.

96° 7′
4

h 24^m 28^s longitude east in time.
5 7 30 apparent time at the ship.

2 43 2 apparent time at Greenwich the 12th Aug.

Given the ship's longitude on the 15th of January, 105° 13′ west, and the apparent time 20^h 5^m 7^s, required the apparent time at Greenwich.

7^h 0^m 52^s longitude west in time.
20 5 7 apparent time at the ship.

3 5 59 apparent time at Greenwich the 16th Jan.

HAVING GIVEN THE APPARENT TIME AT THE SHIP AND THE LONGITUDE, TO FIND THE EQUATION OF TIME, THE SUN'S DECLINATION OR RIGHT ASCENSION.

Find the apparent time at Greenwich, corresponding with the given time at the ship. Take the difference between the required element on the preceding and following noon, from the *Nautical Almanac.* Then say, as 24 hours is to this difference, so is the apparent Greenwich time to the proportional variation from this time, to be added or subtracted from the quantity taken from the *Nautical Almanac* for the preceding noon, according as it increases or diminishes.

EXAMPLES.

Let it be required to find the equation of time on the 2d April, 1823, at 4^h 5^m apparent time.

The difference between the equation of time on the 2d and 3d of April, being $18^s{\cdot}2$, we have $24^h : 18^s{\cdot}2 :: 4^h\ 5^m : 3^s{\cdot}1$, the change in the equation of time, or the difference between apparent and mean time for an interval of 4^h 5^m of apparent time; which being subtracted (because the equation of time is decreasing) from 3^m $48^s{\cdot}6$, the equation at the preceding apparent noon, we have 3^m $45^s{\cdot}5$, the equation of time required.

Let the equation of time be required for the 1st of April, 1823, at 19^h 55^m apparent time.

The diurnal variation in 24 hours being $18^s{\cdot}3$, we have
$24^h : 18^s{\cdot}3 :: 19^h\ 55^m : 15^s{\cdot}2$, the difference for 19^h 55^m; which being subtracted from 4^m $6^s{\cdot}9$, the equation of time for apparent noon on the 1st of April gives 3^m $51^s{\cdot}7$ for the equation of time required.

The same result might have been obtained, by taking the proportional part of $18''{\cdot}3$ for the difference between 19^h 55^m and 24 hours; and increasing the equation of time for noon on the 2d of April by that quantity.

The sun's declination and right ascension for any time at Greenwich may be found in a similar manner; by taking the proportional part of the daily variation of declination or right ascension, and adding it to, or subtracting it from the quantity at noon, according as it increases or decreases.

This method is however so tedious, that various tables have been constructed to facilitate the operation. A table of proportional logarithms, which appears to be by far the most practically convenient for this purpose, was published by Captain Basil Hall, of the Royal Navy, and is added at the conclusion of this article. A description of it will be best given in the author's own words:—

"This table is useful for finding the proportional part of the daily variation of the sun's right ascension and declination, and of the equation of time, for any hour after noon. The numbers at the top to 24 are for hours, and at the side for minutes: or, the first are minutes, and those at the side seconds: or, the first are seconds, and those at the side thirds; or, in the left hand column, the equivalent decimals of a second. The top arguments are carried on to 30, as the daily variation of the equation of time exceeds 30^s on some days in December.

" The proportional logarithms in this table are found by subtracting the logarithms of the minutes in the several arguments, from the logarithm of the minutes in 24 hours; viz. 3·15836; so that the proportional logarithm of 24^h becomes 0. For the proportional logarithms above 24, the arithmetical complements of the logarithms of the arguments have been added to 3·15836, which renders it necessary to subtract 1 from the index of the result, for every proportional logarithm taken from this part of the table."

RULE.

" To the proportional logarithm of the daily variation, add the proportional logarithm of the time from noon; the sum is the proportional logarithm of the correction for that interval, to be added or subtracted according as the element employed is increasing or decreasing.

" When one or both the terms are small, and consequently their proportional logarithms great, it is convenient, in order to prevent the answer falling near the beginning of the table where the differences are large, to take 1 or 2 from the index of the result, which will bring the answer out 10 or 100 times too great, this is readily done by the eye.

" The value of the terms of the answer is regulated by that of the arguments used; and though on this point at first an occasional ambiguity appears to occur, it is easily removed by a due consideration of the question."

EXAMPLES.

Required the equation of time at Greenwich, the 2d of April, 1823, at 4^h 5^m apparent time.

The daily variation being 18^s·2, seek 18 at the top of the table, and ·2 in the left hand side column. Under the first, and in a line with the second, we find the proportional log. of 18^s·2 1201
At the top, and opposite 5^m in the second column
from the left 7692

8893

The sum is the proportional log. of the variation, which is found in the table to be 3^s·1. Subtracting this from 3^m 48^s.6, the equation of time for noon at Greenwich, we have 3^m 48^s·5. the eouation of time sought.

Required the equation of time on the 1st of April, 1823, at 19^h 55^m apparent time, at Greenwich.

Prop. log. of 18^s·3, the variation for 24 hours..1178
Time from noon 19^h 55^m P. log............. 810

— 55^s·2 P. log...............1988

m s
Equation of time at noon.......... 4 6·9
Variation for 19^h 55^m ....— 15·2

Eqnation of time required .... 3 51·7

Required the variation in the equation of time, 23d of July, 1823, for 7^h 35^m.

Daily diff. 1^s·9 prop. log. 1.1015
7^h 35^m prop. log................... ·5004

Variation required 0^s·6 prop. log. ..1·6019

Required the variation in the equation of time, for 9^h 7^m, the 20th Dec. 1823.

Daily diff. 29^s·8 prop. log. 9060
9^h 7^m..........prop. log. 4204

1·3264

Here the daily difference exceeding 24, it is necessary to subtract 1 from the index. The remainder ·3264 is the proportional logarithm of 11^s·3 the variation required.

What is the sun's declination, April 10th, 1823, when on the meridian in longitude, 80° east.

' h m s
Apparent time at the ship 0 0 0
Longitude east in time 5 22 0

Apparent time at Greenwich, 9th April..18 38 0
Difference of declination between 9th and
10th Feb. 22′ 19″ prop. log. 316
18^h 38^m ....prop. log. 1096

Change of declination ..+0° 17′ 19″ prop. log. ·1415
Dec. north at noon on the
9th Feb.......... 7 22 53

Declination required 7 40 12 north.

Required the sun's declination for 12th April, 1823, at 4^h 23^m apparent time at Greenwich.

Daily difference, 21′ 54″ prop. log......... 398
4^h 23^m ....prop. log.........7384

Change of declination + 4° 0′ 0″ prop. log. 7782
⊙ Dec. at apparent noon 8 29 26 north.

Declination required 8 33 26

Required the sun's declination the same day at 12^m 45^s apparent time, at Greenwich.

Daily difference 21′ 54″ prop. log........... 398
12^m 45^s..........prop. log. 2746

Change of declination .. + 11″·6 prop. log. 3144
Declination at noon ....8 29 26 ·0

Declination required ..8 29 37 ·6

Required the sun's declination for 12th April, 1823, at 16^h 50^m apparent time, the ship being in longitude 86° east.

Longitude east.............. 86°
4

Longitude in time east........ 5^h 44^m
Apparent time at the ship......16 50

Apparent time at Greenwich ..11 6

Daily difference of declination 21′ 54″ prop. log... 398
11h 6m.......... prop. log... 3349

Change of declination + 10 8 prop. log. 3747
Dec. at noon, north .. 8 29 26

Declination required 8 39 34

What is the sun's right ascension on the 12*th of April,* 1823, *at* 16h 50m *apparent time at Greenwich.*

	h	m	s	
☉'s right asc. for 12th April,	1	20	27·5	
13th	1	24	8·1	
Difference..		4	40·6	p. log. 7097
	h	m		
	16	50		p. log. 1540
		m	s	
Increase of right ascension .. +		3	17	p. log. 8637
	1	20	27·5	
Right ascension required..	1	23	44·5	

To find the moon's right ascension and declination for any given time, under any given meridian.

Convert the longitude of the ship into time, and find the apparent time at Greenwich corresponding to that of the ship.

As the moon's right ascension and declination are given in the *Nautical Almanac* for noon and midnight, find by proportion the variation in right ascension or declination, for the excess of the given Greenwich time beyond the preceding noon or midnight.

If the moon's motion in right ascension and declination were equable, the quantity thus found applied to that of the preceding noon or midnight would be the right ascension or declination required. But as this is not the case, it becomes necessary to employ the "*Table of second differences,*" given at the end of the *Nautical Almanac,* in the following manner:

Take out of the *Nautical Almanac* the two right ascensions or declinations preceding and following the given Greenwich time, and take the successive differences between the first and second, the second and third, and the third and fourth. The quantities thus obtained are called *first* differences. From the first differences find also the *second* differences, of which take the mean.

In this operation, if the first differences increase and then decrease, or the contrary, half the difference of the two second differences must be taken for the mean.

Having thus found the mean of the second differences, enter the table with it and the given time, and take out the corresponding equation. If the mean of the second differences exceed the limits of the table, the equation may be taken out by two such entries as together make up the mean of the second differences.

If the first of the first differences be *greater* than the third of the first differences, *add* this equation to the proportional part before found; otherwise, subtract it. The result will be, the true variation of the moon's right ascension or declination for the excess of the given time beyond the preceding noon or midnight; and this being applied by addition or subtraction to the right ascension or declination for such noon or midnight, as it may be increasing or decreasing, the sum, or difference, will be the moon's correct right ascension or declination for the given time.

Examples.

Required the moon's right ascension, Jan. 19*th,* 1823, *at* 5h *apparent time.*

		° ′ ″	1st diff. ° ′ ″	2d diff. ′ ″	Mean. ′ ″
☽'s Ɽ Jan.	18th, at midnight	11 31 24			
Ditto	19th, at noon	17 45 12	6 13 48		
Ditto	19th, at midnight	24 11 55	6 26 43	12 55	13 51
Ditto	20th, at noon	30 53 25	6 41 30	14 47	

	″	° ′ ″
As 12h : 6° 26′ 43″ :: 5h to the approximate motion in Ɽ		2 41 7·9
Equation from the table corresponding to.... 5h and 12′ 0′ =	87·5	
5h and 1 50 =	13·4	
Equation of 2d difference	100·9	1 40·9
Correct motion in Ɽ		2 39 27
Ɽ for preceding noon....		17 45 12
☽'s Ɽ required		20 24 39

Required the moon's declination, Jan. 16*th,* 1823, *at* 18h *apparent time.*

18h apparent time will be 6h apparent time beyond midnight.

				1st diff.	2d diff.	Mean.
☽'s Declination, Jan.	16th, at noon,	S.	4° 11′ 22″			
Ditto	16th, at midnight,	S.	1 19 6	2° 52′ 16″		
Ditto	17th, at noon,	N.	1 35 33	2 54 39	2′ 23″ 0 41	1′ 32″
Ditto	17th, at midnight,	N.	4 30 53	2 55 20		

	°	′	″
As 12^h : 2° 54′ 39″ : : 6^h to the approximate change of declination	1	27	19·5
Equation corresponding to 6^h and 1′ 32″		—	11·3
Correct change of declination	1	27	8·2
Declination for preceding midnight, south	1	19	6·0
True declination, north	0	8	2·2

Required the moon's declination, March 19th, 1823, at 19ʰ apparent time.

19^h apparent time will be 7^h apparent time after midnight.

				1st diff.	2d diff.	
☽'s Declination, March	19th, at noon	N	26° 42′ 26″	0° 24′ 14″	0° 14′ 8″	Mean. 30′ 0″
Ditto	19th, at midnight,		26 18 12	0 10 6	1 14 9	
Ditto	20th, at noon,		25 28 18	1 14 15		
Ditto	20th, at midnight,		24 14 3			

		°	′	″
As 12^h : 10′ 6″ : : 7^h to the approximate change of declination		0	5	33
Equation corresponding to 7^h and 12′ =	87″·5			
7^h and 12′ =	87 ·5			
7^h and 6′ =	43 ·8			
	218 ·8 =	—	3	39
Correct change of declination		0	1	54
Declination for preceding midnight		26	18	12
Declination required		26	16	8

The preceding examples will probably be found sufficient; but the reader is recommended to peruse with attention the directions given in the *Nautical Almanac,* for the use of the table of second differences.

To convert apparent time into mean time, and the contrary.

If the apparent time given be that at the ship, convert this into the apparent time at Greenwich. Find the corresponding equation of time, and apply it with its proper sign as directed in the *Nautical Almanac* to the apparent time at the ship. The result will be the mean solar time. If, on the contrary, the time given be mean time, and it is desired to convert it into apparent time, the equation of time, for the time at Greenwich, must be applied to the mean time at the ship with a contrary sign.

Example.

Being in long. 105° 13′ *west, and the apparent time the* 15*th Jan. at the ship* 20ʰ 5ᵐ 7ˢ *required the mean time.*

h	m	s	
7	0	52	Long. west in time.
20	5	7	Apparent time at the ship.
3	5	59	Apparent time at Greenwich, 16th Jan.
20	5	7	Apparent time at the ship.
+ 9	59·2		Equation of time for noon, 16th Jan.
+ 0	2·6		Variation for 3ʰ 6ᵐ.
20 15	8·8		Mean time required.

Had it been required to convert 20ʰ 15ᵐ 8ˢ·8 mean time into apparent time, the equation for noon of the 16th at Greenwich, with its proportional variation, would have been subtracted instead of being added; and the result would have been 20ʰ 5ᵐ 7ˢ of apparent time.

Of the corrections to be applied to the altitude observed at sea, in order to obtain the true altitude.

The dip or depression of the horizon.

In observing an altitude with the sextant, or reflecting circle, the image of the object is made to coincide with the visible horizon; but as the eye is elevated above the surface of the sea, the visible horizon will be below the true horizontal plane.

Let AC (fig. 1. plate 1, Nautical Astronomy,) or CB be the radius of the earth; AE the height of the eye above the level of the sea, EH the horizon. Then E being the place of the observer, EH is the horizontal plane; and the angle HEB will be the *dip* or depression of the horizon, which may be thus found.

The angle HEB is equal to the angle BCE; and in the right angled triangle CBE, the difference of the squares of CE (= the rad. of the earth + the height of the eye above the level of the sea,) and CB is equal to the square of BE; the square root of which therefore gives BE. Then in the same triangle, CE : rad. : : BE to the sine of BCE (=HEB) the depression of the horizon.

The depression thus obtained is the true depression, but this must be lessened by the amount of terrestrial refraction, which is very uncertain. About $\frac{1}{8}$ or $\frac{1}{10}$ of the whole quantity is usually allowed.

A table, No. 1, is added of the *dip* or apparent depression of the horizon for different elevations of the eye, allowing $\frac{1}{10}$ for terrestrial refraction, the manner of using it is sufficiently obvious. The dip must be always subtracted from the observed altitude.

The height of the eye above the sea should be carefully measured at the time the altitude is taken; for a neglect of this precaution might easily occasion an error of one minute, according as the observation is made on the windward or leeward side of the vessel.

Refraction.

The rays of light proceeding from a celestial body having to pass through the atmosphere, are bent in the direction of the vertical from their rectilinear course, and the object in consequence appears to have a greater altitude than it really has. This effect, called *refraction*, is greatest in the horizon, and becomes nothing in the zenith.

When the temperature of the atmosphere is increased, the refraction decreases; and when the density of the air becomes greater, (indicated by an increase in the height of the mercury of the barometer,) the refraction suffers an increase. The corrections for temperature and density have hitherto been little attended to in nautical astronomy; but on some occasions they become too considerable to be neglected. In the *Nautical Almanac* is given a table of refractions with corrections for the barometer and thermometer, and such ample directions for their application, as to render the additional trouble of employing them very trifling.

When the corrections for the depression of the horizon and for refraction have been applied to the observed altitude, the result will be the altitude of the object above the sensible horizon.

Semidiameter.

If the altitude obtained by the preceding corrections be that of the limb of the sun or moon, we have to add or subtract the semidiameter, to obtain the true altitude of the centre. This is readily done for the sun by using the semidiameter taken from the *Nautical Almanac*, and found by proportion, if necessary, for the given time; but for the moon some further consideration is necessary.

When the moon is in the horizon, her distance from the observer is greater than when in the zenith by the semidiameter of the earth. Consequently her apparent semidiameter will be least in the horizon, and greatest in the zenith. This increase is called the augmentation of the moon's semidiameter, and may be thus found.

The distance of the moon being about 60 semidiameters of the earth, the moon's horizontal semidiameter will be increased one sixtieth part in the zenith, and the sine of this quantity (or, as the arc is small, the quantity itself) will vary directly as the sine of the altitude; therefore, to the logarithm of one sixtieth part of the moon's semidiameter in the horizon, add the logarithmic sine of the altitude. The sum will be the logarithm of the augmentation to be added to the horizontal semidiameter before found.

The semidiameter of the moon is given in the *Nautical Almanac* for noon and midnight at Greenwich, from which it may be readily found by proportion for any other time or meridian.

A small table, No. 2, is added, from which the augmentation may be taken out with the given horizontal semidiameter and altitude. The proportional parts may be readily found by inspection.

Of parallax.

Having thus obtained the true altitude of the object above the sensible horizon, the last step is to determine its altitude when referred to the rational horizon, or its situation as seen from the centre of the earth, to which point all observations are reduced.

The *horizontal parallax* is the angle under which the semidiameter of the earth, or its radius at the place of observation, would be seen from the celestial body when in the horizon.

The *parallax in altitude* is the angle subtended by this same radius, at any elevation of the object, and is the quantity to be added to the altitude before found, to reduce it to what it would be at the centre of the earth. The effect of parallax is greatest in the horizon, and is nothing in the zenith.

Let CE (plate 1, fig. 2,) be the semidiameter of the earth, ☽ E *h* the sensible horizon, HC the rational horizon. Then the angle E ☽ C is the horizontal parallax, and the angle C *m* E (=GE *m*,) is the parallax for the altitude *m* E ☽; or the angle by which the object ☽ would appear lower when viewed from the sensible horizon, than it would if seen from the centre of the earth C.

In the triangle ☽ EC, we have ☽ C : sin of ☽ EC (or its equal the sine of ☽ EZ) :: EC : sin. E ☽ C. And in the triangle *m* EC, we have *m* C (= ☽ C) : sin. *m* EC (= sin. *m* EZ) :: EC : sin. E *m* C. The sine of the horizontal parallax varies therefore inversely as the sine of the zenith distance, or directly as the cosine of the altitude.

The moon's horizontal parallax is given in the *Nautical Almanac* for every noon and midnight at Greenwich, and may be found by simple proportion for any other time. But the parallax there found is the equatorial horizontal parallax; and as the earth is a spheroid, having the equatorial to the polar diameter about as 306 to 305, the polar horizontal parallax will be less than the equatorial in the same proportion, and consequently the horizontal parallax for any given latitude, will vary between these limits.

A table, No. 3, is added, containing the correction on this account to be taken out with the equatorial horizontal parallax, given in the *Nautical Almanac* and the latitude; which correction being subtracted from the equatorial parallax before found, the remainder will be the horizontal parallax for the place and time of observation, which is to be employed in the following manner for finding the parallax in altitude.*

Having the horizontal parallax, to find the parallax in altitude.

To the log. sine of the horizontal parallax add the log. cosine of the altitude. The sum rejecting ten from the index, will be the log. sine of the parallax in altitude, to be added to the altitude above the sensible horizon before found. Or,

To the prop. log. of the horizontal parallax add the logarithmic secant of the altitude; the sum will be the proportional logarithm of the parallax in altitude: or,

From the proportional log. of the horizontal parallax, with 10 for its index, subtract the cosine of the altitude. The remainder will be the proportional log. of the parallax in altitude.

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* The same table contains also the correction of the proportional log. of the equatorial parallax, which will be found the most convenient if proportional logarithms are employed.

Either of the two latter methods is preferable to the former, where there is a common table of proportional logarithms; as the secant or cosine of the altitude may be taken to the nearest minute only; but for the sine of the parallax a table of sines, &c. to seconds is required.

Examples.

The observed altitude of Sirius being 20°, the barometer 29.5 inches, thermometer 70°, and height of the eye 22 feet, required the true altitude.

		°	′	″
Observed altitude		20	0	0
Depression of the horizon table 1	—		4	30
		19	55	30
Refraction	—		2	39·4
Correction for barometer	—			2·65
Ditto for thermometer	—			6·44
True altitude		19	52	41·50

In the above example, if the corrections for the barometer and thermometer had been omitted, the result would have been erroneous 9 seconds.

The observed altitude of the sun's lower limb on the 3d of Feb. 1823, being 14°, the height of the eye 22 feet, barometer 29 inches, and thermometer 67° required the true altitude of the sun's centre.

		°	′	″
Observed altitude of L. L.		14	0	0
Depression of the horizon	—		4	30
		13	55	30
Refraction	—		3	51·15
Correction for barometer	—			7·70
Ditto for thermometer	—			7·97
Parallax (table 4,)	+			8·50
Sun's semidiameter on 3d Feb.	+		16	14·80
True altitude of sun's centre		14	7	46·48

In this example, the error, from neglecting the corrections for the barometer and thermometer, would have been 14″·64.

Being in latitude 45° north, and longitude 140° west on the 3d Feb. 1823, at 18^h apparent time, the altitude of the moon's lower limb was observed to be 15°. The height of the eye being 22 feet, the barometer 28·7 inches, and thermometer 40°, required the true altitude of the moon's centre.

	h	m	
Longitude of the ship in time, west	9	20	Feb. 3d.
Apparent time at the ship, add	18	0	
Apparent time at Greenwich	3	20	Feb. 4th.

	°	′	″		
☽'s hor. eq. par. Feb. 4, at 3^h 20^m from *Naut. Alm.*		54	14·3		
Reduction for lat. 45° from table		—	6·0		
True horizontal parallax		54	8·3	P. L.	10·5218
	15	7		cosine	9·9847
Parallax in altitude	0	52	15·5	P. L.	0·5371

		°	′	″
Observed altitude ☽'s L. L.		15	0	0
Depression of horizon	—		4	30
		14	55	30
Refraction	—		3	35·4
Barometer	—		0	9·33
Thermometer	+		0	4·39
Horizontal semidiameter for 4th Feb. at 3^h 20^m from the *Nautical Almanac*	+		14	47·50
Augment. of semidiameter from (table 2)	+		0	8·00
True altitude of ☽'s centre above the sensible horizon		15	6	45·16
Parallax in altitude	+		52	15·50
True altitude of ☽'s centre		15	59	0·66

Having given the true altitude of the centre of the sun or of the moon, to find the apparent altitude.

In clearing the distance of the sun and moon from the effects of parallax and refraction, it is necessary to have the apparent altitudes of their centres. For this purpose, the usual method has been to apply the semidiameter to the apparent altitude of the limb; but as the semidiameter is affected by refraction as well as the limb, this method is erroneous, and if the altitude should be small, the error might sensibly affect the accuracy of the result. This has been noticed in a very useful article, "*On the methods of correcting lunar observations,*" published in the Journal of Science of the Royal Institution, under the title of *Astronomical and Nautical Collections;* and which, though without the name of the author, there is reason to believe is the work of Dr. Thomas Young.

To find the apparent altitude of the centre of the sun or moon from the true altitude.

Subtract the parallax in altitude, and add the refraction answering to the remainder. Apply the corrections for the barometer and thermometer before found, in obtaining the true altitude *with the signs changed.* The result will be the apparent altitude of the centre.

Taking the last example.

		°	′	″
The true altitude of the ☽'s centre was		15	59	0·66
Subtract the parallax in altitude	—		52	15·50
		15	6	45·16
Refraction for 15° 6′ 45″	+		3	32·70
Barometer	+		0	9·33
Thermometer	—		0	4·39
Apparent altitude		15	10	22·8

But the apparent altitude of the centre may be more readily obtained, from having the observed altitude of the limb, by means of table 5, taken from the article in the journal of science before alluded to.

The correction given in this table is the effect of refraction at different altitudes in lessening the true vertical semidiameter either of the sun or moon, which for this purpose may be considered as the same

The observed meridian altitude of the sun's lower limb, on the 5th of Dec. 1823, *was* 25° 29′, *the zenith being to the north of the sun, the ship's longitude* 73° *east, and the height of the eye* 22 *feet; required the latitude.*

		°	′	″
Observed altitude of ⊙'s L. L.	......	25	29	0
Depression of the horizon	....	−	4	30
Apparent altitude of ⊙'s L. L.	......	25	24	30
Refraction	....	−	2	2
Parallax	....	+	0	8
⊙'s semidiameter	..	+	15	54
True altitude of ⊙'s centre	....	25	38	30
Zenith distance, north	....	64	21	30
⊙'s declination, south	....	22	18	14
Latitude, north	....	42	3	16

The ship's longitude being 73°, or $4^h\ 52^m$ east, the corresponding time at Greenwich will be $19^h\ 8^m$ of the 4th of Dec. for which time the sun's declination was computed as before explained.

The meridian altitude of the planet Jupiter was observed on the 5th of Feb. 1823, *to be* 50° 25′, *the zenith being to the north of the planet; the ship's longitude* 24° *west, and the height of the eye* 18 *feet, required the latitude.*

	h	m
Time of the observation at the ship nearly* ..	6	24
Longitude west 24° in time	1	36
Greenwich time nearly........	8	0
	°	′
Jupiter's dec. at Greenwich, 1st Feb. north....	18	44
Jupiter's dec. ...... 7th Feb. north....	18	40
Variation of declination in 6 days....	0	5

Hence, by proportion, the declination of Jupiter, the 5th of Feb. at 8^h is north 18° 47′ 37″.

Observed altitude of Jupiter	50°25′		0″
Depression of the horizon	−	4	4
Apparent altitude ..	50	20	56
Refraction	−	0	48
True meridian altitude	50	20	8
Zenith distance, north	39	39	52
Declination, north	18	47	37
Latitude, north	58	27	29

To find the latitude from the meridian altitude of the moon.

Find in the *Nautical Almanac* the time of the moon's passing the meridian of Greenwich on the given day. Take also the difference between it and the time of the transit on the following day; then say, as 24 hours, augmented by this difference, is to the difference of the daily transit, so is the ship's longitude in time, to a fourth proportional; which, added to the time of the moon's transit at Greenwich on the given day, if the longitude be west, or subtracted from it if east, the sum or difference will be nearly the apparent time of the moon's passage over the meridian of the ship. To this apply the longitude in time as before directed, in order to obtain the corresponding time at Greenwich.

Compute the moon's declination, semidiameter, and horizontal parallax, from the *Nautical Almanac* for the Greenwich time thus obtained; applying, if great accuracy be required, the equation of second differences to the proportional variation of the declination for the excess of the given time beyond noon or midnight, as the omission of this corrrection may occasion an error in the latitude of about 2¼ minutes. Employ the horizontal parallax and semidiameter in the manner before directed, to find the parallax in altitude, and the augmentation of the moon's semidiameter, in order to obtain the true altitude of the moon's centre, and thence the zenith distance; with which, and the declination, proceed as before to find the latitude.

Example.

Feb. 19*th*, 1823, *the ship being in longitude* 40° *west, the observed meridian altitude of the moon's lower limb was* 55° 6′, *the zenith north of the moon, and the height of the eye* 16 *feet; required the latitude.*

	h	m	s
The moon passed the meridian of Greenwich, 19th of February, 1823	6	56	0
Ditto 20th of February, ...	7	59	0
Difference of the ☽'s passage	1	3	0
	h	m	s
As $25^h\ 3^m : 1^h\ 3^m :: 2^h\ 40^m :$	0	6	42
To which add	6	56	0
Apparent time of ☽ s passing the meridian of the ship	7	2	42
Ship's longitude west	2	40	0
Corresponding time at Greenwich	9	42	42

								1st diff.	2d diff.	Mean.
☽'s declination, February	18th,	at midnight,	N	25°	55′	46″		42′ 31″	26′ 9″	15′ 54″
Ditto	19th,	at noon,	N	26	38	17		16 22	5 39	
Ditto	19th,	at midnight,	N	26	54	39		10 43		
Ditto	20th,	at noon,	N	26	43	56				

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* This will be nearly the same as the time of the planet's passing the meridian of Greenwich, and may be taken from the *Naut. Alm.*

than that of M. Delambre. But, as this requires as many references to tables, and a correction on account of the change in the declination of the sun, we have been induced to prefer the direct process.

In the *Astronomical and Nautical collections* before mentioned, under the head of "Parallax" is contained the detail of Delambre's method, of which we shall avail ourselves; adding such remarks as may appear necessary to remove any practical difficulty.

In fig. 7. plate 1, let A and B be the two places of the sun at the times of observation; P the pole, Z the zenith. Then ZB and ZA will be the zenith distances; PA and PB the polar distances, and APB the interval between the observations converted into degrees, &c. In the triangle APB we have the two sides PA and PB, and the included angle, to find the third side AB. In the same triangle APB, the three sides are now given to find either of the angles at A or B, suppose the angle BAP, and in the triangle AZB, we have the three sides to find the angle BAZ.

The difference between the angles PAB and ZAB, (or sometimes their sum between the tropics) will be the angle PAZ, subtended by PZ, the compliment of latitude.

Lastly, in the triangle PAZ, the two sides AP and AZ, and the included angle PAZ, are given to find the third side PZ, the compliment of latitude.

Rules for double altitudes.

From the observed altitudes, find the true altitudes of the sun's centre. Correct the lesser altitude for the change of the ship's place during the interval, in the following manner :—

Find the angle between the ship's course and the sun by compass, at the time of the observation of the lesser altitude. With this angle as a course, and the distance sailed in the interval, enter a traverse table, and find the difference of latitude answering to the distance, which will be the correction of the altitude.

If the lesser altitude be observed in the forenoon, this correction is to be added to it, if the angle between the ship's course and the sun is less than 8 points; but if the angle is greater than 8 points, the correction is to be subtracted from the lesser altitude. If the lesser altitude be observed in the afternoon, the correction is to be subtracted from it, should the angle between the ship's course and the sun be less than 8 points; but if more, the correction is to be added to the lesser altitude.

The change of the ship's longitude in time between the observations must be applied to the elapsed time, by adding it to the interval when sailing eastward, or by subtracting it when sailing westward.

Having thus corrected the lesser altitude and the interval, find the declinations for the times of observation, and thence the polar distances; then,

1. Add together the logarithmic sine of half the sum of the polar distances $\frac{1}{2}$ (PA+PB), and the sine of half the interval converted into space, that is, $\frac{1}{2}t$; and the sum will be the sine of half the distance AB.

2. In the triangle APB, as the sine of AB is to the sine of the opposite angle APB (= t), so is the sine of one of the polar distances, for instance, PB, to the sine of the opposite angle PAB.

3. In the triangle AZB, we have the three sides ZA, ZB, and AB, to find the angle BAZ. For this purpose, add together the two zenith distances ZA, ZB, and the distance AB; and from the half sum subtract in succession the two sides containing the required angle ZA and AB: add together the sines of the two remainders, and the arithmetical compliments of the sines of the sides containing the required angle ZA and AB; half the sum of these four logarithms will be the sine of half the angle sought BAZ.

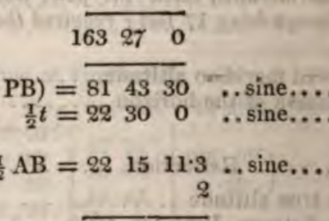
4. The difference of BAZ and PAB, or sometimes the sum between the tropics, will be the angle PAZ, subtended by the compliment of latitude PZ, from the sun's place A. To find the latitude, take out the logarithmic cosines and sines of the sides containing the colatitude, that is, of AP and AZ; and under the sines set down the cosine of the included angle PAZ. Add them separately together, and find the corresponding natural numbers, the sum of which will be the natural sine of the latitude; and its logarithm, of course, the logarithmic sine. But if the angle PAB lies without BAZ, and their sum exceeds a right angle, the cosine becomes negative, and the difference of the natural numbers must be taken for the natural sine of the latitude; and if there is any doubt in the computer's mind, it will be easy to try both suppositions.

Note.—If the declination is very small, it may sometimes be more convenient to find the angle PAB, from the three sides of the triangle, as in the 3d precept, instead of by the 2d.

EXAMPLES.

Let the two zenith distances corrected be ZA = 73° 54′ 13″ ZB = 47° 45′ 51″, *the declinations* 8° 18′ *and* 8° 15′ *north, and the interval of time three hours; or* PA = 81° 42′, PB = 81° 45′, *and* APB = 45°.

Note.—It will be found convenient, and tend much to prevent mistakes, to draw a rough figure, and mark the sides given, thus—



° ′ ″

PA = 81 42 0
PB = 81 45 0

163 27 0

1. $\frac{1}{2}$ (PA + PB) = 81 43 30 ..sine....9·9954547
$\frac{1}{2}t$ = 22 30 0 ..sine....9·5828397

$\frac{1}{2}$ AB = 22 15 11·3 ..sine....9·5782944
2

AB = 44 30 22·6

2. AB = 44° 30′ 22″·6 sine co.ar. 0·1542898
APB = 45 0 0 ..sine....9·8494850
PB = 81 45 0 ..sine....9·9954822

PAB = 86 38 58 ..sine....9·9992570

3. ZB = 47 45 51
ZA = 73 54 13 sine co. ar. 0·0173686
AB = 44 30 23 sine co. ar. 0·1542898

sum..166 10 27

$\frac{1}{2}$ sum 83 5 13·5
$\frac{1}{2}$ sum − ZA = 9 11 0·5 sine .... 9·2030232
$\frac{1}{2}$ sum − AB = 38 34 50·5 sine .... 9·7949174

2)19·1695990

$\frac{1}{2}$ BAZ = 22 36 26·6 sine.... 9·5847995
2

BAZ = 45 12 53
PAB = 86 38 58

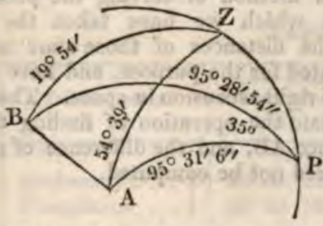
PAZ = 41 26 5

4. PA cosine 9·1594354 PA sine.... 9·9954271
ZA cosine 9·4428780 ZA sine.... 9·9826314
PAZ cosine.. 9·8748934

·040023 Log. 8·6023134 Log... 9·8529519
·712774

·752797 Log. 9·8766779 = log. sine 48° 50′, the latitude required.

Let the two zenith distances corrected be ZA = 54° 39′ ZB = 19° 59′. *The declinations* 5° 31′ 6″, *and* 5° 28′ 54″ *south, and the interval of time* 2ʰ 20ᵐ ; *or* PA = 95° 31′ 6″, PB = 95° 28′ 54″, and APB = 35°.



PA = 95° 31′ 6″
PB = 95 28 54

191 0 0

1. $\frac{1}{2}$ (PA+PB) = 95 30 0 sine ...... 9·99799
$\frac{1}{2}$ t = 17 30 0 sine ...... 9·47814

$\frac{1}{2}$ AB = 17 25 0 sine ...... 9·47613
2

AB = 34 50 0

2. AB = 34 50 0 sine co. ar... 0·24322
APB = 35 0 0 sine ...... 9·75859
PB = 95 28 54 sine ...... 9·99801

88 21 10 sine ...... 9·99982

PAB = 91 38 50 (the supplement.)

3. ZB = 19° 59′ 0″
ZA = 54 39 0 sine co. ar... 0·08851
AB = 34 50 0 sine co. ar... 0·24322

sum 109 28 0

$\frac{1}{2}$ sum 54 44 0
$\frac{1}{2}$ sum − ZA = 0 5 0 sine ...... 7·16269
$\frac{1}{2}$ sum − AB = 19 54 0 sine ...... 9·53196

2)17·02638

$\frac{1}{2}$ BAZ = 1 52 5 sine ...... 8·51319
2

BAZ = 3 44 10
PAB = 91 38 50

PAZ = 87 54 40

4. PA cosine 8·98288 PA sine .... 9·99798
ZA cosine 9·76235 ZA sine .... 9·91149

·055620 Log. 8·74523 PAZ cosine ...... 8·56169

·029591 Log .... 8·47116

·026029 Log. 8·4154575 = log. sine 1° 29′ 28″ the latitude required.

Note. With that zenith distance which was taken furthest from the meridian, suppose PA, the co-latitude ZP, and the polar distance PA, the angle ZPA may be computed by precept 3d ; and this, converted into time, is the apparent time from noon, when the observation was taken ; to which the equation of time being applied, the error of the watch will be known.

To find the latitude from the altitudes of two fixed stars observed at the same time, having given their polar distances, and their difference of right ascension.

This method of determining the latitude has many advantages over that last described ; for, no inaccuracy can arise from an erroneous estimate of the change of the ship's place between the observations as they are completed at the same time ; and the risk of losing the second observation of the sun from unfavourable weather is avoided. The difficulty of seeing the horizon of the sea, at night, is the inconvenience to which this method is liable ; but this may be obviated by using either of the instruments, which will be described, for taking altitudes at sea when the horizon is not visible.

The solution of this problem depends on the same principles as that of the last. The polar distances of the two stars, and their difference of right ascension are given, being the two sides of a spherical triangle and the included angle, to find, in the first place, the third side AB. But as the difference of the polar distances of the two stars may be considerable, it becomes necessary to compute the side AB, more rigorously than in the case of the sun, in which the difference of declination being small, the 1st precept given, although an approximation was sufficiently accurate.

In making the observations, take the altitude of one star, and note the time by a watch; then the altitude of the other star, noting the time, and then again the altitude of the second star with the corresponding time. From the change of altitude of the second star, and the interval, its altitude for the instant when the first star was observed, may be known by simple proportion.

In order to compute the distance of the two stars, proceed as follows.

Rule.

Take out the logarithmic cosines and sines of the two polar distances PA and PB, and under the sines set down the cosine of the difference in right ascension of the two stars converted into degrees, &c. or the angle APB. Add them separately together, and find the corresponding natural numbers, the sum of which, or their difference, if either polar distance exceeds 90°, will be the natural cosine of the distance of the two stars AB; and its logarithm, of course, the logarithmic cosine.

The remainder of the operation for finding the latitude, is precisely the same as in the last problem, commencing with the second precept.

To insure greater accuracy, stars, when near the zenith, should not be used.

Example.

Let the corrected zenith distance of Capella be 29° 14′ 24″ = ZA, *and that of Sirius be* 72° 5′ 48″ = ZB. *The polar distance of Capella* PA = 44° 11′ 39″, *that of Sirius* PB = 106° 28′ 40″, *and the difference of their right ascensions* 1ʰ 33ᵐ 45ˢ *or* APB = 23° 26′ 11″, *required the latitude.*

1. PA cosine .. 9·8555080 PA sine.. 9·8432902

PB cosine .. 9·4527728 PB sine.. 9·9817868

·203367 Log .. 9·3082808 APB cosine 9·9626071

·613316.......................... Log.. 9·7876841

Diff. ·409949 Log .. 9·6127298 = log. cosine 65° 47′ 54″ = AB.

2. AB = 65° 47′ 54 sine″ co. ar. 0·0399536

APB = 23 26 11 sine .... 9·5995891

PB = 106 28 40 sine .... 9·9817868

24 43 04 ...... sine 9·6213295

PAB = 155 16 56 (the supplement.)

3. ZB = 72 5 48

ZA = 29 14 24 sine co. ar. 0·3111631

AB = 65 47 54 sine co. ar. 0·0399536

sum 167 8 06

½ sum 83 34 03

½ sum − ZA = 54 19 39 sine ...... 9·9097504

½ sum − AB = 17 46 09 sine ...... 9·4845601

2)19·7454272

½ BAZ = 48 14 29 ...... sine 9·8727136

2

BAZ = 96 28 58

PAB = 155 16 56

PAZ = 58 47 58

4. PA cosine 9·8555080 PA sine .... 9·8432902

ZA cosine 9·9408059 ZA sine .... 9·6888369

·625619 Log. 9·7963099 PAZ cos .... 9·7143594

·176395.................... Log. .. 9·2464865

·802014 Log. 9·9041820 = Log. sine 53° 19′ 22″, the latitude required.

In the *Nautical Almanac* for 1825, Dr. Brinkley has given a new method of solving the preceding problem, from which we have taken the liberty of extracting the distances of those stars most conveniently situated for the purpose, and have added their difference of right ascension in space. These data will much facilitate the operation of finding the latitude as the distance AB, and the difference of right ascension APB need not be computed.

TABLE I.—*For north latitude.*

Pairs of stars.	North polar distance for 1822.	Change in 10 years.	Distance (AB) for 1822.	Change in 10 years.	Diff. of right ascension (APB.)	Change in 10 years.
Capella ...... Sirius	44° 11′ 39″ 106 28 39	− 46″ + 44	65° 47′ 54″	+0′ 3″	23° 26′ 11″	−270″
Capella ...... Procyon	44 11 39 84 19 33	− 46 + 85	51 6 59	0 0	36 36 23	−189
Procyon ...... Spica Virginis	84 19 33 100 13 41	+ 85 + 190	87 32 33	+0 2	86 27 41	− 1
Procyon ...... Arcturus	84 19 33 69 53 10	+ 85 + 190	96 48 13	+0 2	99 23 29	− 63
Arcturus .... Antares	69 53 10 116 1 33	+ 190 + 86	56 1 25	−0 7	32 44 10	+139
Antares ...... *a* Lyræ	116 1 33 51 22 33	+ 86 − 30	71 41 14	+0 4	33 6 21	−244
Spica Virginis Antares	100 13 41 116 1 33	+ 190 + 86	45 54 15	+0 1	45 39 57	+ 78
Antares ...... Fomalhaut	116 1 33 120 33 48	+ 86 − 191	82 50 50	+0 2	97 19 31	− 48
a Lyræ ...... Fomalhaut	51 22 33 120 33 48	− 30 − 191	91 25 20	+0 3	64 13 10	+196
Arcturus .... *a* Lyræ	69 53 10 51 22 33	+ 190 − 30	59 0 34	+0 24	65 50 31	−105

TABLE II.—*For south latitude.*

Pairs of stars.	South polar distance for 1822.	Change in 10 years.	Distance (AB) for 1822.	Change in 10 years.	Diff. of right ascension (APB)	Change in 10 years.
Archernar .... Fomalhaut	32° 35′ 24″ 59 26 12	+ 186″ + 191	39° 7′ 20″	−0′ 1″	40° 49′ 23″	+260″
Rigel .... Sirius	81 35 10 73 31 21	+ 49 − 44	23 40 35	+0 5	22 49 42	− 40
Canopus...... Procyon	37 19 36 95 40 27	− 16 − 85	60 8 28	−0 10	17 29 33	+270
Sirius .... *β* Argus	73 31 21 21 0 48	− 44 − 148	57 43 43	−0 7	38 27 15	−277
a Crucis ...... Spica Virginis	27 53 12 79 46 19	− 200 − 190	52 58 54	+0 1	14 45 15	− 12
Spica Virginis *a* Centauri	79 46 19 29 53 20	− 190 − 163	51 40 18	+0 2	18 4 30	+190
Spica Virginis Antares	79 46 19 63 58 27	− 190 − 86	45 54 15	+0 1	45 39 57	+ 78

Having the time the sun's diameter takes to pass a horizontal line, to find the latitude of the place of observation.

This problem, which may perhaps sometimes be found useful, is taken from "Selections from the additions that have been occasionally annexed to the *Nautical Almanac.*"

RULE.

From the proportional logarithm of the observed time, the index being increased by 10, subtract the proportional logarithm of the time the sun's diameter takes to pass the meridian, found by doubling the time of the semidiameter set down in the *Nautical Almanac*, the remainder is the logarithmic sine of the angle of position.

To the logarithmic cosine of this angle add the logarithmic cosine of the sun's declination; the sum, rejecting 10 from the index, is the logarithmic sine of the latitude exactly, if the observation was made of the sun's rising through the horizon, otherwise only nearly to be corrected as follows:

Add together the logarithmic sine of the altitude, the logarithmic sine of the declination, and the logarithmic secant of the latitude; the sum, rejecting 20 from the index, is the logarithmic sine of the first correction to be added, if the latitude and declination are both of the same name, otherwise subtracted.

Add together the logarithmic tangent of the latitude, and the logarithmic versed sine of the altitude, the sum rejecting 10 from the index is the logarithmic sine of the second correction to be always subtracted.

EXAMPLE 1.

Suppose December 21st, 1774, the time of the sun's diameter rising through the horizon of the sea was observed to be 7^m 20^s; to find the latitude of the ship.

Observed time 7^m 20^s prop. log............11·3899
Time of ☉'s diam. passing mer. 2^m 22^s·2 p. l. 1·8804

Angle of position, 18° 51′½ sine .... 9·5095
Cosine angle of position ....9·97608
Cosine declination, 23° 28′ ....9·96251

Latitude, 60° 15′ sine ....9·93859

EXAMPLE 2.

Suppose the same day the sun was observed to take 5^m 18^s in rising through a space equal to his diameter, the mean altitude being 5°. To find the latitude.

Observed time, 5^m 18^s prop. log............ 11·5310
Time of ☉'s diameter, passing mer. prop. log. 1·8804

Angle of position, 26° 34′ sine .... 9·6506
Cosine angle of position.........9·95154
Cosine declination..............9·96251

Approx. lat. 55° 7′ 50″ sine....9·91405

Sine alt...........8·9403....V. S. alt.... 7·5804
Sine dec. ....9·6001....tang. lat. ..10·1569

Sec. lat. ....10·2428 2d cor. 18′ 50″.. 7·7373

1st cor. 3° 28′ 50″ sine 8·7832

	°	′	″
Approx. lat.	55	7	50
1st cor.	3	28	50
	51	39	0
2d cor.	—	18	50
Latitude...............	51	20	10

To find the latitude by the altitude of the pole star when not on the meridian.

This method of finding the latitude is so convenient that it deserves particular attention. It is not confined to any precise hour, as is the case with meridian altitudes, but may be employed during the whole night whenever the pole star is visible; for even though the horizon should not be seen, the instrument, which will be described, for taking altitudes under such circumstances may be used; and by employing the mean of several observations, will be found, on most occasions, sufficiently accurate.

The importance of repeatedly ascertaining the latitude on approaching the English channel at night is well known to every seaman; and, we believe, the usual practice in such a case, particularly in rough weather, is to lay-to till the next morning. We have however known an officer of the British navy, with that confidence and decision which superior science confers, run his frigate boldly up the channel at the rate of eleven knots an hour, in a dark squally night, which allowed only occasional glimpses of the pole star between flying clouds, and find himself off Plymouth the next morning at day break.

The following method, which is due to M. Littrow, is equally valuable for its simplicity and accuracy. The data required are the observed altitude of the pole star, and the corresponding apparent time at the ship, from which the latitude may be found in the following manner:

With the apparent time of observation and the longitude, find the corresponding time at Greenwich; for which take the right ascension of the sun from the *Nautical Almanac*, and add it to the apparent time at the ship. The sum rejecting 24 hours, if necessary, will be the sidereal time, or the right ascension of the meridian.

From the right ascension of the meridian subtract the right ascension of the pole star; the remainder will be the distance of the pole star from the meridian in sidereal time.

Enter the annexed table, and seek this time in one of the columns, entitled "Distance from the meridian," and take out the corresponding first correction, subtracting the annual variation if necessary; take out also the logarithm of A.

Apply the first correction with the sign at the top of the column, in which the distance from the meridian was found, to the true altitude of the pole star.

To the logarithm of A add the logarithmic tangent of the altitude; the sum will be the logarithm of a further correction in seconds; which being always added to the number before found, the result will be the latitude.

Note.—The sun's right ascension, for the corresponding time at Greenwich may be taken from the *Almanac* with sufficient accuracy for the present purpose by estimation.*

EXAMPLE.

The 23d of June, 1824, in longitude 30° west, the following altitudes were taken of the pole star, the height of the eye being 20 feet; required the latitude.

	h m s		Observed altitude. ° ′ ″		
Mean time	8 34 24	....	50 38 20		
	8 39 0	....	50 40 20		
	8 40 44	....	50 22 10		
Mean..........	8 38 3	Mean	50 33 37		
Longitude in time, add	2 0 0	Dep. of Horizon..	− 4 17		
Mean time at Greenwich	10 38 3		50 29 20		
Equation time	− 1 51	Refraction ..	− 0 48		
Apparent time at Greenwich	10 36 12	True altitude	50 28 32	tangent ..	10·0989
⊙'s right ascension................	6 10 0	1st correction....	+ 53 14	log. A ..	1·7685
Right ascension of the meridian	16 46 12		51 21 46		
Right ascension of pole star	0 58 1		+ 1 14 = 74″	Log.	1·8674
✳'s distance from the meridian	15 48 11	Latitude required	51 23 0		

25th of June, 1824, in longitude 16° west, the altitude of the pole star was observed to be 47° 5′ at 12ʰ 14ᵐ of mean time, the height of the eye being 20 feet; required the latitude.

	h m s		Observed altitude. ° ′ ″		
Mean time	12 14 0	....	47 5 0		
Longitude in time, add	1 4 0	Dep. of the horizon	− 4 17		
Mean time at Greenwich	13 18 0		47 0 43		
Equation of time	− 2 17	Refraction ..	− 0 54		
Apparent time at Greenwich	13 15 43	True altitude	46 59 49	tangent ..	10·0455
⊙'s right ascension	6 19 0	1st correction....	− 15 43	log. A ..	1·9098
Right ascension of the meridian	19 34 43		46 44 6		
Right ascension of the pole star	0 58 1		+ 1 30 = 90″	log.	1·9553
✳'s distance from the meridian......	18 36 42	Latitude required	46 45 36		

---

* The formula from which the above method is taken, is so admirably adapted to determine the latitude on shore with very great precision and convenience, that we shall here subjoin it.

Let a = the co-latitude sought.
z .. the zenith distance.
p .. the polar distance.
t .. the hour angle. Then
$a = z + p \cos t - A \text{ cotang. } z + B.$
Where $A = \frac{1}{2} p^2 \sin^2 t \sin 1''$.
and $B = \frac{1}{3} p^3 \sin^2 t \text{ cosine } t \sin^2 1''$.

Note.—In the second and third quadrant of t, the value of B is negative.

Table for finding the latitude by the altitude of the pole star.

Distance from the Meridian.				1st Correction.	Annual variation.	Log. of A″.
−	+	+	−		−	
h m	h m	h m	h m	° ′ ″	″	
0 0	12 0	12 0	24 0	1 37 48	19·45	
10	50	10	50	1 37 42	19·41	9·2009
20	40	20	40	1 37 26	19·37	9·8021
30	30	30	30	1 36 58	19·28	0·1529
40	20	40	20	1 36 19	19·15	0·4009
50	10	50	10	1 35 30	18·97	0·5921
1 0	11 0	13 0	23 0	1 34 28	18·80	0·7475
10	50	10	50	1 33 17	18·54	0·8777
20	40	20	40	1 31 54	18·30	0·9897
30	30	30	30	1 30 21	17·99	1·0871
40	20	40	20	1 28 38	17·62	1·1733
50	10	50	10	1 26 45	17·26	1·2503
2 0	10 0	14 0	22 0	1 24 42	16·83	1·3195
10	50	10	50	1 22 29	16·41	1·3819
20	40	20	40	1 20 7	15·92	1·4387
30	30	30	30	1 17 36	15·42	1·4903
40	20	40	20	1 14 55	14·90	1·5377
50	10	50	10	1 12 7	14·36	1·5809
3 0	9 0	15 0	21 0	1 9 9	13·74	1·6205
10	50	10	50	1 6 5	13·15	1·6569
20	40	20	40	1 2 52	12·51	1·6899
30	30	30	30	0 59 32	11·84	1·7205
40	20	40	20	0 56 6	11·17	1·7483
50	10	50	10	0 52 33	10·45	1·7735
4 0	8 0	16 0	20 0	0 48 54	9·73	1·7965
10	50	10	50	0 45 10	8·98	1·8173
20	40	20	40	0 41 19	8·22	1·8361
30	30	30	30	0 37 26	7·45	1·8527
40	20	40	20	0 33 27	6·66	1·8675
50	10	50	10	0 29 25	5·85	1·8803
5 0	7 0	17 0	19 0	0 25 19	5·04	1·8915
10	50	10	50	0 21 10	4·21	1·9007
20	40	20	40	0 16 59	3·38	1·9081
30	30	30	30	0 12 46	2·54	1·9139
40	20	40	20	0 8 32	1·70	1·9181
50	10	50	10	0 4 16	0·85	1·9207
6 0	6 0	18 0	18 0	0 0 0	0·00	1·9215

This Table is calculated for 1824: for any following year the annual variation must be subtracted from the first correction, and the quantity A diminished $\frac{2}{100}$ parts for each minute of diminution in the polar distance of the Pole Star. The variation of A is, however, so small that for many years it may be neglected at sea.

Of longitude.

The longitude of a place has been defined to be an "arc of the equator comprised between the meridian of that place and the first meridian." This arc may be expressed either in degrees or in time. To find the difference of longitude therefore between any two places, is to find their difference of time. When the sun is on the meridian of any place in west longitude, he has evidently passed the first meridian, and it will there be afternoon; and on the contrary, when it is apparent noon in east longitude, the sun will not have arrived at the first meridian, where, consequently, it must be before noon.

The essence of this problem then, so extremely important in nautical astronomy, consists in finding what o'clock it is at two places at the same instant.

However simple this may at first sight appear, it is attended with so many practical difficulties, that large rewards have been, and still are offered for its more perfect attainment.

The time at the ship may be readily and accurately obtained by altitudes of the sun, or of stars, taken when at a distance from the meridian. The difficulty therefore consists in finding the time for the same instant at Greenwich.

The obvious mode of effecting this, it should seem, would be to take a chronometer from Greenwich, which being previously set to the time at that place, would always indicate it when required by simple inspection. Nothing then would be necessary but to determine the time at the ship, and note the corresponding time shewn by the chronometer; the difference of which being converted into degrees, would give at once the longitude of the ship.

But though chronometers have attained a very great degree of perfection, they are still subject to so many possible variations, besides being liable to accidents, that though they are of the utmost importance at sea, it would not be safe to place an implicit reliance upon them, without the occasional check of some method less under the influence of casual derangements.

Suppose some instantaneous phenomenon were known to take place at a particular moment of Greenwich time. If this phenomenon could be observed from the ship and the time noted, the corresponding time at Greenwich being known, the longitude of the ship would be obtained. Thus, if it be known that the beginning of an eclipse of the moon is to take place at 6^h of Greenwich time, and it is observed at the ship to commence at 11 o'clock, the difference 5 hours, or 75 degrees, will be the longitude of the ship east of Greenwich.

The eclipses of Jupiter's satellites, also being calculated for the meridian of Greenwich, may be employed for the same purpose. For the Greenwich time at which the eclipse will take place being known, and the time at which it was observed at the ship, the longitude may be deduced in the same manner as by an eclipse of the moon.

Both these methods of determining the longitude are however of little value. An eclipse of the moon happens but seldom, and then it is very difficult, on account of the penumbra which surrounds the shadow of the earth, to determine the precise time of its commencement; and though Mr. Lecount, a midshipman in the British navy, has succeeded in observing the eclipses of Jupiter's satellites at sea, and has thus proved the possibility of that which was before doubtful; yet, as these eclipses are invisible, a part of the year, during the vicinity of Jupiter to the sun, they are little adapted to supply the every day exigencies of the navigator.

As the distance of the moon from the fixed stars which are situated near her path, varies about half a degree in one hour, her change of place is sufficiently rapid to become perceptible in a very small portion

	h	m	s
Sidereal time elapsed since apparent noon	8	11	25·1
Acceleration of sidereal on mean solar time for $8^h\ 11^m\ 25^s$ from table 6	0	1	20·5
Mean time elapsed since apparent noon	8	10	4·6
Equation of time for apparent noon	− 0	1	15·6
Mean time from mean noon, commonly called the hour of the day	8	8	49·0

At what hour of mean time will Regulus be upon the meridian the 1*st March,* 1823.

	h	m	s
Right ascension of Regulus, or sidereal time	9	58	56·4
Subtract ☉'s right ascension for 1st March at noon	22	46	41·9
Sidereal time elapsed since apparent noon	11	12	14·50
Acceleration of sidereal on mean solar time for $11^h\ 12^m\ 14{\cdot}50^s$ from Table 6	−	1	50·14
Mean time elapsed since apparent noon	11	10	24·36
Equation of time for apparent noon	+	12	43·40
Regulus on the meridian mean time	11	23	7·76

What is the right ascension of the meridian, or the sidereal time at $11^h\ 3^m\ 7{\cdot}76^s$ *of mean time on the* 1*st March,* 1823.

	h	m	s
Mean time from mean noon	11	23	7·76
Equation of time for preceding apparent noon	−	12	43·40
Mean time elapsed since apparent noon	11	10	24·36
Excess of sidereal time on $11^h\ 10^m\ 24^s$ of mean time	+	1	50·12
Sidereal time elapsed since apparent noon	11	12	14·48
Add the ☉'s right ascension for 1st March at noon	22	46	41·90
Right ascension of the meridian or sidereal time	9	58	56·38

To find the longitude by means of a Chronometer.

The chronometer, is an instrument so delicate in its construction, and is consequently so liable to be easily deranged, that great care should be taken to guard against every circumstance unfavourable to its accurate performance. Perhaps the best position for a chronometer would be to place it upon a shelf or tray, suspended freely from the deck of the ship. From this it should never be removed; but in observing, a watch with seconds should be used, which should be compared with the chronometer both before and after the observations. Care should be taken not to place a compass needle, or magnet, near the chronometer as it might impart magnetism to the balance and completely spoil the instrument. If it should be found necessary to move the chronometer, all sudden motion should be avoided, and particularly any circular horizontal movement, as this would accelerate or retard the vibrations of the balance. Attention should be paid to wind up the chronometer carefully, and at the same hour every day, in order that the same portion of the main spring may be constantly employed, and thus the prevalence of any partial irregularity be avoided.

It is necessary before a voyage to know the *rate* of the chronometer; that is, how much it gains or loses in a day. From this, and its ascertained difference from the time at Greenwich, at a certain date the Greenwich time may be obtained in any longitude, by multiplying the rate by the number of days that have elapsed since the error of the chronometer for Greenwich was ascertained, and applying this together with the error to the time shewn by the chronometer. Great care then is necessary in finding the rate of a chronometer, and this is best done in a fixed observatory, but as chronometers have not yet attained perfection, the rate may vary and we shall subsequently give some instructions as to the mode of examining it.

Having the observed altitude of the sun, the declination and the latitude of the place of observation, to find the hour angle or apparent time from noon, and thence if required the mean time.

Rule.

Correct the observed altitude for the depression of the horizon, refraction, parallax, and semi-diameter to obtain the true altitude of the sun's centre, from which find the zenith distance. With the apparent time at the ship nearly, and the estimated longitude, find the corresponding Greenwich time, and to this compute the sun's declination in order to obtain the polar distance. The polar distance, the co-latitude and the zenith distance will form the three sides of a spherical triangle, in which it is required to find the angle at the pole.

Add together the zenith distance, the polar distance, and the co-latitude, and take half the sum. From the half sum subtract the zenith distance (the side opposite to the required angle), then add together the complement arithmetical of the sines of the polar distance and the co-latitude (the sides containing the required angle), the sine of the half sum and the sign of the difference of the half sum, and the zenith distance: half the sum of these four logarithms will be the logarithmic cosine of half the hour angle sought, and this being doubled and converted into time will be the hour angle which, as the sun was taken, is the apparent time from noon.

If the altitude was taken before noon the horary angle must be subtracted from 24 hours, and the equation of time being applied for the corresponding Greenwich time, (if the mean time be required) the result will be the mean time at the place of observation.

Example.

On the 2*d June,* 1823, *the true altitude of the sun's centre was* 30° 2′, *when the chronometer showed* $5^h\ 1^m\ 0^s$ *the latitude being* 40° 5′ *north, and the sun's declination*

22° 9′ 17″ north. The chronometer on the 20th May preceding was 45ˢ slow for Greenwich time and its rate, losing 2ˢ·1 daily. Required the apparent time at the ship and the longitude.

		h	m	s
Time by chronometer		5	1	0
Slow the 20th April	+		0	45
Loss from the 20th April to the 2d June	+		0	25·2
Mean time at Greenwich		5	2	10·2
Corresponding equation of time	+		2	31·4
Apparent time at Greenwich		5	4	41·6

	°	′	″		
Zenith distance	59	58	0		
Polar distance......	67	50	43	co. ar. sine	0·0333098
Co-latitude........	49	55	0	co. ar. sine	0·1162768
Sum	177	43	43		
½ Sum	88	51	51	sine......	9·9999146
½ sum—zen. dist...	28	53	51	sine......	9·6841666
					2)19·8336678
	34	20	17·6	cosine..	9·9168339
			2		
Hour angle ..	68	40	35·2		
			4		
	4ʰ	34	42·3	app. time from noon	

	h	m	s
Apparent time from noon	4	34	42·3
Apparent time at Greenwich........	5	4	41·6
Longitude *west* in time	0	29	59·3
Multiply by ..			10
	4	59	53·0
½ the product..	2	29	56·5
Longitude west in degrees, &c.	7°	29	49·5

Having given the observed altitude of a fixed star, its declination and the latitude, to find the apparent time and the longitude of the ship.

Rule.

Find the hour angle or the distance of the star from the meridian as in the last example. If the star is to the *west* of the meridian, add to this the star's right ascension, but if to the east of the meridian subtract it, and the result will be the right ascension of the meridian. With the mean time at Greenwich, given by the chronometer, find the apparent time at Greenwich, and the sun's right ascension for that time. Subtract the sun's right ascension from the right ascension of the meridian, and the remainder will be the apparent time at the ship, the difference between which, and the apparent time at Greenwich deduced from the chronometer, will be the required longitude in time.

Example.

28th May, 1823, in latitude 0° 50′ north, the mean of several observed altitudes of the star Antares, when eastward of the meridian, was 30° 42′, the height of the eye being 16 feet, and the mean of the corresponding times by the chronometer 9ʰ 35ᵐ 43ˢ, which chronometer was too fast 20th April at Greenwich 1ᵐ 5ˢ at noon, and gaining daily 5ˢ·4. Required the apparent time and the longitude of the ship.

		h	m	s
Time by the chronometer............		9	35	43
Fast 20th April at Greenwich	−		1	5
Gain since the 20th April	−		3	27·5
Mean time at Greenwich		9	31	10·5
Equation of time 28th May at 9ʰ 31ˢ ..	+		3	9·2
Apparent time at Greenwich		9	34	19·7

		°	′	″
Mean of observed altitudes		30	42	0
Depression of the horizon............	−		3	50
		30	38	10
Refraction	−		1	39
Star's true altitude..................		30	36	31

	°	′	″		
Zenith distance	59	23	29		
Polar distance......	116	1	37	co. ar. sine	0·0464395
Co-latitude........	89	10	0	co. ar. sine	0·0000459
Sum......	264	35	6		
½ Sum	132	17	33	sine......	9·8690669
½ Sum—zen. dist...	72	54	4	sine......	9·9803665
					2)19·8959188
	27	29	32	cosine....	9·9479594
			2		

		h	m	s	
Star's dist. from mer. 54 59 4	=	3	39	56·4	of sid. time
Star's right ascension........		16	18	31·2	
Right ascension of the meridian		12	38	34·8	
☉'s right ascension 28th May at 9ʰ 34ᵐ 20ˢ of apparent time		4	19	17·0	
Apparent time at the ship		8	19	17·8	
Apparent time at Greenwich ..		9	34	19·7	
Longitude west in time		1	15	1·9	

In the "Tables requisite to be used with the nautical Ephemeris" a shorter method is given for calculating the hour angle, which will be found very convenient when the observer has tables of natural sines and of the "Logarithmic rising;"* it is as follows:

If the sun's declination and the co-latitude of the ship be one north and the other south, take their difference, but if they be both north or both south, take their sum for the sun's meridianal altitude. If that sum be greater than 90° take it from 180°.

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* The logarithmic rising is the logarithmic versed sine of the hour angle, but with an index of 4, in consequence of the natural sines being considered as whole numbers instead of decimals as is usual, in which case the index would have been 9. The logarithmic versed sine may therefore be used instead of logarithmic rising.

From the natural sine of the meridianal altitude, subtract the natural sine of the true altitude at the time of observation, find the logarithm of the remainder, and add it to the logarithmic secant of the latitude and the logarithmic secant of the declination, their sum will be the "logarithmic rising," and the time corresponding to it will be the hour angle in time from the meridian: or if the object be the sun, the apparent time from the nearest noon, consequently if the observation be made in the morning, the time thus found must be taken from 24 hours, and the remainder will be the apparent time from the noon of the preceding day.

Taking the first Example.

	°	′	″		
Lat. north	40	5	0	secant	0·11628
Co-latitude	49	55	0		
Declin. nor.	22	9	17	secant	0·03330
Merid. altit.	72	4	17	nat. sine 95144	
True altit.	30	2	0	nat. sine 50050	
				Difference 45094 log.	4·65412
Apparent time ..	4^h	34^m	$42^s{\cdot}2$	Log. rising..	4·80370

To find the rate of a chronometer.

The best method of determining the rate of a chronometer, is by comparing it with a good astronomical clock at a fixed observatory. The rate has hitherto been usually given by the maker, but a depot has lately been established at the Royal observatory at Greenwich, on the recommendation of the Board of Longitude, for the chronometers belonging to the Navy, where their rates are determined by the Astronomer Royal. This depot is also open to all chronometer makers, on application at the Admiralty, and superior artists are thus furnished with an opportunity of bringing the merits of their work before the public; and of obtaining from the Board of Longitude a reward for any improvement, *in principle,* attended by practical advantage.

But, however well the rate of a chronometer may be ascertained in the first instance, it is liable to vary from many causes, and particularly unless great care be taken, from the motion in transporting it from the observatory to the ship. Its rate should therefore be examined whenever an opportunity occurs. The most simple method of doing this, when the ship is likely to be stationary for some time, is to determine the error of the chronometer by means of numerous observations of the sun's altitude and the corresponding times, taken when the sun is not less than about 3 hours from the meridian. Having found the error of the chronometer from mean time, this error may be compared with that determined at an interval of not less than a week, and the daily rate may thence be found by simple proportion. The observations may be continued daily, but the rate should not be deduced from a less interval than a week, otherwise it might be too much affected by the unavoidable errors of observation.

If the vessel should be in harbour for some time, observations of the altitude of the sun may be made by means of an artificial horizon, and the error of the chronometer, and consequently its rate, thence determined with much greater accuracy than it could be by employing the horizon of the sea. In using the artificial horizon (which is a surface of quicksilver protected from the wind by a covering of plane glass) the images of the sun's limb are brought into contact, and it will conduce much to accuracy to observe the upper and lower limb alternately, and to take the mean. The result (to which the sextant error if any, must be applied) will be double the apparent altitude of the centre. From the half of this, the difference between the parallax and refraction being subtracted, the remainder will be the true altitude of the sun's centre.

The best method when on shore of determining the rate of a chronometer, is by a portable transit instrument. For this purpose it is by no means necessary that the instrument should be accurately in the meridian, but then the observations must be confined to the stars. The axis, however, should be carefully levelled, and the middle wire (which must be truly vertical) adjusted to a certain mark previous to each observation. It is not *absolute,* but *comparative* time that is required. Now if the time of the passage of a certain star * should be observed every night by the chronometer, and the chronometer keep mean time, should lose $3^m\ 55^s{\cdot}91$ between any two successive transits of the same star. If the loss is less than this, the chronometer is *gaining* the difference daily: if on the contrary, the loss is more than 3′ 55·91″, the difference is the rate which the chronometer *loses* daily. The observations should be continued for at least a week, and the mean taken for the rate. It must be remarked that this rate is for a sidereal day, but may be readily found for a solar day, if thought necessary, by simple proportion.

Having the apparent time from noon, or hour angle, at the ship, and the estimated longitude, to find the true and the apparent altitude of the sun's centre.

This and the following problems may be found useful for computing the altitude of the sun, the moon, or a star, when circumstances have not permitted them to be obtained by observation at the time of taking lunar distances.

RULE.

Find the apparent time at Greenwich, for which compute the sun's declination, with the declination and co-latitude find the meridian altitude as directed in the last article. Then add together the logarithmic rising of the hour angle, or time from noon, the logarithmic cosine of the latitude of the ship and the or their difference, should the polar distance exceed 90°, will be the natural sine of the altitude, and its logarithm of course the logarithmic sine.

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* The reader is supposed to be acquainted with the mode of observing the transit of a star. The telescope is usually furnished with 3 or 5 wires, and the time of the star passing each being noted, the mean of the whole will give the time of its passing the middle wire. It is a good practical check to examine whether the mean of any two wires, equally distant on each side of the centre, gives the same result as the middle wire.

EXAMPLES.

Let the ship's latitude be 40° 5′ north, the sun's declination 22° 9′ 17″ north, and the apparent time, or hour angle 4h 34m 42s = 68° 40′ 35″. *Required the true altitude.*

	°	′	″		
Co-lat. ..	49	55	0	cosine 9·80882	sine .. 9·88372
Polar dist.	67	50	43	cosine 9·57647	sine .. 9·96671
Hour angle	68	40	35	....	cosine 9·56066

,24283 log. 9·38529

,25768 .... log. 9·41109

Sum........ ,50051 log. 9.69941 log. sine 30 2′ the altitude required.

The ship's latitude being 0° 50′ *north, the declination of Antares* 26° 1′ 37″ *south, and the hour angle, or distance of the star from the meridian,* 3h 39m 56s = 54° 59′ 4″ ; *required the true altitude.*

Co-latit. .	89	10	0	cosine 8·16268	sine 9·99995
Polar dist.	116	1	37	cosine 9·64226	sine 9·95356
Hour ang.	54	59	4	....	cosine 9·75876

,00638 log. 7·80494

,51555 .... log. 9·71227

Difference ,50917 log 9·70686 log. sine of 30° 36′ 30″ the altitude.

TO FIND THE LONGITUDE BY LUNAR OBSERVATIONS.

The method of finding the longitude at sea by lunar distances is the most perfect of any at present known. The rate of a chronometer may change, or it may suffer derangement from accidental causes, but the time marked by the varying distance of the moon from the sun, or from the fixed stars, is liable to no irregularities but such as are subject to laws well ascertained, and the effect of which can be computed, and nothing is wanting to the accuracy of the result but good instruments for taking the distances, and careful observers.

The operation depends upon the solution of two spherical triangles, in one of which three sides are given to find an angle, and in the other, two sides and the included angle to find the third side.

In plate 1, fig. 8, let Z be the zenith, Z ☽, the apparent zenith distance of the moon, Z ⊙, that of the sun and ☽ ⊙, the apparent distance of their centres.

Now as the parallax of the moon is always greater than the refraction, the true altitude of the moon, or that seen from the centre of the earth, will be greater than her apparent altitude, and the parallax of the sun being less than the refraction, his true altitude will be less than the apparent altitude. Let *m* be the true place of the moon, and *s* that of the sun. Then Z *m* will be the moon's true zenith distance, Z *s* the true zenith distance of the sun, and *m s*, the true distance of their centres.

In the triangle Z ☽ ⊙, we have given the three sides, viz. the two apparent zenith distances, and the apparent distance of the centres to find the angle ☽ Z ⊙; or the angle at the zenith. And then in the triangle *m* Z *s*, are given the two sides Z *m* and Z *s*, (the true zenith distances) and the included angle *m* Z *s*, from which the side *m s*, the true distance of the centres may be computed.

Three observers are usually employed in finding the longitude by the lunar method, one to take the distance of the limbs of the sun and moon, or of the limb of the moon from a star, and two others to take their altitudes at the same instant. A fourth observer should be added to mark the time by a chronometer, and to note it down together with the corresponding observed distance and altitudes. Several observations should be thus made in succession as quickly as can be done consistent with accuracy, and the means being taken we shall have the observed distance of the limbs of the sun and moon, their apparent altitudes and the corresponding time by the chronometer.

If the sun, or a star, be at a sufficient distance (about three hours) from the meridian, the apparent time at the ship may be computed from the altitude corresponding to the distance. But if it is too near the meridian, a watch is indispensable. The error of the watch should be inferred from altitudes taken specially for that purpose, as near as may be to the time when the distances are observed, and the observed time by the chronometer, after having applied the error, must be reduced to the meridian, under which the lunar distances are taken, by adding to it the difference of longitude in time since the observation of the altitudes, if the ship has made east longitude, or subtracting it if she has been sailing westward.

But as there may not be always a sufficient number of observers, the following method given in Norie's *Navigation* will be found very convenient where there are only two.

Let the observations be taken in the following order, noting the times by a watch: 1. the altitude of the sun or star; 2. the altitude of the moon; 3. any number of distances; 4. the altitude of the moon; 5. the altitude of the sun or a star. Now add together the distances, and the times when they were taken, each of which, being divided by the number observed, will give the mean time and distance: then to reduce the altitudes to the mean time, say, as the difference of times between the observations is to the difference of their altitudes, so is the difference between the time that the first altitude was taken and the mean time, to a fourth number; which added to, or subtracted from the first altitude, according as it is increasing or decreasing, will give the altitude reduced to the mean time.

Of the oblique semi-diameter.

As the vertical semi-diameters of the sun and moon are diminished by refraction, it becomes necessary to find their semi-diameters in the direction of the observed distance in order to obtain the apparent distance of their centres. Dr. Thomas Young in his comparative view of the principal methods of correcting lunar observations before mentioned, has given two very convenient tables for this purpose which we have taken the liberty to copy, and we shall add an account of them nearly in the author's own words.

"The observed distance requires to be corrected, when the altitudes are small for the effect of refraction on the oblique semi-diameters, which may amount to several seconds, especially in low latitudes, where the ecliptic approaches most frequently to a vertical position. The magnitude of this correction depends on the angle of the triangle concerned at the luminary in question, which is required to be known when we employ the tables of this correction which have been published. It will, however, be more convenient to have an easy method of deducing the correction more immediately from the elements observed; and for this purpose the minutes and seconds may be neglected, the nearest degrees only being employed for entering the tables. The first table affords by adding together the numbers taken from it with different arguments, a logarithm of the sine of half the angle at the luminary, multiplied by the cosine of the altitude, and the second table being entered with this argument (A), and with the altitude of the luminary gives the correction which is to be subtracted from the semi-diameter to obtain the oblique semi-diameter.

"The arguments of the first table are d, the distance, or its supplement, h, the half sum of the distance and both altitudes and $h-s$, the same half sum diminished by the altitude of the *opposite* luminary."

USE OF THE TABLES.

EXAMPLE I.

Let h, $= 32°$, $h-s = 21°$, $d = 44°$, and the altitude 11°. Entering table 1 in the column "For h," opposite 32° we find 93, and opposite 21°, in the column $h-s$ on the left hand, we have 55; again with 44° in the column h, we find 16. The sum of these three numbers is 164, with which entering table 2, in the column "sum of A," and under 11° we find O, shewing that the correction required is less than one second.

EXAMPLE II.

Let the ☽'s altitude = 41°, the sun's altitude = 19°, and their distance 104°, required the correction for the oblique semidiameters.

First, h, $=82°$, $h-s = 63°$, $d = 76°$, (the supplement to 104°), and the ☽'s altitude 41°. Then $14+95+1 = 110$, with which, entering table 2, under 41°, we find 0, implying that no correction is necessary to the ☽'s semi-diameter.

Next for that of the sun, $h = 82°$, $h-s = 41°$, $d = 76°$ and the ⊙'s altitude 19°. Then $14+82+1 =97$; with which, entering table 2, under the altitude 19°, we find 1″ the quantity to be subtracted from the ⊙'s semi-diameter.

EXAMPLE III.

Let the ☽'s altitude be 8°, that of the ⊙ 36°, and the distance 31°. Then for the correction of the ☽'s semi-diameter we have, $h= 37°$, $h-s = 1°$, $d=31°$, and the ☽'s altitude 8°, which from table 1 we obtain $90+924+29=1043$ or 43 rejecting the 1000, and from table 2, with 43, and the ☽'s altitude 8°, we get 10″, the correction to be subtracted from the ☽'s semi-diameter.

For the sun we have $h = 37°$, $h-s = 29°$, $d=31°$, and the ⊙'s altitude 36°, with which we get from table 1, $90+69+29=188$, and from table 2, 1″ for the diminution of the ⊙'s semi-diameter.

Having the observed distance of the limb of the moon from that of the sun, or from a fixed star, together with their apparent altitudes, the apparent time at the ship, the latitude and the estimated longitude to find the true distance of their centres.

RULE.

With the apparent time at the ship and the estimated longitude, find the apparent time at Greenwich, for which take from the *Nautical Almanac* the horizontal semidiameters of the sun and moon, and the ☽'s horizontal parallax.

Find the apparent and the true altitudes of the centres in the manner which has been before fully explained.

To the observed distance apply the semi-diameters of the sun and moon, that of the moon being previously corrected for augmentation and both reduced to the oblique semi-diameters. The result will be the apparent distance of the centres.

Add together the apparent distance and the apparent zenith distances, and subtract each of the two latter from the half sum, noting the remainders.

Add together the sines of these two remainders, the arithmetical complements of the sines of the apparent zenith distances, and the sines of the true zenith distances.

From half the sum of these six logarithms subtract the sine of half the difference of the true zenith distances, and the remainder is the tangent of an arc.

Subtract the sine of this arc from the said half sum, and the remainder is the sine of half the true distance.

Example.

Let the latitude be 35°; the observed distance of a star from the moon's nearest limb, 30° 57′ 12″; the observed altitude of the lower limb, 8° 13′ 50″; and that of the star, 35° 43′ 50″; the height of the eye being 16 feet, the barometer 28·7 inches, and Fahrenheit's thermometer 78°; the moon's horizontal semi-diameter at the time being 16′ 22″. and the horizontal parallax, 60′ 0″. Required the true distance.

	°	′	″
Observed altitude of ☽'s L. L. ……	8	13	50
Depression of the horizon (table 1.) ..	−	3	50
	8	10	0
☽'s horizontal semi-diameter ……	+	16	22
Augmentation (table 2.) ……	+	0	2
Diminution of vertical semi-diameter..	−	0	11
Apparent altitude, ☽'s centre ……	8	26	13
Refraction ……	−	6	17
Barometer ……	+	0	35
Thermometer ……	+	0	56
Parallax in altitude ……	+	59	18
☽'s true altitude ……	9	20	45

	°	′	″
☽'s semi-diameter ……	0	16	24
Correction for obliquity, (tables 1 & 2.)	−	0	10
Oblique semi-diameter ……	0	16	14
Observed distance ……	30	57	12
Apparent distance of ✳ from ☽'s centre	31	13	26

	°	′	″
Star's observed altitude ……	35	43	50
Depression of horizon ……	−	3	50
Star's apparent altitude ……	35	40	0
Refraction ……		1	19
Barometer ……	+	0	3·5
Thermometer ……	+	0	4·5
Star's true altitude ……	35	38	49

☽'s equal parallax .. 60′ 0″
Reduction for latitude (table) …… − 4

Horizontal parallax ..	59 56	P. log. ..	10·4776
☽'s altitude 8° 20′		cosine ……	9·9954
Parallax in altitude, 59′ 15″		p log. ….	·4883

	°	′	″		
Apparent distance ……	31	13	26		
☽'s apparent zenith distance ….	81	33	47	co. ar. sine ….	0·0047256
✳'s apparent zenith distance ….	54	20	0	co. ar. sine ….	0·0902179
sum ….	167	7	13		
½ sum ..	83	33	36		
1st remainder ……	1	59	49	sine ….	8·5421554
2d remainder ……	29	13	36	sine ….	9·6886564
☽'s true zenith distance ……	80	39	15	sine ….	9·9941966
✳'s true zenith distance ……	54	21	11	sine ….	9·9098894
				sum ….	38·2298413
				½ sum ….	19·1149206
½ diff. true zenith distances ……	13	9	2	sine ….	9·3570017
Tangent of arc ……	29	47	57	……	9·7579189
Same arc ……	29	47	57	sine ….	9·6963226
½ true distance ……	15	11	58	sine ….	9·4185980
True distance ……	30	23	56		

The preceding method is that given by Dr. Kelly, in his excellent *Practical Introduction to Spherics*. It has been preferred to that of Borda, from sines being employed instead of cosines; as the former in consequence of their increasing instead of decreasing, are perhaps more readily taken from a table than the latter. It is the *direct* method of solution, and the result is perfectly accurate, if the data are correct.

It may here be proper to urge the necessity of an habitual attention to those commonly supposed minor corrections, which have hitherto been neglected, especially those depending on the barometer and thermometer; the omission of which would, in the present instance, have occasioned an error of upwards of one minute in the moon's distance, corresponding to more than 30 minutes of longitude.

As the direct method requires the logarithmic sines, &c. to be taken out to every second, various approximate methods of solution have been devised. Most of these, however, have depended upon tables calculated for the purpose, and many are very inconvenient in practice. The two following are free from these objections, and possess the peculiar, and very desirable, advantage of the result being but little influenced by errors in the observed altitudes. The author of the first is not known to the writer of this article; but the second method may be found in the Journal of the Royal Institution, among the valuable astronomical and nautical collections, which are believed to be due to Dr. Thomas Young.

In the first method, the author observes that it is preferable to attend less to abbreviate the methods of computation, than to investigate a solution, in which considerable errors in the data may produce but a small error in the required distance. When either of the luminaries has a small elevation, its altitude will be affected by the uncertainty of refraction, and the inaccuracy necessarily attending altitudes observed at sea. The sum of these errors, when they tend the same way, may be supposed to amount to at least one minute in altitude; which, in many cases, according to the common rules for computing the true distance will produce an error of about 30 minutes in the longitude. Perhaps the only method of determining the distance, so as not to be affected by the errors of altitude, is that, by first finding the angles at the sun and moon, and by the help of them, the corrections of distance for parallax and refraction.

Rule.

1. Add together the moon's apparent zenith distance, the sun's apparent zenith distance, and the apparent distance of their centres. From the half sum subtract the moon's zenith distance, and call it the first remainder. From the half sum subtract the star's zenith distance, and call it the second remainder. Add the arithmetical complements of the sines (or the co-secants) of the sun's zenith distance, and the apparent distance of the centres, to the sines of the half sum, and of the first remainder: half the sum of these four logarithms is the cosine of half the angle at the sun.

2. Add the arithmetical complements of the sines (or the co-secants) of the moon's zenith distance, and the apparent distance of the centres, to the sines of the half sum, and of the second remainder. Half the sum of these four logarithms is the cosine of half the angle at the moon.

3. To the cosine of the angle at the moon add the logarithm of the difference between the moon's refraction, and her parallax in altitude in seconds. The result will be, the logarithm in seconds of the first correction, which is to be subtracted from the observed distance, if the angle at the moon is acute, but added if it is obtuse.

4. To the cosine of the angle at the sun, add the logarithm of the difference between the sun's refraction and parallax in seconds. The sum is the logarithm in seconds of the second correction. To be added when the angle at the sun is acute, but subtracted when obtuse.

5. To the sine of the angle at the moon, add the beforementioned logarithm of the difference between the moon's parallax and refraction, and multiply the sum by 2. From this subtract the logarithm of twice the distance of the centres in seconds increased by the radius. The result will be the logarithm in seconds of a third small correction, which is also to be added.

Note.—The third correction is generally very minute, except when the distance is small, and the angle at the moon nearly a right angle.

It is sufficient to use the zenith distances, and the apparent distances to the nearest minute only, as a small error in the angles at the sun and moon will very little affect the corrections of distances. If a star is observed, the only change necessary is to employ the refraction of the star, instead of the difference between the refraction and parallax of the sun.

Examples.

Taking, as in the last example, the moon's zenith distance, 81° 34′; *the star's zenith distance,* 54° 20′; *and the apparent distance,* 31° 13′ 26″; *the difference between the* ☽*'s parallax in altitude and refraction,* 54′ 32″; *and the star's refraction,* 1′ 11″; *to find the true distance.*

```
    For ☽'s angle.                              °  ′        For ✱'s angle.
10·00472 sin. co. ar..... ☽'s zen. dist..... 81 34
                          ✱'s zen. dist..... 54 20   sine co. ar. ......10·09022
10·28523 sine co. ar..... Distance ......... 31 14   sine co. ar. ......10·28523
                                            ------
                          Sum...............167  8
 9·99726 sine............ ½ sum ............ 83 34   sine ............  9·99726
                          1st rem. .........  2  0   sine ............  8·54282
 9·68875 sine............ 2d rem. .......... 29 14                     ---------
--------                                                               18·91553
19·97596                                                               ---------
--------                                                   cosine ..   9·45776
 9·98798 cosine
 ½ angle..13° 25′                                      ½ angle..73° 20′
               2                                                     2
           -----                                              --------
           26  50  ☽'s ∠ sine 9·65456                          146  40  ✱'s ∠
     cos.......9·95052                                   cos. ....  9·92194
   3272″ log...3·51481 .... 3·51481                     71″ log. ....  1·85126
                -------    --------                                 -------
  −2920″ log...3·46533     13·16937                − 59″·3 log. ....  1.77320
               -------            2                                 -------
                           --------
                           26·33874    Apparent distance ......  31° 13′ 26″
                   ″                   First correction........  —   48  40
Dist. × 2 = 224880 log.... 15·35195    Second correction......  —    0  59·3
                           --------    Third correction  ......  +    0   9·7
        + 9″·7 log....      0·98679                              -----------
                                          True distance......   30  23  56·4
```

This result is the same as that obtained by the *direct* method.

Let the moon's apparent zenith distance be 80° 22′; the star's apparent zenith distance, 78° 43′; and the apparent distance of the centres, 43° 36′; the difference between the moon's parallax in altitude and refraction, 58′ 10″, (= 2910″); and the star's refraction, 4′ 40″, (= 280″); required the true distance.

```
     For ☽'s angle.                                        °  ′        For ✱'s angle.
 10·00617 sine co. ar..... ☽'s apparent zenith distance ..  80 22
                          ✱'s apparent zenith distance ..  78 43   sine co. ar....... 10·00848
 10·16139 sine co. ar..................Distance   ......  43 36   sine co. ar.  .... 10·16139
                                                          ------
                                     Sum ..........       202 41
  9·99145 sine.......................½ sum..........     101 20   sine............   9·99145
                                     1st rem.........      20 58   sine............   9·55367
  9·58497 sine.......................2d rem..........      22 37                     --------
 --------                                                                            19·71499
 19·74398                                                                            --------
 --------                                                          cosine..........   9·85749
  9·87199 cosine
 ½ angle.. 41° 52′                                                 ½ angle..43° 55′
                2                                                               2
           ------                                                         -------
          83  44′ ☽'s ∠ sine.... 9·99740                                  87  50  ✱'s ∠
     cos..... 9·03805                                               cos.... 8·57757
  2910″ log. .. 3·46389.......... 3·46389                          280″ log. .. 2·44716
               -------          --------                                  -------
  −318″ log. .. 2·50194          13·46129                         +10″·5 log. .. 1·02473
               -------                 2                                  -------
                                 --------                                           °  ′  ″
                                 26·92258    Apparent distance ..............  43 36  0
  Dist. × 2 = 313920″ log.........15·49679    First correction ...............  −  5 18
                                 --------    Second correction ..............  + 10  5
           + 26″·6 ........ 1·42579    Third correction  ..............  + 26  6
                                                                           ---------
                                                  True distance..........  43 31 19·1
```

The second method which is taken from the Journal of the Royal Institution, is as follows:

RULE.

1. Add together the apparent altitude of the star, the apparent altitude of the moon, and the apparent distance; and from half the sum subtract the apparent altitude of the star, and call the result the remainder.

2. To the proportional logarithm of the moon's horizontal parallax, add the logarithmic secant of the moon's altitude; the sum will be the proportional logarithm of the parallax in altitude.

3. Add together the logarithmic secant of the half sum, the sine of the apparent distance, the co-secant of the remainder, the constant logarithm 9·6990, and the proportional logarithm of the horizontal parallax. The sum will be the proportional logarithm of the diminution of parallax; the difference between which, and the parallax in altitude, is the *parallactic correction*, which is to be subtracted from the observed distance, if the parallax in altitude is the greater; but if otherwise, to be added.

4. To the logarithmic sine of the moon's altitude, add the proportional logarithm of the parallactic correction, and subtract the sum from the proportional logarithm of the horizontal parallax. The difference will be the tangent of an angle, which is the refractional distance for the moon. Consider, as zenith distances, both this angle, and the difference between it and the apparent distance; and find the corresponding refractions in a table of refraction. These will be the first and second refractional corrections; the first of which is to be applied to the apparent distance, with a contrary sign to that of the parallactic correction, and the second to be added when the refractional distance for the moon is less than the observed distance; otherwise the second refractional correction must be subtracted.

5. Add together, the proportional logarithm of the diminution of parallax, that of the sum of the parallactic correction and the parallax in altitude, the logarithmic tangent of the apparent distance, and the constant logarithm 1·5870. The sum will be the proportional logarithm of the correction for obliquity, which is always additive.

EXAMPLE.

Taking, as in the last example, the apparent distance, 43° 36′; *the apparent altitude of the moon,* 9° 38′; *that of the star,* 11° 17′; *the moon's horizontal parallax,* 54′ 42″; *and its proportional logarithm,* ·5173: *to find the true distance.*

1	2	3	4	5
	Parallax in altitude.	Parallactic correction.	Refractional correction.	Cor. for obliquity.
° ′ ″				
✳'s apparent altitude 11 17		const. log. . 9·6990		const. log. 1·5870
☽'s apparent altitude 9 38	secant ....0·0612	....	sine ....9·2236	....
Apparent distance . . 43 36	....	sine ......9·8386	....	Tangent 9·9788
Sum........64 31				
½ sum 32 15	....	secant ....0·0728	....	....
Remainder . . 20 58	....	co-secant . . 0·4462	....	....
Horizontal parallax 54 42	P. log. ....0·5173	P. log. ....0·5173	....	....
Parallax in altitude. .	53′55″P.log.0·5785			
	Dimin. of parallax	48′ 1″P.log.0·5739	....	P. log. . . 0·5739
	Parallax in altitude	53 55		
	Diff. = parallactic correction −	5 54	P. log. ......1·4844	
	Parallactic correction + Parallax in altitude	59 49	....	P. log. . . 0·4784
			Sum 0·7080	
			Hor.par. P.log. 0·5173	
	Refractional distance for ☽	32° 48′	Tang........9·8093	
	Apparent distance	43 36		26″ P.log. 2·6181
	Difference . .	10 48		

For zenith distance 32° 48′ tab. refraction = 37″·5 } Refractional corrections.
.... 10 46 .... = 11·0 }

		°	′	″
Apparent distance		43	36	00
Parallactic correction . .	−		5	54
1st refractional correction	+		0	37·5
2d refractional correction	+		0	11·0
Correction for obliquity . .	+		0	26·0
True distance		43	31	20·5

Having the true central distance between the moon and the sun, or a fixed star, and the apparent time at the ship; to find the longitude.

RULE.

Among the true distances of the moon's centre from the sun, or fixed stars, in the *Nautical Almanac*, find those two distances, which are next less and greater than the true given distance, which true distance place under it. Take the difference between the true distance and the first of the two distances taken from the *Almanac*, and also the difference between the two distances. From the proportional logarithm of the first difference subtract the proportional logarithm of the second difference; the remainder will be the proportional logarithm of a portion of time; which, added to the time corresponding to the first of the two differences taken from the *Nautical Almanac*, will give the apparent time of observation at Greenwich.

Take the difference between this time and the apparent time of observation at the ship, which, being converted into degrees, will give the longitude of the ship; east, if the time at the ship be greater than that at Greenwich, otherwise west.

EXAMPLES.

Let the true distance of the moon's centre from α Aquilæ, on the 1st of March, 1823, *be* 82° 25′ 41″; *the corresponding apparent time at the ship being* 12h 4m 16s: *required the longitude.*

	° ′ ″	Diff. ° ′ ″	
True distance......	82 23 41	0 59 4	P. log. 4839
Distance at night, from *Naut. Alm.*	83 34 35	1 0 39	P. log. 4724
Dist. at 15h from Do.	82 14 6		

	h	m	s	
Time from midnight	2	55	18	....P. log...0115
	12	0	0	
App. time at Greenwich	14	55	18	
App. time at the ship....	12	4	16	
Longitude west in time..	2	51	2	= 30° 13′ 15″

The true distance of the moon's centre from Aldebaran the 20th of March, 1823, *was* 35° 50′ 24″; *the corresponding apparent time at the ship being* 7h 5m 21s. *What was the ship's longitude?*

	°	′	″	Diff.	
True distance	35	50	24		
Dist. at 6h from *Naut. Almanac*	35	44	33	0° 5′ 51″	P.log. 1·4881
Dist. at 9h from Do.	37	26	15	1 41 42	P.log. 0·2480

	h	m	s	
Time from midnight....	0	10	21	....P. log. 1·2401
	6	0	0	
App. time at Greenwich	6	10	21	
App. time at the ship ..	7	5	21	
Longitude east in time..	0	55	0	= 13° 45′ 0″

VARIATION OF THE COMPASS.

To find the variation of the compass by observed magnetic amplitude of the sun, or of a star.

The solution of this problem depends upon finding the hypothenuse of a right angled spherical triangle, in which one side (the declination of the sun or star) and an angle (the co-latitude) are given.

RULE.

(From Kerrigan's *Young Navigator's Guide.*)

From the logarithmic sine of the declination, the index being increased by 10, subtract the logarithmic cosine of the latitude, the remainder is the logarithmic sine of the true amplitude, to be reckoned north or south according to the name of the star's declination; then, if the true and observed amplitudes be both north, or both south, their difference is the variation; but if one is north and the other south, their sum is the variation. And to know if it be east or west, let the observer look directly towards the compass representing the true amplitude; then, if the observed amplitude is to the left of the true, the variation is easterly; but if to the right, it is westerly.

Note.—The magnetic bearing of the object must be taken when its apparent altitude is equal to the depression of the horizon.

EXAMPLES.

The latitude being 18° 40′ *north, the star Sirius was observed to set* 24° 10′ *to the southward of the west point of the compass. Required the variation.*

Declination of Sirius..	16° 27′ S	sine ..	19·45206
Latitude............	18 40	cosine	9·97653
True amplitude....W	17° 23′ S	sine ..	9·47553
Magnetic amplitude W	24 10 S		

Variation ...... 6 47 which is east, because the observed amplitude is to the left hand of the true.

In latitude 48° 20′ *north, the star Rigel was observed to set* 9° 50′ *to the northward of the west point of the compass; required the variation.*

Declination of Rigel....	8° 25′ S	sine ..	19·16545
Latitude	48 20	cosine	9·82269
Star's true amplitude W	12° 43′ S	sine ..	9·34276
Magnetic amplitude ..W	9 50 N		

Variation ...... 22 33 which is west, because the observed amplitude is to the right hand of the true amplitude.

To find the variation of the compass, having the true altitude of the sun or of a fixed star, and its magnetic azimuth, the latitude of the ship and the declination being known.

To compute the true azimuth, we have given the three sides of a spherical triangle; viz. the co-latitude, the zenith distance, and the polar distance, to find the angle at the zenith between the zenith distance and the co-latitude.

RULE.

Add together the polar distance, the zenith distance, and the co-latitude, and take half the sum; the difference between which and the polar distance call the remainder.

To the arithmetical complements of the sines (or the co-secants) of the zenith distance and co-latitude, add the sines of the half sum and of the remainder. Half the sum of these four logarithms will be the cosine of half the true azimuth to be reckoned from the north in north latitude, but from the south in south latitude.

Then if the true and observed azimuths are on the same side of the meridian, their difference is the variation; but if on different sides, the sum is the variation. Now, if the observed azimuth is to the left of the true, the variation will be easterly; but if the right, the variation will be westerly.

EXAMPLE.

In latitude 48° 50′ *north, the true altitude of the sun's centre was* 22° 2′; *the declination being* 10° 12′ *south, and its magnetic bearing north* 161° 32′ *east. Required the variation.*

Polar distance	100 12		
☉'s zenith distance ..	67 58	co. ar. sine..	0·03294
Co-latitude	41 10	co. ar. sine..	0·18161
sum	209 20		
½ sum......	104 40	sine	9·98561
Remainder..	4 28	sine	8·89142
			19·09158
	69° 26′	cosine..	9·54579
	2		
☽'s true azimuth ....N	138 52 E		
Magnetic or observed azimuth ....N	161 32 E		

Variation ...... 22 40 which is west, because the observed azimuth is to the right of the true.

THE INSTRUMENTS EMPLOYED IN NAUTICAL ASTRONOMY.

THE SEXTANT.

sextant is an instrument so well known, that
iled account of it would be superfluous; as an
nation of the instrument itself for a few minutes
robably convey a better idea of its construction,
he most elaborate description.

object is reflected from the index-glass to the
n-glass, and from thence to the eye. The
n-glass is half silvered, and a telescope being
ly directed towards it, and parallel to the sex-
he observer is enabled to see the horizon of the
rough the unsilvered part of the glass, at the
time that he views the reflected image of the
in that part which is silvered. By moving the
and consequently the index-glass, the reflected
appears to change its place, and may be made
ntly to coincide with the horizon of the sea.
glasses are placed so that they may be inter-
between the index-glass and the horizon-glass,
ervations of the sun; and there is a screw at
ck of the sextant acting upon the telescope, by
it may be brought nearer to, or removed fur-
rom the instrument, for the purpose of rendering
flected image more or less bright.

glasses of a sextant should be perfectly plane,
e surfaces of each parallel. In order to examine
lace the dark glasses before the horizon-glass
to prevent seeing through it, and interpose as
dark glasses as may be found necessary to
n sufficiently the reflected image of the sun.
the index at zero. Direct the telescope to the
nd make the image as distinct as possible, by
g the eye piece, when the image should appear
nd well defined: shift the index to 120°, keep-
e sun in the telescope during the motion of the
If now the image of the sun appears sharp,
, and well defined, the surfaces of each glass are
el; but if the edge appears indistinct in any
and a projection like a faint second image is
the glasses are bad, and the instrument should
ected.

Adjustments of the sextant.

adjustments of a sextant are to set the index
orizon glasses perpendicular to the plane of the
ment, and their planes parallel to each other
the index division is at O, upon the arch. Also
the axis of the telescope parallel to the plane of
strument.

t the index-glass perpendicular to the plane of the sextant.

ve the index forward to about the middle of the
then holding the instrument with the limb from
bserver, look obliquely into the index-glass, so
see the arch by reflection; and remark if the
ted and the true arch appear in the same plane.
y do, the index-glass is perpendicular to the plane
instrument; but if otherwise, it is to be rectified
eans of the screws at the back of the glass.

the horizon-glass perpendicular to the plane of the sextant.

ew on the dark glass to the eye-piece of the
ope, and directing it to the sun, make the two images pass over each other. If they do not appear perfectly to coincide in passing, turn the screw, which is under the horizon-glass, till this is effected. The same adjustment may be made by means of a star.

To find the index error.

When the index is at O, the images seen by reflection and by direct vision should coincide. In the best sextants there is no adjustment for this purpose, but the error is to be found and to be applied to the observed angle. The best method of determining the index error is to measure the diameter of the sun by moving the index forward and backward on the arch. Half the difference of these diameters will be the index-error, which must be subtracted from the observations when the diameter taken forward is the greater, otherwise it must be added. This correction should be frequently examined.

To set the axis of the telescope parallel to the plane of the sextant.

If the axis of the telescope is not parallel to the plane of the sextant, it may occasion a considerable error when the angle is large. In order to make this adjustment, place the wires which are in the telescope parallel to the plane of the instrument. Then select two well defined objects, of which the best are the moon and a star, the angle between which must not be less than 90 degrees. Bring their images exactly into contact at the wire which is nearest to the sextant; then, by altering a little the position of the instrument, bring them upon the wire furthest from the sextant. If the images still continue in contact, no adjustment is required; but if they appear to separate at the wire which is furthest from the instrument, it is a proof that the object end of the telescope is too near to the sextant, and it must be altered by means of the screws in the ring which holds the telescope.

By repeating this operation a few times, the contact will remain perfect upon both wires, and consequently the axis of the telescope will be parallel to the plane of the sextant.

USE OF THE SEXTANT.

(From Norie's *Navigation.*)

To observe the angular distance between the moon and the sun.

"Screw on the telescope, and place the wires parallel to the plane of the instrument; then turn down one or more of the dark glasses according to the brightness of the sun; place the index at O on the limb, and if the sun's image be very bright, turn up the dark glasses before the index-glass, and with the screw raise the telescope to the transparent part of the horizon-glass. Having done this, hold the sextant so that the plane may be in the right line joining the two objects. If the sun be to the right hand of the moon, the sextant is to be held with its face upwards; but if it be to the left hand, the face is to be held downwards. With the instrument in this position, look directly at the moon through the telescope, and move the index forward till the sun's image comes nearly in contact with the moon's nearest limb. Then fix the index, by the screw under the sextant, and make the contact perfect by means of the adjusting screw; at the same time move the sextant slowly, making the axis of the telescope the centre of motion, by which the objects will pass each other, and the contact be more accurately made. The index will then point out the observed distance of the nearest limbs of the sun and moon.

It will perhaps be more easy for those who are not accustomed to make observations, to find the distance nearly; and setting the index forward to it, to look directly towards the moon, holding the instrument as before; the sun will then appear nearly in contact with it, and is to be made perfect by the method above mentioned. In the *Nautical Almanac* the distances of the sun and moon are set down for the beginning of every third hour of time at Greenwich, on such days as the moon is not less than 35 degrees, nor more than 125 degrees from the sun, and may be found for any intermediate time by proportion. From these distances, their distance at the time of observation may be computed roughly thus:—Find the time at Greenwich nearly; then find by the *Nautical Almanac* the distance nearly at that time, from which subtract 30 minutes, the sum of the semidiameters nearly, and the remainder will give the distance of their nearest limbs at the time of observation.

It will save some trouble, and serve the purpose of finding the reflected image of the sun or moon in the horizon-glass, if you only set the index to the central distance as set down in the *Nautical Almanac* for the nearest three hours without correcting it, to the intermediate time by a rough computation.

To observe the distance between the moon and a star.

Turn down the lightest skreen before the horizon-glass, and direct the telescope to the star, holding the sextant in its proper position; so that if seen edgewise, it may appear to pass through the moon and star, with its face upwards or downwards, according as the star is to the eastward or westward of the moon; then move the index forward till the reflected image of the moon is seen in the telescope. By moving the instrument slowly up and down, the moon will appear to rise and fall by the star; the enlightened limb is then to be brought in contact with the star by means of the adjusting-screw; and the number of degrees and minutes pointed out by the vernier will give the observed distance between the objects.

If the distance between the moon and one of the stars set down in the *Nautical Almanac* for finding the longitude is to be observed, their distance may be roughly calculated as before directed, to which set the index: then look through the telescope, and direct the sight to the star, which is generally a bright one, and lies in a line nearly perpendicular to the horns of the moon either to the eastward or westward, as denoted in the *Almanac*; and holding the sextant in the plane of the two objects, give it a slow motion round the axis of the telescope; and if the moon's image comes into the field of the telescope, it is a proof the right star has been taken, as no other in that direction will correspond to it in distance.

The comparative lightness of the sextant has caused it frequently to be preferred to the reflecting circle, but whenever great accuracy is desired, the latter is far superior; for the sextant having but one vernier, any error which arises from eccentricity remains undiscovered, whilst in the circle this error is corrected by taking a mean of the readings of the several verniers.

DESCRIPTION OF TROUGHTON'S REFLECTING CIRCLE.

This instrument, the production of one of the most celebrated of the English artists, combines considerable strength with lightness. It is composed of a circular rim screwed to a cross bar frame, to the middle of which is fixed a hollow centre A, (plate 2, fig. 1.) upwards of two inches long, and projecting from the back of the frame. This hollow centre receives an axis, to one extremity of which is firmly fixed the index, having three branches at equal distances from each other, and to the other extremity of the axis is attached the index-glass. Each of the three branches terminates in a vernier, and one of them carries the usual apparatus for quick and slow motion, which of course governs also the other two.

On the back of the circle on that side where the index-glass is placed, is a kind of secondary frame BB, attached to the hollow centre, near the index-glass, and also by the pillars to the main frame.

Upon this secondary frame are firmly fixed the horizon-glass and the telescope; on the same frame are also placed two sets of coloured glasses, one between the index and horizon glasses, and the other beyond the horizon-glass. The distance between the two frames affords a relief for the dark glasses to be turned down round a joint, so as to place them out of the way when they are not wanted. The telescope screws into a collar, the square stem of which having its four angles worked into a screw, passes through a barrel, which connects the main and secondary frames. By means of a finger screw upon the stem, the distance of the telescope from the frame of the instrument, may be varied at pleasure; and the two objects observed, may be made of equal brightness, even without removing the eye from the telescope. The collar into which the telescope screws has the usual adjustment for placing the line of collimation parallel to the plane of the circle. A handle, D, appears on the divided side of the instrument, and is attached to a brass tube, which bends over the edge of the circle, and is fixed to the instrument on the same side as the glasses. E, is another handle screwed fast to the frame on the back of the instrument. By means of these handles the circle may be conveniently held in any position.

The telescope being fixed near one edge of the circle, and the horizon-glass near the opposite edge, the rays of light from the object may fall upon the index-glass on either side of the horizon-glass; by which means the great advantage is obtained, of observing on both sides of zero.

It has been before remarked, that in the sextant, from its having only one vernier, any error arising from eccentricity is uncorrected; but in the circle, by taking a mean of the readings of the three verniers, not only is this error removed, but any errors of division are reduced to about one-sixth part. Also, when the angle measured is 120 degrees, in a backward and forward observation, the three readings will just occupy the 720 degrees of this circle.

In Troughton's circles, as they are generally made, the degrees are subdivided into three parts, of 20 ites each, and each vernier subdivides again to 20 nds.

ie following printed instructions for its use are rally given with the instrument.

ctions for observing with Troughton's reflecting circle. Prepare the instrument for observation by screw- he telescope into its place, adjusting the drawer cus, and the wires parallel to the plane exactly u do with a sextant: also set the index forwards e rough distance of the sun and moon, or moon star; and holding the circle by the short handle, t the telescope to the fainter object, and make contact in the usual way. Now read off the ee, minute, and second, by that branch of the x to which the tangent screw is attached; also minute and second shewn by the other two ches; these give the distance taken on three rent sextants; but as yet it is only to be consi- d as half an observation. What remains to be is to complete the whole circle, by measuring angle on the other three sextants. Therefore he index backwards nearly to the same distance, reverse the plane of the instrument, by holding the opposite handle, and make the contact as e, and read off as before what is shewn on the several branches of the index. The mean of all s the true apparent distance, corresponding to the n of the two times at which the observations made.

When the objects are seen very distinctly, so that oubt whatever remains about the contact in both ts being perfect, the above may safely be relied s a complete set; but if, from the haziness of the too much motion, or any other cause, the obser- ns have been rendered doubtful, it will be ad- ble to make more: and if at such times so many ings should be deemed troublesome, six obser- ns, and six readings may be conducted in the wing manner. Take three successive sights ards exactly as is done with a sextant; only take to read them off on different branches of the x: also make three observations backwards, g the same caution; a mean of these will be the nce required. When the number of sights taken ards and backwards are unequal, a mean between means of those taken backwards and those taken ards, will be the true angle.

It need hardly be mentioned that the shades, or glasses apply like those of a sextant for making objects nearly of the same brightness; but it must insisted on, that the telescope should on every sion be raised or lowered by its proper screw for ing them perfectly so.

The foregoing instructions for taking distances ly equally for taking altitudes by the sea or arti- l horizon, they being no more than distances n in a vertical plane. Meridian altitudes cannot ever be taken backwards and forwards in the e day because there is not time: all therefore that be done is, to observe the altitude one way, and the index error; but even here you have a mean hat altitude, and this error taken on three different ants. Both at sea and land, where the observer tationary, the meridian altitude should be observed vards one day, and backwards the next, and so on rnately from day to day; the mean of latitudes, deduced severally from such observations, will be the true latitude; but in these there should be no application of index error; for that being constant, the result would in some measure be vitiated thereby.

"When both the reflected and direct images require to be darkened, as is the case when the sun's diameter is measured, and when his altitude is taken with an artificial horizon, the attached dark glasses ought not to be used; instead of them, those which apply to the eye end of the telescope will answer much better, the former having their errors magnified by the power of the telescope, will, in proportion to this power, and those errors, be less distinct than the latter.

"In taking distances when the position does not vary from the vertical above thirty or forty degrees, the handles which are attached to the circle are generally most conveniently used; but in those which incline more to the horizontal, that handle which screws into a cock on one side, and into the crooked handle on the other, will be found more applicable.

"When the crooked handle happens to be in the way of reading one of the branches of the index, it must be removed for the time, by taking out the finger screw which fastens it to the body of the circle.

"If it should happen that two of the readings agree with each other very well, and the third differs from them, the discordant one must not on any account be omitted, but a fair mean must always be taken.

"It should be stated, that when the angle is about thirty degrees, neither a distance of the sun and moon, nor an altitude of the sun with the sea horizon, can be taken backwards; because the dark glasses at that angle prevent the reflected rays of light from falling on the index-glass; whence it becomes necessary, when the angle to be taken is quite unknown, to observe forwards first where the whole range is without interruption; whereas, in that backwards, you will lose sight of the reflected image about that angle. But in such distances where the sun is out of the question, and when his altitude is taken with an artificial horizon, (the shade being applied to the end of the telescope) that angle may be measured nearly as well as any other; for the rays incident on the index-glass will pass through the transparent half of the horizon-glass, without much diminution of their brightness.

"The advantages of this instrument, when compared with the sextant, are chiefly these:—The observations for finding the index error are rendered useless; all knowledge of that being put out of the question, by observing both forwards and backwards. By the same means, the errors of the dark glasses are also corrected; for if they increase the angle one way, they must diminish it the other way by the same quantity. This also perfectly corrects the errors of the horizon-glass, and those of the index-glass very nearly. But what is of still more consequence, the error of the centre is perfectly corrected, by reading the three branches of the index; while this property combined with that of observing both ways, probably reduces the errors of dividing to one-sixth part of their simple value. Moreover, angles may be measured as far as 150 degrees, consequently the sun's double altitude may be observed when his distance from the zenith is not less than fifteen degrees; at which altitude, the head of the observer begins to intercept the rays of light incident on the artificial horizon; and, of course, if a greater angle could be measured, it would be of no use in this respect.

"This instrument, in common with the sextant, requires three adjustments. First, the index-glass perpendicular to the plane of the circle. This being done by the maker, and not liable to alter, has no direct means applied to the purpose: it is known to be right when, by looking into the index-glass, you see that part of the limb which is next you reflected in contact with the opposite side of the limb, as one continued arc of a circle: on the contrary, when the arc appears broken, where the reflected and direct parts of the limb meet, it is a proof that it wants to be rectified. The second is, to make the horizon-glass perpendicular. This is performed by a capstan screw, at the lower end of the frame of that glass; and is known to be right, when, by a sweep with the index, the reflected image of any object will pass exactly over or cover the image of that object seen directly. The third adjustment is for making the line of collimation parallel to the plane of the circle. This is performed by two small screws, which also fasten the collar into which the telescope screws, to the upright stem on which it is mounted: this is known to be right, when the sun and moon, having a distance of 130 degrees, or more, their limbs are brought in contact, just at the outside of that wire which is next to the circle; and then examining if it be the same, just at the outside of the other wire: its being so is the proof of adjustment."

A DESCRIPTION OF DOLLOND'S IMPROVED REFLECTING CIRCLE.

This instrument is represented at plate 2, fig. 2. It consists of two concentric circles; the one moving within the other, both in the same plane, and nearly in contact. The inner circle has three flat radii, two of which carry at their extremities the telescope and the horizon-glass. The outer circle is divided into degrees, &c. as near as possible to its exterior edge, and numbered in the usual manner. The interior circle has also every tenth degree numbered upon it on both sides of zero to 180, but without any other divisions. A vernier, A, is fixed to the interior circle opposite to its zero, for the purpose of reading off the degrees, &c. on the exterior circle. Near the telescope an apparatus for quick and slow motion is attached to the interior circle. An arm carrying the index-glass, and a vernier, B, moves freely round upon its own centre, and is furnished with an apparatus for slow motion, and a clamp, by means of which it may be attached to the exterior circle. The divisions upon the exterior circle may be read off by means of this vernier, but the figures corresponding to those divisions are concealed, whilst those of the interior circle are visible.

The following observations upon this instrument, and precepts for its use, are given in the maker's own words:

"The reflecting circle being constructed upon the same principles as the sextant, requires the same process for making the observations. The circle having the advantage of enabling the observer to take the angle on what is usually termed the off arc; that is, he may first take the angle precisely in the same manner as with the sextant; and then, by reversing the face of the instrument, take the angle on the off arc, or the other side of the zero. By this method he will correct any error that may arise from a want of parallelism in the glasses, or in the adjustment of the verniers to the zero; and by adding the angles so taken to each other, and dividing them by two, the correct angle will be obtained.

"There are two verniers on each index, which read from the centre division according to the arc upon which the angle is taken, and the one may be read backward as a check upon the other

"There is another and a greater advantage to be derived from the circular construction of the instrument, which is the power of repeating the observation, and thereby correcting the errors arising from the eccentricity of the centering or dividing.

"In the instrument here described there is a very considerable improvement for facilitating the operation of repeating the observations; it principally consists of the two sets of figures, one upon the inner circle, and the other upon the outer or divided circle; the first set are for reading off the angle, which is taken with the index B; and the second for reading off the number of the degrees, &c. which are to be divided by the number of repetitions.

"The method of observing with the instrument is as follows:

"If it is to be used as a sextant, the vernier A, which is attached to the inner circle, must be set to a primary division, and it will be the most correct to set it to 720,* for which purpose there is as a rough guide a line upon the outer chamfered edge of the vernier A, which corresponds with the zero of that vernier. Having adjusted this vernier correctly to 720, the angle must be taken by moving the index B forward, in the same manner as with the sextant, and the angle contained between the two objects will be seen upon the circle by the vernier belonging to that index, and the figures on the inner circle will be the guide for reading off. The index error should in this case be found as with the sextant, by taking a small angle on each side the zero. If there should be time for a second observation, the same angle may be taken with the face of the instrument reversed, by moving the index B the contrary way to the former, or upon the off arc: in this case the index error need not be attended to, as the half of the two angles will be the correct angle.

"The circle may also be used for taking a series of observations, by setting the vernier A to 720, and proceeding in the first instance by moving the index B forward, and then the inner circle, telescope, &c. and so on, each time reading off the angle, or noting the number of observations; and dividing the amount of the degrees, &c. passed through by the number of the observations; and if the number of the angles taken are equal to or nearly so, to the whole circle, it is evident that the errors of division and centering will have been taken into the account.

"When the instrument is used as a repeating instrument, the vernier A should be very correctly adjusted to 720; and the index B moved forward until the two objects, the angle of which is to be taken, are brought into contact; read off, and note the amount, then leaving the index B fast, and unclamping the telescope, &c. move the whole of those parts which are attached

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* In the plate the index A is by a mistake set to 360; this does not affect the reasoning, as it may be equally considered the zero.

inner circle forward, until the figures on the
circle shew the amount of the first angle, on the
ary side of their zero, bring the objects into con-
and it will here be evident that the space passed
will be equal to twice the angle. Note down
and without disturbing the last adjustment move
dex B forward twice the angle, and bring the
ts again into contact; then move the telescope,
rward as before, and note down two in addition
former. In this mode of observing, if the face
instrument is continued always one way, it will
uisite to present the telescope, first to the left
object, and secondly, to the right; but if the
reversed alternately, the telescope should be
ted to the same object. Proceed with the
vations, and each time the telescope, &c. is
forward, note two; and when the vernier A
ther approached or exceeded the whole circle,
, read off by that vernier, and divide the amount
number of the angles taken, and the quotient
e the correct angle; it will here be clearly
t, that all errors of division, centering, and
ation, will have been taken into the account,
herefore the angle will have been correctly
ed."

s instrument, (as well as Troughton's circle,) is
hed with a stand, which will be found very con-
t in making observations on shore.

o place the circle upon the stand, unscrew the
nut which fastens the ball or handle, holding
strument in the left hand with the face down-
, so that it may not slip from the handle; take
handle carefully, and the part upon which the
fitted will fit the hole in the end of the bent
near to the quick adjusting screw. The collar
oes on before the brass nut which held the
in its place, must then be put on the same
and the counterpoise screwed after it, which
rs the purpose of the nut; it should not be
ed tight, only sufficiently so as to prevent shake,
low the circle to turn pleasantly."

IPTION OF AN INSTRUMENT FOR TAKING ALTITUDES T SEA, WHEN THE HORIZON IS NOT VISIBLE.

s instrument, which is represented at plate I.
is a common Hadley's octant of wood, having
, A, fixed above the horizon-glass, parallel to
ane of the instrument, in such a manner that
bble may be seen by reflexion through the tube
he fluid of the level. A projecting frame of
B, is also fixed to the instrument, and a part
rms a white image, or line, in the horizon-
The image of the white line is always visible,
at of the bubble of the level may be seen bisected
white line when the instrument is held verti-
n the requisite position.

very ingenious arrangement is due to Mr.
son, mathematical instrument maker, of Devon-
treet, Portland-place. He generally attaches
el to an octant of wood, which is sufficiently
te for the purpose, as the angle cannot be
ed to be obtained by this method nearer than
ut two minutes of the truth. But it is evident
e same contrivance may be applied to a sextant.
his instrument may be employed in the same
r as a sextant, the adjustments are similar; that is to say, the index and the horizon glasses must be placed perpendicular to the plane of the instrument.

To find the index error.

There is no telescope used with this instrument, but merely a plain tube. Screw this tube into its place, and setting the index near O, direct the tube towards the horizon of the sea, and move the instrument vertically, till the bubble of the level appears bisected by the white line. Move the index now a little backwards or forwards, till the horizon of the sea seen by reflexion appears to coincide with the white line, taking care that it at the same time bisects the bubble of the level. The index ought now to shew, (counting *backwards* from O) the depression of the horizon, known from the height of the eye and table 1. If the reading is less than the depression, the difference between them must be added, but if otherwise subtracted from the observed altitude. But should the reading be *forward*, or in the order of the degrees from O, the sum of the reading and the depression is the index error to be subtracted from the observed altitude. The mean of a number of observations should be used.

Manner of using the instrument.

Place the index at O, and keeping the plane of the instrument in a vertical position, look at the sun or star, and cause the white line to bisect it. Then move the index gradually, keeping the reflected image in view, and always bisected by the line till the bubble of the level is seen, when the index must be clamped, and gradually moved by the tangent screw till the bubble and the reflected image of the object are bisected at the same instant by the white line, when the degrees, &c. are to be read off. In this manner several observations should be made, and the mean taken; and we may here caution the observer against being discouraged by the differences he will perceive in his readings, and which may sometimes amount to 10 or 12 minutes; notwithstanding which, by taking the mean of four or six observations, the result will seldom differ more than two or three minutes from the truth. To this mean the index error and other corrections, with the exception of that for the depression of the horizon are to be applied. In observing the sun, it is preferable to take the centre instead of the limb. When altitudes of stars are taken, the level and white line must be illumined by a lanthorn held by an assistant; the light employed should be no more than sufficient to render the level and line visible.

An instrument for the same purpose was contrived some time since by Captain Kater for the arctic expedition, and was found to answer very well, though the use of it was attended with some difficulties. This instrument is made by Mr. Dollond. It consists of a circle six inches diameter divided into degrees, and is furnished with a vernier reading off to minutes. At right angles to the index, and near the vernier, a level is fixed; and upon the centre of the instrument is placed a small plane mirror, which is moveable about an axis, the mirror continuing perpendicular to the plane of the circle. To the back of the circle a telescope is fixed, furnished with cross wires, and a skreen is so placed as to shield the eyes from the sun. To use this instrument, direct the telescope to the sun or star; (if to the former, previously putting on a dark glass,) and keep the intersection of the cross wires on the centre of the object. Move the index till the level is nearly horizontal, and turn the mirror till the image of the bubble is seen in it by the left eye, whilst the sun or star is viewed with the right. Clamp the index, and move it by its tangent screw till the bubble of the level which is very active, is seen in the mirror to fluctuate backwards and forwards, whilst the cross wires are kept as steadily as possible upon the centre of the sun or star. The degrees, &c. shewn by the index are then to be registered, and the mean of several observations is to be taken. The index error may be found by taking the depression of the horizon of the sea, and inferring the error in the same manner as for the instrument last described. With this instrument, the altitude of the sun has been taken at sea, when it was so much obscured by fog as not to be visible by reflexion; and the latitude deduced from the observations made with it seldom differed more than two minutes from the truth.

In concluding this article, we may remark that care has been taken to give all that is necessary, and to avoid every thing superfluous. From among a variety of methods which tend only to embarras the student, those have been selected which are the most accurate, and at the same time practically convenient. No other tables are necessary but the few which follow, and a common table of logarithms to minutes. In the direct method, however, of computing lunar distances, the logarithms are required to single seconds; but unless the altitudes can be obtained very correctly, the approximate method given is perhaps to be preferred.

Among the works which may be consulted on nautical astronomy, may be mentioned Ducom's *Cours d' Observations Nautiques*, and Kerrigan's *Young Navigator's Guide*, both of which contain tables of logarithmic sines, &c. conveniently arranged, so that they may be readily obtained to single seconds. Kelly's *Introduction to Spherics*, *Tables requisite to be used with the Nautical Ephemeris*, and Professor Lax's Tables, by the aid of which last all the problems of nautical astronomy may be solved in a very neat and expeditious manner.

Proportional Logarithms to twenty-four hours, for correcting the daily differences.

Tenths.	M.or S.	H.orM 0	..1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	0		1·3802	1·0792	9031	7782	6812	6021	5351	4771	4260	3802	3388	3010	2663	2341	2041
	1	3·1584	1·3730	1·0756	9007	7764	6798	6009	5341	4762	4252	3795	3382	3004	2657	2336	2036
	2	2·8573	1·3660	1·0720	8983	7745	6784	5997	5331	4753	4244	3788	3375	2998	2652	2331	2031
	3	2·6812	1·3590	1·0685	8959	7728	6769	5985	5320	4744	4236	3781	3369	2992	2646	2325	2027
	4	2·5563	1·3522	1·0649	8936	7710	6755	5973	5310	4735	4228	3773	3362	2986	2641	2320	2022
	5	2·4594	1·3455	1·0615	8912	7692	6741	5961	5300	4726	4220	3766	3355	2980	2635	2315	2017
.1	6	2·3802	1·3388	1·0580	8889	7674	6726	5949	5290	4717	4212	3759	3349	2974	2629	2310	2012
	7	2·3133	1·3323	1·0546	8865	7657	6712	5937	5279	4708	4204	3752	3342	2968	2624	2305	2008
	8	2·2553	1·3259	1·0512	8842	7639	6698	5925	5269	4699	4196	3745	3336	2962	2618	2300	2003
	9	2·2041	1·3195	1·0478	8819	7622	6684	5913	5259	4691	4188	3738	3329	2956	2613	2295	1998
	10	2·1584	1·3133	1·0444	8796	7604	6670	5902	5249	4682	4180	3730	3323	2950	2607	2289	1993
	11	2·1170	1·3071	1·0411	8773	7587	6656	5890	5239	4673	4172	3723	3316	2945	2602	2284	1989
.2	12	2·0792	1·3010	1·0378	8751	7570	6642	5878	5229	4664	4164	3716	3310	2939	2596	2279	1984
	13	2·0444	1·2950	1·0345	8728	7552	6628	5867	5219	4655	4156	3709	3304	2933	2591	2274	1979
	14	2·0122	1·2891	1·0313	8706	7535	6614	5855	5209	4646	4149	3702	3297	2927	2585	2269	1974
	15	1·9823	1·2833	1·0280	8683	7518	6601	5843	5199	4638	4141	3695	3291	2921	2580	2264	1969
	16	1·9542	1·2776	1·0248	8661	7501	6587	5832	5189	4629	4133	3688	3284	2915	2575	2259	1965
	17	1·9279	1·2719	1·0216	8639	7484	6573	5821	5179	4620	4125	3681	3278	2909	2569	2255	1960
.3	18	1·9031	1·2663	1·0185	8617	7467	6559	5809	5169	4611	4117	3674	3271	2903	2564	2249	1955
	19	1·8796	1·2607	1·0154	8595	7451	6546	5797	5159	4603	4110	3667	3265	2897	2558	2244	1951
	20	1·8573	1·2553	1·0122	8573	7434	6532	5786	5149	4594	4102	3660	3258	2891	2553	2239	1946
	21	1·8361	1·2499	1·0091	8552	7417	6519	5774	5139	4585	4094	3653	3252	2886	2547	2234	1941
	22	1·8159	1·2446	1·0061	8530	7401	6505	5763	5129	4577	4086	3646	3246	2880	2542	2229	1936
	23	1·7966	1·2393	1·0030	8509	7384	6492	5752	5120	4568	4079	3639	3239	2874	2537	2224	1932
.4	24	1·7782	1·2341	1·0000	8487	7368	6478	5740	5110	4559	4071	3632	3233	2868	2531	2219	1927
	25	1·7604	1·2289	9970	8466	7351	6465	5729	5100	4551	4063	3625	3227	2862	2526	2214	1922
	26	1·7434	1·2239	9940	8445	7335	6451	5718	5090	4542	4056	3618	3220	2856	2520	2209	1918
	27	1·7270	1·2188	9911	8424	7319	6438	5707	5081	4534	4048	3611	3214	2850	2515	2203	1913
	28	1·7112	1·2139	9881	8403	7302	6426	5695	5071	4525	4040	3604	3208	2845	2510	2198	1908
	29	1·6960	1·2090	9852	8382	7286	6413	5684	5061	4517	4033	3597	3201	2839	2504	2193	1904
.5	30	1·6812	1·2041	9823	8361	7270	6399	5673	5052	4508	4025	3590	3195	2833	2499	2188	1899
	31	1·6670	1·1993	9794	8341	7254	6385	5662	5042	4499	4017	3583	3189	2827	2493	2183	1894
	32	1·6532	1·1946	9765	8320	7238	6372	5651	5032	4491	4010	3577	3183	2822	2488	2178	1890
	33	1·6399	1·1900	9737	8300	7222	6359	5640	5023	4483	4002	3570	3176	2816	2483	2173	1885
	34	1·6269	1·1852	9708	8280	7206	6346	5629	5013	4474	3995	3563	3170	2810	2477	2169	1880
	35	1·6143	1·1806	9680	8259	7190	6333	5618	5004	4466	3987	3556	3164	2804	2472	2164	1876
.6	36	1·6021	1·1761	9652	8239	7175	6320	5607	4994	4457	3979	3549	3158	2798	2467	2159	1871
	37	1·5902	1·1716	9625	8219	7159	6307	5596	4985	4449	3972	3542	3151	2793	2461	2154	1866
	38	1·5786	1·1671	9597	8199	7143	6295	5585	4975	4440	3964	3535	3145	2787	2456	2149	1862
	39	1·5673	1·1627	9570	8179	7128	6282	5574	4966	4432	3957	3529	3139	2781	2451	2144	1857
	40	1·5563	1·1584	9542	8159	7112	6269	5563	4956	4424	3949	3522	3133	2776	2446	2139	1852
	41	1·5456	1·1540	9515	8140	7097	6256	5552	4947	4415	3942	3515	3127	2770	2440	2134	1848
.7	42	1·5351	1·1498	9489	8120	7081	6243	5541	4937	4407	3934	3508	3120	2764	2435	2129	1843
	43	1·5249	1·1455	9462	8101	7066	6231	5531	4928	4399	3927	3502	3114	2758	2430	2124	1839
	44	1·5149	1·1413	9435	8081	7051	6218	5520	4919	4390	3920	3495	3108	2753	2424	2119	1834
	45	1·5052	1·1372	9409	8062	7035	6205	5509	4909	4382	3912	3488	3102	2746	2419	2114	1829
	46	1·4956	1·1331	9383	8043	7020	6193	5498	4900	4374	3905	3481	3096	2741	2414	2109	1825
	47	1·4863	1·1290	9357	8023	7005	6180	5488	4891	4366	3897	3475	3089	2736	2409	2104	1820
.8	48	1·4771	1·1249	9331	8004	6990	6168	5477	4881	4357	3890	3468	3083	2730	2403	2099	1816
	49	1·4682	1·1209	9305	7985	6975	6155	5466	4872	4349	3883	3461	3077	2724	2398	2095	1811
	50	1·4594	1·1170	9279	7966	6960	6143	5456	4863	4341	3875	3455	3071	2719	2393	2090	1806
	51	1·4508	1·1130	9254	7948	6945	6131	5445	4853	4333	3868	3448	3065	2713	2388	2085	1802
	52	1·4424	1·1092	9228	7929	6930	6118	5435	4844	4325	3860	3441	3059	2708	2382	2080	1797
	53	1·4341	1·1053	9203	7910	6915	6106	5424	4835	4316	3853	3435	3053	2702	2377	2075	1793
.9	54	1·4260	1·1015	9178	7892	6900	6094	5414	4826	4308	3846	3428	3047	2696	2372	2070	1788
	55	1·4180	1·0977	9153	7873	6885	6081	5403	4817	4300	3839	3421	3041	2691	2367	2065	1784
	56	1·4102	1·0939	9129	7855	6871	6069	5393	4808	4292	3831	3415	3035	2685	2362	2061	1779
	57	1·4025	1·0902	9104	7836	6856	6057	5382	4799	4284	3824	3408	3028	2679	2356	2056	1775
	58	1·3949	1·0865	9079	7818	6842	6045	5372	4789	4276	3817	3401	3022	2674	2351	2051	1770
	59	1·3875	1·0828	9055	7800	6827	6033	5362	4780	4268	3809	3395	3016	2668	2346	2046	1766
.10	60	1·3802	1·0792	9031	7782	6812	6021	5351	4771	4260	3802	3388	3010	2663	2341	2041	1761

TABLE—*continued.*

Tenths.	M.or S.	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
	0	1761	1498	1249	1015	792	580	378	185	0000	9823	9652	9488	9330	9178	9031
	1	1756	1493	1245	1011	788	577	375	182	9997	9820	9650	9486	9328	9176	9028
	2	1752	1489	1241	1007	785	573	371	179	9994	9817	9647	9483	9325	9173	9026
	3	1747	1485	1237	1003	781	570	368	175	9991	9814	9644	9480	9323	9171	9024
	4	1743	1481	1233	999	777	566	365	172	9988	9811	9641	9478	9320	9168	9021
	5	1738	1476	1229	996	774	563	361	169	9985	9808	9638	9475	9318	9166	9019
.1	6	1734	1472	1225	992	770	559	358	166	9982	9805	9636	9472	9315	9163	9016
	7	1729	1468	1221	988	767	556	355	163	9979	9802	9633	9470	9312	9161	9014
	8	1725	1464	1217	984	763	552	352	160	9976	9799	9630	9467	9310	9158	9012
	9	1720	1460	1213	980	759	549	348	157	9973	9797	9627	9464	9307	9156	9009
	10	1716	1455	1209	977	756	546	345	153	9970	9794	9625	9462	9305	9153	9007
	11	1711	1451	1205	973	752	542	342	150	9967	9791	9622	9459	9302	9151	9004
.2	12	1707	1447	1201	969	749	539	339	146	9964	9788	9619	9456	9300	9148	9002
	13	1703	1443	1197	965	745	535	335	144	9961	9785	9616	9454	9297	9146	8000
	14	1698	1438	1193	962	741	532	332	141	9958	9782	9614	9451	9294	9143	8997
	15	1694	1434	1190	958	738	528	329	138	9955	9779	9611	9448	9292	9141	8995
	16	1689	1430	1186	954	734	525	326	135	9952	9777	9608	9446	9289	9138	8992
	17	1685	1426	1182	950	731	522	322	132	9949	9774	9605	9443	9287	9136	8990
.3	18	1680	1422	1178	947	727	518	319	129	9946	9771	9603	9440	9284	9133	8988
	19	1676	1417	1174	943	724	515	316	125	9943	9768	9600	9438	9282	9131	8985
	20	1671	1413	1170	939	720	511	313	122	9940	9765	9597	9435	9279	9128	8983
	21	1667	1409	1166	935	716	508	309	119	9937	9762	9594	9433	9277	9126	8980
	22	1663	1405	1162	932	713	505	306	116	9934	9760	9592	9430	9274	9124	8978
	23	1658	1401	1158	928	709	501	303	113	9931	9757	9589	9427	9271	9121	8976
.4	24	1654	1397	1154	924	706	498	300	110	9928	9754	9586	9425	9269	9119	8973
	25	1649	1393	1150	920	702	495	296	107	9925	9751	9583	9422	9266	9116	8971
	26	1645	1388	1146	917	699	491	293	104	9922	9748	9581	9419	9264	9114	8969
	27	1641	1384	1142	913	695	488	290	101	9919	9745	9578	9417	9261	9111	8966
	28	1636	1380	1138	909	692	484	287	98	9916	9742	9575	9414	9259	9109	8964
	29	1632	1376	1134	905	688	481	283	94	9913	9740	9572	9411	9256	9106	8961
.5	30	1627	1372	1130	902	685	478	280	91	9910	9737	9570	9409	9254	9104	8959
	31	1623	1368	1126	898	681	474	277	88	9907	9733	9567	9406	9251	9101	8957
	32	1619	1364	1123	895	677	471	274	85	9905	9731	9564	9403	9249	9099	8954
	33	1614	1359	1119	891	674	468	271	82	9902	9728	9561	9401	9246	9096	8952
	34	1610	1355	1115	887	670	464	267	79	9899	9725	9559	9398	9243	9094	8950
	35	1605	1351	1111	883	667	461	264	76	9896	9723	9556	9396	9241	9092	8947
.6	36	1601	1347	1107	880	663	458	261	73	9893	9720	9553	9393	9238	9089	8945
	37	1597	1343	1103	876	660	454	258	70	9890	9717	9551	9390	9236	9087	8942
	38	1592	1339	1099	872	656	451	255	67	9887	9714	9548	9388	9233	9084	8940
	39	1588	1335	1095	868	653	447	251	64	9884	9711	9545	9385	9231	9082	8938
	40	1584	1331	1092	865	649	444	248	61	9881	9708	9542	9383	9228	9079	8935
	41	1579	1327	1088	861	646	441	245	58	9878	9706	9540	9380	9226	9077	8933
.7	42	1575	1322	1084	858	642	438	242	55	9875	9703	9537	9377	9223	9075	8931
	43	1571	1318	1080	854	639	434	239	52	9872	9700	9534	9375	9221	9072	8928
	44	1566	1314	1076	850	635	431	235	48	9869	9697	9532	9372	9218	9070	8926
	45	1562	1310	1072	847	632	427	232	45	9866	9694	9529	9369	9216	9067	8924
	46	1558	1306	1068	843	628	424	229	42	9863	9691	9526	9367	9213	9065	8921
	47	1553	1302	1064	839	625	421	226	39	9860	9689	9523	9364	9211	9062	8919
.8	48	1549	1298	1061	836	621	418	223	36	9858	9686	9521	9362	9208	9060	8916
	49	1545	1294	1057	832	618	414	220	33	9855	9683	9518	9359	9206	9057	8914
	50	1540	1290	1053	828	614	411	216	30	9852	9680	9515	9356	9203	9055	8912
	51	1536	1286	1049	824	611	408	213	27	9849	9677	9513	9354	9201	9053	8910
	52	1532	1282	1045	821	608	404	210	24	9846	9675	9510	9351	9198	9050	8907
	53	1528	1278	1041	817	604	401	207	21	9843	9672	9507	9349	9196	9048	8905
.9	54	1523	1274	1037	814	601	398	204	18	9840	9669	9505	9346	9193	9045	8902
	55	1519	1270	1034	810	597	394	201	15	9837	9666	9502	9343	9191	9043	8900
	56	1515	1265	1030	806	594	391	197	12	9834	9663	9499	9341	9188	9041	8898
	57	1510	1261	1026	803	590	388	194	9	9831	9661	9496	9338	9186	9038	8895
	58	1506	1257	1022	799	587	384	191	6	9828	9658	9494	9336	9183	9036	8893
	59	1502	1253	1018	795	583	381	188	3	9826	9655	9491	9333	9181	9033	8891
.10	60	1498	1249	1015	792	580	378	185	0	9823	9652	9488	9330	9178	9031	8888

Nautical Astronomy.

TABLE I.

Depression of the horizon.

Height.	Depression.		Height.	Depression.	
Feet.	′	″	Feet.	′	″
1	0	58	19	4	11
2	1	21	20	4	17
3	1	40	21	4	23
4	1	56	22	4	30
5	2	9	23	4	36
6	2	21	24	4	42
7	2	33	26	4	52
8	2	44	28	5	5
9	2	53	30	5	15
10	3	2	35	5	39
11	3	10	40	6	4
12	3	19	45	6	27
13	3	27	50	6	46
14	3	36	60	7	25
15	3	42	70	8	1
16	3	50	80	8	34
17	3	57	90	9	6
18	4	4	100	9	35

TABLE II

Augmentation of moon's semi-diameter.

Altitude	Aug.
°	″
0	0
5	1
10	3
15	4
20	6
25	7
30	8
35	9
40	10
45	11
50	12
55	13
60	14
70	15
80	15
90	16

TABLE III.

Diminution of equatorial parallax.

Subtract	Equatorial Parallax.					Subtract
	53′	55′	57′	59′	61′	
1″	Lat. 18°	18°	17°	17°	17°	1″
2	26	25	25	24	24	2
3	32	31	31	30	30	3
4	38	37	36	36	35	4
5	43	42	41	41	40	5
6	49	48	47	46	45	6
7	55	54	52	51	50	7
8	61	59	58	56	55	8
9	69	66	66	62	60	9
10	79	74	71	68	66	10
11			83	71	74	11
12					88	12

For the proportional logarithm.

Lat.... 0° 11° 19° 25° 30° 34° 39° 43° 47° 51° 56° 60° 61° 71° 79° 90°
Add.... 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14

TABLE IV.

Sun's parallax in altitude.

Altitude.	Parallax.
0°	9″
10	9
20	8
30	8
40	7
50	6
55	5
60	4
65	4
70	3
75	2
80	2
85	1
90	0

TABLE V.

Vertical semi-diameter.

Altitude.	Correction.
5°	25″
6	19
7	14
8	11
9	9
10	8
11	7
12	6
13	5
14	4
15	4
16	3
18	3
20	2
30	1
45	1

TABLE VI.

For reducing sideral to mean time.

Hours.	Min.	Sec.	Min.	Seconds.	Sec.	Seconds.
1	0	9·83	1	0·16	1	0·00
2	0	19·16	2	0·33	2	0·01
3	0	29·49	3	0·49	3	0·01
4	0	39·32	4	0·66	4	0·01
5	0	49·15	5	0·82	5	0·01
6	0	58·58	6	0·98	6	0·02
7	1	8·81	7	1·15	7	0·02
8	1	18·64	8	1·31	8	0·02
9	1	28·47	9	1·47	9	0·02
10	1	38·30	10	1·64	10	0·03
11	1	48·13	11	1·80	11	0·03
12	1	57·96	12	1·97	12	0·03
13	2	7·78	13	2·13	13	0·04
14	2	17·61	14	2·29	14	0·04
15	2	27·44	15	2·46	15	0·04
16	2	37·27	16	2·62	16	0·04
17	2	47·10	17	2·78	17	0·05
18	2	56·93	18	2·95	18	0·05
19	3	6·76	19	3·11	19	0·05
20	3	16·59	20	3·28	20	0·05
21	3	26·42	30	4·91	30	0·08
22	3	36·25	40	6·55	40	0·11
23	3	46·08	50	8·19	50	0·14
24	3	55·91	60	9·83	60	0·16

TABLE VII. *For reducing mean to sideral time.*

s.	Min.	Sec.	Min.	Seconds.	Sec.	Seconds.
	0	9·86	1	0·16	1	0·00
2	0	19·71	2	0·33	2	0·01
3	0	29·57	3	0·49	3	0·01
4	0	39·43	4	0·66	4	0·01
5	0	49·28	5	0·82	5	0·01
6	0	59·14	6	0·99	6	0·02
7	1	8·99	7	1·15	7	0·02
8	1	18·85	8	1·31	8	0·02
9	1	28·71	9	1·48	9	0·02
10	1	38·56	10	1·64	10	0·03
11	1	48·42	11	1·82	11	0·03
12	1	58·28	12	1·97	12	0·03
13	2	8·13	13	2·14	13	0·04
14	2	17·99	14	2·30	14	0·04
15	2	27·85	15	2·46	15	0·04
16	2	37·70	16	2·63	16	0·04
17	2	47·56	17	2·79	17	0·05
18	2	57·42	18	2·96	18	0·05
19	3	7·27	19	3·12	19	0·05
20	3	17·13	20	3·28	20	0·05
21	3	26·98	30	4·93	30	0·08
22	3	36·84	40	6·57	40	0·11
23	3	46·70	50	8·21	50	0·14
24	3	56·55	60	9·86	60	0·16

Correction for the oblique semi-diameter. TABLE I. *For argument A.*

For *h* *h—s* °	A	*d* °	A	For *h* *h—s* °	A	*d* °	A	For *h* *h—s* °	A	*d* °	A
89	924	1	176	59	71	31	29	29	94	61	6
88	954	2	146	58	72	32	28	28	95	62	5
87	972	3	128	57	74	33	26	27	95	63	5
86	984	4	116	56	75	34	25	26	95	64	5
85	994	5	106	55	76	35	24	25	96	65	4
84	2	6	98	54	77	36	23	24	96	66	4
83	9	7	91	53	78	37	22	23	96	67	4
82	14	8	86	52	79	38	21	22	97	68	3
81	19	9	81	51	80	39	20	21	97	69	3
80	24	10	76	50	81	40	19	20	97	70	3
79	28	11	72	49	82	41	18	19	98	71	2
78	32	12	68	48	83	42	17	18	98	72	2
77	35	13	65	47	83	43	17	17	98	73	2
76	38	14	62	46	84	44	16	16	98	74	2
75	41	15	59	45	85	45	15	15	98	75	2
74	44	16	56	44	86	46	14	14	99	76	1
73	47	17	53	43	86	47	14	13	99	77	1
72	49	18	51	42	87	48	13	12	99	78	1
71	51	19	49	41	88	49	12	11	99	79	1
70	53	20	47	40	88	50	12	10	99	80	1
69	55	21	45	39	89	51	11	9	99	81	1
68	57	22	43	38	90	52	10	8	100	82	0
67	59	23	41	37	90	53	10	7	100	83	0
66	61	24	39	36	91	54	9	6	100	84	0
65	63	25	37	35	91	55	9	5	100	85	0
64	64	26	36	34	92	56	8	4	100	86	0
63	66	27	34	33	92	57	7	3	100	87	0
62	67	28	33	32	93	58	7	2	100	88	0
61	69	29	31	31	93	59	7	1	100	89	0
60	70	30	30	30	94	60	6	0	100	90	0

Correction for the oblique semi-diameter. TABLE II. *Diminution of the semi-diameter. Argument* A (*h*) + A (*h—s*) + A (*d*).

	Altitude.													
Sum of A	5°	6°	7°	8°	9°	10°	11°	12°	14°	16°	18°	20°	30°	45°
0″	25″	19″	14″	11″	9″	8″	6″	5″	4″	3″	3″	2″	1″	1″
20	24	18	14	11	9	7	6	5	4	3	2	2	1	0
40	23	17	13	10	8	7	6	5	4	3	2	2	1	0
60	21	16	12	9	8	6	5	5	3	3	2	2	1	0
70	20	15	12	9	8	6	5	5	3	3	2	2	1	0
80	19	14	11	8	7	6	5	4	3	2	2	2	1	0
99	17	13	10	8	7	6	5	4	3	2	2	1	1	0
100	16	12	9	7	6	5	4	4	3	2	1	1	1	0
110	14	10	8	6	5	4	3	3	2	2	1	1	0	0
120	11	9	7	5	4	3	3	2	2	1	1	1	0	0
130	9	7	5	4	3	3	2	2	2	1	1	1	0	0
135	7	6	4	3	2	2	2	2	1	1	0	0	0	0
140	6	5	4	3	2	2	2	1	1	1	0	0	0	0
145	5	4	3	2	2	1	1	1	1	0	0	0	0	0
150	3	3	2	2	1	1	1	1	1	0	0	0	0	0
155	3	2	2	1	1	1	1	1	0	0	0	0	0	0
160	1	1	1	0	0	0	0	0	0	0	0	0	0	0
170	0	0	0	0	0	0	0	0	0	0	0	0	0	0
178	1	1	1	0	0	0	0	0	0	0	0	0	0	0
180	2	1	1	1	1	1	0	0	0	0	0	0	0	—
182	3	2	2	1	1	1	1	0	0	1	1	1	0	—
184	4	3	2	2	1	1	1	1	1	1	1	1	1	—
186	5	4	3	2	2	2	1	1	1	1	1	1	1	—
188	7	6	4	3	2	2	2	1	1	1	1	1	1	—
190	9	7	5	4	3	3	2	2	1	2	1	1	1	—
191	10	8	6	4	4	3	3	2	2	2	2	2	1	—
192	11	9	7	5	4	4	3	3	2	2	2	2	1	—
193	12	9	7	5	5	4	3	3	2	2	2	2	1	—
194	14	10	8	6	5	5	4	3	2	3	2	2	—	—
195	15	11	9	6	6	5	4	4	3	3	3	—	—	—
196	17	13	10	7	6	6	5	4	3	3	3	—	—	—
197	19	14	11	8	7	6	5	5	3	—	—	—	—	—
198	21	16	12	9	8	7	6	5	3	—	—	—	—	—
199	23	17	13	10	8	8	6	5	4	—	—	—	—	—
200	25	19	14	11	9	—	—	—	—	—	—	—	—	—
Alt.	5°	6°	7°	8°	9°	10°	11°	12°	14°	16°	18°	20°	30°	4

PHYSICAL ASTRONOMY.

The object of the philosopher in the investigation of nature is to arrange and classify facts and phenomena, with a view to trace the agency of their remote, or, at least, their proximate causes, and ascend as high as the imperfection of human means of observation, and the limited powers of the human intellect will allow us in the scale of generalization.

To beings endowed with more perfect faculties, and more comprehensive intelligence than ourselves, much of that complication we observe in natural phenomena would disappear: many effects, which seem to us independent of each other, and linked by no natural connection, would be in their eyes collateral results of one and the same principle: much, that to us seems fortuitous would to them appear predisposed and regularly arranged. The laws of nature would at once be reduced in number and enlarged in extent; and that higher order of generalization which would consist in classifying together laws of the same kind, and referring them to others yet more universal, would exercise *their* power and constitute *their* science. That such would be the case with more perfect beings, our own experience, limited as it is, amply shews. The man who has learnt to regard the fall of a leaf and the precession of the equinoxes as results equally certain and unavoidable of a law capable of being stated in three lines, and understood by a child of ten years old, has made already a considerable step in this way—the patient exercise of his natural reason has stood him in the stead of a sharper intellect; and secrets which an angel might penetrate perhaps at a glance, become revealed to man by the slow, yet sure, effects of persevering thought.

The progress of modern science has done more than the keenest metaphysical reasoning, and has given us the most convincing proofs of the agency of one general and intelligent cause throughout the whole system of nature. When we see on all sides phenomena grouping themselves under laws intelligible and simply expressed, which are themselves subordinate to others, yet more simple and extensive; when we see every anomaly which threatened destruction to a theory, becoming, in the progress of our knowledge, its firmest support; every inequality disappearing when viewed from a higher level; every exception proving a rule of greater generality; all, in short, conveying more and more towards order and simplicity the more severely we scrutinize it; it is impossible not to allow that that last great step, which unites all the phenomena of the universe under one general head, and refers them to one all-pervading agency—however inconceivably remote, and surpassing probably the utmost limits of the human intellect to comprehend, if explained, would still be but the continuation and final completion of a chain of reasoning whose first links we hold within our grasp—the consummation of a process actually begun—the termination of a career into which we are fairly entered.

It is difficult to avoid such contemplations at the outset of an essay on physical astronomy; they crowd upon us; and in rejecting them we should reject the noblest use of the sublimest of sciences. For scarcely in any are the phenomena presented by nature more various and more complicated; in none is the generalization so complete, the final result so simple, or the object more imposing.

From what has just been said, it may be gathered that the object of physical astronomy is to reduce, under general laws, the motions and phenomena of the heavenly bodies, and investigate their causes; to trace the history of what has already happened in our own system, and to ascertain what changes the causes demonstrably in action (unless interfered with by others we have no knowledge of,) will superinduce in the course of ages, and thus to appreciate the stability of the present order of creation. In a more limited and practical point of view, the physical astronomer is called on to furnish formulæ deduced from theory for determining the state of the system at any assigned instant; and adapted for the purposes of the observer, so as to serve as a basis for the construction of tables; and to descend, by the application of his general principles, to those more refined inequalities which, owing to their minuteness, or the length of their periods, would escape or mislead the observer unassisted by theory.

There is one feature in physical astronomy which renders it remarkable among the sciences, and has been the chief, if not the only, source of the perfection it has attained. It is this—that the fundamental law embracing all the minutiæ of the phenomena so far as we yet know them, presents itself at once, on the consideration of broad features and general facts, deduced by observations even of a rude and imperfect kind, in such a form as to require no modification, extension, or addition when applied in minute detail. In other sciences, when an induction of a moderate extent has led us to the knowledge of a law which we conceive to be general, the further progress of our inquiries frequently obliges us either to limit its extent or modify its expression. To those who are familiar with the history of chemistry, instances of this will present themselves at every turn. In physical optics, the general representation of all the series of polarised tints and the colours of natural bodies by a certain universal scale—the Cartesian law of refraction when applied to the extraordinary ray in crystallized media, and even to the ordinary, if the reports of some recent experiments are to be relied on—together with innumerable other laws, simple, natural, and resting on extensive inductions, have all been either overset, extended, or materially modified by the progress of the science.

In physical astronomy, however, when taken in that limited acceptation, which restricts it to the explanation of the planetary motions, our first conclusion is our last. The law on which all its phenomena depend, flows naturally and easily from the simplest among them, as presented by the rudest observation; and, in point of fact, such has really been the order of investigation in this science. The rude supposition of the uniform revolution of the moon in a circle about the earth as a centre, led Newton at once to the true law of gravity, as extending from the earth to its companion. The uniform circular motions of the planets about the sun, in times following the progression assigned by observation in Kepler's rule, confirmed the law, and extended its influence to the boundaries of our system. Every thing more refined than this—the elliptic motions of the planets and satellites — their mutual perturbations — the slow changes of their orbits and motions, denominated secular variations—the deviation of their figures from the spherical form—the oscillatory motions of their axes, which produce nutation and the precession of the equinoxes—the theory of the tides, both of the ocean and the atmosphere, have all in succession been so many trials for life and death in which this law has been, as it were, pitted against nature; trials, whose event no human foresight could predict, and where it was impossible even to conjecture what modifications it might be found to need. Even at this moment, if among the innumerable inequalities of the lunar or planetary motions any one, however small, should be discovered decidedly not explicable on the hypothesis of a force varying as the inverse square of the distance, that hypothesis must be modified till it accounts for it. It is hardly necessary to add, however, that in the present state of science, this is a case not to be contemplated.

Still, these are refinements. The deviations of the planetary orbits from circles are small, their deviations from ellipses excessively minute: the lunar orbit alone presents results of perturbation so large as to strike us at once with the appearance of irreconcilable anomalies, but it is only by a refinement of calculation that we can trace them all to the laws of gravity; but the motions of comets put the truth and generality of this law to a severe and rude test, by giving it a trial under the greatest varieties of distance, position, and velocity of motion, and instancing its influence on matter of a rarity almost spiritual, and differing so utterly from that of which our planet consists, as scarcely to authorize the admittance of any property in common.

The above observations have been made in conformity with the general language of natural philosophers, and the customary acceptation of the term physical astronomy, and are, no doubt, strictly applicable when we confine ourselves to the celestial phenomena of our own immediate system, and the motions of those larger masses of matter of which planets and their satellites consist. The cautious philosopher however will still regard it as worthy inquiry, whether, at enormous distances, like those of the fixed stars, or at such comparatively microscopic intervals as those we are ordinarily conversant with on the surface of our planet, the rigorous law of a force as the inverse square of the distance may not suffer some modification. An emanation, like light, traversing in succession every part of space, may be conceived to go on without acceleration or retardation, without loss or change, to the remotest regions; but an active and immediate intercourse carried on between points infinitely distant, is not only incapable of demonstration, but, could it be proved, must, I suppose, be referred to direct spiritual agency. At the same time, it is worthy attention how strict and indissoluble a bond gravity establishes between remote objects—to see this in its real light, we must compare it with the most effectual of our ordinary means of transmitting power. If the earth and sun were connected by a rod of cast iron, in one piece, an impulse or pull, however violent and sudden, applied at the sun, would not begin to be felt at the earth till after a lapse of eighteen months from the moment of its communication, while a change in the sun's attraction, such as might arise from a sudden alteration of its figure or density, would demonstrably* affect our planet in an instant of time many thousand times less than the least interval perceptible to our senses.

The subsistence of sidereal clusters, in which the compression or crowding of the stars is carried to the extent we have instances of in many parts of the heavens, seems hardly compatible with a gravitating force, unopposed by some principle of conservation, unless we suppose them in a state rapidly verging to a catastrophe. On the other hand, with regard to small distances, we have no distinct proof, that within a few inches, or even miles, from a material point, the law of gravity may not begin to deviate appreciably from the Newtonian law. The experiments of Maskelyne and Cavendish, which may perhaps be adduced as supporting its rigorous application, are far too gross, and differ too widely in their results, to be cited in so delicate a matter, besides which, their results, as applied to such an inquiry, are affected with an unknown element, the mean density of the earth. At much closer intervals we are certain of the existence of attractive and repulsive forces following a widely different law; and by what imperceptible gradations these shade into that of gravity, or whether they are to be regarded as distinct from it in their nature and origin, is a point whose consideration seems reserved for a much higher state of science than we can boast of having yet attained.

But it is quite sufficient for the purposes of physical astronomy to know, that as far as the motions of the great masses of matter connected with our system either in the heavens or on our globe are concerned, observation and theory present no difference capable of being made an objection to the strict expression of Newton's law, and we shall therefore wave all further discussion of the subject, and proceed to the object of the present essay, in which the reader will be presumed acquainted with the general *facts* of astronomy, with the principles of mechanical philosophy, and so much of analysis, of the differential and integral calculus, plane and spherical trigonometry, as shall render it unnecessary for us to interrupt the general chain of our reasoning to demonstrate such theorems, &c. as we shall have occasion to call to our aid.

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* Laplace, *Système du Monde*, p. 286.

SECTION I.

ON THE CIRCULAR AND ELLIPTIC MOTIONS OF THE PLANETS AND SATELLITES.

By observing the places of the sun, moon, and planets in the heavens at different times, and measuring their angular diameters, we have learnt, that, provided certain excessively small inequalities, to be hereafter considered, are disregarded, their motions are all compatible with the supposition of the planets revolving about the sun in elliptic orbits of small eccentricity, and having the sun's centre in one focus; each orbit lying wholly in a fixed plane peculiar to itself, and slightly inclined to the ecliptic, or plane, in which the sun appears to revolve about the earth. We learn, moreover, that if the apparent motion of the sun be transferred in a contrary direction to the earth, and the sun be supposed at rest in space, the motion so assigned to the earth will be such as to include it in the expression of the same law. The supposition then, of the sun at rest, and the earth in motion, being agreeable to this general analogy, and supported by incontrovertible arguments drawn from the great magnitude of the former in comparison with the latter body, is assumed as a demonstrated truth. Observation moreover assisted, it is true, by calculation, but independent of all theory, (i.e. of all reasoning from causes,) has taught us the truth of the following remarkable laws, in which also the earth is included among the planets.

The areas described about the sun's centre by the radius vector of any one of the planetary orbits (or the line drawn from the sun to the place of the planet,) are proportional to the times of their description.

Let S be the sun, and APP′ part of the orbit of any planet. Then, A being assumed as a point of departure, the area ASP is to the area ASP′ as the time of the planet's describing the arc AP of its orbit is to the time of its describing AP′.

The squares of the periodic times of different planets, (or of the times of a complete revolution of each about the sun,) are as the cubes of their mean distances from the sun, or of the greater semiaxes of their respective ellipses.

The periodic time of the earth is 365·2564, and that of Mars 686·9796. The greater semiaxis of the earth's orbit being 1, that of Mars is 1·523693; and we may easily satisfy ourselves, by executing the computation, that $(365{\cdot}2564)^2 : (686{\cdot}9796)^2 :: 1^3 : (1{\cdot}523693)^3$.

These three laws, viz. 1st. The elliptic motion of the planets about the sun as a focus: 2dly. The proportionality of the areas to the times: and 3dly. The law of the periodic times, which have immortalized the name of Kepler, and whose discovery, and the manner of it, afford at once matter of humiliation and triumph to the human intellect, were all deduced immediately from observation, as insulated, and, for aught their discoverer knew, unaccountable facts. It shall now be our business to demonstrate their mutual dependence, and to shew how the general law of attraction may be derived from them most simply.

The analogy observed between the motions of the other planets and of the earth, affords a reasonable presumption of their being masses of matter subject to the same mechanical laws of rest, impulse, and resistance, as that of which our own planet consists. Moreover, from what we know of the constitution of our own atmosphere, and its rapid diminution of density as we recede from the earth, we have every reason to believe, that the immense space in which their revolutions are performed, is either completely void, or at least free from any material substance capable of sensibly resisting or impeding their motions, or preventing any external impulse they may receive from acting on them with its full effect. Setting out with these assumptions (the strict truth of which will be best tried by the conclusions they will lead to,) it is obvious that as the planets, instead of moving continually forward in straight lines, as masses of inert matter would do if projected in space and left to themselves, are, in fact, constantly deviating from this rectilinear progression—they must be under the perpetual influence of some agency external to themselves, which, (by the second law of motion,) can be no other than that of a mechanical force acting in a direction inclined to that in which they move at any instant.

The enormous distance at which the planets are from the sun, and their own minuteness, compared with it, permit us at present to regard them as points; and it will be shewn hereafter that this supposition introduced here merely for simplification, is strictly legitimate. Let us then consider the motion of a material point perpetually deflected from a straight line by the action of an external force; and to this end let us conceive the curve OPQ described by the planet to be replaced by a polygon of an infinite number of sides OP, PQ, &c. and setting out from O, let it describe the chord OP in the first instant of time $d\,t$. Fig. 2. In an equal subsequent instant it would, if left to itself, go on describing PR equal to OP, and in the same straight line. But since we have regarded the curve as replaced by an elementary polygon, we must (on the principles of the differential calculus) conceive the deflecting force to act by interrupted impulses at the angles of that polygon. Let the first impulse therefore be conceived to take place at P. Then, since the material point P, in virtue of the motion inherent in it at P, would have described PR in the instant $d\,t$; but, in virtue of that motion, combined with the new motion it receives at P, does actually describe PQ in the same time, that new motion (by the composition of motions,) must be such as, *alone*, would carry it from P over a space Pv equal and parallel to RQ; and as the change of motion takes place in the direction in which the moving force acts, Pv must be the direction of the deflecting force. Prolong Pv indefinitely, and take in it any point S; join OS, SQ, SR. Then, since OP = PR, the area OSP = SPR = SPQ, because QR is parallel to PS.

A force may be conceived to *tend to* any point in the line of its direction. We see, therefore, that any point to which the force acting at P tends is characterised by this remarkable property, that the areas described about it in equal evanescent instants on either side of P are equal. This property belongs to every point in the line PvS, but (as is obvious) to no point situated out of that line; and any point possessing this property may be regarded (at least for that moment) as a point of tendency, or centre of the force acting on the body at P.

Now, as we have seen, it is matter of observation that each planet describes areas proportional to the times, and consequently equal areas in equal infinitely small times before and after any given instant, about a fixed point in the system coincident with the sun's centre. This point, therefore, possesses at all times and in all positions of the planets, the property above demonstrated to belong to a point situated in the direction of the deflecting force; or, in other words, the forces deflecting the planets in their orbits are invariably directed to the centre of the sun.

The moon, (neglecting periodical inequalities,) describes about the centre of the earth, and the satellites of Jupiter and Saturn about their respective primaries, areas proportional to the times of their description. The forces, therefore, which deflect them in their orbits, are directed (small causes of inequality being neglected) to the centres of the earth, of Jupiter, and of Saturn respectively.

Having ascertained the directions of the forces which deflect the planets from their rectilinear paths, and retain them in their orbits, we come now to estimate their intensity, and investigate the laws of their action. In order to this, we shall find it more simple to abandon the supposition of the interrupted impulsive action of the deflecting force, and consider the body P as deflected from the *tangent* PR, and describing not the chord PQ, but the infinitesimal arc, the force being supposed to act during the whole time dt. This time, however, being infinitely small, the force may be regarded as constant; and since the angle PSQ between its first and last directions is also evanescent, it must be considered as acting constantly in a direction parallel to PS or QR. If then we take F to represent the force at P, and $g = 32^{\text{ft}}{\cdot}1908$ (or double the space through which a heavy body falls in the first second at the earth's surface,) and suppose unity to represent the force of gravity, we shall have by mechanics,

$$RQ = \tfrac{1}{2} g \,.\, F\, d t^2$$

and*
$$F = \frac{2QR}{g \,.\, d t^2}.$$

If, therefore, we know by observation the nature of the orbit, and the velocity of the body at any point P, we may thence calculate the magnitude of the deflection QR produced in any very minute time dt: and thus the intensity of the deflecting force will become known. Let us, for instance, take the case of the moon; and, supposing her orbit a circle, with the earth in the centre, let us inquire the actual magnitude of QR, the deflection from the tangent produced in some extremely minute portion of time as 1″, by the force retaining it in its orbit.

Call the mean radius of the moon's orbit R, her period (in seconds of mean time) T; then will her velocity (being equal to the circumference of her orbit, divided by the number of seconds in the time of one revolution,) be represented by $\frac{2\pi R}{T}$, where $\pi = 3{\cdot}14159$, &c.; and this is the actual length of the arc described in 1″. Now, since (neglecting higher powers of the arc than the square)

$$\text{versed sine} = \frac{(\text{arc})^2}{\text{diameter}};$$

and since QR or Pv is in this case equal to the versed sine of PQ, we must have,

$$QR = \frac{1}{2R} \times \left(\frac{2\pi R}{T}\right)^2 = \frac{2\pi^2 R}{T^2};$$

and, finally, (since $d t = 1''$)

$$F = \frac{4\pi^2}{g} \times \frac{R}{T^2};$$

and this is the general expression for the force perpetually urging a body to the centre of a circle. To reduce it to numbers in the case of the moon, we have

$R = 238783^{\text{m}} = 39165700 \times g$ (since $g = 32^{\text{ft}}{\cdot}1908$)

and $\frac{R}{g} = 39165700$

moreover,

$T = 27^{\text{d}}{\cdot}32167 = 2360592''$.

Calculating from which data, we find

$$F = 0{\cdot}00028394 = \frac{1}{3522}.$$

So that the force by which the moon is retained in its orbit, is about 3522 times feebler than that of gravity at the earth's surface.

When we observe a tendency in all bodies at the earth's surface to approach or fall towards its centre, and if hindered from approaching, still to *press* towards that point, we express these phenomena, by saying, that they are attracted towards the earth. At all moderate elevations above its surface, and in the same geographical situation this tendency seems invariable; but at great elevations, the delicate indications of modern instruments will detect a decrease in its energy;* and indeed the gradual enfeebling of attraction towards any body by an increase of distance, is not only a natural supposition of itself, but is borne out by the strong analogy of magnetic and electrical attractions. At vast elevations then, like that of the moon, there is reasonable ground to expect a considerable diminution of attraction; and if the force by which the moon be retained in her orbit be nothing more than this same attraction modified by the remoteness of the two bodies, we see that an increase of the distance to about 60 times the earth's radius from its centre, is sufficient to weaken it more than 3500 times. As the distance then from the earth's centre increases, the attraction diminishes in a much more rapid progression. What the exact nature of this progression is, we must satisfy ourselves by other phenomena; but even from the rude calculation already made, (in which every correction has been neglected) we may perceive, that a law of decrease as the squares of the distances (the next in simplicity to the distances themselves,) has a *primâ facie* probability. In fact, $60^2 : 1^2 :: 3600 : 1$.

Having only one attendant satellite, however, we

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* QR here represents the deflection from the *tangent*, and is only half the length of QR in the last figure, which represents the deflection from the preceding chord prolonged.

* At an elevation of a mile above the surface of the earth, the intensity of gravity is diminished $\frac{1}{1977{\cdot}291}$; and a pendulum clock, beating seconds at the level of the sea, would lose 21·898 seconds a day at this altitude, a quantity not to be overlooked. Any traveller having leisure, and the proper apparatus, might try the experiment in the barrack on Mont Cenis, or at the Hospice of St. Bernard.

e no means of obtaining any further verification of
a law, in this way; but if we regard the earth
ell as the other planets, as so many satellites of
sun, we have here ample room to satisfy ourselves,
ng a progression of no less than eleven distances,
that of Mercury to that of Uranus, on which to
nd the assumption of a law. And here we have
dvantage of dispensing altogether with numerical
lation, and substituting in its stead the third law
epler: for if we call R and r the radii of the
es described by any two planets round the sun,
d t their periodic times, F and f the forces re-
ng them in their orbits, we have (as a result of
rvation,)

$$\frac{T^2}{t^2}=\frac{R^3}{r^3},$$

h, combined with the equations,

$$F=\frac{4\pi^2}{g}\times\frac{R}{T^2},\qquad f=\frac{4\pi^2}{g}\times\frac{r}{t^2}$$

h give

$$\frac{F}{f}=\frac{R}{r}\cdot\frac{t^2}{T^2},$$

hall find

$$\frac{F}{f}=\frac{r^2}{R^2}.$$

us, then we encounter the same rate of diminu-
in the attractive tendencies of each of the planets
rds the sun, which the lunar motions had given
n to surmise in the case of the earth and moon;
n the case now under consideration, the verifica-
is much more satisfactory, and the numerical
idences, when the calculations are gone through,
lete; the third law of Kepler, on which the
e is founded, being almost rigorously exact.

e may now, with great confidence, presume the
se proportion of the squares of the distances to
e law of variation of that force which retains the
s of our system in their orbits; but previous to
ning its generality, it will be right to compare
orce retaining the moon in its path round the
, and that deflecting the earth in its orbit about
un. Calling R, r, the respective radii of the
's and moon's orbits, and T, t, their periodic
, we have still

$$\frac{F}{f}=\frac{R}{r}\cdot\left(\frac{t}{T}\right)^2.$$

we have

$$\frac{r}{R}=\frac{60{\cdot}23799}{23405};\quad\frac{T}{t}=\frac{365{\cdot}25638}{27{\cdot}32167}$$

at, executing the calculation,

$$\frac{F}{f}=2{\cdot}17399:$$

un then, although more than 380 times the dis-
of the moon, exerts a force of more than double
tensity on the earth compared with the earth's
tion on the moon. At equal distances, then,
rces exerted by the sun and earth would be in
tio of

$$2{\cdot}17399\times\left(\frac{R}{r}\right)^2 : 1 \text{ or } 328196 : 1.$$

s enormous difference in the attractive energies of the two bodies, must evidently be owing to some equally striking difference in the bodies themselves; and when we consider the immense magnitude of the sun (in comparison with our planet) we shall not be at a loss to what cause to assign it. Whatever be the cause of attraction, we may fairly conclude that, if it be the result of a force inherent in matter, two equal and similar bodies (i. e. each containing the same quantity of attracting matter,) placed close together, will each attract a third placed at a distance, with equal forces; and both together, with double the force of either separately, and pursuing the same idea—that 328196 such bodies as the earth, if placed close together, and forming one mass in the place of the sun, would attract as the sun actually does: in other words, that the sun only attracts other bodies with more energy than the earth, by reason of its being a greater body, and containing a greater quantity of attracting or *gravitating* matter.

By such reasonings we are led to assume, as a general law, that similar and equal particles of matter, however situated in space, attract, or tend to each other with a force directly proportional to their *masses* or quantities of gravitating substance, and inversely proportional to the squares of their distances from each other; and having arrived at this law by the steps described, we must now proceed to verify its rigorous exactness, by applying it in succession to the phenomena as presented by nature in our system, which will be the object of the following sections.

SECTION II.

On the attractions of spherical bodies

The earth, sun, and planets, as well as their satellites, being shewn by observation to be spherical bodies of great magnitude, it becomes necessary to examine, *in limine*, whether the law of attraction above stated be compatible with this fact—in other words, whether from a knowledge that the gross attractions of the whole masses follow that law of decrease, we can argue that the attraction of each elementary molecule follows the same.

Let BDCE be the attracting sphere, m the body attracted, which at present we will suppose to be a single particle, taking its mass as unity. Suppose mBAC the axis of the sphere passing through the molecule m, and let M, M′ be two equal and similar molecules, similarly, but oppositely situated with respect to the axis. Each of these molecules will attract m with a force represented by $\frac{M}{(Mm)^2}$, but the directions of their attractions not coinciding, we must resolve them into others, whose effects may directly assist or counteract each other. Draw MPM′ (which will of course be perpendicular to mAC), and if we take $Mm=f$, and $MP=\rho$, we shall have $\frac{M}{f^2}$ to represent the force of M on m, which, reduced to the directions mC and PM, give the partial forces $\frac{M}{f^2}\times\frac{mP}{mM}$ and $\frac{M}{f^2}\times\frac{MP}{mM'}$, that is, (if we call AP, x, and Am, a)

Fig. 3.

$$\frac{M(a-x)}{f^3}\text{ and }\frac{M\cdot\rho}{f^3}.$$

The partial forces of M′ are represented by the same quantities, but the latter of them, acting in the direction PM′ destroys the partial force of M in the opposite direction PM, while the former conspires with the corresponding force of M, and doubles its effect, so that we have

$$\frac{2\,(a-x)\cdot \mathrm{M}}{f^3}$$

for the attraction of this pair of molecules in the direction m A; and the sum of all such pairs throughout the sphere being found by the ordinary rules of the integral calculus will express the whole attraction of the sphere. The simplest way will be to regard the molecule M as a parallelopiped included, first, between two consecutive positions of the plane DE perpendicular to the axis A m, separated from each other by the interval $d\,x$; 2dly. Between two cylindrical surfaces, having for their bases the circle MM′, whose radius is PM $(=\rho)$ and the same circle in its consecutive position, when its radius varies from ρ to $\rho+d\,\rho$; and 3dly. Between two consecutive positions of the plane PM m, assumed during its rotation about P m as an axis. In virtue of this, if we put the angle FPM $=\theta$ the dimensions of the molecule in a direction perpendicular to PM will be $\rho\,d\,\theta$, and its dimensions in other two directions being respectively $d\,x$ and $d\,\rho$, we have

$$\mathrm{M}=\rho\,d\,\rho\,.\,d\,\theta\,.\,d\,x\,;$$

so that the whole attraction (A) will be expressed by the triple integral

$$\iiint\frac{2\,(a-x)\,.\,\rho\,d\,\rho\,.\,d\,\theta\,.\,d\,x}{f^3}.$$

The variables ρ, θ, and x, are here independent; and it is therefore indifferent with which we begin, we will commence with θ, because f being $=\sqrt{(a-x)^2+\rho^2}$ is independent on θ. Thus we have

$$\mathrm{A}=\iint\frac{2\,(a-x)\,\rho\,d\,\rho\,.\,d\,x}{f^3}\,(\theta+\text{Const.})$$

This integral must be extended only from $\theta=o$ to $\theta=\pi$, or over only *half* the circumference of the circle MM′, otherwise the attraction of each molecule M, M′ (having been grouped in pairs,) would be repeated twice over. Then we have

$$\mathrm{A}=\iint\frac{2\,\pi\,(a-x)\,d\,x\,.\,\rho\,d\,\rho}{\{(a-x)^2+\rho^2\}^{\frac{3}{2}}}.$$

If we now perform the integration relative to ρ regarding x as constant, we get

$$\mathrm{A}=\int 2\,\pi\,(a-x)\,d\,x\,.\left\{\text{Const.}-\frac{1}{\sqrt{(a-x)^2+\rho^2}}\right\}$$

But the integral in this case is to be extended from $\rho=o$ to $\rho=\mathrm{PD}=\sqrt{r^2-x^2}$, r being the radius of the sphere, so that it becomes

$$\mathrm{A}=\int 2\,\pi\,(a-x)\,d\,x\,.\left\{\frac{1}{a-x}-\frac{1}{\sqrt{(a^2+r^2)-2\,a\,x}}\right\}$$

$$=2\,\pi\,x+\pi\int\frac{(a-x)\,d\,.\,(a-x)}{\sqrt{2\,a\,(a-x)-(a^2-r^2)}}$$

$$=\text{Const.}+2\,\pi\,x+\frac{2\,\pi}{3}\left\{2\,a^2-r^2-a\,x\right\}.\frac{\sqrt{a^2+r^2-2\,a\,x}}{a^2}$$

This integral must be extended from $x=\mathrm{AC}=-r$ to $x=\mathrm{AB}=+r$, when it finally becomes, after all reductions,

$$\mathrm{A}=\frac{4}{3}\cdot\frac{\pi\,r^3}{a^2}.$$

Now, $\frac{4}{3}\pi\,r^3$ represents the mass of the sphere, which being called S, we have

$$\mathrm{A}=\frac{\mathrm{S}}{a^2},$$

an equation which shews, that *the attraction of the sphere is expressed by the whole mass, divided by the square of the distance of its centre from the attracted molecule, and is therefore precisely the same as if the whole sphere were condensed into its centre.*

The hypothesis, then, which refers the observed attractions of the great masses composing our system to the effect of the mutual attraction of their ultimate molecules, varying according to the same law, has nothing in it incompatible with mathematical reasoning; but it is a very remarkable coincidence that this should so happen, as the only mathematical laws of attraction which would lead to a similar conclusion, are that of nature, and that in which the force is directly proportional to the distance, or one resulting from the combination of these two laws.

Let us next examine the case when the attracted body is also a sphere of sensible magnitude. S and s being the two spheres, and a the distance of their centres, it has been shewn that S will attract every molecule of s with the same force as if it were condensed into its centre. Now the mass of a single molecule having been regarded as unity, S, the mass of the first sphere will be proportional to, and represent, the number of molecules it consists of. The attraction then of s on S will be the same as if the latter sphere were removed, and in its centre a single molecule placed, endowed with an attractive energy S times as great as that of any molecule, such as S actually consists of. But the attraction of s on one molecule of the last named kind has been shewn to be $\frac{s}{a^2}$, therefore its attraction on S being S times as forcible, will therefore be represented by $\frac{\mathrm{S}\times s}{a^2}$.

This expression represents the absolute force with which the two spheres tend to each other, or the number of pounds, grains, or other units, which must be opposed to it in order to hinder their approach. This in mechanical language, is called the *moving force;* and we therefore see, that *the moving force with which two homogeneous spheres attract each other, is as the product of their masses directly, and the square of the distances of their centres inversely.*

The moving force then is, of course, the same on each sphere; and in consequence of the equality of action and re-action, (which always refers to *moving force,*) it ought to be so. Were the spheres allowed to approach each other, however their velocities would obviously be different, the greater moving slower than the less. In fact, the accelerating force on any body being equal to the moving force divided by the mass moved, we have

$$\text{accelerating force on S}=\frac{s}{a^2}$$

$$\text{accelerating force on } s=\frac{\mathrm{S}}{a^2}.$$

Suppose now the two spheres at liberty in space, and moving, in consequence of their mutual attraction and any projectile force. If to both of them we apply an accelerating force $\frac{s}{a^2}$ equal to that exerted by s, but in an opposite direction, or towards S, the sphere S will be urged by forces destroying each other, and will therefore either remain at rest, or move uniformly in a right line: but the sphere s will now be urged by an accelerating force equal to $\frac{S+s}{a^2}$. Now, it is shown in mechanics, that the application of a common accelerating force to all the bodies of a system, does not alter their relative motions. Hence, if we refer the motions of our spheres, not to a fixed point in space, but to the centre of one of them S; or take that centre as the origin of our co-ordinates, we must then put $\frac{S+s}{a^2}$ for the accelerating force animating the other.*

Let us now consider the attractions of spheres not homogeneous, but composed of concentric strata varying in density according to any law of the distance from their centres. There is every reason to suppose this the actual constitution of the sun and planets; and it therefore becomes necessary to examine this case. Now, any stratum of infinitesimal thickness $d\,r$, may be regarded as the difference of two spheres s, and $s + d\,s$, homogeneous, and of the same density as the stratum, their radii being r and $r + d\,r$. The attraction of s on a molecule equal to 1 placed at a distance a, is $\frac{S}{a^2}$, and that of $s + d\,s$ is $\frac{s+d\,s}{a^2}$: consequently the attraction of the stratum is $\frac{s+d\,s}{a^2} - \frac{s}{a^2} = \frac{d\,s}{a^2}$; and therefore the same as if the stratum were collected in its centre. As this is true of every stratum separately, whatever be its density, and their attractions do not interfere, it will be true of all together; so that, whether the sphere be homogeneous, or composed of concentric layers, or strata of different density, the same property still holds good; and all we have demonstrated in the case of homogeneous spheres, remains true in this.

SECTION III.

Theory of elliptic motion.

We are now enabled to enter on the general theory of the planetary motions, but we will still confine ourselves to a case of comparative simplicity. The vast mass of the central body of our system, compared with those which circulate round it, permits us to regard their motions as influenced by it alone, and to neglect, in a first approximation, all the minute effects arising from the mutual attractions of the planets and satellites on each other. The case then we propose to consider in this section is that of the sun and a single planet, or a primary and one of its satellites.

Let M represent the mass of the sun or central body, and m that of the planet; and, fixing the origin of the co-ordinates in the centre of M, let x, y, z, represent the co-ordinates of m. Also let r be *the radius vector*, or line joining the centres of the two bodies; so that $r^2 = x^2 + y^2 + z^2$, and t the time (in seconds of mean solar time,) elapsed since any fixed epoch.

Fig. 4.

The accelerating force of M on m is $\frac{M}{r^2}$, and that of m on M $\frac{m}{r^2}$; and since we regard M as fixed, the latter quantity must be added to the force animating m, according to the observation made in the last section; so that the *relative* accelerating force acting on m in the direction of the radius vector m M will be $\frac{M+m}{r^2}$ which being resolved into forces in the directions m P, PQ, QM, of the three co-ordinates, becomes multiplied by $\frac{x}{r}$, $\frac{y}{r}$, and $\frac{z}{r}$, respectively, and produces the partial forces

$$P = \frac{(M+m)\,x}{r^3}, Q = \frac{(M+m)\,y}{r^3}, R = \frac{(M+m)\,z}{r^3}.$$

The effect of an accelerating force P acting, during an instant of time $d\,t$, on a body in a direction parallel to any given axis, that of the x, is to produce a variation in its velocity in that direction, which is to the variation gravity on the earth's surface would produce in the same time, as the force P is to the accelerating force of gravity which we will represent by unity. Now gravity producing the variation $g\,.\,d\,t$ in that time, the variation produced by the force P, will be $P\,.\,g\,d\,t$; or, if instead of taking one foot, as we have hitherto done, for the unit of linear measure, we take g $(= 32^{\text{ft.}}\cdot1908)$ for our standard unit, simply $P\,d\,t$. But to this the negative sign should be prefixed, as the force P tends to *diminish* the co-ordinate x. Again, the velocity in the direction x being $\frac{d\,x}{d\,t}$, its variation is $d\,\frac{d\,x}{d\,t}$; we have therefore $d\,\frac{d\,x}{d\,t} = -P\,d\,t$; or, supposing

$$R = \frac{M+m}{r^3}$$

and writing for P its value R x

$$d\frac{d\,x}{d\,t} + R\,x = o; \qquad (1)$$

similarly,

$$d\frac{d\,y}{d\,t} + R\,y = o; \qquad (2)$$

and

$$d\frac{d\,z}{d\,t} + R\,z = o; \qquad (3)$$

These equations contain the whole theory of the planetary motions, neglecting their mutual perturbations; and if, instead of supposing $R = \frac{M+m}{r^3}$, we suppose it equal to $\frac{\phi\,(r)}{r}$, the same equations will express the motion of a point m about a centre of force M, attracting it with a force represented by any function $\phi\,(r)$ of the distance M m.

If we eliminate R from the two first of these equations, by multiplying the first by y, and the second by $-x$, and adding, we get

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* If, however, we suppose the sphere S forcibly retained in its place by some external agency *not* acting on s, the case will be different; and $\frac{S}{a^2}$ will continue to represent the accelerating force on S.

$$y\,d\frac{dx}{dt} - x\,d\frac{dy}{dt} = o$$

that is,

$$\frac{y\,d^2x - x\,d^2y}{dt} = \frac{(y\,dx - x\,dy)\,d^2t}{dt^2}$$

or

$$\frac{y\,d^2x - x\,d^2y}{y\,dx - x\,dy} = \frac{d^2t}{dt}$$

Each member of this equation being a complete differential, because $y\,d^2x - x\,d^2y = d\,.\,(y\,dx - x\,dy)$, we get by integration

$$y\,dx - x\,dy = h\,dt; \qquad (4)$$

and similarly,

$$z\,dx - x\,dz = h'\,dt; \qquad (5)$$

$$z\,dy - y\,dz = h''\,dt; \qquad (6)$$

If, now, we multiply the first of these three equations (in which h, h', and h'', represent arbitrary constant quantities) by z, the second by $-y$, and the third by x, and add all together, we get

$$h\,z - h'\,y + h''\,x = o; \qquad (7)$$

which is the equation of a plane passing through the origin of the co-ordinates. Consequently, the curve described by the body is one of simple curvature, and its plane passes through the centre of attraction.

For simplicity, let us suppose the plane of the orbit to be coincident with that of the x and y, we have then $z = o$, and our equations are reduced to,

$$d\frac{dx}{dt} + R\,x\,dt = o$$

$$d\frac{dy}{dt} + R\,y\,dt = o$$

$$y\,dx - x\,dy = h\,dt$$

Fig. 5. The area of the elementary sector $M\,m\,m'$, described by the radius vector $M\,m$ in the instant dt, is equal to

$$M\,m\,m'\,P' - M\,m'\,P'$$

$$= M\,m\,P - M\,m'\,P' + P\,m\,m'\,P'$$

$$= -d\cdot\frac{xy}{2} + y\,dx = \frac{y\,dx - x\,dy}{2}$$

If, then, we call A the area described by the radius vector since the commencement of the time t, we have $y\,dx - x\,dy = 2\,d\,A$, and consequently

$$2\,d\,A = h\,dt;\ A = \frac{h}{2}\,t; \qquad (8)$$

this equation expresses the proportionality between the areas and times in Kepler's law; and since the process by which it is deduced, is independent of any particular value of R, R having been eliminated to obtain it, the analytical demonstration here given applies generally for all possible laws of central force.

In order next to investigate the nature of the curve described, we must eliminate t, which will be easiest done after a transformation of the co-ordinates. Let then the angle AM m described by the radius vector, since the origin of the time t, be called θ, and we have $m\,\mu = r\,d\theta$, and the elementary sector $m\,M\,m' = \frac{r \times r\,d\theta}{2}$, or $d\,A\,\frac{1}{2}\,r^2\,d\theta$, so that

$$r^2\,d\theta = h\,dt; \qquad (9)$$

Now, since $r^2 = x^2 + y^2$, we have

$$r\,dr = x\,dx + y\,dy$$

$$r\,d^2r + dr^2 = (x\,d^2x + y\,d^2y) + (dx^2 + dy^2)$$

so that (putting $ds = m\,m' = \sqrt{dx^2 + dy^2}$)

$$x\,d^2x + y\,d^2y = r\,d^2r + dr^2 - ds^2.$$

But $ds^2 - dr^2 = m\,m'^2 - m'\,\mu^2 = m\,\mu^2 = r^2\,d\theta^2$

whence, $x\,d^2x + y\,d^2y = r\,d^2r - r^2\,d\theta^2.$

This premised, since our equations (1) and (2) give

$$o = d^2x - \frac{dx}{dt}\,d^2t + R\,x\cdot dt^2$$

$$o = d^2y - \frac{dy}{dt}\,d^2t + R\,y\cdot dt^2$$

if we multiply the first of them by x, and the second by y, and add, we have

$$o = (x\,d^2x + y\,d^2y) - \frac{x\,dx + y\,dy}{dt}\,d^2t + R\,r^2\,dt^2 \qquad (10)$$

but if we take the logarithmic differential of the equation $h\,dt = r^2\,d\theta$, and suppose $d\theta$ constant, (which we are at liberty to do, having as yet taken no differential constant,) we get

$$\frac{d^2t}{dt} = 2\cdot\frac{dr}{r}.$$

So that, making this substitution for $\frac{d^2t}{dt}$, and for $x\,d^2x + y\,d^2y$, writing its value $r\,d^2r - r^2\,d\theta^2$, and $r\,dr$ for $x\,dx + y\,dy$, we get

$$o = r\,d^2r - r^2\,d\theta^2 - 2\,dr^2 + R\,r^2\,dt^2,$$

in which we have put for R its value $\frac{M + m}{r^3}$, and for dt, $\frac{r^2\,d\theta}{h}$, it becomes

$$o = r\,d^2r - r^2\,d\theta^2 - 2\,dr^2 + \frac{M + m}{h^2}\cdot r^3\,d\theta^2$$

Put $r = \frac{1}{u}$, and since $dr = -\frac{du}{u^2}$, and $d^2r = -\frac{d^2u}{u^2} + \frac{2\,du^2}{u^3}$, it becomes

$$\frac{d^2u}{d\theta^2} + u - \frac{M + m}{h^2} = o; \qquad (11)$$

This equation (being the simplest case of an equation of the second order of a linear form,) is immediately integrable, and gives

$$u = f\cos(\theta + g) + \frac{M + m}{h^2}; \qquad (12)$$

whence,

$$r = \frac{\frac{h^2}{M + m}}{1 + \frac{h^2 f}{M + m}\cdot\cos(\theta + g)}; \qquad (13)$$

f and g being two arbitrary constants.

In any conic section, if we call a the semiaxis major, $a\,(1 - e^2)$ the semiparameter, and $\theta + g$ the angle included between the radius r and the vertex nearest the focus from which r is supposed to take its origin, we have

$$r = \frac{a\,(1 - e^2)}{1 + e\cdot\cos(\theta + g)} \qquad (14)$$

Consequently we see that the curve described by the body must necessarily be a conic section, having the body M in the focus, and the relation between the arbitrary constants f, h, and the axis and semiparameter of this conic section will be (if we call p the semiparameter, or put $p = a\,(1 - e^2)$

$$f = \frac{e}{a\,(1 - e^2)} = \frac{e}{p}; \qquad (15)$$

$$h = \sqrt{a\,(1 - e^2)\,(\mathrm{M} + m)} = \sqrt{p\,(\mathrm{M} + m)}\,; \qquad (16)$$

If we suppose the angle θ to commence from the vertex nearest the focus, or from the *nearer apside*, or *perihelion* of the orbit, we have $g = o$, and

$$r = \frac{a\,(1 - e^2)}{1 + e\,.\cos\theta} = \frac{p}{1 + e\cos\theta}; \qquad (17)$$

If the value of e be less than unity, the conic section described is an ellipse, if equal and a infinite, a parabola, if greater, and a negative, an hyperbola.

But to complete the theory of the planetary motions, it is necessary to know, not only the nature of the orbit, in general, but also whereabouts in it the body will be at any moment assigned. For this purpose we must obtain a finite equation involving t, and either r or θ, or some functions of them. Now, the equation (9) gives $h\,d\,t$, or

$$\sqrt{a\,(1 - e^2)\,(\mathrm{M} + m)} \times d\,t = r^2\,d\,\theta$$

and substituting for r, its value in equation (17)

$$d\,t = \frac{a^{\frac{3}{2}}(1 - e^2)^{\frac{3}{2}}}{\sqrt{\mathrm{M} + m}} \cdot \frac{d\,\theta}{(1 + e\,.\cos\theta)^2} \qquad (18)$$

and

$$t + \mathrm{C} = \frac{a^{\frac{3}{2}}\,(1 - e^2)^{\frac{3}{2}}}{\sqrt{\mathrm{M} + m}} \cdot \int \frac{d\,\theta}{(1 + e\,.\cos\theta)^2}$$

To integrate this, take another variable v, such, that

$$\cos\theta = \frac{\cos v - e}{1 - e\,.\cos v}; \qquad (19)$$

whence,

$$\sin\theta = \frac{\sqrt{1 - e^2}\,.\sin v}{1 - e\,.\cos v}$$

$$d\,.\cos\theta = -\,d\,v\,.\,\frac{(1 - e^2)\,.\sin v}{(1 - e\,.\cos v)^2}$$

$$d\,\theta = -\,\frac{d\,.\cos\theta}{\sin\theta} = \frac{d\,v\,.\sqrt{1 - e^2}}{1 - e\,.\cos v}$$

$$1 + e\,.\cos\theta = \frac{1 - e^2}{1 - e\,.\cos v}$$

and substituting these expressions in the value of t above given, it will be found to reduce itself to the following very simple form:

$$t + \mathrm{C} = \frac{a^{\frac{3}{2}}}{\sqrt{\mathrm{M} + m}} \int d\,v\,(1 - e\,.\cos v)$$

so that, making

$$\frac{a^{\frac{3}{2}}}{\sqrt{\mathrm{M} + m}} = \frac{1}{n}; \qquad (20)$$

and taking the integral to commence, when $v = o$, or $\theta = o$, that is, from the instant of the body leaving the lower apside,

$$n\,t = v - e\,.\sin v\,; \qquad (21)$$

This equation fixes the relation between t and v; but that between v and θ may be expressed more conveniently for the purposes of calculation than by equation (19), as follows. By the equation last mentioned, we get

$$1 - \cos\theta = \frac{(1 + e)\,(1 - \cos v)}{1 - e\,.\cos v}$$

$$1 + \cos\theta = \frac{(1 - e)\,(1 + \cos v)}{1 - e\,.\cos v}$$

whence,

$$\frac{1 - \cos\theta}{1 + \cos\theta} = \frac{1 + e}{1 - e} \cdot \frac{1 - \cos v}{1 + \cos v}$$

or,

$$\tan\tfrac{1}{2}\,\theta = \sqrt{\frac{1 + e}{1 - e}} \cdot \tan\tfrac{1}{2}\,v\,; \qquad (22)$$

Finally, we obtain immediately the relation between r and v, by substituting, in equation (17) for $\cos\theta$, its value in (19), when we find

$$r = a\,(1 - e\,.\cos v)\,; \qquad (23)$$

These equations comprise the whole theory of the motions of bodies in conic sections. Equation (17) exhibits the relation between r and θ, or the polar equation to the curve; while (21), (22), and (23), express relations between the time, t, the *true anomaly* (as it is called) θ, and the *radius vector*, r, respectively, and an auxiliary angle v, to which the name of the *eccentric anomaly* has been given. The quantity $n\,t$, when reduced into angular measure, by multiplying it by 180°, and dividing it by $\pi = 3{\cdot}14159$, &c. is called also the *mean anomaly*. The eccentric anomaly may be exhibited geometrically, as follows:

On the major axis AB of the ellipse, let a semicircle A m' B be described, and draw the ordinate P $m\,m'$ through m. Then, AM m will be the true anomaly of the body m, and AC m' the eccentric. For, if we take CP $= x$, Fig. 6.

$$\cos \mathrm{AC}\,m' = -\,\frac{\mathrm{CP}}{\mathrm{C}\,m'} = -\,\frac{x}{a}$$

$$\cos \mathrm{AM}\,m = -\,\frac{\mathrm{MP}}{\mathrm{M}\,m} = -\,\frac{a\,e + x}{a + e\,x}$$

because, by the property of the ellipse, $\mathrm{M}\,m = a + e\,x$. Hence, we have

$$\frac{1 - \cos \mathrm{AM}\,m}{1 + \cos \mathrm{AM}\,m} = \frac{(1 + e)\,(a + x)}{(1 - e)\,(a - x)}$$

$$= \frac{1 + e}{1 - e} \times \frac{1 + \dfrac{x}{a}}{1 - \dfrac{x}{a}}$$

$$= \frac{1 + e}{1 - e} \cdot \frac{1 - \cos \mathrm{AC}\,m'}{1 + \cos \mathrm{AC}\,m'}$$

and, consequently,

$$\tan\tfrac{1}{2}\,\mathrm{AM}\,m = \frac{1 + e}{1 - e} \times \tan\tfrac{1}{2}\,\mathrm{AC}\,m'$$

Let T represent the time of one entire revolution in the ellipse, or the periodic time, then, as θ increases from 0 to 360° (or $2\,\pi$) t increases from 0 to 360 ($2\,\pi$) also. Consequently, by (21) we get

$$n\,\mathrm{T} = 2\,\pi\,; \qquad \mathrm{T} = \frac{2\pi}{n}$$

or

$$\mathrm{T} = \frac{2\,\pi\,.\,a^{\frac{3}{2}}}{\sqrt{\mathrm{M} + m}}; \qquad (24)$$

The periodic times, then, of several bodies revolving about the same central body, are, in the sesquiplicate ratio of the major axes of their orbits, (or mean distances from the central body,) directly, and in the subduplicate ratio of the sum of the masses of the revolving and central body inversely. The mass of the sun being enormously great compared with those of the planets, we may neglect m in comparison with M; and M being the same for the whole system, we have $T \propto a^{\frac{3}{2}}$; $T^2 \propto a^3$ which is no other than Kepler's third law.

The periodic times of the planets being very exactly known, we might expect to find in equation (24) the means of ascertaining the masses of the planets, supposing that of the sun, and any one of them, known. Thus, if m, m', be the masses of two planets, and T, T′, their periodical times, we have

$$\left(\frac{T'}{T}\right)^2 = \left(\frac{a'}{a}\right)^3 \times \frac{M+m}{M+m'} \qquad (25)$$

On applying calculation, however, all we learn from this relation is, that the resulting masses are so small, as to be incapable of accurate determination by this method; their values, as deduced from it, being materially affected by the small uncertainties still prevailing as to the lengths of the periods, and by the mutual perturbations of the sun and planets. There is a case, however, where it may be used with advantage, viz. in that of a planet accompanied by a satellite. If we call M the sun, and m the planet, and neglect the mass of the latter in comparison with the former, and that of the satellite in comparison with the primary, we have at once

$$\left(\frac{T'}{T}\right)^2 = \left(\frac{a'}{a}\right)^3 \times \frac{M}{m}; \quad \frac{m}{M} = \left(\frac{a'}{a}\right)^3 \times \left(\frac{T}{T'}\right)^2$$

Thus, in the case of the earth, we have

$$\frac{a'}{a} = \frac{60{\cdot}23799}{23405}; \quad \frac{T}{T'} = \frac{365{\cdot}25638}{27{\cdot}32167}$$

So that, by executing the computations, we find

$$\frac{m}{M} = 0{\cdot}00000304697 = \frac{1}{328196}$$

To find v in terms of t, or to calculate the eccentric (and thence the true) anomaly at any given instant, we must resolve the transcendental equation,

$$n\,t = v - e\,.\sin v$$

This can only be done in a series; and fortunately, in the case of the planets, e is so small, that a series ascending by powers of e will converge sufficiently. Now we have

$$v = n\,t + e\,.\sin v$$

and, since e is small, and $\sin v$ necessarily less than 1, $n\,t$ itself expresses the value of v within a limit less than e, and is therefore a first approximation. Again, if in $\sin v$ we write its value for v or $(n\,t + e\,.\sin v)$, we get

$$v = n\,t + e\,.\sin(n\,t + e\,.\sin v)$$

So that,

$$v = n\,t + e\,.\sin n\,t$$

is an approximation carried one step farther, or to the first power of e. Let this be again substituted for e, and we have

$$v = n\,t + e\,.\sin\{n\,t + e\,.\sin n\,t\}$$

But (neglecting the squares and higher powers of e) we have

$$\sin\{n\,t + e\,.\sin n\,t\} = \sin n\,t + e\,.\sin n\,t\,.\cos n\,t$$
$$= \sin n\,t + \frac{e}{2}\sin 2\,n\,t$$

So that we find

$$v = n\,t + e\,.\sin n\,t + \frac{e^2}{2}\sin 2\,n\,t \qquad (26)$$

If we again repeat the process, we get an approximation, pushed to the third power of e, and so on, as far as we please.

For numerical calculation, however, the equation $v = n\,t + e\,.\sin v$ furnishes the readiest solution; as we have only to reduce e into seconds, (taking 57° 17′ 44″·8 for the arc equal to radius or 1, or adding the logarithm 5·3144251 to the logarithm of e, which gives at once that of the number of seconds e is equal to,) and assuming $n\,t$ (the given mean anomaly) for a first approximation, correct it successively, as in the following example.

Required Jupiter's eccentric anomaly corresponding to 53° of mean anomaly.

Here $n\,t = 53°$, and in Jupiter's orbit we have

$e = 0{\cdot}048077$ log. . 8·6819374
5·3144251

$e = 9916''{\cdot}6$; log. . 3·9963625
Take $v = 53°$. . . . . . log sin 53° 9·9023486

$e\,.\sin v =$ 2° 12′ 0″ = 7919″·8; 3·8987111
$n\,t = 53$

$v = 55$ 12 0 Corrected value, with which resuming the process.

log sin v = 9·9144221
e. . . . . . 3·9963625

2° 15′ 43″ = 8143″·0. . . . log. . . . 3·9107846
53° $= n\,t$

55 15 43. . . . Second corrected value.

Another repetition of the very same process gives v = 55° 15′ 49″·1, which is true within 0″·2.

But even this process, simple as it is, becomes tedious for the orbit of Mercury, and those of the new planets, Pallas and Juno, in which the value of e is not *very* small; and here we must have recourse to the well known method of trial and error, which may be applied in this case as follows:—Having assumed by estimation a value of v, (neglecting minutes and seconds,) by noticing whether the term $e\,.\sin v$ is additive or subtractive, and increasing or diminishing $n\,t$ accordingly; calculate the value of $v - e\,.\sin v$ for that and the next *subsequent* degree; and let the values so found be called V, and V′: should either of these be exactly equal to the proposed value of $n\,t$, the corresponding value of v will be the truth; but as this will probably never happen, we have only to say,

$$V' - V : V' - n\,t :: -3600'' : x = -\frac{3600''}{V' - V} \times (V' - n\,t)$$

which correction being applied with its proper sign to the *latter* of the two assumed values of v, will give an approximate value. Let the value of $v - e\,.\sin v$ be again computed with this value of v, and call the result V″. This will always be found *very* nearly equal to $n\,t$; but if not exactly so, the correction

$$-\frac{3600}{V' - V}(V'' - n\,t)$$

must be computed and applied to the new value, and so on.

For instance: Let $n\,t = 332°\ 28'\ 65''$, and $e = 50600'' = 14°\ 3'\ 20''$. Here $\sin n\,t$ is negative; so that $e\,.\sin n\,t$ is subtractive, and v must be less than $n\,t$. Take then for the two values of v, 325° and 326° respectively, and compute as follows:—

$$\sin 325^\circ - 9\cdot7585913 \text{ (neg)}$$
$$\log e'' \ldots\ldots 4\cdot7041513$$
$$4\cdot4627426 \text{ (neg)}$$
$$e \,.\, \sin 325^\circ = -\,29023''\cdot0 = -\,8^\circ\, 3'\, 43''\cdot0$$

Hence
$$V = 333^\circ\, 3'\, 43''\cdot0$$
$$V' - V = 2872''\cdot2$$

$$\sin 326^\circ - 9\cdot7475617 \text{ (neg)}$$
$$4\cdot7041513$$
$$4\cdot4517130 \text{ (neg)}$$
$$e \,.\, \sin 326^\circ = -\,28295\cdot2 = -\,7^\circ\, 51'\, 35''\cdot2$$
$$V' = 333^\circ\, 51'\, 35''\cdot2$$
$$V' - n\,t = -\,4960\cdot2$$

$$\text{Log } 3600 \ldots\ldots\ldots\ldots 3\cdot5563025$$
$$\text{Log } 2872\cdot2 \ldots\ldots\ldots 3\cdot4582147$$
$$0\cdot0980878$$
$$\text{Log } 4960\cdot2 \ldots\ldots\ldots 3\cdot6954992$$
$$\text{Log } 6217\cdot1 \ldots\ldots\ldots 3\cdot7935870$$
$$v = 326^\circ - 6217''\cdot1 = 324^\circ\, 16'\, 23''$$

Taking this for a new value of v, we find, by another repetition of the process,

$$V'' = 332^\circ\, 28'\, 49''\cdot6\,; \quad V'' - n\,t = -\,5''\cdot4$$

and, $-\dfrac{3600}{V' - V} \times (V'' - n\,t) = +\,6''\cdot768$

so that $\quad v = 324^\circ\, 16'\, 29''\cdot768$

which is true to the hundredth of a second; and this case is nearly the worst that can occur in the theory of the planets.

If we substitute the value of v given in equation (26) in the expression for r, $r = a\,(1 - e\,.\cos v)$; (23), and develope in powers of e, we obtain r in a series of powers of e, and cosines of $n\,t$, and its multiples,

$$r = a\left\{1 - e\,.\cos n\,t + \frac{e^2}{2}(1 - \cos 2\,n\,t) - e^3 \times \&c.\right\};$$

Finally, if we resolve the equation

$$\tan \tfrac{1}{2}\,\theta = \sqrt{\frac{1+e}{1-e}}\,.\,\tan \tfrac{1}{2}\,v$$

we get

$$\theta = v + e\,.\,\sin v + \frac{e^2}{4}\,.\,\sin 2\,v + \&c.$$

wherein, if we write for v its value, we shall obtain

$$\theta = n\,t + 2\,e\,.\,\sin n\,t + \frac{5}{4}\,e^2\,.\,\sin 2\,n\,t + \&c.$$

To carry these series to the higher powers of e, and to ascertain their law, we must have recourse to particular theorems for facilitating such developements, but these will suffice for our present purpose; and the reader who wishes to proceed farther in the investigation, will find in the second book of the *Mécanique Céleste*, arts. xx. xxi. xxii. every information he can desire.

We shall content ourselves here with merely setting down the formulæ, which are as follows:—

$$\begin{aligned} v = n\,t + e\,.\,\sin n\,t &+ \frac{e^2}{1\,.\,2\,.\,2}\,.\,2\,.\,\sin 2\,n\,t \\ &+ \frac{e^3}{1\,.\,2\,.\,3\,.\,2^2}\left\{3^2\,.\,\sin 3\,n\,t - 3\,.\,\sin n\,t\right\} \\ &+ \frac{e^4}{1\,.\,2\,.\,3\,.\,4\,.\,2^3}\left\{4^3\,.\,\sin 4\,n\,t - 4\,.\,2^3\,.\,\sin 2\,n\,t\right\} \\ &+ \frac{e^5}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,2^4}\left\{5^4\,.\,\sin 5\,n\,t - 5\,.\,3^4 \sin 3\,n\,t + \frac{5\,.\,4}{1\,.\,2}\,.\,\sin n\,t\right\} \\ &+ \&c.\,; \end{aligned} \qquad (27)$$

$$\begin{aligned} \frac{r}{a} = 1 + \frac{e^2}{2} - e\,.\cos n\,t &- \frac{e^2}{2}\cos 2\,n\,t \\ &- \frac{e^3}{2\,.\,4}\left\{3\,.\,\cos 3\,n\,t - 3\,.\,\cos n\,t\right\} \\ &- \frac{e^4}{2\,.\,4\,.\,6}\left\{4^2\,.\,\cos 3\,n\,t - 4\,.\,2^2\,.\,\cos 2\,n\,t\right\} \\ &- \frac{e^5}{2\,.\,4\,.\,6\,.\,8}\left\{5^3\,.\,\cos 5\,n\,t - 5\,.\,3^3\,.\,\cos 3\,n\,t + \frac{5\,.\,4}{1\,.\,2}\,.\,\cos n\,t\right\} \\ &- \&c.\,; \end{aligned} \qquad (28)$$

$$\theta = v + 2\,\epsilon : \sin v + \frac{2\epsilon^2}{2}\,.\,\sin 2\,v + \frac{2\epsilon^3}{3}\,.\,\sin 3\,v + \&c.\,; \qquad (29)$$

where $\epsilon = \dfrac{e}{1 + \sqrt{1 - e^2}}$ and is, of course, a fraction smaller than e. Also, if we neglect powers of e higher than the fifth,

$$\left.\begin{aligned} \theta = n\,t + \left\{2\,e - \frac{1}{4}e^3 + \frac{5}{96}e^5\right\}.\,\sin n\,t + \left\{\frac{5}{4}\,e^2 - \frac{11}{24}e^4\right\}\sin 2\,n\,t \\ + \left\{\frac{13}{12}\,e^3 - \frac{43}{64}\,e^5\right\}\sin 3\,n\,t + \frac{103}{96}\,e^4\,.\,\sin 4\,n\,t + \frac{1097}{960}\,e^5\,.\,\sin 5\,n\,t \end{aligned}\right\}; \qquad (30)$$

SECTION IV.

On the velocities of the planetary motions, and the determination a priori of the elements of their orbits.

The angular velocity of a body is measured by the angle which it appears to describe in any very small time to the eye of a spectator. In fact, if we call θ the angle, and t the time, we have

$$\text{Angular velocity about the sun} = \frac{d\theta}{dt}$$

Now, by our 9th equation, we have $r^2 d\theta = h\,dt$, and, consequently, $\frac{d\theta}{dt} = \frac{h}{r^2}$; that is to say, the angular velocity in any orbit is inversely as the square of its distance from the centre; and this law is general for all central forces.

The paracentric velocity is the approach to, or recess from, the centre; and is measured by $\frac{dr}{dt}$. Now, $\frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt}$, but

$$\frac{dr}{d\theta} = \frac{a(1-e^2) \cdot e \sin\theta}{(1+e \cdot \cos\theta)^2} = \frac{e \cdot \sin\theta}{a(1-e^2) \cdot r^2}$$

and, consequently,

$$\frac{dr}{dt} = \frac{h e \cdot \sin\theta}{a(1-e^2)\,r^4} = \sqrt{\frac{M+m}{a(1-e^2)}} \cdot \frac{e \cdot \sin\theta}{r^4}; \qquad (31)$$

To complete our knowledge of the body's motion, we must inquire its linear velocity at any instant. To this end, if we call V the velocity, $V = \frac{ds}{dt}$, ds being the element of the curve. Now, first, we must remark, that if we write for dt its value $\frac{y\,dx - x\,dy}{h}$, this gives

$$V = h \cdot \frac{ds}{y\,dx - x\,dy}; \qquad (32)$$

But $\frac{y\,dx - x\,dy}{ds}$ expresses the length of a *perpendicular dropped, or a tangent to the curve from the origin of the co-ordinates*. Thus we see, that *the velocity is inversely as the perpendicular so let fall, and directly as the quantity* h, *or the area described in a given time.*

Moreover, we have $ds^2 = dr^2 + r^2 d\theta^2$; so that, writing for dr its equal $-\frac{du}{u^2}$, or $-r^2 du$,

$$\left(\frac{ds}{dt}\right)^2 = r^2 \left(\frac{d\theta}{dt}\right)^2 \left\{ 1 + r^2 \cdot \left(\frac{du}{d\theta}\right)^2 \right\}$$

Now, by reason of the equation $r^2 d\theta = h\,dt$, we have

$$r^2 \left(\frac{d\theta}{dt}\right)^2 = \frac{h^2}{r^2}$$

Again, if we differential the equation (12), we find

$$\begin{aligned} \left(\frac{du}{d\theta}\right)^2 &= f^2 \cdot \sin(\theta+g)^2 \\ &= f^2 - f^2 \cdot \cos(\theta+g)^2 \\ &= f^2 - \left(u - \frac{M+m}{h^2}\right)^2, \quad \text{(by equation 12)} \\ &= \frac{e^2}{p^2} - \left(u - \frac{1}{p}\right)^2, \quad \text{(by equations 15, 16)} \\ &= \frac{e^2-1}{p^2} + \frac{2}{pr} - \frac{1}{a} \end{aligned}$$

writing for u its value $\frac{1}{r}$. Consequently, we find

$$\begin{aligned} 1 + r^2\left(\frac{du}{d\theta}\right)^2 &= \left(\frac{2}{pr} - \frac{1-e^2}{p^2}\right) \cdot r^2 \\ &= \frac{r^2}{p}\left(\frac{2}{r} - \frac{1}{a}\right) \end{aligned}$$

because $p = a(1-e^2.)$ Hence, we finally obtain

$$\left(\frac{ds}{dt}\right)^2 = \frac{h^2}{p}\left(\frac{2}{r} - \frac{1}{a}\right)$$

or

$$V^2 = (M+m) \cdot \left(\frac{2}{r} - \frac{1}{a}\right); \qquad (33)$$

On the determination of the elements of the planetary orbits, *a priori*, there is no occasion to enter into any very extensive discussion, in a practical point of view. Since, however, it is a subject generally touched upon in astronomical works, and is not without its interest, when we consider what changes in our own system may have taken place, or may yet take place from the action of violent causes, we shall devote a part of our space to its consideration.

Let us suppose, then, a body of a given mass, to be launched in space from a given point, with a velocity and direction also given; and to be attracted by another body, whose mass is also given with a force inversely as the square of the distance: it is required to determine the form, magnitude, and position of the conic section it will describe.

The plane passing through the attracting body and the primitive direction of projection will, of course, be that in which the orbit will lie, there being no force to draw the body out of this plane. Taking then the central body for the origin of two co-ordinates, x and y, lying in this plane, and retaining all the other denominations of the foregoing pages, and considering, as the unit of velocity, that with which a body would describe a space equal to 32·1908 feet in one second, we shall have, by (29),

$$v^2 = (M+m)\left(\frac{2}{r} - \frac{1}{a}\right)$$

and, consequently,

$$\frac{1}{a} = \frac{2}{r} - \frac{v^2}{M+m} \qquad (34)$$

Now, by hypothesis, the masses of the central and revolving body are given: and the distance from the centre, as well as the velocity with which the latter is projected. If, then, we suppose the quantities in this equation to correspond to the point of projection, r is this distance, and V, M, m, are known; so that we have at once the value of a, the semiaxis major of the orbit.

This result is a remarkable one. It shews us, that the major axis of the orbit is independent of the angle of inclination to the radius at which the original projection takes place: in other words, that any number of bodies (of equal, or of exceedingly small magnitudes compared with the central one,) launched from one point with equal velocities, but in any different directions, will all describe orbits having equal major axes. Another result, not less curious, follows from this—that they will all describe conic sections of the same nature, that is, all ellipses, or all parabolas, or all hyperbolas—for the nature of the conic section depends only on the algebraic sign of its major axis.

Thus, if a be positive, or if V, the velocity of projection, be *less* than $\sqrt{\frac{2(M+m)}{r}}$, the orbit will be an ellipse. If equal, a will be infinite $\left(\frac{1}{a} = o\right)$, and the orbit will be a parabola; but if greater, a is negative, and the orbit will be an hyperbola

In the two latter cases, the bodies will never return to their original point of departure, but in the former they will do so: and since the periodic time depends only on the major axes and masses, if their masses be either all equal, or all extremely small, they will all return in the same time, and a collision will take place; after which it is impossible to say what will happen. If, on the other hand, there exists any sensible proportion between the revolving and central bodies, and any considerable inequality in their masses, their periods will be unequal, and each may perform its orbit undisturbed.

This is supposing their mutual attractions to be neglected. In fact, however, at the instant of their departure, these may be incomparably greater than that of the central body, and will then materially change their velocities, unless the latter be so great as speedily to carry them beyond the sphere of their mutual influence. If a small portion of the earth, for instance, were suddenly projected from its surface, the attraction of the earth on it would, at the moment of its departure, exceed that of the sun in the ratio of 3522 : 2·17399, or upwards of 1600 : 1. So that, in the first instants of its motion, it would move as if influenced by the earth alone. But this effect would diminish rapidly; by the time the projectile had reached the distance of the moon, the sun's action would already have a preponderance (as we have seen in section I,) in the ratio of 2·17399 : 1, and it would depend entirely on the relative velocity of projection, whether such a space could be described in a time small enough to escape the influence of the earth or not.

It has been a matter of some speculation, whether the small planets between Mars and Jupiter may not have had their origin in the destruction, by violence, of some larger mass once revolving in the situation they now occupy. The very considerable approximation of their periodic times, which, in the case of Ceres and Pallas is singularly near (within $\frac{1}{300}$ of the whole period,) and the equally remarkable fact of the mutual intersections of their orbits falling all in the same part of the heavens, (in a general way,) have given rise to this surmise; and it has even been conjectured, that an explosive rupture of a former planet may have scattered its fragments far and wide over our system, and produced these singular bodies. There is no limit to conjecture; but if any such event have taken place, we are forced to conclude, that the mass of the ruptured planet must have been very small, or the fragments must have collapsed by their mutual attraction; or, at least, their velocities would have been so materially modified by it, as to obliterate all traces of their once having had a common velocity about the sun. The smallness of the ruptured mass renders the supposition of an explosion less revolting; and we know, at least from observation, that the fragments (if such) are extremely minute.

If the orbit be a circle, we have $r = a$, and $\frac{2}{r} - \frac{1}{a} = \frac{1}{r}$; so that, for the velocity in a circle, we have $V^2 = \frac{M+m}{r}$, $V = \frac{\sqrt{M+m}}{\sqrt{r}}$; that is to say, the velocity in different circles is in the sub-duplicate ratio of the sum of the masses, or the absolute force, as it is sometimes called, directly, and of the radii inversely. Moreover, if we denote by V the velocity in a circle of the radius r, or $V = \sqrt{\frac{M+m}{r}}$, we have

$$\frac{v^2}{V^2} = \frac{\frac{2}{r} - \frac{1}{a}}{\frac{1}{r}} = 2 - \frac{r}{a} = \frac{2a - r}{a}$$

or $$v^2 : V^2 :: 2a - r : a.$$

Now, if APM be an ellipse, S, H, the foci, AM = $2a$ and SP = r, we have HP = $2a - r$; so that Fig. 7.

$$v : V :: \sqrt{HP} : \sqrt{AR}$$

by which property the velocity in a conic section may be immediately compared with that in a circle at the same distance.

Hence, when SP = AC, or at the extremities of the conjugate axis, the velocity is equal to that in a circle.

In the parabola, we have $\frac{v^2}{V^2} = 2$, or $v = V \cdot \sqrt{2}$, so that in this curve the velocity bears a constant ratio to that in a circle at the same distance, $\sqrt{2} : 1$.

In the hyperbola, HP increases without limit, and the velocity bears continually a greater and greater ratio to that in a circle.

As the velocity depends only on the distance and major axis of the conic section, and not at all on its form, we may conceive the conjugate axis so diminished that the conic section shall pass into a straight line. In this case, the extremity of the axis will coincide with the focus, and the velocity at any distance r will be that acquired by falling from a distance $2a$ from the centre to the distance r. The expression for this velocity is therefore still the same with that in the conic section. Hence, "The velocity in a conic section at any point is that which would be acquired by falling freely towards the centre, from a distance equal to the longer axis, to that point." In the parabola the longer axis is infinite, and the velocity at any point is, therefore, that acquired by falling from an infinite distance. In the hyperbola the axis is negative, and even an infinite fall is not sufficient to give a body all the velocity requisite for the description of this curve.

We have then the following expressions :—

Velocity in a conic section, semiaxis = a, distance = r; $v = \sqrt{(\mathrm{M}+m)\left(\frac{2}{r}-\frac{1}{a}\right)}$

———— circle, whose radius is r. . . . . $\mathrm{V} = \sqrt{\frac{\mathrm{M}+m}{r}}$

———— parabola at any distance r, ... $v = \sqrt{\frac{2(\mathrm{M}+m)}{r}}$ (35)

Let us now consider the effect on the form of the conic section and on the position of its major axis, arising from a change not in the velocity, but in the *angle of projection*, i. e. the angle made with the radius vector by the direction in which the body is projected.

Fig. 7. APM being any conic section, and S, H, its foci, SY, SZ, perpendiculars on a tangent at P, the angle of projection SPy, which we will call A, is equal to HPZ, and therefore

$$\mathrm{SY} = \mathrm{SP} \,.\, \sin \mathrm{A}, \text{ and } \mathrm{HZ} = \mathrm{HP} \,.\, \sin \mathrm{A}.$$

consequently,

$$\mathrm{SY} \times \mathrm{HZ} = \mathrm{SP} \,.\, \mathrm{HP} \,.\, \sin \mathrm{A}^2;$$

but, by the property of the conic sections, SY × HZ = CD^2; and therefore CD^2 = SP . HP . sin A^2. Now, we have. SP = r, HP = $2a - r$, CD$^2 = a^2(1-e^2)$;

hence, $a^2(1-e^2) = r(2a-r) \,.\, \sin \mathrm{A}^2$

and $a(1-e^2) = r \times \left(2 - \frac{r}{a}\right) . \sin \mathrm{A}^2.$

But $a(1-e^2)$ is the semiparameter of the conic section. Moreover, calling v the velocity in the curve, and V that in a circle; and denoting by n the ratio $\left(\frac{v}{\mathrm{V}}\right)^2$ we have already seen, that $\left(2 - \frac{r}{a}\right) = n^2$. Hence, we have

$$a(1-e^2) = r \times n^2 \,.\, \sin \mathrm{A}^2; \qquad (36)$$

and therefore is given, when the angle and distance of projection are given, and also the velocity. When the angle of projection varies, other circumstances remaining, we see hence that the parameter varies as the square of its sine.

The eccentricity is easily found; for we have

$$e = \sqrt{1 - \frac{r}{a}\left(2 - \frac{r}{a}\right) . \sin \mathrm{A}^2}$$

Now $\frac{r}{a} = 2 - \frac{r v^2}{\mathrm{M}+m}$; but since $v^2 = n^2 \mathrm{V}^2 = n^2 \,.\, \frac{\mathrm{M}+m}{r}$, we have $\frac{r v^2}{\mathrm{M}+m} = n^2$, and $\frac{r}{a} = 2 - n^2$; $2 - \frac{r}{a} = n^2$. so that we get by substitution

$$\begin{aligned} e &= \sqrt{1 - n^2(2-n^2) \,.\, \sin \mathrm{A}^2} \\ &= \sqrt{\cos \mathrm{A}^2 + \sin \mathrm{A}^2 (1-n^2)^2}; \end{aligned} \qquad (37)$$

We see, therefore, that the ratio e of the eccentricity to the semiaxis, or the figure of the ellipse, or hyperbola, depends solely on the angle of projection and the ratio of the velocity of projection to that in a circle at the same distance; and if this latter ratio remain the same, the distance may be varied to any extent without changing the figure of the conic section.

It only remains to determine its position, or the angle made by the greater axis (or line of apsides) with the distance SP.

Now, if we call ASP, θ, we have

$$r = \frac{a(1-e^2)}{1 + e \,.\, \cos \theta};$$

$$\cos \theta = \frac{1}{e} \times \left\{\frac{a(1-e^2)}{r} - 1\right\}.$$

in which, substituting for e and $a(1-e^2)$, their values before found, we shall obtain

$$\cos \theta = \frac{n^2 \,.\, \sin \mathrm{A}^2 - 1}{\sqrt{\cos \mathrm{A}^2 + (1-n^2)^2 \,.\, \sin \mathrm{A}^2}} \qquad (38)$$

on which value we may make the same remark as on that of e, and in both which it will be recollected, that $n^2 = \frac{r v^2}{\mathrm{M}+m}$.

SECTION V.

On certain peculiar cases of the celestial motion; viz. when the orbit is of very great, or very small eccentricity. Of circular and parabolic motion.

The circumstances of circular motion are too simple to need much consideration. The velocity is uniform and equal to that which would be acquired by falling down half the radius with the force at the circumference continued uniformly. This is evident, if we recollect that the velocity in a circle, by what has been just proved, is $\sqrt{\frac{\mathrm{M}+m}{r}}$, but that acquired by the action of a constant force F, acting through a space $\frac{1}{2} r$, is given by the equation $v^2 = 2 \mathrm{F} s = \mathrm{F} r$ in the present case $= \frac{(\mathrm{M}+m) r}{r^2} = \frac{\mathrm{M}+m}{r}$. The periodic time is equal to the circumference divided by this constant velocity, or to $\frac{2 \pi r^{\frac{3}{2}}}{\sqrt{\mathrm{M}+m}}$, as we have before shewn by a more general reasoning.

In a conic section of so small an eccentricity that its square may be neglected, it may be worth while to recapitulate some of the chief formulæ, developed to the first power. We have then

$$\left.\begin{aligned} r &= a(1 - e \,.\, \cos \theta) \\ &= a(1 - e \,.\, \cos v) \\ &= a(1 - e \,.\, \cos n t) \end{aligned}\right\}; \qquad (39)$$

$$\left.\begin{aligned} \theta &= v - e \,.\, \sin v \\ \theta &= n t + 2 e \,.\, \sin n t \end{aligned}\right\}; \qquad (40)$$

$$v = n t + e \,.\, \sin n t; \qquad (41)$$

$$n t = \theta - 2 e \,.\, \sin \theta; \qquad (42)$$

Let us consider the motion in a parabola, and since in this case we have $a = \infty$, $e = 1$, and $p = a(1-e^2)$, if we call the perihelion distance D, we have $\mathrm{D} = \frac{1}{2} p$, and $p = 2 \mathrm{D}$. Then will the equations (17) and (18) become

$$r = \frac{2 \mathrm{D}}{1 + \cos \theta} = \frac{\mathrm{D}}{\left(\cos \frac{\theta}{2}\right)^2}; \qquad (43)$$

$$dt = \frac{(2\,\mathrm{D})^{\frac{3}{2}}}{\sqrt{\mathrm{M}+m}} \cdot \int \frac{d\,\theta}{(1+\cos\theta)^2}$$

To integrate the latter, put $\tan\frac{\theta}{2} = x$, and we have

$$\tan\theta = \frac{2\,x}{1-x^2}$$

$$\cos\theta = \frac{1-x^2}{1+x^2}$$

$$d\,\theta = \frac{2\,d\,x}{1+x^2}\,,$$

which substituted, give

$$d\,t = \frac{(2\,\mathrm{D})^{\frac{3}{2}}}{\sqrt{\mathrm{M}+m}} \times \left(\frac{d\,x}{2} + \frac{x^2\,d\,x}{2}\right)$$

$$t = \frac{\mathrm{D}^{\frac{3}{2}}\,.\,\sqrt{2}}{\sqrt{\mathrm{M}+m}} \left(x + \frac{x^3}{3}\right)$$

or,

$$t = \frac{\mathrm{D}^{\frac{3}{2}}\,.\,\sqrt{2}}{\sqrt{\mathrm{M}+m}} \times \left\{\tan\frac{\theta}{2} + \frac{1}{3}\left(\tan\frac{\theta}{2}\right)^3\right\}; \quad (44)$$

This equation gives $\tan\frac{\theta}{2}$, and consequently θ, by the resolution of a cubic equation of the form

$$x^3 + 3\,x = \mathrm{A}$$

where $\mathrm{A} = \frac{3\,t\,.\,\sqrt{\mathrm{M}+m}}{\mathrm{D}^{\frac{3}{2}}\,.\,\sqrt{2}}$. This cubic, it is easily shewn, has but one real root; and since the co-efficient of x is an absolute number, and therefore x a function of A only, its root may be found in any proposed case by a table of single entry. In fact, suppose we have formed such a table, containing the values of x (or, for greater convenience, of 2 × arc ($\tan = x$) or θ) for every value of t on the supposition of $\mathrm{D} = 1$, then will this table serve for all cases, and for every other value of D; for we have only to multiply t by $\mathrm{D}^{-\frac{3}{2}}$; and calling the product T, look out in the table for the value of θ corresponding to the time T. A comet, describing a parabola, whose perihelion distance is 1, will describe 90° of anomaly from the perihelion in about 109 days. Hence, the use of the *table of a comet of* 109 *days* given in works on astronomy, Lalande's *Astron.* 2d edit. vol. iii. p. 335; Delambre's *Astronomy*, vol. iii. p. 434, from which we extract it as subjoined.

Table of a comet of 109 *days.*

Days	True anomaly.	Days	True anomaly.	Days	True anomaly.	Days	True anomaly.	Days	True anomaly.	Days	True anomaly.
0	0° 0′ 0″0	35	43°58′34″8	70	71°51′23″2	105	88°20′42″9	140	98°56′22″5	175	106°20′14″2
1	1 23 37·4	36	44 59 57·5	71	72 27 4·4	106	88 42 42·6	141	99 11 12·9	176	106 30 59·4
2	2 47 11·9	37	46 0 26·6	72	73 2 13·4	107	89 4 26·0	142	99 25 54·3	177	106 41 39·4
3	4 10 40·4	38	47 0 2·2	73	73 36 51·0	108	89 25 53·4	143	99 40 26·9	178	106 52 14·1
4	5 34 0·1	39	47 58 44·8	74	74 10 57·8	109	89 47 5·1	144	99 54 50·9	179	107 2 43·5
5	6 57 8·0	40	48 56 34·5	75	74 44 34·3	110	90 8 1·3	145	100 9 6·4	180	107 13 7·7
6	8 20 1·3	41	49 53 32·0	76	75 17 41·2	111	90 28 42·4	146	100 23 13·4	181	107 23 26·9
7	9 42 37·2	42	50 49 37·6	77	75 50 18·9	112	90 49 8·7	147	100 37 12·2	182	107 33 41·1
8	11 4 53·0	43	51 44 51·8	78	76 22 28·1	113	91 9 20·3	148	100 51 2·8	183	107 43 50·2
9	12 26 45·9	44	52 39 15·2	79	76 54 9·3	114	91 29 17·6	149	101 4 45·4	184	107 53 54·6
10	13 48 13·4	45	53 32 48·2	80	77 25 23·1	115	91 49 0·7	150	101 18 20·1	185	108 3 54·1
11	15 9 13·1	46	54 25 31·6	81	77 56 10·1	116	92 8 30·0	151	101 31 47·1	186	108 13 48·8
12	16 29 42·5	47	55 17 25·8	82	78 26 30·6	117	92 27 45·7	152	101 45 6·5	187	108 23 38·8
13	17 49 39·4	48	56 8 31·5	83	78 56 25·3	118	92 46 48·0	153	101 58 18·2	188	108 33 24·2
14	19 9 1·5	49	56 58 49·4	84	79 25 54·6	119	93 5 37·3	154	102 11 22·6	189	108 43 5·0
15	20 27 46·8	50	57 48 20·1	85	79 54 59·1	120	93 24 13·6	155	102 24 19·6	190	108 52 41·3
16	21 45 53·4	51	58 37 4·3	86	80 23 39·1	121	93 42 37·2	156	102 37 9·4	191	109 2 13·1
17	23 3 19·4	52	59 25 2·7	87	80 51 55·4	122	94 0 48·4	157	102 49 52·1	192	109 11 40·4
18	24 20 3·1	53	60 12 15·9	88	81 19 48·1	123	94 18 47·3	158	103 2 27·8	193	109 21 3·5
19	25 36 2·9	54	60 58 44·8	89	81 47 17·9	124	94 36 34·2	159	103 14 56·6	194	109 30 22·2
20	26 51 17·3	55	61 44 29·9	90	82 14 25·2	125	94 54 9·2	160	103 27 18·5	195	109 39 36·7
21	28 5 45·1	56	62 29 32·1	91	82 41 10·3	126	95 11 32·7	161	103 39 33·8	196	109 48 46·9
22	29 19 25·0	57	63 13 52·0	92	83 7 33·7	127	95 28 44·7	162	103 51 42·4	197	109 57 53·1
23	30 32 15·8	58	63 57 30·3	93	83 33 35·9	128	95 45 45·4	163	104 3 44·6	198	110 6 55·1
24	31 44 16·7	59	64 40 27·7	94	83 59 17·2	129	96 2 35·2	164	104 15 40·3	199	110 15 53·1
25	32 55 26·7	60	65 22 45·0	95	84 24 38·0	130	96 19 14·0	165	104 27 29·7	200	110 24 47·1
26	34 5 45·2	61	66 4 22·8	96	84 49 38·7	131	96 35 42·2	166	104 39 12·8		
27	35 15 11·4	62	66 45 22·0	97	85 14 19·8	132	96 51 59·8	167	104 50 49·8		
28	36 23 44·8	63	67 25 43·1	98	85 38 41·6	133	97 8 7·1	168	105 2 20·7		
29	37 31 24·9	64	68 5 26·9	99	86 2 44·4	134	97 24 4·2	169	105 13 45·6		
30	38 38 11·5	65	68 44 34·0	100	86 26 28·7	135	97 39 51·3	170	105 25 4·7		
31	39 44 4·2	66	69 23 5·1	101	86 49 54·7	136	97 55 28·6	171	105 36 17·9		
32	40 49 2·9	67	70 1 1·0	102	87 13 2·8	137	98 10 56·2	172	105 47 25·4		
33	41 53 7·6	68	70 38 22·1	103	87 35 53·3	138	98 26 14·3	173	105 58 27·2		
34	42 56 18·2	69	71 15 9·3	104	87 58 26·5	139	98 41 23·0	174	106 9 23·4		
35	43 58 34·8	70	71 51 23·2	105	88 20 42·9	140	98 56 22·5	175	106 20 14·2		

2dly. Let us now consider the circumstances of the motion of a body in an ellipse of great eccentricity, or approaching very near to a parabola. This is necessary to complete the theory of such comets as are known, or may be suspected, to revolve in very elongated ellipses.

Since e, the ratio of the eccentricity, approaches very near to unity, $1 - e$ must be a very small quantity; and if we put $\beta = \frac{1-e}{1+e}$, we shall have $e = \frac{1-\beta}{1+\beta}$, $1 - e = \frac{2\beta}{1+\beta}$, $1 + e = \frac{2}{1+\beta}$, $(1 - e^2) = \frac{4\beta}{(1+\beta)^2}$; now, if D = the perihelion distance, we have $D = a\,(1 - e)$, consequently

$$r = \frac{D\,(1+e)}{2 + e\cos\theta} = \frac{2\,D}{(1+\beta) + (1-\beta)\,.\cos\theta} = \frac{D}{\frac{1+\cos\theta}{2} + \beta\,.\frac{1-\cos\theta}{2}}$$

$$= \frac{\frac{D}{\left(\cos\frac{\theta}{2}\right)^2}}{1 + \beta\,.\,\frac{1-\cos\theta}{1+\cos\theta}} = \frac{D}{\left(\cos\frac{\theta}{2}\right)^2\left(1+\beta\,.\,\left(\tan\frac{\theta}{2}\right)^2\right)}$$

and developing this in powers of β, we obtain

$$r = \frac{D}{\left(\cos\frac{\theta}{2}\right)^2}\left\{1 - \beta\,.\left(\tan\frac{\theta}{2}\right)^2 + \beta^2\,.\left(\tan\frac{\theta}{2}\right)^4 - \&c.\right\}; \qquad (45)$$

which is a very simple expression for the radius vector in the elongated ellipse.

To find the time, we must make the same substitution in the expression,

$$t + C = \frac{a^{\frac{3}{2}}\,(1-e^2)^{\frac{3}{2}}}{\sqrt{M+m}}\,.\int^{\circ}\frac{d\,\theta}{(1+e\,.\cos\theta)^2}$$

Now $\{a\,.\,(1-e^2)\}^{\frac{3}{2}} = \left(\frac{2\,D}{1+\beta}\right)^{\frac{3}{2}}$ and since

$$(1 + e\,.\,\cos\theta)^2 = \frac{1}{(1+\beta)^2}\{1 + \beta + (1-\beta)\,.\cos\theta\}^2$$

$$= \frac{4}{(1+\beta)^2}\left\{\frac{1+\cos\theta}{2} + \beta\,.\frac{1-\cos\theta}{2}\right\}^2$$

$$= \frac{4\,.\left(\cos\frac{\theta}{2}\right)^2\left\{1 + \beta\,.\left(\tan\frac{\theta}{2}\right)^2\right\}^2}{(1+\beta)^2}$$

if we put $\tan\frac{\theta}{2} = T$, we shall have $\frac{\frac{1}{2}\,d\,\theta}{\left(\cos\frac{\theta}{2}\right)^2} = d\,T$; and the integral will become $\frac{(1+\beta)^2}{2}\,.\int\frac{d\,T}{(1+\beta\,.\,T^2)^2}$

whence we get after all substitution,

$$t + C = \sqrt{\frac{2\,(1+\beta)}{M+m}}\,.\,D^{\frac{3}{2}}\,.\int\frac{d\,T\,(1 + T^2)}{(1+\beta\,.\,T^2)^2}$$

and the part of this under the integral sign being developed in powers of β, and the whole integrated, we get

$$t + C = D^{\frac{3}{2}}\,.\sqrt{\frac{2\,(1+\beta)}{M+m}} \times \left\{\left(T + \frac{T^3}{3}\right) - \frac{2}{1}\,\beta\left(\frac{T^3}{3} + \frac{T^5}{5}\right) + \frac{2\,.\,3}{1\,.\,2}\,\beta^2\left(\frac{T^5}{5} + \frac{T^7}{7}\right) - \&c.\right\} \qquad (46)$$

By the aid of this series, we may express very readily the difference between the true anomaly in a very eccentric ellipse, and that in a parabola of equal perihelion distance, for, T still expressing the tangent of half the former true anomaly, and τ that of the latter, if we develope according to powers of β, pursuing the process only as far as β^2, we get

$$t + C = \frac{D^{\frac{3}{2}}\sqrt{2}}{\sqrt{M+m}}\left\{\left(T + \frac{T^3}{3}\right) + \beta\left(\frac{T}{2} - \frac{T^3}{2} - \frac{2T^5}{5}\right) - \beta^2\left(\frac{T}{8} + \frac{3\,T^3}{8} - \frac{2\,T^5}{5} - \frac{3\,T^7}{7}\right)\right\}; \qquad (47)$$

Now, the time being the same both in the parabola and ellipse, this must equal $\frac{D^{\frac{3}{2}}\sqrt{2}}{\sqrt{M+m}}\left\{\tau + \frac{\tau^3}{3}\right\}$, and therefore the series within the brackets on both sides must be equal.

Let

$$\tau + \frac{\tau^3}{3} = a;\ \frac{\tau}{2} - \frac{\tau^3}{2} - \frac{2\,\tau^5}{5} = b;\ -\frac{\tau}{8} - \frac{3\,\tau^3}{8} + \frac{2\,\tau^5}{5} + \frac{3\,\tau^7}{7} = c$$

$$T + \frac{T^3}{3} = A,\ \frac{T}{2} - \frac{T^3}{2} - \frac{2\,T^5}{5} = B;\ -\frac{T}{8} - \frac{3\,T^3}{8} + \frac{2\,T^5}{5} + \frac{3\,T^7}{7} = C$$

then we have
$$A + B\beta + C\beta^2 = a;$$
and the first member of this equation, though apparently a function of β, cannot be really so, as the second does not contain it. Hence, if we call Δ the difference between the elliptic and parabolic anomaly, or suppose the former anomaly $\theta + \Delta$ the latter θ, since $T = \tan \frac{\theta + \Delta}{2}$, and $\tau = \tan \frac{\theta}{2}$, we may express A, B, and C, thus :—

$$A = a + \frac{da}{d\theta} \cdot \frac{\Delta}{1} + \frac{d^2 a}{d\theta^2} \cdot \frac{\Delta^2}{1 \cdot 2} + \&c.$$
$$B = b + \frac{db}{d\theta} \cdot \frac{\Delta}{1} + \frac{d^2 b}{d\theta^2} \cdot \frac{\Delta^2}{1 \cdot 2} + \&c.$$
$$C = c + \frac{dc}{d\theta} \cdot \frac{\Delta}{1} + \frac{d^2 c}{d\theta^2} \cdot \frac{\Delta^2}{1 \cdot 2} + \&c.$$

but Δ itself is required to be expressed in terms of β and its powers; and, consequently we may suppose $\Delta = P\beta + Q\beta^2 + \&c.$

If then we make these substitutions, our equation becomes
$$\begin{aligned} a = a + P\frac{da}{d\theta}\beta + \left(Q\frac{da}{d\theta} + \frac{P^2}{2} \cdot \frac{d^2 a}{d\theta^2}\right)\beta^2 + \&c. \\ + b\beta + P\frac{db}{d\theta} \cdot \beta^2 + \&c. \\ + c\beta^2 + \&c. \end{aligned}$$
whence we obtain by equating the co-efficients of β to zero,
$$o = P\frac{da}{d\theta} + b; \; o = Q\frac{da}{d\theta} + \frac{P^2}{2} \cdot \frac{d^2 a}{d\theta^2} + P \cdot \frac{db}{d\theta} + c; \; \&c.$$
Now we have $\frac{da}{d\theta} = \frac{da}{d\tau} \cdot \frac{d\tau}{d\theta}$, and since $a = \tau + \frac{\tau^3}{3}$, $\frac{da}{d\tau} = 1 + \tau^2$, moreover $\frac{d\tau}{d\theta} = \frac{1 + \tau^2}{2}$, whence $\frac{da}{d\theta} = \frac{(1 + \tau^2)^2}{2}$, and therefore $P = \frac{1}{(1 + \tau^2)^2}\left\{-\tau + \tau^3 + \frac{4}{5}\tau^5\right\}$.

Again,
$$\frac{d^2 a}{d\theta^2} = \frac{d}{d\theta} \cdot \frac{da}{d\theta} = 2\tau(1 + \tau^2) \cdot \frac{d\tau}{d\theta} = \tau(1 + \tau^2)^2$$
$$\frac{db}{d\theta} = \frac{d}{d\theta}\left\{\frac{\tau}{2} - \frac{\tau^3}{2} - \frac{2}{5}\tau^5\right\} = \left\{\frac{1}{2} - \frac{3\tau^2}{2} - 2\tau^4\right\} \cdot \frac{1 + \tau^2}{2}$$
whence, after all reductions, we obtain
$$Q = \frac{\frac{3}{4}\tau - \frac{5}{4}\tau^3 + \frac{1}{20}\tau^5 + \frac{167}{140}\tau^7 + \frac{24}{35}\tau^9 + \frac{18}{175}\tau^{11}}{(1 + \tau^2)^4}$$
and, finally,
$$\left.\begin{aligned} \Delta = \frac{-\tau + \tau^3 + \frac{4}{5}\tau^5}{(1 + \tau^2)^2} \cdot \beta \\ + \frac{\frac{3}{4}\tau - \frac{5}{4}\tau^3 + \frac{1}{20}\tau^5 + \frac{167}{140}\tau^7 + \frac{24}{35}\tau^9 + \frac{18}{175}\tau^{11}}{(1 + \tau^2)^4} \cdot \beta^2; \end{aligned}\right\} \quad (48)$$

SECTION VI.

On the determination of the planetary orbits *a posteriori*, or from observation.

The determination of the elements of the orbit of a planet, or comet, is justly regarded as one of the most difficult problems of astronomy. By the *elements* of the orbit are meant those constant data which determine the position, form, and magnitude of the conic section described, and the body's place in it at a given epoch, and are six in number, viz.

The inclination of its plane to the ecliptic,
The longitude of its ascending node,
The position of its axis, or the longitude of the perihelion,
The length of its axis,
The eccentricity, or the length of the semiparameter, or the perihelion distance,
The moment of passing the perihelion, or the heliocentric longitude at any assigned time,

which correspond to, and are functions of the six arbitrary constants introduced by the integration of the equations (1), (2), and (3). If we knew, and could subject to calculation, the causes which originally determined the motions of the heavenly bodies, we could assign, *a priori*, by the means already pointed out, the values of these constants from the conditions which subsisted at the commencement of their motions; but as these are, and must ever remain, unknown, we can only discover the values in question, *a posteriori*, by observation of the apparent places of each body at different times.

In this research, the great difficulties arise from our not observing from a fixed station in the centre of our system. To an observer stationed in the sun, the elements would offer themselves with comparative ease.

But the motion and eccentric position of the earth render the whole affair infinitely more complicated; and to analyse the problem, it becomes necessary to take into consideration the relations between the heliocentric and geocentric places of a heavenly body.

To this effect, let S be the sun, E the earth, P the planet, or comet, projected into p upon the ecliptic by the perpendicular Pp. Let $S\gamma$ be the line of equinoxes, parallel to which draw $E\gamma$. Then will

Fig. 8.

$L = \gamma SE$ represent the earth's *heliocentric* longitude.
$l = \gamma Sp$ ——— the planet's heliocentric longitude.
$\lambda = \gamma Ep$ ——— the planet's geocentric longitude.
$b = PSp$ ——— the planet's heliocentric latitude.
$\beta = PEp$ ——— its geocentric latitude.
$R =$ the earth's distance from the sun.
$r =$ the planet's distance from the sun.
$\rho =$ the planet's distance from the earth.
$d, \delta =$ the respective *curtate*, or projected distances of the planet from the sun and earth.

It is obvious, then, that we shall have the following relations :—

1st. $Pp = r \,.\, \sin b$, also the same $Pp = \rho \,.\, \sin \beta$, so that,

$$r \,.\, \sin b = \rho \,.\, \sin \beta; \tag{49}$$

2dly. $$\begin{cases} d = r \,.\, \cos b & (50) \\ \delta = \rho \,.\, \cos \beta & (51) \end{cases}$$

3dly. Since the angle PSE of the triangle PSE is equal to $l - L$, and its sides are R, d, and δ

$$\delta^2 = R^2 + d^2 - 2 R d \cos (l - L); \tag{52}$$

Lastly producing SE to C, the angle $p EC = p E \gamma - CS\gamma = \lambda - L$, but the angle $p E c = 180° - p E S$, consequently the equation

$$d^2 = R^2 + \delta^2 - 2 R \delta \,.\, \cos p E S$$

gives

$$d^2 = R^2 + d^2 + 2 R \delta \,.\, \cos (\lambda - L); \tag{53}$$

Thus then we have five distinct and independent relations among the ten quantities L, l, λ, &c. Now, it is evident, that if R and L be given, (or the place of E fixed) the geocentric latitude and longitude, β and λ, will be determined when the place of P is, but this is not determined unless l, d, and b, are so. But when all the five quantities L, R, l, d, b, are given, the five equations above deduced suffice to determine the others, and therefore express all the relations subsisting between the heliocentric and geocentric places. The last may, however, be simplified by substituting for δ^2 its value given in (52), when the whole becomes divisible by 2 R, and gives

$$o = R + \delta \,.\, \cos (\lambda - L) - d \,.\, \cos (l - L); \tag{54}$$

Another equation (included, of course, in the foregoing,) may also be obtained if we consider, that in the triangle PSE, we have

$$r^2 = R^2 + \rho^2 - 2 R \rho \,.\, \cos PES$$

but $\cos PES = \cos p ES \times \cos PE p$ by spherical trigonometry, the plane PEp being at right angles to pES. Hence, we get (writing $\frac{\rho'}{\cos \beta}$,) or $\rho' \,.\, \text{Sec} \beta$ for ρ, and $- \cos (\lambda - L)$ for $\cos p ES$

$$r^2 = R^2 + \delta^2 \,.\, \text{Sec} \beta^2 + 2 R \delta \,,\, \cos (\lambda - L); \tag{55}$$

In like manner, eliminating ρ from (49), it becomes

$$r \,.\, \sin b = \delta \,.\, \tan \beta; \tag{56}$$

and putting in (54) for d, its value $r \,.\, \cos b$,

$$o = R + \delta \,.\, \cos (\lambda - L) - r \,.\, \cos b \,.\, \cos (l - L) \tag{57}$$

This equation may also be derived independently, if we drop the perpendicular $p q$, for we then have

$$\begin{aligned} S q &= SP \,.\, \cos PS p \,.\, \cos p S q \\ &= r \,.\, \cos b \,.\, \cos (l - L) \end{aligned}$$

and also

$$\begin{aligned} S q &= SE + E q = \\ &= R + \delta \,.\, \cos (\lambda - L) \end{aligned}$$

which equated, give the equation in question. We may also derive another, which will be useful in the sequel, by equating two values of the perpendicular itself.

$$\begin{aligned} p q &= E p \,.\, \sin p E q \\ p q &= S p \,.\, \sin p S q \end{aligned}$$

which equated, give

$$r \,.\, \cos b \,.\, \sin (l - L) = \delta \,.\, \sin (\lambda - L); \tag{58}$$

Thus we have reduced the five equations to four, 55, 56, 57, 58, which contain only the radius vector of the earth, that of the planet, and its curtate distance from the earth, and the heliocentric and geocentric latitudes and longitudes of the two bodies.

The problem of determining the elements of P's orbit from observation requires these relations to be combined with the conditions of P's motion and of the earth's. Now, if we call ☊ the longitude of P's ascending node, and i the inclination of its orbit, we have, by spherical trigonometry, for the heliocentric latitude.

$$\tan b = \tan i \,.\, \sin (l - \text{☊}) \tag{59}$$

Moreover, if we call π the longitude of the perihelion on the orbit, θ the true anomaly, and ψ the angle ☊ SA, or the distance of the node from the perihelion on the orbit, (see fig. 9) we shall have Fig. 9.

$$\cos A \text{☊} a = \tan \text{☊} S a \,.\, \cot \text{☊} SA$$

or $$\tan (\pi - \text{☊}) = \cos i \,,\, \tan \psi \tag{60}$$

and also

$$\sin b = \sin i \,.\, \sin (\psi + \theta) \tag{61}$$

these seven equations, of which the four first, 55, 56, 57, 58, express the relations between the heliocentric and geocentric places of a body any how situated, the other three, 59, 60, 61, merely express the condition of every point in the orbit lying in a plane passing through the sun. These must be combined with the dynamical results peculiar to the planetary motions, viz. 1st. The proportionality of the areas to the times; and, 2dly, The nature of the orbit, regarded as an ellipse or parabola of unknown position, form, and magnitude; all which are included in the three equations, 21, 22, 23, or others equivalent to them. Thus we have a system of nine equations, representing, in general, the relations between the time elapsed since a given epoch, and the observed geocentric latitude and longitude.

These equations involve, as constant quantities, the elements of the orbit, i, ☊, π, a, e, E, while the variable ones, being either given by observation, or expressed in functions of others that are so, the whole system of equations may be regarded as expressing a known (though a very complicated) relation among the elements; and as each observation affords a similar on, it is evident that the complete determination latter cannot require more than six observations. t, however, from the peculiar nature of some of quations, fewer will suffice, half that number being necessary.

e complication of the relations in question pres all idea of a direct solution of the problem, o apply them to any particular case indirect and ximative ones have been invented, but with assistance from such simplification; and, after e force of analysis has been exercised upon it, it emains a very difficult problem, and one which mits will by no means permit us to enter upon full extent. In fact, the case of an elliptical to be determined is one of very rare occurrence; e shall therefore content ourselves with pointing e course to be pursued to obtain the readiest ximation to the elements of parabolic motion. frequent visits of comets to our system render n important case. These singular bodies, with r two exceptions, have always been found to ibe parabolas, or ellipses of such extreme eccen-, as to be undistinguishable from parabolas n considerable distances on either side their elion.

paratory to this research, we must premise the ing remarkable theorem.

a parabola, if r, r' denote any two distances from cus, k the chord of the arc they include, and t me of a body describing that arc, and g a certain ant the same for all parabolas, we shall have

$$t = (r + r' + k)^{\frac{3}{2}} - (r + r' - k)^{\frac{3}{2}}; \qquad (62)$$

ew this, let θ and θ' be the true anomalies cornding to the radii r, r', and t being the time of bing the intermediate arc, and ϵ the time elapsed the perihelion passage to the commencement of ime t, we have, by (44)

$$\epsilon = q\left\{\tan\frac{\theta}{2} + \frac{1}{3}\left(\tan\frac{\theta}{2}\right)^3\right\}$$

$$t + \epsilon = q\left\{\tan\frac{\theta'}{2} + \frac{1}{3}\left(\tan\frac{\theta'}{2}\right)^3\right\}$$

$q = \frac{D^{\frac{3}{2}}\sqrt{2}}{\sqrt{M+m}}$. Consequently, we have

$$t = q \,.\left\{\left(\tan\frac{\theta'}{2} - \tan\frac{\theta}{2}\right) + \frac{1}{3}\left(\left(\tan\frac{\theta'}{2}\right)^3 - \left(\tan\frac{\theta}{2}\right)^3\right)\right\}$$

by the property of the parabola, we have also

$$r = \frac{D}{\left(\cos\frac{\theta}{2}\right)^2} \quad r' = \frac{D}{\left(\cos\frac{\theta'}{2}\right)^2}$$

f we put $\frac{\theta' - \theta}{2} = \phi$, we shall have

$$\frac{r}{r'} = \frac{r}{\sqrt{r r'}} = \frac{\cos\frac{1}{2}\theta'}{\cos\frac{1}{2}\theta} = \frac{\cos(\frac{1}{2}\theta + \phi)}{\cos\frac{1}{2}\theta}$$

$$\frac{r}{\sqrt{r r'}} = \cos\phi - \tan\tfrac{1}{2}\theta \,.\, \sin\phi$$

ce we get

$$\frac{\theta}{2} = \text{cosec.}\,\phi\left\{\cos\phi - \frac{r}{\sqrt{r r'}}\right\} \qquad (63)$$

Similarly, since $\frac{1}{2}\theta = \frac{1}{2}\theta' - \phi$, we have,

$$\sqrt{\frac{r'}{r}} \text{ or } \frac{r'}{\sqrt{r r'}} = \frac{\cos(\frac{1}{2}\theta' - \phi)}{\cos\frac{1}{2}\theta'}$$

and therefore

$$\tan\frac{\theta'}{2} = -\,\text{cosec.}\,\phi\left\{\cos\phi - \frac{r'}{\sqrt{r r'}}\right\}$$

but k being the side of a triangle opposite to the angle $(\theta' - \theta)$ or 2ϕ and r and r' the sides adjacent, we have, by trigonometry,

$$\cos\phi = \frac{\sqrt{(r + r')^2 - k^2}}{2\sqrt{r r'}}$$

$$\sin\phi = \frac{\sqrt{k^2 - (r - r)^2}}{2\sqrt{r r'}}$$

or, if we put

$$\sqrt{(r + r')^2 - k^2} = R \text{ and } \sqrt{k^2 - (r - r)^2} = S$$

$$R = 2\sqrt{r r'} \,.\, \cos\phi \text{ and } S = 2\sqrt{r r'} \,.\, \sin\phi$$

$$\cos\phi = \frac{R}{2\sqrt{r r'}} \quad \sin\phi = \frac{S}{2\sqrt{r r'}}$$

Hence, we obtain

$$\tan\frac{\theta}{2} = \frac{R - 2r}{S}\,; \quad \tan\frac{\theta'}{2} = -\frac{R - 2r'}{S}$$

$$\tan\frac{\theta'}{2} - \tan\frac{\theta}{2} = \frac{2(r + r' - R)}{S}$$

also

$$\left(\cos\frac{\theta}{2}\right)^2 = \frac{1}{1 + \left(\tan\frac{\theta}{2}\right)^2} = $$

$$= \frac{S^2}{(R^2 + S^2) - 4rR + 4r^2}$$

but $R^2 + S^2 = (r + r')^2 - (r' - r)^2 = 4rr'$, so that we get

$$\left(\cos\,.\,\frac{\theta}{2}\right)^2 = \frac{1}{4r} \,.\, \frac{S^2}{r + r' - R}$$

and, in consequence,

$$D = r\,.\left(\cos\frac{\theta}{2}\right)^2$$

$$= \frac{1}{4} \,.\, \frac{S^2}{r + r' - R}$$

Now $(r + r' - R)(r + r' + R) = (r + r')^2 - R^2 = k^2$; so that

$$D = \frac{S^2}{4k^2} \,.\, (r + r' + R)$$

and $$D\left\{\tan\frac{\theta'}{2} - \tan\frac{\theta}{2}\right\} = \frac{S}{2}$$

consequently

$$\frac{D^{\frac{3}{2}} \,.\, \sqrt{2}}{\sqrt{M + m}}\left\{\tan\frac{\theta'}{2} - \tan\frac{\theta}{2}\right\} = \frac{S^2 \,.\, \sqrt{2(r + r' + R)}}{4k \,.\, \sqrt{M + m}};$$

Now, if we put T and T′ for $\tan\frac{\theta}{2}$ and $\tan\frac{\theta'}{2}$, we have $T'^3 - T^3 = (T' - T)\{T'^2 + TT' + T^2\}$ and therefore

$$t = \frac{D^{\frac{3}{2}}\sqrt{2}}{\sqrt{M + m}}(T' - T)\left\{1 + \frac{T^2 + TT' + T'^2}{3}\right\}$$

The part without the brackets we have already considered. Let us next examine that within. Now, if in $T^2 + TT' + T'^2$, we substitute for T and T′, their values $\frac{R - 2r}{S}$ and $\frac{-R + 2r'}{S}$, it will become

$$\frac{R^2 - 2R(r + r') + 4(r^2 - rr' + r'^2)}{S^2}$$

and substituting this, the quantity within the brackets becomes

$$\frac{3S^2 + R^2 + 4(r^2 - rr' + r'^2) - 2R(r + r')}{3S^2}$$

But $R^2 + S^2 = 4rr'$, therefore $3S^2 + R^2 = 12rr' - 2R^2 = 2k^2 - 2r^2 + 8rr' - 2r'^2$. So that our expression reduces itself to

$$\frac{2}{3S^2}\{(r + r')^2 + k^2 - (r + r').R\}$$

$$= \frac{2}{3S^2}\{k^2 + (r + r')(r + r' - R)\}$$

and therefore we have, finally,

$$t = \frac{\sqrt{2(r + r' + R)}}{6k.\sqrt{M + m}}\{k^2 + (r + r')(r + r' - R)\}$$

but $\sqrt{r + r' + R}.(r + r' - R) =$

$$= \sqrt{(r + r' + R).(r + r' - R)^2}$$

$$= k.\sqrt{r + r' - R}$$

because $(r + r' + R)(r + r' - R) = k^2$. So that substituting, we get

$$= \frac{k.\sqrt{r + r' + R} + (r + r'):\sqrt{r + r' - R}}{6.\sqrt{M + m}}.\sqrt{2}$$

Now $R = \sqrt{(r + r')^2 - k^2}$, so that the two radicals in the above expression are of the form

$$\sqrt{a \pm \sqrt{a^2 - k^2}}$$

putting a for $r + r'$. Hence the usual rule for extracting the root of a binomial surd applies to them, the root being, as it is shewn in all books of algebra,

$$\frac{\sqrt{a + k} \pm \sqrt{a - k}}{\sqrt{2}}$$

Consequently, we have

$$t = \frac{1}{6\sqrt{M + m}}\left\{\begin{array}{l} k(\sqrt{r + r' + k} + \sqrt{r + r' - k}) \\ + (r + r')(\sqrt{r + r' + k} - \sqrt{r + r' - k}) \end{array}\right\}$$

or,

$$6\sqrt{M + m} \times t = (r + r' + k)^{\frac{3}{2}} - (r + r' - k)^{\frac{3}{2}}; \quad (62)$$

which agrees with the proposition as announced, if we take $g = 6\sqrt{M + m}$.

The relation just proved to subsist between the time of describing any parabolic arc, and the rectilinear distances of its two extremities from each other, and from the sun, is very useful, as it enables us from any three of these quantities given, to find the fourth without knowing either the perihelion distance, or the position of the perihelion with respect to the arc described. In an analytical point of view, it is certainly remarkable for the length and complexity of the transformations by which it is obtained from the general principles of parabolic motion. It appears to admit of no simple demonstration, and that above given, tedious as it may seem, can hardly be replaced by one materially shorter.

We shall now proceed to show how this relation may be rendered available in computing the elements of a comet's orbit.

Let x, y, z, be the co-ordinates of the comet's place at the epoch of a first observation, x', y', z', their values after any time t and x'', y'', z'', their values after any other times t' has elapsed. Let X, Y, Z, X', Y', Z', and X'', Y'', Z'', be the corresponding co-ordinates of the earth's place. (If the plane of the ecliptic be chosen for that of the X and Y, we shall have $Z = o$, $Z' = o$, $Z'' = o$.) Also let

$$t' - t = t''$$

then will t'' be the time elapsed between the second and third observations.

Since the comet moves in a plane passing through the sun, we have

$$z = ax + by \qquad (64, 1)$$

$$z' = ax' + by' \qquad (64, 2)$$

$$z'' = ax'' + by'' \qquad (64, 3)$$

from which, if we eliminate a and b; first, from the first and second; secondly, from the first and third; and lastly, from the second and third; we shall get the three following

$$o = x(y''z' - y'z'') + x'(yz'' - y''z) + x''(y'z - yz') \qquad (65, 1)$$

$$o = y(z''x' - z'x'') + y'(zx'' - xz'') + y''(xz' - zx') \qquad (65, 2)$$

$$o = z(x''y' - x'y'') + z'(xy'' - x''y) + z''(x'y - xy') \qquad (65, 3)$$

Now $zy' - z'y$ represents the double area of the projection of the plane triangle included between the radii, r, r' drawn to the place of the comet at the first and second observations, and the chord k of the parabolic arc joining their extremities on the plane of the z and y; similarly, $zy'' - z''y$ represents double the triangle formed by the projections of the radii r, r'' and the chord k' joining their extremities on the same plane; and if we denote by k'' the chord joining the extremities of the radii r', r'', the double of the projection of the triangle included between r', r'', and k'' will be represented by $z'y'' - z''y'$. If, therefore, we call the surfaces of these triangles respectively

$$S, S', S''$$

Since the projection of each is equal to the surface multiplied by the cosine of its inclination to the plane of the z, y, which is the same for all, we shall have the following equation instead of (65,1),

$$xS'' - x'S' + x''S = o \qquad (66, 1)$$

and similarly,

$$yS'' - y'S' + y''S = o \qquad (66, 2)$$

$$zS'' - z'S' + z''S = o \qquad (66, 3)$$

Now, if we retain the denominations of the foregoing pages, and assume the line of the equinoxes S♈ for the axis of the x, we have

$$x = S\pi = Se + e\pi$$

$$= R.\cos L + \delta.\cos\lambda; \qquad (67, 1)$$

$$x' = R'.\cos L' + \delta'.\cos\lambda'; \qquad (67, 2)$$

$$x'' = R''.\cos L'' + \delta''.\cos\lambda''; \qquad (67, 3)$$

also

$$y = Ee + pq = R.\sin L + \delta.\sin\lambda; \qquad (67, 4)$$

$$y' = R'.\sin L' + \delta'.\sin\lambda'; \qquad (67, 5)$$

$$y'' = R''.\sin L'' + \delta''.\sin\lambda''; \qquad (67, 6)$$

We have also

$$z = \delta \,.\, \tan\beta \qquad (67,7)$$

$$z' = \delta' \,.\, \tan\beta' \qquad (67,8)$$

$$z'' = \delta'' \,.\, \tan\beta'' \qquad (67,9)$$

and by the substitution of these values we shall obtain equations, in which, instead of the co-ordinates x, y, z, &c. the curtate distances δ, δ', δ'', will be involved, viz.

$$\left.\begin{array}{l} o = S''(\delta \,.\, \cos\lambda + R \,.\, \cos L) \\ \;\; - S'(\delta' \,.\, \cos\lambda' + R' \,.\, \cos L') \\ \;\; + S\,(\delta'' \,.\, \cos\lambda'' + R'' \,.\, \cos L'') \end{array}\right\} ; \qquad (68,1)$$

$$\left.\begin{array}{l} o = S''(\delta \,.\, \sin\lambda + R \,.\, \sin L) \\ \;\; - S'(\delta' \,.\, \sin\lambda' + R' \,.\, \sin L') \\ \;\; + S\,(\delta'' \,.\, \sin\lambda'' + R'' \,.\, \sin L'') \end{array}\right\} \qquad (68,2)$$

$$o = S'' \,.\, \delta \,.\, \tan\beta - S' \,.\, \delta' \,.\, \tan\beta' + S \,.\, \delta'' \,.\, \tan\beta'' ; \qquad (68,3)$$

R, L, λ, and β, and their accented values, are all quantities given by the solar tables, or by observation at the three assigned instants, so that the only unknown quantities contained in these equations are the three curtate distances, δ, δ', δ'', and the areas S, S', S''; and by elimination, any one of the three former may be expressed in terms of the latter. Let this process be executed then in succession for each of the quantities δ, δ', δ'', and we shall have the following results:

$$(a \,.\, \delta + AR) \,.\, S'' - A'R' \,.\, S' + A''R'' \,.\, S = o ; \qquad (69,1)$$

$$BR \,.\, S'' - (a \,.\, \delta' + B'R')\, S' + B''R''S = o ; \qquad (69,2)$$

$$CR \,.\, S'' - C'R' \,.\, S' + (a \,.\, \delta'' - C''R'')\, S = o ; \qquad (69,3)$$

where, for brevity's sake, we have made the following substitutions—

$$\left.\begin{array}{l} a = \tan\beta \;\,.\, \sin(\lambda'' - \lambda') \\ \;\;\; - \tan\beta' \,.\, \sin(\lambda'' - \lambda) \\ \;\;\; + \tan\beta'' \,.\, \sin(\lambda' - \lambda) \end{array}\right\} ; \qquad (70,1)$$

$$A\;\; = \tan\beta \;\,.\, \sin(L\;\, - \lambda'') - \tan\beta'' \,.\, \sin(L\;\, - \lambda') ; \qquad (70,2)$$

$$A'\; = \tan\beta' \,.\, \sin(L'\, - \lambda'') - \tan\beta'' \,.\, \sin(L'\, - \lambda') ; \qquad (70,3)$$

$$A'' = \tan\beta' \,.\, \sin(L'' - \lambda'') - \tan\beta'' \,.\, \sin(L'' - \lambda') ; \qquad (70,4)$$

$$B\;\; = \tan\beta'' \,.\, \sin(L\;\, - \lambda\,) - \tan\beta \;\,.\, \sin(L\;\, - \lambda'') ; \qquad (70,5)$$

$$B'\; = \tan\beta'' \,.\, \sin(L'\, - \lambda\,) - \tan\beta \;\,.\, \sin(L'\, - \lambda'') ; \qquad (70,6)$$

$$B'' = \tan\beta'' \,.\, \sin(L'' - \lambda\,) - \tan\beta \;\,.\, \sin(L'' - \lambda'') ; \qquad (70,7)$$

$$C\;\; = \tan\beta \;\,.\, \sin(L\;\, - \lambda'\,) - \tan\beta' \,.\, \sin(L\;\, - \lambda\,) ; \qquad (70,8)$$

$$C'\; = \tan\beta \;\,.\, \sin(L'\, - \lambda'\,) - \tan\beta' \,.\, \sin(L'\, - \lambda\,) ; \qquad (70,9)$$

$$C'' = \tan\beta \;\,.\, \sin(L'' - \lambda'\,) - \tan\beta' \,.\, \sin(L'' - \lambda\,) ; \qquad (70,10)$$

In these equations, the right hand members are composed entirely of quantities given either by observation, or by the solar tables; and their calculation, though long, is greatly facilitated by the regularity of their composition. The values of a, A, B, C, &c. obtained, we have, by (69, 1) (69, 2)

$$\delta = -\,\frac{ARS'' - A'R'S' + A''R''S}{a\,S''}$$

$$\delta' = +\,\frac{BRS'' - B'R'S' + B''R''S}{a\,S'}$$

whence we obtain

$$\frac{\delta}{\delta'} = -\,\frac{S'}{S''} \cdot \frac{A'}{B'} + \frac{(B'A - BA') \,.\, RS'' + (B'A'' - B''A')\, R''S}{B' \left\{ BRS'' - B'R'S' + B''R''S \right\}} \times \frac{S'}{S''} \qquad (71)$$

So far, we have nothing but rigorous equations, and it does not immediately appear how these can become serviceable in the question before us; but if we consider attentively the composition of the numerator of the fraction which forms the second term of $\frac{\delta}{\delta'}$ here statec., we shall find that the equation last arrived at, affords a means of obtaining a first approximation. Let us consider, first, the term $AB' - BA'$:. If we substitute in this the values of A, B, A', B', in equations (70, 2, 3, 5, 6,) execute all the multiplications, strike out such terms as mutually destroy each other, and then reduce as much as possible, we shall find for its value

$$\tan\beta'' \left\{ \begin{array}{l} \;\;\;\tan\beta \;\,.\, \{\sin(L - \lambda') \,.\, \sin(L' - \lambda'') - \sin(L' - \lambda') \,.\, \sin(L - \lambda'')\} \\ + \tan\beta' \,.\, \{\sin(L - \lambda'') \,.\, \sin(L' - \lambda) - \sin(L' - \lambda'') \,.\, \sin(L - L)\} \\ + \tan\beta'' . \{\sin(L - \lambda) \,.\, \sin(L' - \lambda') - \sin(L' - \lambda) \,.\, \sin(L - \lambda')\} \end{array} \right\}$$

Now, by trigonometry, we have, (since $\sin A \,.\, \sin B = \frac{1}{2}\{\cos(A - B) - \cos(A + B)\}$

$$\sin(L - \lambda) \,.\, \sin(L' - \lambda') = \tfrac{1}{2}\cos\{(L - L') - (\lambda - \lambda')\} - \tfrac{1}{2}\cos\{(L + L') - (\lambda + \lambda')\}$$

$$\sin(L' - \lambda) \,.\, \sin(L - \lambda') = \tfrac{1}{2}\cos\{(L - L') + (\lambda - \lambda')\} - \tfrac{1}{2}\cos\{(L + L') - (\lambda + \lambda')\}$$

and subtracting, we find

$$\begin{array}{l} \sin(L - \lambda) \,.\, \sin(L' - \lambda') - \sin(L' - \lambda) \,.\, \sin(L - \lambda') = \\ \qquad = \tfrac{1}{2}\cos\{(L - L') - (\lambda - \lambda')\} - \tfrac{1}{2}\cos\{(L - L') + (\lambda - \lambda')\} \end{array}$$

which another application of the same trigonometrical formula already used converts back again into

$$\sin(L - L') \,.\, \sin(\lambda - \lambda')$$

a similar transformation applies to the co-efficients of tan β and tan β'; so that our expression becomes

$$AB' - BA' = \tan\beta'' \times \begin{cases} \sin(L - L') \, . \sin(\lambda' - \lambda'') \, . \tan\beta \\ + \sin(L - L') \, . \sin(\lambda'' - \lambda) \, . \tan\beta' \\ + \sin(L - L') \, . \sin(\lambda - \lambda') \, . \tan\beta'' \end{cases}$$

or

$$AB' - BA' = \sin(L - L') \, . \tan\beta'' \, . \left\{ \begin{array}{l} \tan\beta \, . \sin(\lambda' - \lambda'') \\ + \tan\beta' \, . \sin(\lambda'' - \lambda) \\ + \tan\beta'' \, . \sin(\lambda - \lambda') \end{array} \right\}$$

or, if we put

$$\left. \begin{array}{l} \gamma = \tan\beta \, . \sin(\lambda' - \lambda'') \\ \quad + \tan\beta' \, . \sin(\lambda'' - \lambda) \\ \quad + \tan\beta'' \, . \sin(\lambda - \lambda') \end{array} \right\} ; \quad \ldots\ldots\ldots\ldots \quad (72)$$

$$AB' - BA' = \gamma \, . \sin(L - L') \, . \tan\beta'' \qquad (73, 1)$$

Now, it is evident that γ is a symmetrical function, and is not changed by putting at the same time β' for β, and λ' for λ, and reciprocally; so that, by pursuing a process of reduction exactly similar, we should obtain

$$AB'' - BA'' = \gamma \, . \sin(L - L'') \, . \tan\beta'' \qquad (73, 2)$$

$$A'B'' - B'A'' = \gamma \, . \sin(L' - L'') \, . \tan\beta'' \qquad (73, 3)$$

and were we to pursue similar processes, combining A and C, B and C, we should get

$$CA' - AC' = \gamma \, . \sin(L - L') \, . \tan\beta' \qquad (73, 4)$$

$$CA'' - AC'' = \gamma \, . \sin(L - L'') \, . \tan\beta' \qquad (73, 5)$$

$$C''A' - A''C' = \gamma \, . \sin(L' - L'') \, . \tan\beta' \qquad (73, 6)$$

and

$$BC' - CB' = \gamma \, . \sin(L - L') \, . \tan\beta \qquad (73, 7)$$

$$BC'' - CB'' = \gamma \, . \sin(L - L'') \, . \tan\beta \qquad (73, 8)$$

$$B'C'' - C'B'' = \gamma \, . \sin(L' - L'') \, . \tan\beta \qquad (73, 9)$$

Thus the numerator of the fraction in $\frac{\delta}{\delta'}$ above referred to, becomes.

$$\gamma \, . \tan\beta'' \, . \left\{ RS'' \, . \sin(L - L') - R''S \, . \sin(L' - L'') \right\} ; \qquad (74)$$

Now, let us suppose the three observations made at small intervals of time. The supposition almost always holds good in the case of comets, which usually excite sufficient interest about the time of their first appearance, to induce astronomers to observe them as frequently as possible; so that observations separated by an interval of a few days at most may generally be obtained for calculation. On this supposition, then, the variations $\lambda - \lambda'$, $\lambda' - \lambda''$ and $\lambda'' - \lambda$ of the observed geocentric longitudes may be regarded as very small quantities of the first order; so that γ is also a very small quantity of the first order, especially as β (the observed geocentric latitude) is usually below 45°; so that tan β, tan β', &c. are less than unity. Moreover, $L - L'$, $L' - L''$, &c. which represent the sun's motion in the intervals between the observations, are also very small quantities of the first order; so that the expression (74) is, on these accounts, a very small quantity of the second order.

Fig. 10. But besides this, since S, S'' represent the rectilinear triangles SPP' and $SP'P''$ described about the sun in the times $t' - t$ and $t'' - t'$ (which for the moment we will call θ and $n\,\theta$) they will be nearly proportional to those times, being extremely near to equality with the sectors $SP\,n\,P'$, $SP'\,n'\,P''$, so that we have very nearly $S'' = n \, . \, S$. Again, the earth's orbit being nearly a circle, and its motion in it nearly uniform, we have

$$R = R'' \text{ and } (L' - L'') = n \, . \, (L - L')$$

whence $(L - L')$ being very small, $\sin(L' - L'') = n \, . \sin(L - L')$ If these suppositions were rigorously exact, we should have

$$R \, . \, S'' \, . \sin(L - L') = n\,RS \, . \sin(L - L')$$
$$= R''S \, . \sin(L' - L'')$$

and the quantity within the brackets of (74) would be rigorously zero; and in all cases we see that it must be a very small quantity of the second order at least, so that the whole expression (74) must be a very small quantity of the third order; and that, whether the intervals between the consecutive observations approach to equality or not.*

Let us next consider the denominator of the fraction, and retaining the substitution $n = \frac{\theta'}{\theta} = \frac{t'' - t'}{t' - t}$

it becomes, on the same hypotheses as before

$$B' \, . \, RS \left\{ n\,B - B'\,(1 + n) + B'' \right\}$$

because $S' = S + S''$ nearly. We have, therefore, first to consider the values of $B'' - B'$ and of $B - B'$. Now, if in $B'' - B'$ we substitute the values of B', B'', and reduce as far as possible, we shall find

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* Mr. Littrow, from whose excellent and most useful work on theoretical and practical astronomy (*Theoretische und Practische Astronomie*, von J. J. Littrow, director der Sternwarte und professor der astron. an der K. K. universität in Wien.—Wien, 1821, 2 vols. 8vo.) I have taken this exposé of the determination of a comet's orbit and the example which follows, has merely remarked that the quantity $\frac{\delta}{\delta'}$ is equal to $-\frac{A'}{B'} \times \frac{S}{S''}$ + a remainder, and "this remainder is of the order $AB' - A'B$, and therefore may be neglected in a first approximation, when the intervals are small *and nearly equal*," (vol. ii. p. 129.) This is not, however, a satisfactory view of the subject, and is, in fact, slurring over a very considerable difficulty and evading one of the most difficult and delicate points in the theory of comets. I have therefore judged it better to enter, though somewhat more at length than is absolutely consonant with the nature of this work into this part of the subject, rather than leave a doubt on the mind of the student, or a mystery attached to a fundamental proposition. Meanwhile I am happy to have to call the attention of the lovers of astronomy to the interesting work alluded to, of which a translation by any competent hand would be a real accession to our stock of elementary works.

$$- B' = 2\tan\beta \,.\, \tan\beta'' \,.\, \sin\frac{L'-L''}{2} \,.\, \left\{\cot\beta \,.\, \cos\left(\frac{L'+L''}{2}-\lambda\right) - \cot\beta'' \,.\, \cos\left(\frac{L'+L''}{2}-\lambda''\right)\right\}$$

l the quantity within the brackets being the difference between two values of the same function at a small erval of time is of the first order, as is also sin $\frac{L'-L''}{2}$, so that B″−B′ is of the second order. Similarly, we have

$$- B' = -2 \,.\, \tan\beta \,.\, \tan\beta'' \sin\frac{L-L'}{2} \,.\, \left\{\cot\beta \,.\, \cos\left(\frac{L+L'}{2}-\lambda\right) - \cot\beta'' \,.\, \cos\left(\frac{L+L'}{2}-\lambda''\right)\right\}$$

w, $\sin\frac{L'-L''}{2} = n \,.\, \sin\frac{L-L'}{2}$, and therefore we have

$$3-B')+(B''-B')=2n \,.\, \tan\beta \,.\, \tan\beta'' \,.\, \sin\left(\frac{L-L'}{2}\right) \,.\, \left\{\begin{matrix} \cot\beta\left(\cos\left(\frac{L'}{2}-\lambda+\frac{L''}{2}\right)-\cos\left(\frac{L'}{2}-\lambda+\frac{L}{2}\right)\right) \\ -\cot\beta''\left(\cos\left(\frac{L'}{2}-\lambda''+\frac{L''}{2}\right)-\cos\left(\frac{L'}{2}-\lambda''+\frac{L}{2}\right)\right)\end{matrix}\right\}$$

et this be reduced as much as possible; and, putting M for $\frac{L+2L'+L''}{4}$, and $(1+n) \,.\, \sin\left(\frac{L-L'}{4}\right)$ sin $\left(\frac{L-L''}{4}\right)$ it will become

$$R \,.\, S \times 2n\,(1+n)^2 \,.\, \tan\beta^2 : \tan\beta''^2 \,.\, \sin\frac{L-L'}{2} \,.\, \sin\frac{L-L'}{4} \times$$
$$\times \left\{\sin(L'-\lambda) \,.\, \cot\beta - \sin(L'-\lambda'') \,.\, \cot\beta''\right\} \times$$
$$\times \left\{\sin(M-\lambda) \,.\, \cot\beta - \sin(M-\lambda'') \,.\, \cot\beta''\right\}$$

is is evidently a quantity of the fifth order, the two last factors of it being each the difference of conutive values.

Granting then the hypotheses above employed, viz. . The strict circularity and uniformity of the earth's tion.—2. The proportionality of the rectilinear, tead of the parabolic, sectors to the times we see t the numerator of the fraction vanishes absolutely, ile the denominator on the same hypotheses does strictly vanish, but only reduces itself to a quan- of the fifth order. Hence, we are entitled (at st in making a first approximation to the elements) neglect the fraction, and suppose

$$\frac{\delta}{\delta'} = -\frac{S'}{S''} : \frac{A'}{B'}; \qquad (75)$$

l similarly

$$\frac{\delta}{\delta''} = +\frac{S}{S''} \cdot \frac{A'}{C'}; \qquad (76)$$

l

$$\frac{\delta'}{\delta''} = -\frac{B'}{C'} \cdot \frac{S}{S'}; \qquad (77)$$

s true, from the minuteness of the denominator of fraction in (71) the deviation from truth of the otheses assumed becomes *magnified*. But these otheses themselves are so very nearly correct, t unless in extremely unfavourable cases the ations (75) and (76) may safely be employed; and vill be observed too, that *independent of both hyposes*, the numerator would be of the fourth order at st, for if we throw it into the form

$$(\,.\, \tan\beta'' \,.\, SS'' \,.\, \left\{R \,.\, \frac{\sin(L-L')}{S} - R'' \,.\, \frac{\sin(L'-L'')}{S''}\right\}$$

$$(\,.\, \tan\beta'' \,.\, SS'' \,.\, \left\{R \,.\, \frac{\sin(L-L')}{S} - R' \,.\, \frac{\sin(L'-L'')}{S''}\right\}$$

$$+ \gamma \,.\, \tan\beta'' \,.\, SS'' \,.\, (R'-R'') \,.\, \frac{\sin(L'-L'')}{S''}$$

second term is evidently so; and since the factor the first within the brackets is the difference of consecutive values, and therefore a very small quantity,—the first is also.

To apply these equations to the determination of a comet's elements, we will first suppose the sectors P *n* SP′, &c. equal to the plane triangles SPP′, &c. or at least in the same ratio with them. This amounts to substituting $\frac{t'-t}{t''-t'}$ or $\frac{\theta'}{\theta''}$ for $\frac{S'}{S''}$; when we obtain, instead of the equations (69, 1, 2, 3,) Fig. 10.

$$(a\,\delta + AR) \times \theta'' - A'R' \,.\, \theta' + A''R''\,\theta = o \qquad (78,1)$$
$$BR\,\theta'' - (a\,\delta' + B'R')\,\theta' + B''R''\,\theta = o \qquad (78,2)$$
$$CR\,\theta'' - C'R'\,\theta' + (a\,\delta'' - C'R'')\,\theta = o \qquad (78,3)$$

which give the values of δ, δ', δ''.

Now we get, by the equation (55)

$$r'^2 = R'^2 + \delta'^2 \,.\, \sec\beta'^2 + 2R'\delta' \,.\, \cos(L'-\lambda'); \qquad (79)$$

whence the value of r' is obtained.

r' once obtained, the values of δ, δ', &c. and r' may be corrected by the following process, let τ represent the time between the second observation and any assigned instant. If then x, y, z, represent the coordinates at the first observation, and x'', y'', z'', at the third, we have

$$x = x' - \frac{\theta}{1} \,.\, \frac{dx'}{d\tau} + \frac{\theta^2}{1\,.\,2} \,.\, \frac{d^2x'}{d\tau^2} - \&c. \qquad (80,1)$$

$$x'' = x' + \frac{\theta''}{1} \,.\, \frac{dx'}{d\tau} + \frac{\theta''^2}{1\,.\,2} \,.\, \frac{d^2x'}{d\tau^2} - \&c. \qquad (80,2)$$

$$y = y' - \frac{\theta}{1} \,.\, \frac{dy'}{d\tau} + \frac{\theta^2}{1\,.\,2} \,.\, \frac{d^2y'}{d\tau^2} - \&c. \qquad (80,3)$$

$$y'' = y' + \frac{\theta''}{1} \,.\, \frac{dy'}{d\tau} + \frac{\theta''^2}{1\,.\,2} \,.\, \frac{d^2y'}{d\tau^2} - \&c. \qquad (80,4)$$

and similar expressions for z, z''. Now, assuming τ as the independent variable, or $d\tau$ constant, we have, by (1), (2), (3),

$$\left.\begin{aligned}\frac{d^2 x'}{d\tau^2} &= -(\mathrm{M}+m)\cdot\frac{x'}{r'^3}\\ \frac{d^2 y'}{d\tau^2} &= -(\mathrm{M}+m)\cdot\frac{y'}{r'^3}\\ \frac{d^2 z'}{d\tau^2} &= -(\mathrm{M}+m)\cdot\frac{z'}{r'^3}\end{aligned}\right\}\qquad(81,\ 1,\ 2,\ 3)$$

The numerical value of $\mathrm{M}+m$ is easily found, since it is the same for the whole solar system except in so far as the masses of the planets and comets denoted by m differ, and these bear so minute a proportion to the mass M of the sun, that they may be neglected: let τ denote the period of any one plane, as the earth, and a the semi-major axis of its orbit, then we shall have, by (24)

$$\frac{\mathrm{T}^2}{a^3} = \frac{4\,\pi^2}{\mathrm{M}+m}$$

so that $\frac{\tau^2}{a^3}$ is a quantity very nearly constant for the whole planetary system; and the period of the earth being $365^{\mathrm{d}}.256384$, and the semiaxis of its orbit 1, its value for the earth is $\frac{(365{\cdot}256384)^2}{1^3}$, and the same quantity expresses its value for all the other planets and comets. Hence we have

$$\mathrm{M}+m = \frac{4\times(3{\cdot}141592)^2}{(365{\cdot}256384)^2} = 0{\cdot}0002959122\,;$$

and $\qquad \log(\mathrm{M}+m) = 6{\cdot}4711628$

To correct the values of δ, δ', &c. we must correct the supposition on which they were obtained, of the proportionality of the rectilinear triangles instead of the parabolic sectors to the times, or the values of S, S′, S″. Now, if c be the inclination of the plane of the orbit to the plane of the x and y, we have

$$\begin{aligned}2\,\mathrm{S}\ .\cos c &= y\,x' - x\,y'\\ 2\,\mathrm{S}'\,.\cos c &= y'\,x'' - x'\,y''\\ 2\,\mathrm{S}''.\cos c &= y\,x'' - x\,y''\end{aligned}$$

In the first of these, for y and x write their values as given in (80, 1) and (80, 3), and we find

$$\begin{aligned}2\,\mathrm{S}\,.\cos c = {}&\frac{\theta}{1}\cdot\frac{y'\,d\,x' - x'\,d\,y'}{d\,\tau}\\ &-\frac{\theta^2}{1\,.\,2}\cdot\frac{y'\,d^2x' - x'\,d^2y'}{d\,\tau^2}+\\ &+\frac{\theta^3}{1\,.\,2\,.\,3}\cdot\frac{y'\,d^3x' - x'\,d^3y'}{d\,\tau^3} - \&\mathrm{c.}\end{aligned}$$

But if the values of $\frac{d^2\,x'}{d\,\tau^2}$ and $\frac{d^2\,y'}{d\,\tau^2}$ be written in the second term of this, it will vanish. Again, if we differentiate the equations (81, 1) and (81, 2) we get

$$\frac{d^3\,x'}{d\,\tau^3} = -(\mathrm{M}+m)\,.\left(\frac{1}{r'^3}\cdot\frac{d\,x'}{d\,\tau} - \frac{3\,x'}{r'^4}\cdot\frac{d\,r'}{d\,\tau}\right)$$

$$\frac{d^3\,y'}{d\,\tau^3} = -(\mathrm{M}+m)\,.\left(\frac{1}{r'^3}\cdot\frac{d\,y'}{d\,\tau} - \frac{3\,y'}{r'^4}\cdot\frac{d\,r'}{d\,\tau}\right)$$

so that, by substitution, we obtain

$$\frac{y'\,d^3\,x' - x'\,d^3\,y'}{d\,\tau^3} = -\,\frac{\mathrm{M}+m}{r'3}\cdot\frac{y'\,d\,x' - x'\,d\,y'}{d\,\tau}$$

If we neglect the higher powers of θ than the third, and put, for the present, p for $\frac{x'\,d\,y' - y'\,d\,x'}{d\,\tau}$, we shall have

$$2\,\mathrm{S}\,.\cos c = p\,\theta\left\{1 - \frac{\mathrm{M}+m}{r'^3}\,\theta^2\right\};\qquad(82,\ 1)$$

and similarly,

$$2\,\mathrm{S}'.\cos c = p\,\theta'\left\{1 - \frac{\mathrm{M}+m}{r'^3}\,\theta'^2\right\};\qquad(82,\ 2)$$

$$2\,\mathrm{S}''.\cos c = p\,\theta''\left\{1 - \frac{\mathrm{M}+m}{r'^3}\,\theta''^2\right\};\qquad(82,\ 3)$$

If we now employ the value of r' as above approximately determined, and substitute it in the right hand members of these equations, we shall obtain, dividing one by the other the *ratios* of S, S′, S″, much more accurately than on the original hypothesis, for we have

$$\frac{\mathrm{S}}{\mathrm{S}''} = \frac{\theta}{\theta''}\times\frac{1 - \frac{\mathrm{M}+m}{r'^3}\,\theta^2}{1 - \frac{\mathrm{M}+m}{r'^3}\,\theta''^2}\qquad(83,\ 1)$$

$$\frac{\mathrm{S}'}{\mathrm{S}''} = \frac{\theta'}{\theta''}\times\frac{1 - \frac{\mathrm{M}+m}{r'^3}\,\theta'^2}{1 - \frac{\mathrm{M}+m}{r'^3}\,\theta''^2}\qquad(83,\ 2)$$

Now, in the equations for determining δ, δ', &c. (69, 1, 2, 3) it is only the ratio of S, S′, S″, that are required, and thus we are enabled to correct the values of their quantities, and thence again that of r'. To carry the process to a greater degree of precision, however, the series expressing the value of $2\,\mathrm{S}\,.\cos c$ must be continued further. Without, however, going through the process as here set down we may content ourselves, after computing first values of δ' and r' by the equations (78, 2) and (79) with calculating the quantity

$$\begin{aligned}\frac{\mathrm{M}+m}{r'^3}\Big\{\mathrm{B}'\mathrm{R}'\,.\,\frac{\theta}{\theta'}\,(\theta'^2 - \theta^2)&\\ -\,\mathrm{BR}\,.\,\frac{\theta''}{\theta'}\,(\theta'^2 - \theta''^2)&\Big\};\end{aligned}\qquad(84)$$

which must be applied as a correction to the first found value of δ' with its proper sign.

The values of r, r', r'', δ, δ', δ'', so obtained, the remaining unknown quantities may be found as follows:

The equation (58)

$$r\,.\cos b\,.\sin(l - \mathrm{L}) = \delta\,.\sin(\lambda - \mathrm{L})$$

divided by the equation (38)

$$r\,.\cos b\,.\cos(l - \mathrm{L}) = \mathrm{R} + \delta\,.\cos(\lambda - \mathrm{L})$$

member for member, gives

$$\tan(l - \mathrm{L}) = \frac{\delta\,.\sin(\lambda - \mathrm{L})}{\mathrm{R} + \delta\,.\cos(\lambda - \mathrm{L})}\qquad(85)$$

the right hand member consists only of known quantities, so that the value of $l - \mathrm{L}$ is easily found, and thus the heliocentric longitude becomes known at each of the three observations. Again, we have by the equation (56)

$$r\,.\sin b = \delta\,.\tan\beta$$

and by (58)

$$r\,.\cos b = \delta\,.\frac{\sin(\lambda - \mathrm{L})}{\sin(l - \mathrm{L})}$$

omy.

which, divided member for member, give

$$\tan b = \tan \beta \, . \, \frac{\sin (l - \mathrm{L})}{\sin (\lambda - \mathrm{L})}; \qquad (86)$$

and thus the heliocentric latitudes may be computed.

The values of l and b known, the same equations afford a value of r, viz.

$$r = \delta \, . \, \frac{\tan \beta}{\sin b}$$

whence r, r', r'' may be found; and if these values agree with those before determined, it will be a proof of the correctness of the previous work. If l' and l'' be greater than l, the comet's motion is direct; if less, retrograde.

The heliocentric longitudes and latitudes thus obtained at each of the three observations, we easily obtain the inclination and place of the node. In fact, (taking the first and last observations, to embrace a greater arc of the great circle) our equation (40) gives

$$\tan b = \tan i \, . \sin (l - \text{☊})$$
$$\tan b'' = \tan i \, . \sin (l'' - \text{☊})$$

so that we have

$$\frac{\sin (l - \text{☊})}{\sin (l'' - \text{☊})} = \frac{\tan b}{\tan b''}$$

which gives

$$\tan b'' (\sin l \, . \cos \text{☊} - \cos l \, . \sin \text{☊})$$
$$= \tan b \, (\sin l'' \, . \cos \text{☊} - \cos l'' . \sin \text{☊})$$

or

$$\sin \text{☊} \left\{ \tan b \, . \cos l'' - \cos l \, . \tan b'' \right\} =$$
$$= \cos \text{☊} \left\{ \sin l \, . \tan b'' - \sin l'' . \tan b \right\}$$

and consequently

$$\tan \text{☊} = \frac{\tan b \, . \cos l'' - \tan b'' \, . \cos l}{\sin l \, . \tan b'' - \sin l'' \, . \tan b}; \qquad (87)$$

and

$$\tan i = \frac{\tan b}{\sin (l - \text{☊})} \qquad (88)$$

If the motion of the comet be *direct*, the value of ☊ given by the above equation, will be the longitude of the *ascending node*; but if retrograde, it is evidently that of the *descending*; and to get that of the ascending node, 180° must be added. The equation (61) or the corresponding one,

$$\tan (l - \text{☊}) = \cos i \, . \tan (\psi + \theta) \qquad (89)$$

gives the values of $\psi + \theta$ the longitudes on the orbit reckoned from the node, and these being obtained, ($\psi + \theta$, $\psi + \theta'$, and $\psi + \theta''$)* the angle $\theta'' - \theta$ between the first and last places of the comet at the sun becomes known.

If we denote this by ϕ, we have at once, by the equation

$$\tan \frac{\theta}{2} = \operatorname{cosec} \phi \left\{ \cos \phi - \sqrt{\frac{r}{r''}} \right\} \qquad (90)$$

demonstrated in the foregoing pages (p. 665) the value of θ, or the distance (on the orbit) of the perihelion from the place of the first observation; so that, subtracting the value found, from $\psi + \theta$ before determined, we get ψ, and thence π, the longitude of the perihelion on the ecliptic, by the equation (60)

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It only remains to find the perihelion distance and time of perihelion passage. The equation of the parabola

$$\mathrm{D} = r \, . \left(\cos \frac{\theta}{2} \right)^2$$

gives the former immediately, r and θ being known, and the latter is found as follows. In the table of a comet of 109 days subjoined, seek the number of days (by interpolation) corresponding to the anomaly θ, and call it M, then, by equation (44) we have, (the perihelion distance of such a comet being unity)

$$\mathrm{N} = \sqrt{\frac{2}{\mathrm{M} + m}} \times \left\{ \tan \frac{\theta}{2} + \frac{1}{3} \, . \tan \left(\frac{\theta}{2} \right)^3 \right\}$$

and consequently, if T be the time of the perihelion passage *before* the first observation,

$$\mathrm{T} = \mathrm{N} \times \mathrm{D}^{\frac{3}{2}}$$

If we institute a similar operation for the time T″ of the perihelion passage *before* the third observation, we shall find

$$\mathrm{T}'' = \mathrm{N}'' \, . \, \mathrm{D}^{\frac{3}{2}}$$

Now, we ought to have $\mathrm{T}'' - \mathrm{T}$ = the interval between the first and last observations. If this be found on trial to be the case, it is a proof of the correctness of the work. The perihelion passage will be before or after the first observation (in the case of direct motion) according as the longitude π of the perihelion is less or greater than the heliocentric longitude l at the first observation; the reverse is the case in retrograde motion.

The chief difficulty in the solution of the problem of a comet's orbit, consists in the determination of the distances from the sun and earth, at the moments of the several observations. Our equations above derived afford a great variety of means for accomplishing this, among which we shall only select one, proposed by Dr. Olbers, which in many cases has particular advantages. It depends on the property already demonstrated of the chords of the parabolic arcs.

Let k represent the chord of the arc included between the two extreme places, then we have

$$k^2 = (x'' - x)^2 + (y'' - y)^2 + (z'' - z)^2$$

or

$$k^2 = r^2 + r''^2 - 2\,(x x'' + y y'' + z z'')$$

In this equation, let the values of x, y, z, x'', y'', z'', in (67, 1, 2, 3, 4, &c.) in terms of R and δ, &c. be substituted; and if we put

$$m = \frac{\mathrm{C}'}{\mathrm{A}'} \times \frac{\theta''}{\theta}$$

so that, by (76) $\delta'' = m \, \delta$

we shall have

$$\begin{aligned} k^2 &= r^2 + r''^2 \\ &- 2\, m\, \delta^2 \, . \left\{ \cos (\lambda - \lambda'') + \tan \beta \, . \tan \beta'' \right\} \\ &- 2\, m\, \mathrm{R}\, \delta \, . \cos (\lambda'' - \mathrm{L}) \\ &- 2\, \mathrm{R}''\, \delta \, . \cos (\lambda - \mathrm{L}'') \qquad (91) \\ &- 2\, \mathrm{R}\mathrm{R}'' . \cos (\mathrm{L} - \mathrm{L}'') \end{aligned}$$

This equation combined with the equation (63)

$$6 \, . \sqrt{\mathrm{M} + m} \, . \, \theta' = (r + r'' + k)^{\frac{3}{2}} - (r + r'' - k)^{\frac{3}{2}}$$

and the two equations

$$r^2 = \mathrm{R}^2 + \delta^2 \, . \sec \beta^2 + 2\, \mathrm{R}\, \delta \, . \cos (\lambda - \mathrm{L})$$
$$r''^2 = \mathrm{R}''^2 + m^2 \delta^2 \, . \sec \beta''^2 + 2\, m\, \mathrm{R}''\, \delta \, . \cos (\lambda'' - \mathrm{L})$$

gives four equations for determining the four unknown quantities. To resolve them, we must use the well known *rule of false*, one of the simplest and most widely useful rules mathematical science affords. Assume a value for δ, as near the truth as can be conjectured, compute m and thence $m\,\delta$, or δ''; and having used these to obtain k, r, and r'', substitute these in the expression for $g\,\theta'$. Call the result A. Assume then a second value for δ, which call Δ, and executing the same process, call the result B. Then we shall have

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* The reader will not confound θ here with the same letter before used to denote the time.

$$\mathrm{B} - \mathrm{A} : g\,t - \mathrm{A} :: \Delta - \delta : x - \delta$$

where x is the corrected value of δ, and hence we get

$$x = \delta + (\Delta - \delta) \cdot \frac{g\,t - \mathrm{A}}{\mathrm{B} - \mathrm{A}}$$

and substituting this again, and employing that one of the former values of δ which gave a result nearest to $g\,t$ for a corresponding value, we shall obtain a yet nearer approximation, and so on.

It remains to exemplify the method of computation by an instance. Let us therefore take the following observations of the comet of 1779, taken at Paris with a transit and a repeating circle.

Mean Time, Paris.	h ′ ″		° ′ ″		° ′ ″
1779. Aug. 30	11 9 42,	$\lambda =$	125 48 39·3	$\beta =$	41 53 52·2
Sep. 2	10 36 8,	$\lambda' =$	132 53 48·5	$\beta' =$	45 54 48·1
Sep. 4	10 7 51,	$\lambda'' =$	138 56 31·2	$\beta'' =$	48 32 27·8

$\mathrm{L} = 337\ 29\ 8{\cdot}7 \qquad \mathrm{R} = 1{\cdot}0087218$

$\mathrm{L}' = 340\ 22\ 26{\cdot}9 \qquad \mathrm{R}' = 1{\cdot}0079991$

$\mathrm{L}'' = 342\ 17\ 47{\cdot}8 \qquad \mathrm{R}'' = 1{\cdot}0074854$

The intervals therefore between the first and second, first and third, and second and third observations are respectively* $\theta = 2{\cdot}976690$ days, $\theta' = 4^{\mathrm{d}}{\cdot}957049$, $\theta'' = 1^{\mathrm{d}}{\cdot}980350$

Consequently, we have by the equations expressing the values of A′ and C′ reduced into numbers

$$\frac{\mathrm{C}'}{\mathrm{A}'} = 1{\cdot}18408; \quad m = \frac{\delta''}{\delta} = 0{\cdot}787752$$

and substituting this value of m, and the values of λ, λ'', β, β'', R, R″, we get the following equations, noticing that $g = 0{\cdot}103212$

$$r^2 = 1{\cdot}01752 - 1{\cdot}71693 \,.\, \delta + 1{\cdot}80493 \,.\, \delta^2; \qquad r''^2 = 1{\cdot}01503 - 1{\cdot}45724 \,.\, \delta + 1{\cdot}41564 \,.\, \delta^2$$

$$k^2 = 0{\cdot}00716 - 0{\cdot}04739 \,.\, \delta + 0{\cdot}08627 \,.\, \delta^2; \quad 135{\cdot}84219 = \left(\frac{r + r'' + k}{2}\right)^{\frac{3}{2}} - \left(\frac{r + r'' - k}{2}\right)^{\frac{3}{2}}$$

These equations are very nearly satisfied by taking $\delta = 0{\cdot}71469$, whence we find

$$r = 0{\cdot}8440; \quad r'' = 0{\cdot}8346; \quad k = 0{\cdot}1317; \quad \delta'' = m\,\delta = 0{\cdot}562998$$

whence we next deduce the heliocentric longitudes and latitudes at the extreme observations from equations (85) and (86)

$$l = 20^\circ\ 37'\ 40''{\cdot}8 \quad \text{and } b = 49^\circ\ 26'\ 23''{\cdot}1$$

$$l'' = 6\ \ 45\ \ 39\ {\cdot}7 \qquad b'' = 49\ \ 46\ \ 46\ {\cdot}9$$

Since l'' is less than l, the comet's motion is retrograde. Moreover, we have by equations (87) and (88)

$$☊ = 100^\circ\ 51'\ 53''{\cdot}4; \quad i = 49^\circ\ 51'\ 7''{\cdot}9$$

we obtain also, by following the process pointed out in equations (89) and (90) $\theta = 12^\circ\ 39'\ 35''{\cdot}6$, $\pi = 4^\circ\ 32'\ 8''{\cdot}2$ for the longitude of the perihelion, and $\mathrm{D} = 0{\cdot}830761$ for the perihelion distance. Consequently the time between the first observation and the perihelion passage is

$$\frac{\mathrm{D}^{\frac{3}{2}} \,.\, \sqrt{2}}{\sqrt{\mathrm{M} + m}} \cdot \left\{\tan\frac{\theta}{2} + \frac{1}{3}\left(\tan\frac{\theta}{2}\right)^3\right\} = +\ 6{\cdot}97118 =$$

$$= 6^{\mathrm{d}}\ 23^{\mathrm{h}}\ 18'\ 30''$$

30th Aug. 11 9 42

10 28 12 Sept. 6, 1799 = time of perihelion passage.

Of the two solutions given above, the latter only supposes the orbit to be determined a parabola; the former may be applied to the determination of elliptic elements, but the latter part of the process from equation (70) must be varied accordingly; but as the various methods of determining and correcting elliptic as well as parabolic elements would lead us a great deal beyond our proposed limits, we must content ourselves with referring to Mr. Gauss's *Theoria Motus Corporum Celestium*, in which the whole subject is treated with the utmost generality. The reader may also consult Sir H. Englefield, *On Comets;* the *Mécanique Céleste* of Laplace, vol. I. c. iv.; Lagrange, *Méc. Analytique* (the new edition.) The German scholar will find it well worth his while to consult the paper of the same eminent writer in the *Berlin Ephemeris* for 1785; and the volume of the same work for 1789, p. 197; the papers of Gauss and Olbers, in the *Monatliche Correspondenz;* as well as the treatise of the latter, *Ueber die leichteste und bequemste Methode den Bahn eines Cometen zu berechnen.*

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* The reader is requested to take the numbers in this example upon Mr. Littrow's authority, from whose work the example is taken.

PART II. SECTION I.

OF THE PERTURBATION OF THE ELLIPTIC MOTIONS OF THE HEAVENLY BODIES, ARISING FROM THEIR MUTUAL ATTRACTION

IN the preceding investigations we have supposed only two attracting bodies to exist in space; and on this supposition have succeeded in representing all the phenomena of their motions in finite equations, by the aid of which their relative situations may be assigned at any past or future moment. The resulting formulæ reduced into tables, and compared with actual observation, manifest an agreement sufficiently complete to leave no doubt of the correctness of the principles from which they have been deduced, at least, when observations separated only by moderately long intervals of time are compared together. Yet, on descending to a more rigorous nicety, and especially on comparing together observations embracing a very long series of years, it is found that the results of the calculations, founded on the assumption of elliptic motion according to the laws above demonstrated, do not represent the observations perfectly. Minute irregularities are still detected, and the planets are found sometimes a little in advance, and sometimes a little falling short, sometimes a little to the right or left, above or below, their calculated places in their orbits. Moreover, if the elements of the orbits themselves, as deduced from modern observations, be compared with those similarly deduced from ancient ones, they will be found not to correspond exactly, but to differ by small variations, which are however greater, as the observations themselves compared are more distant in their dates. In a word, though the elliptic theory agrees very nearly with observation; yet, to make it tally rigorously with them, it is necessary to introduce modifications of this kind.

1st. The ellipse in which each planet moves, must be conceived to change its eccentricity and position in space, by exceedingly slow gradations—so slow indeed, as to be insensible in a single revolution of the planet, and only discoverable by a comparison, such as we have described, of its nature in past and present ages.

2dly. The planet itself is not always found exactly in its place in the ellipse *even* when so varied; nor indeed is it always found in the exact periphery of the ellipse at all. But if we conceive an imaginary point to describe this ellipse according to the rigorous laws of elliptic motion, the real planet will never be very far distant from it, but will oscillate or revolve round it like a secondary about its primary in an orbit of extremely small dimensions, yet according to laws of a very complicated nature, too complicated indeed for observation alone to unravel.

These variations are much more sensible in the orbit and motions of the moon, where, in fact, they amount to very considerable quantities. If the moon, for instance, at its greatest northern latitude be observed to pass over a certain star, it should continue to do so each revolution, if its motion were strictly conformable to the elliptic hypothesis. So far, however, is this from being the case, that it will be observed to deviate visibly southward every lunation; and after a lapse of nine years and a half, its path, in the same part of the heavens as to longitude, will pass not less than 10° south of the star; after which it will again advance northward, and in nine years and a half more will once more pass over, or at least very near the star. Deviations like these must have some positive and decided cause; as much so as the elliptic motion itself; and if they are to be accounted for on the theory of universal attraction, it is manifest that the same mechanical principles applied in the same way to the case of several bodies abandoned in free space to their mutual attractions, ought to lead us to their explanation.

But the mathematical difficulties to be encountered in this research, are of a much higher order than in the case of two bodies only. *There* we had no difficulty in integrating the differential equations of the problem. Here, on the other hand, the equations are too complicated to allow of their integrals being exhibited otherwise than in series; and even then we have no means of ascertaining their laws. Fortunately, however, this is not necessary; the enormous preponderance of the sun's mass in our system being such, that the fractions, representing those of each of the other planets, are small enough to allow of their squares and products being neglected without any fear of inducing appreciable errors.

It is a well known theorem, which extends to all the applications of mathematical reasoning to natural phenomena, that when several causes of motion act together, if their effects taken singly are of such an order of minuteness, that their squares and products may be disregarded, then their joint effect will be the sum of the effects which would be produced by each acting alone. This principle allows us to regard the disturbance or perturbation of any one planet produced by the joint action of all the others, as the sum of the effects each would produce separately; so that we may simplify the investigation by leaving out of consideration all but a single disturbing planet, and applying the analytical formulæ so investigated to each in succession, we shall thus obtain expressions for the perturbations produced by each; which, added together, give the total perturbation.

Investigation of the forces exerted by one body to disturb the orbit of another revolving about a common central body, and of the differential equations of their motions.

Taking unity for the mass of the central body, and its centre for the origin of the co-ordinates, let m and m' represent the masses of two revolving bodies, and suppose x, y, z, r to denote the three co-ordinates and radius vector of the body m, whose perturbations we would investigate; and x', y', z', r', the corresponding quantities relative to the disturbing body m'. Moreover, let λ represent the distance of the two bodies, m, m', on each other. Then will the several attractive forces exerted by the bodies of the system on each other be as follows:—

1st. The sun (or central body) attracts m and m' with forces represented respectively by $\frac{1}{r^2}$ and $\frac{1}{r'^2}$.

2dly. m attracts the sun with a force $\frac{m}{r^2}$, and it attracts m' with a force $\frac{m}{\lambda^2}$.

3dly. m' attracts the sun with the force $\frac{m'}{r'^2}$, and it attracts m with the force $\frac{m'}{\lambda^2}$.

Fig. 11. These are the forces exerted in the directions of the lines joining the several bodies in the system; but to estimate their effects, they must be reduced to the directions AS, BS, CS, of the three co-ordinates. This done, we shall find

For the forces acting on the sun,

$$\left.\begin{array}{l}\text{Force of } m = -\dfrac{m\,x}{r^3}\\[2ex] \text{Force of } m' = -\dfrac{m'\,x'}{r'^3}\end{array}\right\}\text{ in the direction AS}$$

and similarly,

$$-\frac{m\,y}{r^3} \text{ and } -\frac{m'\,y'}{r'^3} \text{ in the direction BS}$$

$$-\frac{m\,z}{r^3} \text{ and } -\frac{m'\,z'}{r'^3} \text{ in the direction CS.}$$

So that the three forces acting on the sun, are respectively

$$-\left(\frac{m\,x}{r^3}+\frac{m'\,x'}{r'^3}\right)$$
$$-\left(\frac{m\,y}{r^3}+\frac{m'\,y'}{r'^3}\right)$$
$$-\left(\frac{m\,z}{r^3}+\frac{m'\,z'}{r'^3}\right)$$

Again, the forces acting on m are

$$\text{In the direction AS}\ldots\ldots\ldots\ldots\left\{\begin{array}{l}\text{Force of the sun} = +\dfrac{x}{r^3}\\[2ex] \text{Force of } m'\ldots\ldots = -\dfrac{m'\,(x'-x)}{\lambda^3}\end{array}\right.$$

$$\text{In the direction BS}\ldots\ldots\ldots\left\{\begin{array}{l}\text{Force of the sun} = +\dfrac{y}{r^3}\\[2ex] \text{Force of } m'\ldots\ldots = -\dfrac{m'\,(y'-y)}{\lambda^3}\end{array}\right.$$

$$\text{In the direction CS}\ldots\ldots\ldots\left\{\begin{array}{l}\text{Force of the sun} = +\dfrac{z}{r^3}\\[2ex] \text{Force of } m'\ldots\ldots = -\dfrac{m'\,(z'-z)}{\lambda^3}\end{array}\right.$$

So that the aggregate forces acting on m in these directions are

$$+\frac{x}{r^3}-m'\,.\,\frac{x'-x}{\lambda^3}$$
$$+\frac{y}{r^3}-m'\,.\,\frac{y'-y}{\lambda^3}$$
$$+\frac{z}{r^3}-m'\,.\,\frac{z'-z}{\lambda^3}$$

and in the same manner, the forces acting on m' will be found to be respectively

$$+\frac{x'}{r'^3}-m\,.\,\frac{x-x'}{\lambda^3}$$
$$+\frac{y'}{r'^3}-m\,.\,\frac{y-y'}{\lambda^3}$$
$$+\frac{z'}{r'^3}-m\,.\,\frac{z-z'}{\lambda^3}$$

From these three sets of forces we might determine separately the motions of the sun, of m, and of m'; but as we have chosen to suppose the sun at rest, and fix the origin of our co-ordinates in its centre, we must transfer to both m and m' in a contrary direction the forces which act upon it. This done, the forces animating m in the respective directions AS, BS, CS, will become

$$\frac{(1+m) \,.\, x}{r^3} + m' \left\{ \frac{x'}{r'^3} - \frac{x' \quad x}{\lambda^3} \right\}$$

$$\frac{(1+m) \,.\, y}{r^3} + m' \left\{ \frac{y'}{r'^3} - \frac{y' - y}{\lambda^3} \right\}$$

$$\frac{(1+m) \,.\, z}{r^3} + m' \left\{ \frac{z'}{r'^3} - \frac{z' - z}{\lambda^3} \right\}$$

and if, in these expressions, we exchange the accented letters for the corresponding unaccented ones, and *vice versâ*, (λ only excepted) we shall evidently have the forces acting on m'.

The first terms of these expressions, as will be observed, are the same as if the mass m' of the disturbing planet were nothing, or as if there were no such planet; the other parts, those multiplied by m', express the *disturbing forces*, or the effect of the attraction of m' to derange the orbit of m from its elliptical form. These, it will also be observed, consist each of two parts; the one expressing the action of m' on the sun, and the other its action on m with an opposite sign. It is only therefore in virtue of the difference of the attractions of m' on the sun, and on m that the orbit of the latter is deranged.

These forces are susceptible of a very simple mode of expression; and the equations of m's motion in consequence admit of a very compendious form, if we consider, that owing to the form of the function λ, which is equal to $\sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}$, we have

$$\left.\begin{aligned} \frac{x'-x}{\lambda^3} &= \frac{d}{dx} \,.\, \frac{1}{\lambda} \\ \frac{y'-y}{\lambda^3} &= \frac{d}{dy} \,.\, \frac{1}{\lambda} \\ \frac{z'-z}{\lambda^3} &= \frac{d}{dz} \,.\, \frac{1}{\lambda} \end{aligned}\right\} \cdots\cdots\cdots\cdots (92)$$

and that moreover, if we put U for the function $\frac{x\,x' + y\,y' + z\,z'}{r'^3}$, we shall have

$$\frac{x'}{r'^3} = \frac{d\,\mathrm{U}}{d\,x}; \quad \frac{y'}{r'^3} = \frac{d\,\mathrm{U}}{d\,y}; \quad \frac{z'}{r'^3} = \frac{d\,\mathrm{U}}{d\,z}$$

so that if we assume

$$\begin{aligned} \Omega &= \mathrm{U} - \frac{1}{\lambda} \\ &= \frac{x\,x' + y\,y' + z\,z'}{(x'^2 + y'^2 + z'^2)^{\frac{3}{2}}} - \frac{1}{\sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}} \end{aligned} \qquad (93)$$

the disturbing forces will be represented by

$$m' \,.\, \frac{d\,\Omega}{d\,x}; \quad m' \,.\, \frac{d\,\Omega}{d\,y}; \quad m' \,.\, \frac{d\,\Omega}{d\,z}$$

and the equations representing the motion of m, will be (if we put $1 + m = \mu$) and suppose $d\,t$ the element of the time invariable.

$$\left.\begin{aligned} o &= \frac{d^2 x}{d\,t^2} + \mu \,.\, \frac{x}{r^3} + m' \,.\, \frac{d\,\Omega}{d\,x} \\ o &= \frac{d^2 y}{d\,t^2} + \mu \,.\, \frac{y}{r^3} + m' \,.\, \frac{d\,\Omega}{d\,y} \\ o &= \frac{d^2 z}{d\,t^2} + \mu \,.\, \frac{z}{r^3} + m' \,.\, \frac{d\,\Omega}{d\,z} \end{aligned}\right\} \cdots\cdots\cdots\cdots (94)$$

Similarly, if we put $\mathrm{U}' = \frac{x\,x' + y\,y' + z\,z'}{r^3}$ and take $\mu' = 1 + m'$, and

$$\Omega' = \mathrm{U}' - \frac{1}{\lambda} \qquad (95)$$

the motion of m will be represented by the equations

$$\left.\begin{aligned} o &= \frac{d^2 x'}{d\,t^2} + \mu' \,.\, \frac{x'}{r'^3} + m \,.\, \frac{d\,\Omega'}{d\,x'} \\ o &= \frac{d^2 y'}{d\,t^2} + \mu' \,.\, \frac{y'}{r'^3} + m \,.\, \frac{d\,\Omega'}{d\,y'} \\ o &= \frac{d^2 z'}{d\,t^2} + \mu' \,.\, \frac{z'}{r'^3} + m \,.\, \frac{d\,\Omega'}{d\,z'} \end{aligned}\right\} \cdots\cdots\cdots\cdots (96)$$

The joint integration of these six equations would determine the motions of both m and m', but their rigorous integration being impracticable in the present state of analysis, we are driven to have recourse to methods of approximation, of which the principle may be explained as follows :—

Suppose $A = o$ to be any equation, or system of equations, susceptible of rigorous integration, and let the process to be followed for this purpose, consist in deducing in succession from $A = o$, by any analytical artifices, or transformations, the equations, or systems of equations,

$$B = o, \quad C = o, \ldots\ldots\ldots K = o$$

the last of which, $K = o$, is integrable by known methods. Now, suppose, instead of $A = o$, we had proposed the equation

$$A + m' . a = o$$

where $m' . a$ is a minute term of the order of the disturbing forces, m' being an extremely small constant quantity, and a any assigned function of the co-ordinates. If then we pursue with this equation the very same process by which $B = o$ was derived from $A = o$, it is clear that we shall obtain an equation which can differ only from $B = o$ by a quantity which vanishes when $m' = o$, and which therefore must be of the form $m' \times b$: So that instead of $B = o$, we have

$$B + m' \times b = o$$

where b is some explicit function of the co-ordinates, their differential co-efficients, and of m', and either developed, or at least developable in a series of ascending powers of m'. In like manner we may deduce equations $C + m' . c = o; \ldots\ldots K + m' . k = o$
in the place of the equations $C = o, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots K = o$

In order then to obtain a final equation for the solution of the problem of three bodies, we have only to follow out any system of processes and transformations which, in the case of two only, would prove successful in reducing the differential equations to an integrable form.

This may be accomplished in a great variety of ways, as the equations of undisturbed motion are integrable by a great many different artifices, besides those which we have employed in the former part of this essay; and it is therefore necessary to select among them such as lead to final equations best adapted to the nature of the case under consideration. Now two courses have been adopted by geometers; the former adapted to the theory of the planetary, and the latter to that of the lunar perturbations.

In the theory of the planets, the disturbing force is so extremely minute, that its square and higher powers may be neglected with safety. This simplification being permitted, it becomes practicable (as we shall presently see) so to conduct the investigation as to make the time, or *mean* longitude of the disturbed planet our independent variable, and thus to express the perturbations at once in functions of the time, or of angles proportional to it, a simplification of the utmost moment in the construction of tables. In the more complicated theory of the moon, in which the part of the perturbations depending on higher powers of the disturbing force than the first is very conspicuous, it is no longer permitted to neglect them, and the necessity of preserving, as far as possible, the rigorous expressions of the forces, &c. at least in the *differential* equations, obliges us to employ, not the mean, but the true longitude of the moon for our independent variable, as by this means we are enabled to arrive at final equations perfectly rigorous, and can thus estimate the influence which the neglect of small quantities is capable of producing in their integration, with less likelihood of being misled.

The theory of the moon then differs entirely from that of the planets in its treatment. The general principles of approximation, however, are the same in both. Both theories, as we shall see, lead to final equations of the form

$$\frac{d^2 u}{d t^2} + n^2 u + m' . k = o \qquad (97)$$

where n^2 is constant, and k an explicit function of the several co-ordinates, distances, and angles, of the problem, and m' a very small quantity (which, in the lunar theory, also enters into the composition of k,) of the order of the disturbing forces. To approximate then to the value of u, we first suppose $m = o$, and we get a value of u corresponding to $m' = o$, and which we call its *elliptic value*.

2dly. We deduce from this the *elliptic values* of all the variables which enter into the composition of the term $m' k$, in terms of the independent variable t, whether t represent the mean, or the true longitude. These being substituted in the last term, it will become an explicit function of the independent variable.

3dly. If we now again integrate the differential equation so prepared, the value of u will consist of two parts; the first will be the same as before, viz. the elliptic value, and the second will be a correction which must be of the order of disturbing forces, and will express the perturbation of u with a degree of precision corresponding to their first power.

If the same process of substitution be repeated, and the equation again integrated, another set of terms will be added to the value of u, which carry the approximation a step farther, or to the squares of the disturbing forces; and were the same process continued to infinity, the series of terms so obtained would be a rigorous analytical expression for u.

The final equation (97) is of the form so well known in analysis, under the name of linear differential equations, and (as we have observed) almost all the equations on which the planetary motions depend being of this nature, it will be right to premise some few points relative to their theory, to which to refer hereafter.

It is demonstrated in all works in the differential and integral calculus, and will be so in our article on that subject, that any linear differential equation of the second order is integrable, provided we can find two, or even one particular value capable of satisfying it when deprived of its last term. Thus, if

$$\frac{d^2 u}{d t^2} + M . \frac{d u}{d t} + N u + \Pi = o \qquad (98)$$

be any such equation, M and N being functions of t, and if u' and u'' be any two functions of t which satisfy the equation

$$\frac{d^2 u}{d t^2} + \text{M} \cdot \frac{d u}{d t} + \text{N} u = o$$

then the complete integral will be

$$\text{C}' u' + \text{C}'' u'' - u' \int d \frac{u''}{u'} \int \frac{\Pi\, d t^2}{u'\, d \frac{u''}{u'}} \tag{99}$$

Now, the last term of this, being integrated by parts, becomes

$$u' \left\{ \frac{u''}{u'} \int \frac{\Pi\, d t^2}{u'\, d \frac{u''}{u'}} - \int \frac{\Pi\, d t^2 \times u''}{u'^2\, d \frac{u''}{u'}} \right\}$$

$$= u'' \int \frac{\Pi\, d t^2}{u'\, d \frac{u''}{u'}} + u' \int \frac{\Pi\, d t^2}{u''\, d \frac{u'}{u''}}$$

Consequently, the complete integral of the equation (98) will be

$$u = \text{C}' u' + \text{C}'' u'' - \left\{ u' \int \frac{\Pi\, d t^2}{u''\, d \frac{u'}{u''}} + u'' \int \frac{\Pi\, d t^2}{u'\, d \frac{u''}{u'}} \right\} \tag{100}$$

Suppose we have the equation

$$\frac{d^2 u}{d t^2} + n^2 u + \Pi = o \tag{101}$$

then u' and u'', the particular integrals of $\frac{d^2 u}{d t^2} + n^2 u = o$ offer themselves readily, being no other than $\sin n t$ and $\cos n t$; and if these be substituted in the general expression above given, we get at once

$$\begin{aligned} u &= \text{C}' . \cos n t + \text{C}'' . \sin n t \\ &+ \frac{1}{n} \left\{ \cos n t \int \Pi\, d t . \sin n t - \sin n t \int \Pi\, d t \cos n t \right\}; \end{aligned} \tag{102}$$

and if $n = 1$, or in the case of

$$\frac{d^2 u}{d t^2} + u + \Pi = o \tag{103}$$

the integral is

$$\begin{aligned} u &= \text{C}' . \cos t + \text{C}'' . \sin t \\ &+ \cos t . \int \Pi\, d t . \sin t - \sin t \int \Pi\, d t . \cos t \end{aligned} \tag{104}$$

These values of u are rigorous, whatever be the value of Π, and independent of any approximation, as it is easy for the reader to satisfy himself by substituting them in the differential equations from which they were deduced, when the whole will be found to vanish, independent of any particular value of Π, or any supposition made as to its magnitude. The equation (102) may however be obtained perhaps easier as follows.

Let $\frac{d^2 u}{d t^2} + n^2 u = - \Pi$ be multiplied by $d t^2 . \cos n t$ and it becomes

$$d^2 u . \cos n t + n^2 u\, d t^2 . \cos n t = - \int \Pi\, d t^2 . \cos n t$$

and integrating,

$$d u . \cos n t + n u\, d t . \sin n t = - \int \Pi\, d t^2 . \cos n t$$

If we again multiply this by $\frac{1}{\cos n t^2}$ we get

$$\begin{aligned} \frac{d u}{\cos n t} + n u\, d t . \frac{\sin n t}{(\cos n t)^2} &= - \frac{d t}{(\cos n t)^2} \int \Pi\, d t . \cos n t \\ &= - \frac{d . \tan n t}{n} \int \Pi\, d t . \cos n t \end{aligned}$$

and again integrating,

$$\begin{aligned} \frac{u}{\cos n t} &= - \frac{1}{n} \int d . \tan n t \int \Pi\, d t . \cos n t \\ &= \frac{1}{n} \left\{ - \tan n t \int \Pi\, d t . \cos n t + \int \Pi\, d t . \sin n t \right\} \end{aligned}$$

by integrating by parts. Hence we have

$$u = \frac{1}{n} \left\{ \cos n t \int \Pi\, d t . \sin n t - \sin n t \int \Pi\, d t . \cos n t \right\} \tag{105}$$

or, expressing the arbitrary constants by C′ and C″

$$u = \mathrm{C}' \,.\, \cos n\,t + \mathrm{C}'' \,.\, \sin n\,t$$

$$+ \frac{1}{n}\left\{ \cos n\,t \int \Pi\, d\,t \,.\, \sin n\,t - \sin n\,t \int \Pi\, d\,t \,.\, \cos n\,t \right\}$$

as before.

If Π be an explicit function of t, this value of u is always assignable, at least by quadratures, and, whenever the integrations can be executed, in finite terms. In the lunar and planetary theories, Π is always reducible to a series of sines or cosines of the form ${}^{\cos}_{\sin}\,(\mathrm{A}\,t + \mathrm{B})$. Let us therefore consider this case more closely. Now any term of Π, such as $a \times \sin(\mathrm{A}\,t + \mathrm{B})$ will introduce into u the term

$$\frac{a}{n}\left\{ \cos n\,t \int \sin n\,t \,.\, \sin(\mathrm{A}\,t + \mathrm{B})\, d\,t - \sin n\,t \int \cos n\,t \,.\, \sin(\mathrm{A}\,t + \mathrm{B})\, d\,t \right\}$$

But we have

$$\sin n\,t \,.\, \sin(\mathrm{A}\,t + \mathrm{B}) = \tfrac{1}{2}\left\{ \cos(\overline{\mathrm{A} - n} \,.\, t + \mathrm{B}) - \cos(\overline{\mathrm{A} + n} \,.\, t + \mathrm{B}) \right\}$$

whence

$$\int d\,t \,.\, \sin n\,t \,.\, \sin(\mathrm{A}\,t + \mathrm{B}) = \frac{\sin(\overline{\mathrm{A} - n} \,.\, t + \mathrm{B})}{2\,(\mathrm{A} - n)} - \frac{\sin(\overline{\mathrm{A} + n} \,.\, t + \mathrm{B})}{2 \,.\, (\mathrm{A} + n)}$$

and similarly,

$$\int d\,t \,.\, \cos n\,t \,.\, \sin(\mathrm{A}\,t + \mathrm{B}) = -\frac{\cos(\overline{\mathrm{A} + n} \,.\, t + \mathrm{B})}{2 \,.\, (\mathrm{A} + n)} - \frac{\cos(\overline{\mathrm{A} - n} \,.\, t + \mathrm{B})}{2\,(\mathrm{A} - n)}$$

So that the term introduced into u will become

$$a \,.\, \frac{\cos n\,t \,.\, \sin(\overline{\mathrm{A} - n} \,.\, t + \mathrm{B}) + \sin n\,t \,.\, \cos(\overline{\mathrm{A} - n} \,.\, t + \mathrm{B})}{2\,n\,(\mathrm{A} - n)}$$

$$- a \,.\, \frac{\cos n\,t \,.\, \sin(\overline{\mathrm{A} + n} \,.\, t + \mathrm{B}) - \sin n\,t \,.\, \cos(\overline{\mathrm{A} + n} \,.\, t + \mathrm{B})}{2\,n\,(\mathrm{A} + n)}$$

$$= a \,.\, \sin(\mathrm{A}\,t + \mathrm{B}) \left\{ \frac{1}{2\,n\,(\mathrm{A} - n)} - \frac{1}{2\,n\,(\mathrm{A} + n)} \right\}$$

$$= \frac{a \,.\, \sin(\mathrm{A}\,t + \mathrm{B})}{\mathrm{A}^2 - n^2}$$

Similarly, if $a \,.\, \cos(\mathrm{A}\,t + \mathrm{B})$ were any term of Π, the corresponding term in u would be $\dfrac{a \,.\, \cos(\mathrm{A}\,t + \mathrm{B})}{\mathrm{A}^2 - n^2}$

Π therefore consisting of a series of terms, such as

$$\Pi = a \,.\, {}^{\cos}_{\sin}\,(\mathrm{A}\,t + \mathrm{B}) + a' \,.\, {}^{\cos}_{\sin}\,(\mathrm{A}'\,t + \mathrm{B}') + \&c. \tag{106}$$

we shall have

$$u = \frac{a}{\mathrm{A}^2 - n^2} \,.\, {}^{\cos}_{\sin}\,(\mathrm{A}\,t + \mathrm{B}) + \frac{a'}{\mathrm{A}'^2 - n^2} \,.\, {}^{\cos}_{\sin}\,(\mathrm{A}'\,t + \mathrm{B}') + \&c. \tag{107}$$

$$+ \mathrm{C} \,.\, \cos n\,t + \mathrm{C}' \,.\, \sin n\,t$$

C and C′ being two arbitrary constants.

Terms of the form $a \,.\, {}^{\cos}_{\sin}(\mathrm{A}\,t + \mathrm{B})$ being of perpetual occurrence in physical astronomy, it is necessary to designate them and their several parts by names. The whole term is called an equation, or inequality: the part $(\mathrm{A}\,t + \mathrm{B})$ within the sign sin or cos is called the *argument;* and the co-efficient a, the maximum. The *period* of the inequality, or the time (in units of time such as t consists of) which it occupies in passing through all its gradations of magnitude and sign, is equal to $\dfrac{360}{\mathrm{A}}$, A being expressed in degrees. Hence, the period of an inequality is longer or shorter, according as the co-efficient of the time in its argument is less or greater.

The arguments of all the inequalities in u then are the same as those in Π, one remarkable case only excepted, in which $\mathrm{A} = n$; for, in this case, $\mathrm{A}^2 - n^2 = o$; and the term having $(\mathrm{A}\,t + \mathrm{B})$ for its argument, changes its form. In fact, since the constants C and C′ are arbitrary, we may change them into $\mathrm{C} - \dfrac{a \,.\, \sin \mathrm{B}}{\mathrm{A}^2 - n^2}$ and $\mathrm{C}' - \dfrac{a \quad \cos \mathrm{B}}{\mathrm{A}^2 - n^2}$ respectively, in which case $\mathrm{C} \,.\, \cos n\,t + \mathrm{C}' \,.\, \sin n\,t$ will be changed to $\mathrm{C} \,.\, \cos n\,t + \mathrm{C}' \,.\, \sin n\,t - \dfrac{a}{\mathrm{A}^2 - n^2} \,.\, \sin(\mathrm{A}\,t + \mathrm{B})$. Thus u will contain the terms

$$\frac{a}{\mathrm{A}^2 - n^2} \,.\, \sin(\mathrm{A}\,t + \mathrm{B}) - \frac{a}{\mathrm{A}^2 - n^2} \,.\, \sin(n\,t + \mathrm{B})$$

But when $A = n$, $a \, . \, \frac{\sin (A\,t + B) - \sin (n\,t + B)}{A^2 - n^2} = \frac{o}{o}$ and differentiating numerator and denominator with respect to A, it becomes simply (on making $A = n$)

$$\frac{a\,t}{2} \, . \, \cos (n\,t + B) \qquad (108)$$

Similarly, if Π contained the term $\cos (n\,t + B)$, this would introduce into u the term

$$-\frac{a\,t}{2} \, . \, \sin (n\,t + B) \qquad (108)$$

Terms of this kind form an exception to the law of periodicity observed by all the rest, having t disengaged from the sign sin or cos. If t represent either the time, or the mean or true longitude, of the disturbed body, they will represent inequalities, whose maxima go on continually increasing without limit. Such inequalities, if they really had an existence in our system, must end in its destruction, or at least in the total subversion of its present state; but we shall see hereafter, that when they do occur, they have their origin, not in the nature of the differential equations, but in the imperfection of our analysis, and in the inadequate representation of the perturbations, and are to be got rid of, or rather included in more general expressions, of a periodical nature, by a more refined investigation than that which led us to them. The nature of this difficulty will be easily understood from the following reasoning. Suppose that a term, such as $a \sin (A\,t + B)$ should exist in the value of u, in which A being extremely minute, the period of the inequality denoted by it would be of great length; then, whatever might be the value of the co-efficient a, the inequality would still be always confined within certain limits, and after many ages would return to its former state. Suppose now that our peculiar mode of arriving at the value of u, led us to this term, not in its real analytical form $a \, . \sin (A\,t + B)$, but by the way of its developement in powers of t, $\alpha + \beta\,t + \gamma\,t^2 + \&c.$; and that, not at once, but piecemeal, as it were; a first approximation giving us only the term α, a second adding the term $\beta\,t$, and so on. If we stopped here, it is obvious that we should mistake the nature of this inequality, and that a really periodical function, from the effect of an imperfect approximation, would appear under the form of one not periodical.

This is what actually takes place in the theory of the problem of three bodies. These terms in the value of u, when they occur, are not superfluous; they are essential to its expression, but they lead us to erroneous conclusions as to the stability of our system and the general laws of its perturbations, unless we keep in view that they are only parts of series; the principal parts, it is true, when we confine ourselves to intervals of moderate length, but which cease to be so after the lapse of very long times, the rest of the series acquiring ultimately the preponderance, and compensating the want of periodicity of its first terms.

SECTION II.

General theory of the planetary perturbations depending on their mutual configurations.

OUR first object in the theory of the planets is to transform the differential equations of the disturbed orbit, so as to obtain final equations in which the radius vector, and true longitude, or those parts of them arising from the action of the disturbing forces, shall be expressed in terms of the time; and to reduce them to the general form of the linear equation of the second order, whose theory we have just considered. To this end,

Let the equations (94) of m's motion be respectively multiplied by x, y, z, and added together, and we get

$$o = \frac{d\,x\,d^2\,x + d\,y\,d^2\,y + d\,z\,d^2\,z}{d\,t^2} + \mu \, . \, \frac{x\,d\,x + y\,d\,y + z\,d\,z}{r^3}$$
$$+ \, m' \, . \left\{ \frac{d\,\Omega}{d\,x}\,d\,x + \frac{d\,\Omega}{d\,y}\,d\,y + \frac{d\,\Omega}{d\,z}\,d\,z \right\}$$

The portion within the brackets of the last term is the differential of Ω taken on a supposition of the co-ordinates of m only varying. Let this be represented by the Roman character d, so that

$$\mathrm{d}\,\Omega = \frac{d\,\Omega}{d\,x}\,d\,x + \frac{d\,\Omega}{d\,y}\,d\,y + \frac{d\,\Omega}{d\,z}\,d\,z$$

bearing this in mind, and that $\mathrm{d}\,\Omega$ is only an abbreviated expression for this function, we have by integration

$$o = \frac{d\,x^2 + d\,y^2 + d\,z^2}{d\,t^2} - \frac{2\mu}{r} + \frac{\mu}{a} + 2\,m' \int \mathrm{d}\,\Omega; \qquad (109)$$

in which we must be careful not to confound $\int \mathrm{d}\,\Omega$ with $\int d\,\Omega$ or Ω, $\mathrm{d}\,\Omega$ being only an incomplete differential, and the characteristic $\int$ denoting an integration relative to t, supposes the variation of the co-ordinates of the disturbing as well as the disturbed body.

Again, if we multiply the same equations (94) by x, y, z, respectively, we shall get

$$o = \frac{x\,d^2\,x + y\,d^2\,y + z\,d^2\,z}{d\,t^2} + \frac{\mu}{r} + m' \left\{ x\,\frac{d\,\Omega}{d\,x} + y\,\frac{d\,\Omega}{d\,y} + z\,\frac{d\,\Omega}{d\,z} \right\};$$

If we add this to the former, observing that

$$d\,x^2 + d\,y^2 + d\,z^2 + x\,a^2\,x + y\,d^2 y + z\,d^2 z = \tfrac{1}{2}\,d^2\,(x^2 + y^2 + z^2) = \frac{d^2\,.\,r^2}{2}$$

we get

$$o = \tfrac{1}{2}\,\frac{d^2 r^2}{d\,t^2} - \frac{\mu}{r} + \frac{\mu}{a} + 2\,m' \int d\ \Omega + m' \left\{ x\,\frac{d\,\Omega}{d\,x} + y\,\frac{d\,\Omega}{d\,y} + z\,\frac{d\,\Omega}{d\,z} \right\}$$

Now, if we put $x = \rho\,.\cos\theta$, $y = \rho\,.\sin\theta$ and $z = \rho\,.\,s$, or suppose $\rho =$ the projected radius vector r, (see fig. 11.) θ the angle it makes with the axis of the x, and $s =$ the tangent of the latitude of m, we have

$$\rho = \frac{r}{\sqrt{1 + s^2}}\,;\ r\,\frac{d\,\rho}{d\,r} = r\,.\,\frac{1}{\sqrt{1 + s^2}} = \rho$$

But

$$r\,\frac{d\,x}{d\,r} = r\,\frac{d\,x}{d\,\rho}\,.\,\frac{d\,\rho}{d\,r} = \rho\,\frac{d\,x}{d\,\rho} = \rho\,.\cos\theta = x$$

and similarly,

$$r\,\frac{d\,y}{d\,r} = y \quad \text{and}\ r\,\frac{d\,z}{d\,r} = z$$

Consequently, substituting for x, y, z, these values

$$x\,\frac{d\,\Omega}{d\,x} + y\,\frac{d\,\Omega}{d\,y} + z\,\frac{d\,\Omega}{d\,z} = r \left\{ \frac{d\,\Omega}{d\,x}\,.\,\frac{d\,x}{d\,r} + \frac{d\,\Omega}{d\,y}\,.\,\frac{d\,y}{d\,r} + \frac{d\,\Omega}{d\,z}\,.\,\frac{d\,z}{d\,r} \right\} = r\,.\,\frac{d\,\Omega}{d\,r}$$

and it will be observed, that this property is altogether independent of the nature of the function Ω, and belongs to every possible function of the co-ordinates x, y, z, x', y', z'.

Thus we see, that

$$2\,m' \int d\,\Omega + m' \left\{ x\,\frac{d\,\Omega}{d\,x} + y\,\frac{d\,\Omega}{d\,y} + z\,\frac{d\,\Omega}{d\,z} \right\} = m'\,.\left\{ 2 \int d\ \Omega + r\,\frac{d\,\Omega}{d\,r} \right\}$$

Hence, if we put

$$Q = 2 \int d\,\Omega + r\,\frac{d\,\Omega}{d\,r}, \tag{110}$$

our differential equation becomes

$$o = \tfrac{1}{2}\,\frac{d^2\,.\,r^2}{d\,t^2} - \frac{\mu}{r} + \frac{\mu}{a} + m'\,Q\,; \tag{111}$$

Let us now suppose that r represents only the elliptic value of r, and x, y, z, Ω, the elliptic values of the co-ordinates, and the value of the function Ω, which would arise from writing the elliptic values of x, y, z, x', y', z', in their expressions; and let $r + m'\,\delta\,r$, $x + m'\,\delta\,x$, $y + m'\,\delta\,y$, $z + m'\,\delta\,z$, $\Omega + m'\,\delta\,\Omega$, &c. represent the disturbed values of these quantities; then, if we neglect m'^2, we shall get by substitution

$$o = \tfrac{1}{2}\,\frac{d^2\,.\,r^2}{d\,t^2} - \frac{\mu}{r} + \frac{\mu}{a} + m'\,.\left\{ \frac{d^2\,.\,r\,\delta\,r}{d\,t^2} + \frac{\mu\,.\,r\,\delta\,r}{r^3} + Q \right\}$$

But since r represents the elliptic value of r, the first part of this equation vanishes of itself, and to determine $\delta\,r$ or $r\,\delta\,r$, we have the differential equation

$$o = \frac{d^2\,.\,r\,\delta\,r}{d\,t^2} + \frac{\mu}{r^3}\,.\,r\,\delta\,r + Q\,; \tag{112}$$

This equation being linear, and of the second order, is immediately integrable by our general formula, equation (100) provided we can find the two particular integrals u' and u'' of the equation

$$\frac{d^2\,u}{d\,t^2} + \frac{\mu}{r^3}\,u = o$$

but since r, on the supposition of the term Q bearing zero, may be taken for the radius vector on the hypothesis of elliptic motion, it is obvious that the elliptic values of either x, or y, or z, will satisfy this equation, because these values are, in fact, no other than what are derived from the integration of equations precisely similar, viz.

$$\frac{d^2\,x}{d\,t^2} + \frac{\mu}{r^3}\,x = o$$

$$\frac{d^2\,y}{d\,t^2} + \frac{\mu}{r^3}\,y = o$$

$$\frac{d^2\,z}{d\,t^2} + \frac{\mu}{r^3}\,z = o$$

Consequently, we may take $u' = x$ and $u'' = y$, whence we get $u''\,\dfrac{d\,u'}{u''} = \dfrac{y\,d\,x - x\,d\,y}{y} = \dfrac{h\,d\,t}{y}$ because on the hypothesis of elliptic motion $y\,d\,x - x\,d\,y = h\,d\,t$, and it is of the elliptic values of x and y that we are now speaking. Similarly, $u'\,d\,\dfrac{u''}{u'} = \dfrac{x\,d\,y - y\,d\,x}{x} = -\dfrac{h\,d\,t}{x}$, and the formula (100) gives

$$C'\,.\,x + C''\,.\,y - x \int \frac{Q\,y\,d\,t}{h} + y \int \frac{Q\,x\,d\,t}{h}$$

for the complete value of u or $r\,\delta\,r$. The constants then being included under the integral sign, we have

$$r\,\delta\,r = \frac{y\int Q\,x\,d\,t - x\int Q\,y\,d\,t}{h}; \qquad (113)$$

Such is the value of $r\,\delta\,r$ when we consider only the first power of the disturbing force. It would be absolutely exact, but that x and y are only particular values of u on the hypothesis of r having its elliptic value. If this supposition were not made, we should have $u' = x + m'\,\delta\,x$ and $u'' = y + m'\,\delta\,y$; and substituting these values, we should obtain terms in the expression of $m'\,\delta\,r$ depending on the square of the disturbing force, but with these we have no concern.

The perturbation in longitude ($m'\,\delta\,\theta$) is easily obtained when the value of $\delta\,r$ is found. In fact, we have

$$d\,x^2 + d\,y^2 + d\,z^2 = d\,r^2 + r^2\,d\,\theta^2$$

whence we get

$$o = \frac{r^2\,d\,\theta^2 + d\,r^2}{d\,t^2} - \frac{2\,\mu}{r} + \frac{\mu}{a} + 2\,m'\int d\,\Omega \qquad (114)$$

and if we subtract this from the equation (111), we find

$$o = \frac{r^2\,d\,\theta^2}{d\,t^2} - \frac{r\,d^2\,r}{d\,t^2} - \frac{\mu}{r} - m'\,.\,r\,\frac{d\,\Omega}{d\,r} \qquad (115)$$

If in this we substitute $r + m'\,\delta\,r$ for r and $\theta + m'\,\delta\,\theta$ for θ we get, (after obliterating the terms which destroy each other by reason of the properties of elliptic motion, and those which contain m'^2)

$$o = \frac{2\,r^2\,d\,\theta\,d\,\delta\,\theta}{d\,t^2} + \frac{2\,r\,\delta\,r\,d\,\theta^2}{d\,t^2} - \frac{r\,d^2\,\delta\,r}{d\,t^2} - \frac{d^2\,r\,.\,\delta\,r}{d\,t^2} + \frac{\mu\,r\,\delta\,r}{r^3} - r\,\frac{d\,\Omega}{d\,r}; \qquad (116)$$

From this, let the term multiplied by $\frac{d\,\theta^2}{d\,t^2}$ be eliminated by means of equation (115) and we get

$$\frac{2\,r^2\,d\,\theta\,d\,\delta\,\theta}{d\,t^2} = \frac{r\,d^2\,\delta\,r - \delta\,r\,.\,d^2\,r}{d\,t^2} - \frac{3\,\mu\,r\,\delta\,r}{r^3} + r\,\frac{d\,\Omega}{d\,r}; \qquad (117)$$

And if in this we substitute for $\frac{\mu\,r\,\delta\,r}{r^3}$ its value given by the equation (112) we obtain, (restoring the value of Q)

$$d\,\delta\,\theta = \frac{d\,\{d\,r\,\delta\,r + 2\,r\,d\,\delta\,r\} + \left(3\int d\,\Omega + 2\,r\,\frac{d\,\Omega}{d\,r}\right).\,d\,t^2}{r^2\,d\,\theta}; \qquad (118)$$

but $r^2\,d\,\theta = h\,d\,t$, elliptic values only being considered in the second member of this equation, and consequently integrating, we have

$$h\,\delta\,\theta = \frac{d\,r}{d\,t}\,\delta\,r + 2\,r\,\frac{d\,\delta\,r}{d\,t} + \int\left(3\int d\,\Omega + 2\,r\,\frac{d\,\Omega}{d\,r}\right)d\,t; \qquad (119)$$

Now, we have $h = \sqrt{\mu\,a\,(1 - e^2)}$, and if we put $n\,t$ for the mean motion of the disturbed planet m, we have $n = \sqrt{\frac{\mu}{a^3}}$ so that $h = n\,a^2\,.\,\sqrt{1 - e^2}$; and $\frac{1}{h} = \frac{n\,a}{\mu\,\sqrt{1 - e^2}}$

Consequently we get, for the perturbation of the radius vector,

$$m'\,\delta\,r = \frac{m'\,a\left\{\cos\theta\int r\,.\,\sin\theta\,.\,Q\,n\,d\,t - \sin\theta\int r\,.\,\cos\theta\,.\,Q\,n\,d\,t\right\}}{\mu\,\sqrt{1 - e^2}} \qquad (120)$$

and the formula expressing the perturbation in longitude, will become

$$m'\,\delta\,\theta = \frac{m'}{n\,a^2\,\sqrt{1 - e^2}}\left(\frac{d\,r}{d\,t}\,\delta\,r + 2\,r\,\frac{d\,\delta\,r}{d\,t}\right)$$
$$+ \frac{a\,m'}{\mu\,\sqrt{1 - e^2}}\left\{2\int r\,\frac{d\,\Omega}{d\,r}\,.\,n\,d\,t + 3\iint d\,\Omega\,.\,n\,d\,t\right\}; \qquad (121)$$

It only remains to determine the amount of the perturbation in latitude, or the value of z or of $\delta\,s$, if we put $z = r\,\delta\,s$, in which case $\delta\,s$ represents the tangent of the heliocentric latitude of m in its disturbed orbit above the plane of its elliptic motion. Now, if we treat the equation

$$o = \frac{d^2\,z}{d\,t^2} + \frac{\mu\,z}{r^3} + m'\,\frac{d\,\Omega}{d\,z}$$

in the same manner as the equation by which the value of $r\,\delta\,r$ was found, viz. regarding x and y as two particular integrals of the equation $\frac{d^2\,z}{d\,t^2} + \frac{\mu\,z}{r^3} = o$, and then completing the integration by the general formula (100)* we shall find (putting $r \,.\, \cos\theta$ and $r \,.\, \sin\theta$, for x and y)

$$m'\,\delta s = \frac{m'\,a\left\{\cos\theta\int y\,\frac{d\,\Omega}{d\,z}\,.\,n\,d\,t - \sin\theta : \int x\,\frac{d\,\Omega}{d\,z}\,.\,n\,d\,t\right\}}{\mu\,\sqrt{1-e^2}} \qquad (122)$$

This is the latitude of m above its primitive orbit; and if we denote by s its undisturbed latitude above any fixed plane (as that of the ecliptic) slightly inclined to this orbit, $s + \delta s$ will be its latitude when subjected to the action of the disturbing forces.

The equation (121) gives the perturbation in longitude when that of the radius vector is known, and the latter may be computed from the expression (120) which is general; and considering the complication of the subject, as simple as can be expected. Its form enables us to compute the amount of perturbation even in the most difficult cases, as in that of a comet, by the application of the method of quadratures. Meanwhile, in the theory of the planets, where it is required to develope the value of δr in series of sines and cosines of arcs depending on the configurations of the disturbed and disturbing planet, it will be found much simpler to set out immediately from the differential equations for the disturbed radius, and proceed in the manner now to be explained.

Since the form of this equation is not precisely that of the equation $\frac{d^2 u}{d\,t^2} + n^2 u + \Pi = o$, the co-efficient of the second term instead of being constant, being $\frac{\mu}{r^3}$ a variable quantity, we must first endeavour to transform it by substitution into one of this form. Assuming then that u is such a quantity that its elliptic value shall satisfy the equation $\frac{d^2 u}{d\,t^2} + n^2 u = o$, and its disturbed value (or $u + m'\,\delta u$) the equation $\frac{d^2 (u + m'\,\delta u)}{d\,t^2} + n^2 (u + m'\,\delta u) + m'\,\Pi = o$, which gives $\frac{d^2\,\delta u}{d\,t^2} + n^2\,\delta u + \Pi = o$ we must inquire first, the relation between r and u; and secondly, the value of Π.

A satisfactory relation between r and u is easily found. In fact, since u is to satisfy $\frac{d^2 u}{d\,t^2} + n^2 u = o$, it must be of the form $u = \text{const.}\cos(n\,t + \text{const.})$. Now the developement of r in terms of the mean longitude gives, putting ϵ for the longitude at the epoch of the commencement of the time t, and ϖ for the longitude of the perihelion,

$$r = a\left\{1 - e\,.\cos(n\,t + \epsilon - \varpi) + e^2 \sin^2(n\,t + \epsilon - \varpi) + \&c.\right\}$$

So that, if we take $u = e\,.\cos(n\,t + \epsilon - \varpi)$, we shall have

$$r = a\left\{(1 + e^2) - u\,(1 - \tfrac{1}{2}e^2) - u^2 + \&c.\right\} \qquad (123)$$

This gives at once

$$\begin{aligned}\delta r &= -\,a\,\delta u\left\{1 + 2\,u + e^2 \times \&c.\right\}\\ &= -\,a\,\delta u\left\{1 + 2\,e\,.\cos(n\,t + \epsilon - \varpi) + e^2\,.\,\&c.\right\};\end{aligned} \qquad (124)$$

by which, when δu is found, δr may be had at once.

It only remains to discover Π. Now, since our equation (111), if we put $r^2 = v$ and $m'\left\{2\int d\,\Omega + r\frac{d\Omega}{d\,r}\right\} = m'\,Q$, becomes

$$\frac{d^2 v}{d\,t^2} = \frac{2\,\mu}{\sqrt{v}} - \frac{2\,\mu}{a} - 2\,m'\,Q \qquad (125)$$

if we multiply by $2\,d\,v$, and integrate, we shall find

$$\left(\frac{d\,v}{d\,t}\right)^2 = 8\,\mu\,\sqrt{v} - \frac{4\,\mu\,v}{a} - 4\,m'\int Q\,d\,v \qquad (126)$$

but because u is a function of r, and therefore of r^2 or v, we have

$$\frac{d\,u}{d\,t} = \frac{d\,u}{d\,v}\,.\,\frac{d\,v}{d\,t}$$

$$\frac{d^2 u}{d\,t^2} = \frac{d^2 u}{d\,v^2}\left(\frac{d\,v}{d\,t}\right)^2 + \frac{d\,u}{d\,v}\,.\,\frac{d^2 v}{d\,t^2}$$

so that

$$\begin{aligned}\frac{d^2 u}{d\,t^2} + n^2 u = {}& \frac{d^2 u}{d\,v^2}\left\{8\,\mu\,\sqrt{v} - \frac{4\,\mu\,v}{a} - 4\,m'\int Q\,d\,v\right\}\\ &+ \frac{d\,u}{d\,v}\left\{\frac{2\,\mu}{\sqrt{v}} - \frac{2\,\mu}{a} - 2\,m'\,Q\right\} + n^2 u.\end{aligned}$$

Now u is a certain function of v (or of r) whose form is determined by the reversion of the series in (123) and is independent of the disturbing forces. But were these forces zero, we should have $\frac{d^2 u}{d\,t^2} + n^2 u = o$.

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* This formula, which perhaps is new, and which has stood us in some stead in the explanation of that chapter of the *Mécanique Céleste*, (cap. vi. liv. 2.) which I have adopted for the groundwork of this part of the present essay, may be deduced at once from the general theory of linear equations, in my paper *On various points of Analysis.—Phil. Trans.* 1814.

Hence the portion of the right hand member of the equation just deduced, which does not depend on the disturbing forces, must be identically zero, in virtue of the relation between u and v; and that it is so, we may assure ourselves by actual substitution. Consequently we must have, when the disturbing forces are regarded,

$$\frac{d^2 u}{d t^2} + n^2 u = \left(-2 Q \frac{d u}{d v} - 4 \int Q \, d v \times \frac{d^2 u}{d v^2}\right) \times m'$$

that is (putting $u + m' \delta u$ for u, and disregarding terms depending on m'^2)

$$\frac{d^2 \delta u}{d t^2} + n^2 \delta u = -2 Q \frac{d u}{d v} - 4 \frac{d^2 u}{d v^2} \int Q \, d v$$

Comparing this with the equation $\frac{d^2 \delta u}{d t^2} + n^2 \delta u + \Pi = 0$ we have

$$\Pi = 2 Q . \frac{d u}{d v} + 4 \frac{d^2 u}{d v^2} \int Q \, d v$$

It is desirable to express this in terms of v. Now, as u is a function of v, v is reciprocally a function of u, and

$$\frac{d u}{d v} = \frac{1}{\frac{d v}{d u}}; \quad \frac{d^2 u}{d v^2} = -\frac{\frac{d^2 v}{d u^2}}{\left(\frac{d v}{d u}\right)^3}$$

and

$$\Pi = \frac{2}{\frac{d v}{d u}} Q - 4 \frac{\frac{d^2 v}{d u^2}}{\left(\frac{d v}{d u}\right)^3} \int Q \, d v \qquad (127)$$

But since $v = r^2$, if for r we put its value in terms of u (123), we get, neglecting higher powers of u than the first, $v = a^2 (1 - 2 u + \&c.)$ and substituting this in Π, and after the differentiations writing for u its value $e . \cos (n t + \epsilon - \varpi)$ we get

$$\Pi = -\frac{Q}{a^2}\left(1 - e . \cos (n t + \epsilon - \varpi)\right) - \frac{1}{4} e^2 . \cos (2 n t + 2 \epsilon - 2 \varpi)$$

$$- \frac{2 e}{a^2} \int Q \, n \, d t . \sin (n t + \epsilon - \varpi) \{1 + e . \cos (n t + \epsilon - \varpi)\} \qquad (128)$$

and this being the value of Π, we find δu from the equation

$$\frac{d^2 \delta u}{d t^2} + n^2 \delta u + \Pi = 0. \qquad (129)$$

SECTION III.

Reduction of the perturbative function Q *or* $2 \int d \Omega + r \frac{d \Omega}{d r}$ *to a series of sines and cosines, and investigation of the perturbations, neglecting the eccentricities and inclinations of both orbits.*

WE have now reduced the investigation of the perturbation to the integration of the linear equation (129) and we have before seen that this is accomplished without difficulty, when Π, the last term, is reducible to sines and cosines of the independent variable and its multiples. All then that remains to be done to get their actual expressions, is to execute this reduction. This, however, is by no means a simple process; and in an essay like the present it is not possible to pursue it into all its details: we shall therefore only carry it to a certain extent necessary for our future reference, and point out the principles by which it may be, if required, carried farther.

Let us consider then the value of the function Ω. If we write in it $r . \cos \theta$, $r . \sin \theta$, $r' . \cos \theta'$ and $r' . \sin \theta'$ for x, y, x', y', and neglect $z z'$, z'^2, $(z - z')^2$, which are either of the order of the squares of the disturbing forces, or of the products of these forces by the mutual inclination of the orbits, and put $\theta - \theta' = w$, we get

$$x x' + y y' + z z' = r r' . \cos w$$

$$\lambda = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} = \sqrt{r^2 - 2 r r' . \cos w + r'^2}$$

and

$$\Omega = \frac{r}{r'^2} . \cos w - \frac{1}{\sqrt{r^2 - 2 r r' . \cos w + r'^2}}$$

Let us conceive this function developed in a series of cosines of w, and its positive and negative multiples to infinity; then, since the cosines of the negative are equal to those of the positive multiples, we may represent Ω as follows:

$$\Omega = R + R' . \cos w + R'' . \cos 2 w + R''' . \cos 3 w + \&c.; \qquad (130)$$

where R, R′, R″, &c. are certain given, explicit functions of r and r' and of these alone, depending solely on the peculiar form of Ω, let A, A′, A″, &c. represent the same functions of a and a'. Then, if the eccentricities of the orbits were nothing, we should have

$$\left.\begin{aligned} \Omega &= \frac{a}{a'^2} . \cos w - \frac{1}{\sqrt{a^2 - 2\,a\,a' . \cos w + a'^2}} \\ &= \text{A} + \text{A}' . \cos w + \text{A}'' . \cos 2\,w + \&\text{c.} \end{aligned}\right\} ; \ldots\ldots\ldots\ldots\ldots\ldots (131)$$

Were the eccentricities nothing, the orbits would be circles, and the motion in them uniform. We should therefore have $\theta = n\,t$ and $\theta' = n'\,t$, whence $w = \theta - \theta' = (n - n')\,t$, so that Ω in this case would be expressed in the very simple series

$$\Omega = \text{A} + \text{A}' . \cos\,(n - n')\,t + \text{A}'' . \cos 2\,(n - n')\,t + \&\text{c.} \qquad (132)$$

Moreover, since d Ω represents the differential of Ω taken on the hypothesis that only the disturbed body moves, we should then have

$$\text{d}\,\Omega = \frac{d\,\Omega}{d\,r}\,d\,r + \frac{d\,\Omega}{d\,\theta}\,d\,\theta = \frac{d\,\Omega}{d\,\theta}\,d\,\theta = n\,d\,t . \frac{d\,\Omega}{d\,\theta},$$

because, in the case of circular orbits $d\,r = o$. Thus we should have

$$\text{d}\,\Omega = -\,n\,d\,t . \{\text{A}' . \sin\,(n - n')\,t + 2 . \text{A}'' . \sin\,(2\,n - 2\,n')\,t + \&\text{c.}\}$$

and integrating relative to t,

$$\int \text{d}\,\Omega = \frac{g}{a} + \frac{n}{n - n'}\{\text{A}' . \cos\,(n - n')\,t + \text{A}'' . \cos 2\,(n - n')\,t + \&\text{c.}\}; \qquad (133)$$

$\frac{g}{a}$ being an arbitrary constant.

Again, since R, R′, R″, &c. are explicit functions of r, r', and Ω is only so far a function of r, as this symbol is contained in them, we must have

$$r\,\frac{d\,\Omega}{d\,r} = r\,\frac{d\,\text{R}}{d\,r} . + r\,\frac{d\,\text{R}'}{d\,r} . \cos w + \&\text{c.} \qquad (134)$$

and in the case of circular orbits, denoting by $\frac{d\,\text{A}}{d\,a}$, &c. the same functions of a, a' that $\frac{d\,\text{R}}{d\,r}$ denote of r, r',

$$r\,\frac{d\,\Omega}{d\,r} = a\,\frac{d\,\text{A}}{d\,a} + a\,\frac{d\,\text{A}'}{d\,a} . \cos w + \&\text{c.}$$

The values of $\frac{d\,\text{A}}{d\,a}$, $\frac{d\,\text{A}'}{d\,a}$, &c. are easily had in functions of a, a', when those of A, A′, &c. are found. Now, to obtain these, we may proceed as follows:

Take $c = \cos w + \sqrt{-1} . \sin w$. Then (by trigonometry) we shall have $\frac{1}{c} = \cos w - \sqrt{-1} . \sin w$; now, let us consider the function $(a^2 - 2\,a\,a' . \cos w + a'^2)^{-s}$ which agrees with the second term of Ω if $s = \frac{1}{2}$. This equals $a^{-2s}\left(1 - 2\,\frac{a'}{a} . \cos w + \left(\frac{a'}{a}\right)^2\right)^{-s}$ or $a^{-2s}\,(1 - 2\,\alpha . \cos w + \alpha^2)^{-s}$ putting $\alpha = \frac{a'}{a}$. But we have

$$1 - 2\,\alpha . \cos w + \alpha^2 = (-\,\alpha\,c)\left(1 - \alpha . \frac{1}{c}\right)$$

because $c + \frac{1}{c} = 2 . \cos w$. Hence

$$(a^2 + 2\,a\,a' . \cos w + a'^2)^{-s} = a^{-2s} . (1 - \alpha\,c)^{-s} . \left(1 - \frac{\alpha}{c}\right)^{-s}$$

$$= a^{-2s} \times \left\{1 + \frac{s}{1} . \alpha\,c + \frac{s\,(s + 1)}{1 . 2}\,\alpha^2\,c^2 + \&\text{c.}\right\} \times \left\{1 + \frac{s}{1} . \frac{\alpha}{c} + \frac{s\,(s + 1)}{1 . 2} . \frac{\alpha^2}{c^2} + \&\text{c.}\right\}$$

$$= a^{-2s}\left\{1 + \left(\frac{s}{1}\right)^2 . \alpha^2 + \left(\frac{s\,(s + 1)}{1 . 2}\right)^2 \alpha^4 + \&\text{c.}\right\}$$

$$+ \, a^{-2s}\left\{\frac{s}{1}\,\alpha + \frac{s\,(s + 1)}{1 . 2} . \frac{s}{1}\,\alpha^3 + \frac{s\,(s + 1)\,(s + 2)}{1 . 2 . 3} \times \frac{s\,(s + 1)}{1 . 2}\,\alpha^5 + \&\text{c.}\right\}\left(c + \frac{1}{c}\right)$$

$$+ \, a^{-2s}\left\{\frac{s\,(s + 1)}{1 . 2}\,\alpha^2 + \frac{s\,(s + 1)\,(s + 2)}{1 . 2 . 3} \times \frac{s}{1} . \alpha^4 + \&\text{c.}\right\}\left(c^2 + \frac{1}{c^2}\right)$$

$$+ \, \&\text{c.}$$

but $c + \frac{1}{c} = 2 \cos w$; $c^2 + \frac{1}{c^2} = 2 . \cos 2\,w$, &c. (by trigonometry); consequently we have

$$(a^2 - 2\,a\,a' . \cos w + a'^2)^{-s} = a^{-2s}\left\{1 + \left(\frac{s}{1}\right)^2 . \alpha^2 + \&c.\right\} \quad (135)$$
$$+ 2a^{-2s}\left\{\frac{s}{1}\,\alpha + \frac{s\,(s+1)}{1\,.\,2} . \frac{s}{1}\,\alpha^3 + \&c.\right\} . \cos w$$
$$+ 2\,a^{-2s}\left\{\frac{s\,(s+1)}{1\,.\,2}\,\alpha^2 + \&c.\right\} . \cos 2\,w + \&c.$$

In the case before us, where $s = \frac{1}{2}$, this gives at once

$$\text{A} = \frac{-1}{a}\left\{1 + \left(\frac{1}{2}\right)^2 \alpha^2 + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2 \alpha^4 + \&c.\right\} \quad (136,\,1)$$
$$\text{A}' = \frac{-2}{a}\left\{\frac{1}{2}\,\alpha + \frac{1^2\,.\,3}{2^2\,.\,4}\alpha^3 + \frac{1^2\,.\,3^2\,.\,5}{2^2\,.\,4^2\,.\,6}\,\alpha^5 + \&c.\right\} + \frac{a}{a'^2} \quad (136,\,2)$$
$$\text{A}'' = \frac{-2}{a}\left\{\frac{1\,.\,3}{2\,.\,4}\alpha^2 + \frac{1^2\,.\,3\,.\,5}{2^2\,.\,4\,.\,6}\,\alpha^4 + \&c.\right\} \quad (136,\,3)$$
&c.

If α be less than unity, these series are convergent; but if greater, we have only to throw the expression into the form $a'^{-2s}\left(1 - 2\frac{a}{a'} . \cos w + \left(\frac{a}{a'}\right)^2\right)^{-s}$ previous to developement, and taking $\alpha = \frac{a}{a'}$, instead of $\frac{a'}{a}$, α will now be less than 1, and we shall have $\text{A} = \frac{1}{a'}\left(1 + \left(\frac{1}{2}\right)^2 \alpha^2 + \&c.\right)$ &c.

In the former case, when $\frac{a}{a'}$ is less than 1, or the orbit of the disturbed planet is *interior* to that of the disturbing, we have

$$-\text{A} = \frac{1}{a} + \left(\frac{1}{2}\right)^2 . \frac{a'^2}{a^3} + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2 . \frac{a'^4}{a^5} + \&c.; \quad (137,\,1)$$
$$\frac{d\,\text{A}}{d\,a} = + \frac{1}{a^2}\left\{1 + 3 . \left(\frac{1}{2}\right)^2 . \left(\frac{a'}{a}\right)^2 + \&c.\right\}; \quad (137,\,2)$$
$$\text{A}' = -\,2\left\{\frac{1}{2}\,\frac{a'}{a^2} + \frac{1^2\,.\,3}{2^2\,.\,4} . \frac{a'^3}{a^4} + \&c.\right\} + \frac{a}{a'^2}; \quad (137,\,3)$$
$$\frac{d\,\text{A}'}{d\,a} = + \frac{2}{a^2}\left\{1 . \frac{a'}{a} + \frac{1^2\,.\,3}{2^2} . \left(\frac{a'}{a}\right)^3 + \&c.\right\} + \frac{1}{a'^2} \quad (137,\,4)$$

and so on; and similar expressions are also readily obtained in the case when the orbit of the disturbed planet is the exterior. Thus, when the mean distances of the two planets are given, the values of A, A′, &c. and their differential co-efficients $\frac{d\,\text{A}}{d\,a}$, &c. are reducible to numerical evaluation, and may therefore be regarded as known quantities. The properties of the series on which they depend, afford many resources for facilitating their evaluation, and rules for deriving one of these quantities from another, but these we shall not stay to explain. The reader will find them with every developement in the second book of Laplace's *Mécanique Céleste,* art. 49.

These values once determined, we have Q, or $2\int d\,\Omega + r\frac{d\,\Omega}{d\,r}$, expressed as follows:

$$\left.\begin{aligned}\text{Q} = \frac{2g}{a} + a\,\frac{d\,\text{A}}{d\,a} + \left\{a\,\frac{d\,\text{A}'}{d\,a} + \frac{2\,n\,\text{A}'}{n - n'}\right\}\cos\,(n - n')\,t \\ + \left\{a\,\frac{d\,\text{A}''}{d\,a} + \frac{2\,n\,\text{A}''}{n - n'}\right\}\cos 2\,(n - n')\,t + \&c.\end{aligned}\right\}\ldots\ldots \quad (138,\,1)$$

Such is the value of the perturbative function when the eccentricities and inclinations of the orbits are neglected. Let us, for the present, confine ourselves to this case; and, writing M, M′, M″, &c. for the successive co-efficients, we have

$$\text{Q} = \text{M} + \text{M}' . \cos\,(n - n')\,t + \text{M}'' . \cos 2\,(n - n')\,t + \&c. \quad (138,\,2)$$

Now the equation (128) gives, when $e = o$,

$$\Pi = -\frac{\text{Q}}{a^2} = -\,n^2\,a\,\text{Q}\left(\text{because } n^2 = \frac{1}{a^3}\right) \text{ or,}$$
$$\Pi = -\,n^2\,a\,\{\text{M} + \text{M}' . \cos\,(n - n')\,t + \&c.\}$$

so that the differential equation in $\delta\,u$ becomes

$$o = \frac{d^2\,\delta\,u}{d\,t^2} + n^2\,\delta\,u - n^2\,a\,\{\text{M} + \text{M}' . \cos\,(n - n')\,t + \&c.\} \quad (139)$$

and integrating,

$$\delta u = a\,\mathrm{M} - \frac{n^2 a\,\mathrm{M}'}{(n-n')^2 - n^2} . \cos (n-n')\,t - \frac{n^2 a\,\mathrm{M}''}{4\,(n-n')^2 - n^2} \cos 2\,(n-n')\,t - \&c.$$

Consequently, since by equation (124) $\delta r = - a\,\delta u$ when $e = o$,

$$\delta r = - a^2 . \mathrm{M} + \frac{n^2 a^2\,\mathrm{M}'}{(n-n')^2 - n^2} . \cos\,(n-n')\,t - \frac{n^2 a^2\,\mathrm{M}''}{4\,(n-n')^2 - n^2} . \cos 2\,(n-n')\,t + \&c. \qquad (140)$$

The value of δr thus obtained, $\delta \theta$ is easily got from (121); for in this case $\frac{d\,r}{d\,t} = o$,

$$2\,r\,\frac{d\,\delta r}{d\,t} = - 2\,n^2 a^3\,(n-n') . \Sigma \frac{i\,\mathrm{M}^i}{i^2\,(n-n')^2 - n^2} . \sin i\,w$$

where Σ is used to express the sum of all similar terms from $i = 1$ to $i = \infty$ inclusive, and M^i represents the i^{th} in order of the co-efficients M', M'', M''', &c. Moreover

$$\int r\,\frac{d\,\Omega}{d\,r}\,n\,d\,t = a\,\frac{d\,\mathrm{A}}{d\,a} . n\,t + \Sigma \frac{a\,n}{i\,(n-n')}\,\frac{d\,\mathrm{A}^i}{d\,a} . \sin i\,w$$

$$\int n\,d\,t \int d\,\Omega = \frac{g}{a} . n\,t + \frac{n^2}{(n-n')^2}\,\Sigma\,\frac{\mathrm{A}^i}{i} \sin i\,w$$

Uniting therefore these several parts, we get

$$\begin{aligned} \delta \theta &= \frac{1}{n\,a^2} \times 2\,r\,\frac{d\,\delta r}{d\,t} + 2\,a \int r\,\frac{d\,\Omega}{d\,r} . n\,d\,t + 3\,a \int n\,d\,t \int d\,\Omega \\ &= \left(3\,g + 2\,a^2 . \frac{d\,\mathrm{A}}{d\,a}\right) \times n\,t \\ &\quad + \Sigma \left\{ \frac{3\,a\,n^2\,\mathrm{A}^i}{i\,(n-n')^2} + \frac{2\,a^2\,n}{i\,(n-n')}\,\frac{d\,\mathrm{A}^i}{d\,a} - \frac{2\,n\,a\,(n-n')\,i}{i^2\,(n-n')^2 - n^2}\,\mathrm{M}^i \right\} \sin i\,w \end{aligned}$$

Now, first, since $n\,t$ represents the mean longitude of m as deduced from observation, the quantity n is already affected with the whole influence of the planetary perturbation, and consequently the part multiplied by $n\,t$ in this expression of the perturbation in longitude, and which, if allowed to remain, would express an additional perturbation, is superfluous. This furnishes the condition

$$g = - \frac{2}{3}\,a^2\,\frac{d\,\mathrm{A}}{d\,a}; \qquad (141)$$

which determines the constant g. Moreover, in the latter part of this expression, if we write for M^i its value

$$\mathrm{M}^i = a . \frac{d\,\mathrm{A}_i}{d\,a} + \frac{2\,n\,\mathrm{A}_i}{n-n'}$$

it will admit reductions, and the value of $\delta \theta$ will at length be found as follows:

$$\delta \theta = \Sigma \left\{ \frac{2\,n^3\,a^2 . \frac{d\,\mathrm{A}_i}{d\,a}}{i\,(n-n')\,(i^2\,(n-n')^2 - n^2)} + \frac{(3\,n^2 + i^2\,(n-n')^2) . n^2\,a\,\mathrm{A}^i}{i\,(n-n')^2\,(i^2\,(n-n')^2 - n^2)} \right\} \sin i\,w \qquad (142)$$

If these expressions of δr and $\delta \theta$ be each multiplied by m', we have the values of $m'\,\delta r$ and $m'\,\delta \theta$ the perturbations of the radius vector and the longitude, *i. e.* those parts of them which are independent of the eccentricities of the orbits. These expressions give room for some remarks. The perturbation in longitude as we observe is wholly periodical and dependent on a single angle w and its multiples. In forming then a table of the values of $m'\,\delta \theta$, the numerical co-efficients being computed, and the value of $m'\,\delta \theta$ thus reduced to the form $p . \sin w + q . \sin 2\,w + \&c.$ we may include the whole of this in one column, entered under the general argument w, instead of regarding it as consisting of an infinite number of separate inequalities.

The same remark extends to the periodical part of $m'\,\delta r$, its arguments are the same as in the formula for $m'\,\delta \theta$; but besides this, δr includes, as we have seen, a constant part $- a^2 . \mathrm{M}$ or $- 2\,g\,a - a^3 . \frac{d\,\mathrm{A}}{d\,a}$, which becomes by substituting for g its value in (141)

$$\text{const. part of } \delta r = \frac{1}{3}\,a^3 . \frac{d\,\mathrm{A}}{d\,a},$$

and therefore

$$\delta r = \frac{a^3}{3} . \frac{d\,\mathrm{A}}{d\,a} + n^2\,a^2 . \Sigma \frac{1}{i^2\,(n-n')^2 - n^2} \left(a\,\frac{d\,\mathrm{A}^i}{d\,a} + \frac{2\,n\,\mathrm{A}^i}{n-n'} \right) \cos i\,w; \qquad (143)$$

In the formation of a table of $m'\,\delta r$ this constant part is of course included with the variable one, but the effect is remarkable. It appears that the action of the disturbing planet alters the mean distance from the sun of the disturbed, and, of course, its mean motion and periodical time from what they would have been had the disturbing planet no existence. At the same time, it will be demonstrated in the following pages, that these alterations once produced, are permanent and unchangeable in their quantity by the subsequent actions of the bodies composing the system.

The angle w is the difference of longitudes of the two planets, or their heliocentric elongation from each other. If we call ϵ and ϵ' their epochs, or their actual longitudes at the commencement of the time t, we have

$$\begin{aligned} w &= n\,t + \epsilon - (n'\,t + \epsilon') \\ &= n\,t - n'\,t + \epsilon - \epsilon' \end{aligned}$$

and this, in fact, is the argument of the perturbations when we neglect the eccentricities and inclinations of the orbits.

Let us next examine the perturbation in latitude, we have

$$\frac{d\,\Omega}{d\,z} = \frac{z'}{r'^3} - \frac{z' - z}{\lambda^3}$$

Hence, it is evident, that if we neglect the inclinations and eccentricities, we have $\frac{d\,\Omega}{d\,z} = o$, and the plane of the disturbed orbit does not change.

We have thus determined the effect of the action of a third body on the orbit and motion of m, on the simplest supposition, and our results (to recapitulate them) amount to this.

1st. That the radius vector undergoes a permanent change in its mean value, and, of course, that the period and mean motion of m are permanently altered.

2d. That the elliptic value of the radius vector receives an accession of terms, of the form

$$p + q\,.\cos w + r\,.\cos 2\,w + s\,.\cos 3\,w + \&c.$$

and that of the true longitude, a series of terms of the form

$$q'\,.\sin w + r'\,.\sin 2\,w + s'\,.\sin 3\,w + \&c.$$

w being the difference of longitudes, or mutual elongation of the planets, from each other.

3dly. That to express the several co-efficients of these formulæ in numbers, nothing more is required than a knowledge of the mass of the disturbing planet, and the mean distances and mean motions of both.

In the cases then where the disturbing planet has satellites, as in those of Jupiter, Saturn, and Uranus, the mass is known, and the reduction of the formulæ to numbers is complete. It is fortunate that these are by far the most considerable bodies of our system, but proximity to a certain extent supplies the place of intrinsic energy; and, in the case of the perturbations of the earth, our uncertainty of the masses of Mars and Venus leaves us in some degree at a loss. But physical astronomy furnishes us in this dilemma with considerable aid. Regarding the masses of these planets as unknown quantities, we may calculate in general terms their effect, either on the places of the other planets at assigned instants, or, on the elements of their orbits themselves, which are susceptible of much more accurate determination, by bringing a long series of observations to bear on them, and comparing the variations in their values after long intervals, as computed by theory, and as given by observation, we obtain data for the determination of these delicate quantities, so much the more precise as the variations observed in the elements are greater, or, in other words, as the interval of time in which they are observed is longer. It is thus that the lapse of ages is necessary to give precision to the numerical data of our system, and that continual and patient observation must ultimately lead us to the knowledge of points which elude the direct cognizance of our senses, and defy any investigation but those in which successive generations of mankind bear a part.

In fact, if we regard the masses of all the planets as unknown quantities, but their mean distances and periodic times as known ones;—the latter afford us the means of computing the values of $\delta\,r$ and $\delta\,\theta$ in any assigned case, independent of the value of m', which does not enter into their expressions. Let us therefore represent by $m'\,\delta'\,r$ and $m'\,\delta'\,\theta$, the perturbations of the radius vector and longitude produced by the planet m'; by $m''\,\delta''\,r$ and $m''\,\delta''\,\theta$ those produced by m'', and so on. Then will the true values of these quantities at any assigned instant be

$$r + m'\,\delta'\,r + m''\,\delta''\,r + m'''\,\delta'''\,r + \&c.$$

$$\theta + m'\,\delta'\,\theta + m''\,\delta''\,\theta + m'''\,\delta'''\,\theta + \&c.$$

in which r, $\delta'\,r$, $\delta''\,r$, &c. and θ, $\delta'\,\theta$, $\delta''\,\theta$, &c. are quantities susceptible of calculation from theory. Suppose now that we construct tables of the values of θ, $\delta'\,\theta$, $\delta''\,\theta$, &c. (or, as we will for a moment write these latter quantities, θ, ϕ, ψ, &c.) then, at any assigned instant, we have only to take out of these tables the values of θ, ϕ, ψ, &c.; and the true longitude of m will be

$$\theta + m'\,\phi + m''\,\psi + \&c. = \mathrm{L}$$

Suppose now we compare this formula with a great multitude of observations, and thus obtain a series of equations,

$$\begin{aligned} m'\,.\,\phi_1 + m''\,.\,\psi_1 + m''\,.\,\chi_1 + \&c. &= \mathrm{L}_1 - \theta_1 \\ m'\,.\,\phi_2 + m''\,.\,\psi_2 + m'''\,.\,\chi_2 + \&c. &= \mathrm{L}_2 - \theta_2 \\ \&c. &= \&c. \end{aligned}$$

The only unknown quantities in these will be the masses of the disturbing planets m', m'', &c. and by resolving these, we may determine their values, and thus estimate the masses of the planets by the perturbations they produce.

In this, as in almost all such delicate inquiries, where the quantities to be determined are exceedingly small, and the observations from which they are to be discovered liable to inaccuracies, bearing a sensible proportion to the thing observed, (which in this case is $L - \theta$, or the total perturbation arising from the united action of all the planets,) we are obliged to employ a great many more observations than would be, mathematically speaking, sufficient, if each were perfect, with a view to destroy the errors of observation in the mean result. The number of disturbing planets in our system at present known does not exceed ten; and it would therefore appear that ten observations of the longitude of one disturbed planet would enable us to determine the masses of all the rest; and so they would, were the observations mathematically exact, the elements of the orbits exactly known, and the theory by which the values of ϕ, ψ, χ, &c. are computed, complete. But each of these conditions is far from being fulfilled in the present state of astronomy; and if we would use this method to determine the masses of the planets, we must accumulate many hundreds of observations made, not on one, but on all of them, especially on those subject to the greatest perturbations.

The method of treating a series of equations more numerous than the unknown quantities they contain, so as to give them all their proper influence on the result, and obtain from them a set of values which, though satisfying neither of them separately, yet when substituted in all of them, shall, on the whole, give more satisfactory results than any other, depends on the theory of probabilities and may be found in Laplace's *Theorie Analytique des Probabilités.*

If the mass of any one or more of the planets (m') for instance, be regarded as sufficiently known from other methods, we need only employ this mode for the rest, and regarding the perturbation $m' \delta' \theta$ produced by it as known, place it on the known side of the equation, which will thus become

$$m'' \delta'' \theta + m''' \delta''' \theta + \&c. = L - \theta - m' \delta' \theta$$

Thus we may determine for instance, the masses of Mars and Venus, by means of an extensive series of observations of the sun's longitude, or (which is the same thing) by employing to that end the perturbations they produce on the earth. For the masses of Jupiter, Saturn, and Uranus, being known from the periods of their satellites; and those of Mercury, and the four new planets—Ceres, Pallas, Juno, and Vesta, being so small, as to have little or no influence, we have only two unknown quantities (m', m'') to determine.

This method is laborious, certainly; but considering the perfection of modern observations, the great multitude of them which may be brought to bear upon this point, and the considerable degree of exactness which the theory of the planetary perturbations has now attained, it is not impossible that it may one day be made to render the best service in determining the masses even of those planets which have satellites. At all events, it is highly desirable that it should be applied for that purpose, as its results would lead us to judge how far the latter method can be depended on in cases like that of Jupiter and Saturn, where the great deviation from sphericity of the central body renders the application of the formula $t = \dfrac{2\pi \times a^{\frac{3}{2}}}{\sqrt{M + m}}$ somewhat liable to error. In fact, this formula is derived on the hypothesis of a force represented by $\dfrac{M + m}{(\text{dist})^2}$ but in the case of spherical bodies only does the total attraction vary precisely in that ratio[*]. This alone, however, will not account for the great difference which Mr. Gauss has lately found between the mass of Jupiter, as obtained from observations of its satellites, and that deduced from the perturbations of the small planets intermediate between Jupiter and Mars, so that the subject must be regarded as open to further investigation, should the calculations of the last named eminent geometer be found to coincide with a more extensive series of observations of those interesting bodies than the shortness of the time they have been known has hitherto allowed.

SECTION IV.

Of the method of taking into account the effect of the eccentricities of the orbits on the planetary perturbations, and of the origin of the secular equations of their motions.

WHEN we regard the orbits as elliptical, the whole of the foregoing investigations require modification, the value of the perturbative function, and, of course, of the perturbations themselves, receiving accessions of terms depending on the powers and products of the eccentricities. We will here endeavour to explain the manner in which these terms originate; and to a certain (though limited) extent, the course pursued by geometers in determining their form and value.

The functions Ω, $\int d\,\Omega$, $r\dfrac{d\,\Omega}{d\,r}$, are explicitly given in terms of r, r', and w or $\theta - \theta'$, and contain no other symbols. Hence it arose, that when r, r' were supposed constant, the only cause of the variation of these functions consisted in that of w; and θ and θ' being in this case each expressed by an arc proportional to the time, it was sufficient to develope them in cosines and sines of w, to have at once the expression of the function Π in such a form as we required for integrating our equations. When, however, the eccentricities are introduced, all these facilities are at an end; r, r', and w, are no longer constant quantities and simple functions of the time, but each of them branches out into a series of powers of the eccentricities, and sines and cosines of variable arcs.

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[*] Laplace (*Theorie des Satellites de Jupiter*, p. 102.) makes the deviation of the attraction of the first, compared with the fourth satellite from the law of the inverse squares of the distances, only $\frac{1}{5000}$ of the whole attraction of the former, supposing Jupiter homogeneous. In Saturn, the attraction of the ring must cause a much more considerable deviation from that law.

Our object being to reduce Ω, &c. to sines and cosines of arcs proportional to the time, or of the form $A\,t + B$, it is evident that we must substitute for r, r', and w, their values so expressed, and then develope each term of Ω to the extent we wish. At present we will confine ourselves to the first powers of the eccentricities.

Now we have
$$\Omega = R + R' . \cos w + R'' . \cos 2\,w + \&c.$$
in which R, R′, R″, &c. and w are explicit functions of r, r', &c. and $w = \theta - \theta'$ putting θ and θ' for the true longitudes of the two planets. Now, if we call ϵ, and ϵ', the longitudes at the commencement of the time t, $n\,t + \epsilon$ and $n'\,t + \epsilon'$ will be their mean longitudes after the lapse of that time, and calling π and π' the longitudes of the perihelion, the mean anomalies will be $n\,t + \epsilon - \pi$ and $n'\,t + \epsilon' - \pi'$. Hence, the *true anomalies* will be (by equation 30),
$$n\,t + \epsilon - \pi + 2\,e . \sin (n\,t + \epsilon - \pi) + e^2 \times \&c.$$
$$n'\,t + \epsilon' - \pi' + 2\,e' . \sin (n'\,t + \epsilon' - \pi') + e^2 \times \&c.$$
and the true *longitudes* of course are
$$(n\,t + \epsilon) + 2\,e . \sin (n\,t + \epsilon - \pi) + e^2 \times \&c.$$
$$(n'\,t + \epsilon') + 2\,e' . \sin (n'\,t + \epsilon' - \pi') + e^2 \times \&c.$$
Hence, if we neglect the eccentricities, we have simply $w = (n\,t - n'\,t + \epsilon - \epsilon')$ and as R, R′, &c. in this case assume their circular values A, A′, &c., the terms of Ω not depending on the eccentricities will remain as before,
$$A + A' . \cos (n\,t - n'\,t + \epsilon - \epsilon') + A'' . \cos 2\,(n\,t - n'\,t + \epsilon - \epsilon') + \&c.$$
On the other hand, the terms depending on the eccentricities have their origin,

1st. In the developement of the functions R, R′, &c.; when, instead of r, r', we put their elliptic values,
$$r = a + \Delta\,r, \text{ and } r' = a' + \Delta\,r'$$
denoting by $\Delta\,r$ and $\Delta\,r'$ the parts of r, r' arising from the eccentricities.

2d. In the substitution of $W + \Delta\,w$ for w in $\cos w$, $\cos 2\,w$, &c. W being the part of w independent of the eccentricities, or
$$W = (n - n') . t + (\epsilon - \epsilon')$$
and $\Delta\,w$ being the part depending on them, or
$$\Delta\,w = 2\,e . \sin (n\,t + \epsilon - \pi) - 2\,e' . \sin (n'\,t + \epsilon' - \pi') + e^2 \times \&c. \&c.$$

3rd. In the multiplication of these terms together.

Now, if we still continue to denote by Δ that variation in Ω r, r', &c. which arises from the eccentricities, we have
$$\Delta\,\Omega = \frac{d\,\Omega}{d\,r}\,\Delta\,r + \frac{d\,\Omega}{d\,r'}\,\Delta\,r' + \frac{d\,\Omega}{d\,w}\,\Delta\,w$$
in which the differential co-efficients $\frac{d\,\Omega}{d\,r}$, $\frac{d\,\Omega}{d\,r'}$, $\frac{d\,\Omega}{d\,w}$, are to have their *circular* values. If we would pursue the investigation further, we must add to $\Delta\,\Omega$ the terms $\frac{d^2\,\Omega}{d\,r^2} . \frac{(\Delta\,r)^2}{1 . 2}$, &c.

Now, in general, we have
$$\frac{d\,\Omega}{d\,r} = \frac{d\,R}{d\,r} + \frac{d\,R'}{d\,r} . \cos w + \frac{d\,R''}{d\,r} . \cos 2\,w + \&c.$$
$$\frac{d\,\Omega}{d\,r'} = \frac{d\,R}{d\,r'} + \frac{d\,R'}{d\,r'} . \cos w + \frac{d\,R''}{d\,r'} . \cos 2\,w + \&c.$$
the circular values of which are respectively
$$\frac{d\,A}{d\,a} + \frac{d\,A'}{d\,a} . \cos W + \frac{d\,A''}{d\,a} . \cos 2\,W + \&c. \qquad (144, 1)$$
and
$$\frac{d\,A}{d\,a'} + \frac{d\,A'}{d\,a'} . \cos W + \frac{d\,A''}{d\,a'} . \cos 2\,W + \&c. \qquad (144, 2)$$
and, in like manner, the circular value of $\frac{d\,\Omega}{d\,w}$ is
$$- \{ 1 . A' . \sin W + 2 . A'' . \sin 2\,W + 3 . A''' . \sin 3\,W + \&c. \} \qquad (144, 3)$$

The values of $\Delta\,r$ and $\Delta\,r'$ are given by equation (28) if we substitute merely for $n\,t$ the expressions $n\,t + \epsilon - \pi$, and $n'\,t + \epsilon' - \pi'$; which in this case are the mean anomalies, because the mean longitudes $n\,t$, $n'\,t$, and the constants ϵ, ϵ', π, π', are reckoned, not from the perihelion of the orbits, as in that equation, but from the line of the equinoxes. If then we put $V = n\,t + \epsilon - \pi$, $V' = n'\,t + \epsilon' - \pi'$, we have
$$\left.\begin{aligned} \Delta\,r &= - a\,e . \cos V + e^2 \times \&c. \\ \Delta\,r' &= - a'\,e' . \cos V' + e'^2 \times \&c. \\ \text{and} \quad \Delta\,w &= 2\,e . \sin V - 2\,e' . \sin V' + e^2 \times \&c. \end{aligned}\right\} ; \;\ldots\ldots\ldots\ldots\ldots\ldots (145)$$
Substituting therefore in $\Delta\,\Omega$ for the differential co-efficients, their circular values in (144, 1, 2, 3,) and for $\Delta\,r$, $\Delta\,r'$, and $\Delta\,w$, the values just now found, it will become

$$\begin{aligned}\Delta\,\Omega = &- a\,e\,.\cos V\left\{\frac{d\,A}{d\,a} + \frac{d\,A'}{d\,a}.\cos W + \frac{d\,A''}{d\,a}.\cos 2\,W + \&c.\right\}\\ &- a'\,e'.\cos V'\left\{\frac{d\,A}{d\,a'} + \frac{d\,A'}{d\,a'}.\cos W + \frac{d\,A''}{d\,a'}.\cos 2\,W + \&c.\right\}\\ &- (2\,e\,.\sin V - 2\,e'\,.\sin V')\left\{A'\,.\sin W + 2\,A''\ \sin 2\,W + \&c.\right\}\\ &\quad + e^2 \times \&c.\end{aligned} \tag{146}$$

Now these series are not precisely in the form we want them; the cosines and sines of V and W being multiplied together; and to disengage them, we must employ the well known trigonometrical formula

$$\begin{aligned}\cos A\,.\cos B &= \tfrac{1}{2}\left\{\cos(A + B) + \cos(A - B)\right\}\\ -\sin A\,.\sin B &= \tfrac{1}{2}\left\{\cos(A + B) - \cos(A - B)\right\}\end{aligned} \tag{A}$$

By the aid of this, we get

$$\begin{aligned}&\cos V = \cos V; && \cos V' = \cos V'\\ &\cos V\,.\cos W = \tfrac{1}{2}\left\{\cos(V + W) + \cos(V - W)\right\}; && \cos V'\,.\cos W' = \tfrac{1}{2}\left\{\cos(V' + W') + \cos(V' - W')\right\}\\ &\cos V\,.\cos 2\,W = \tfrac{1}{2}\left\{\cos(V + 2\,W) + \cos(V - 2\,W)\right\}; && \cos V'\,.\cos 2\,W' = \tfrac{1}{2}\left\{\cos(V' + 2\,W') + \cos(V' - 2\,W')\right\}\\ &\&c. = \&c. && \&c. = \&c.\end{aligned}$$

$$\begin{aligned}&\sin V\,.\sin W = \tfrac{1}{2}\left\{\cos(V - W) - \cos(V + W)\right\}; && \sin V'\,.\sin W = \tfrac{1}{2}\left\{\cos(V' - W) - \cos(V' + W)\right\}\\ &\sin V\,.\sin 2\,W = \tfrac{1}{2}\left\{\cos(V - 2\,W) - \cos(V + 2\,W)\right\}; && \sin V'\,.\sin 2\,W = \tfrac{1}{2}\left\{\cos(V' - 2\,W) - \cos(V' + 2\,W)\right\}\\ &\&c. = \&c. && \&c. = \&c.\end{aligned}$$

Thus we see that $\Delta\,\Omega$ (if we confine ourselves to the first powers of the eccentricities) is reducible to a series of sines and cosines, whose arguments are all comprehended in the forms $V \pm i\,W$, and $V' \pm i\,W$; or, (since $\cos - A = -\cos A$) in the forms $i\,W \pm V$ and $i\,W \pm V'$. That is, (since $V = n\,t + \epsilon - \pi$, and $W = n\,t - n'\,t + \epsilon - \epsilon'$) in the forms

$$\begin{aligned}&i\,(n\,t - n'\,t + \epsilon - \epsilon') \pm (n\,t + \epsilon - \pi)\\ &i\,(n\,t - n'\,t + \epsilon - \epsilon') \pm (n'\,t + \epsilon' - \pi')\end{aligned}$$

If we actually execute the substitutions (still, for brevity, preserving the denominations W and V) we shall obtain for the value of $\Delta\,\Omega$ the following expression—

$$\begin{aligned}\Delta\,\Omega = &- a\,e\,.\frac{d\,A}{d\,a}.\cos V && - a'\,e'\,.\frac{d\,A}{d\,a'}.\cos V';\\ &- e\left\{\frac{a}{2}\,\frac{d\,A'}{d\,a} - A'\right\}.\cos(W + V) && - e'\left\{\frac{a'}{2}\,\frac{d\,A'}{d\,a'} + A'\right\}\cos(W + V')\\ &- e\left\{\frac{a}{2}\,\frac{d\,A'}{d\,a} + A'\right\}.\cos(W - V) && - e'\left\{\frac{a'}{2}\,\frac{d\,A'}{d\,a'} - A'\right\}\cos(W - V')\\ &- e\left\{\frac{a}{2}\,\frac{d\,A''}{d\,a} - 2\,A''\right\}.\cos(2\,W + V) && - e'\left\{\frac{a'}{2}\,\frac{d\,A''}{d\,a'} + 2\,A''\right\}\cos(2\,W + V')\\ &- e\left\{\frac{a}{2}\,\frac{d\,A''}{d\,a} + 2\,A''\right\}.\cos(2\,W - V) && - e'\left\{\frac{a'}{2}\,\frac{d\,A''}{d\,a'} - 2\,A''\right\}\cos(2\,W - V')\\ &- \&c. && - \&c.\end{aligned} \tag{147}$$

But our object is to obtain the developement of Q the perturbative function, or $2\int\!d\,\Omega + r\,\frac{d\,\Omega}{d\,r}$. The part of this independent of the eccentricities is already found, and we have now only to consider that depending on them; which, in the notation above adopted, will be expressed by Δ Q, or

$$2\,\Delta\int\!d\,\Omega + \Delta\left(r\,\frac{d\,\Omega}{d\,r}\right) = 2\int\!d\,\Delta\,\Omega + \Delta\left(r\,\frac{d\,\Omega}{d\,r}\right); \tag{148}$$

Let us first consider the value of the first part of this expression $2\int\!d\,\Delta\,\Omega$, and let any term of $\Delta\,\Omega$ be represented by

$$M\,.\cos(i\,W + k\,V + l\,V')$$

in which i may be any integer from o to infinity, and k and l either ± 1 or o, an expression which obviously comprehends all the terms of which $\Delta\,\Omega$ consists. This premised, it is obvious that the co-efficient M being constant, the variation of $\Delta\,\Omega$ can only arise from the variation of the angle $i\,W + k\,V + l\,V'$; and as this angle is supposed only to vary by the motion of m, we are to suppose $n\,t$ only to vary, and $n'\,t$ to remain constant. So that we have, for that part of the variation of $\Delta\,\Omega$ which arises from the term in question,

$$-M\,(i\,d\,W + k\,d\,V)\ \sin(i\,W + k\,V + l\,V')$$

because $d\,V' = o$, since $V' = n'\,t + \epsilon' - \pi'$. Now $d\,W = n\,d\,t$ and $d\,V = n\,d\,t$ also. Consequently, this becomes

$$-(i+k)\,\mathrm{M}\,.\,n\,d\,t\,.\,\sin\,(i\,\mathrm{W}+k\,\mathrm{V}+l\,\mathrm{V}')$$

and the part of the expression $2\int \mathrm{d}\,\Delta\,\Omega$ arising from this term, will therefore be

$$\frac{2\,(i+k)\,n}{i\,(n-n')+k\,n+l\,n'}\,.\,\mathrm{M}\cos\,(i\,\mathrm{W}+k\,\mathrm{V}+l\,\mathrm{V}') \qquad (149)$$

We see therefore by the foregoing reasoning, that in order to obtain that part of the perturbative function which originates in the term $2\int \mathrm{d}\,\Omega$, and depends on the eccentricities, we have only to take the terms of the expression for $\Delta\,\Omega$ (147) in their order, and with their proper signs, and multiply each of them respectively by that value of the fraction

$$\frac{2\,(i+k)\,.\,n}{i\,(n-n')+k\,n+l\,n'}$$

which corresponds to the values of i, k, l, in its argument, represented by $i\,\mathrm{W}+k\,\mathrm{V}+l\,\mathrm{V}'$. For instance, the term which has simply V for its argument, must be multiplied by $\frac{2\,n}{1\,.\,n}=2$, that which has V′ by *o*.—Again, the terms, whose respective arguments are $\mathrm{W}+\mathrm{V}$, $\mathrm{W}-\mathrm{V}$, $\mathrm{W}+\mathrm{V}'$, $\mathrm{W}-\mathrm{V}'$, are to be multiplied, according to this rule, by the respective co-efficients $\frac{4\,n}{2\,n-n'}$, o, 2, and $\frac{2\,n}{n-2\,n'}$. Similarly, the terms which have $2\,\mathrm{W}+\mathrm{V}$, $2\,\mathrm{W}-\mathrm{V}$, $2\,\mathrm{W}+\mathrm{V}'$, $2\,\mathrm{W}-\mathrm{V}'$, for their arguments, acquire the co-efficients $\frac{6\,n}{3\,n-2\,n'}$, $\frac{2\,n}{n-2\,n'}$, $\frac{4\,n}{2\,n-n'}$, and $\frac{4\,n}{2\,n-3\,n'}$; and so on.

The co-efficients thus acquired by integration depend solely on the ratios of the mean motions, or periodic times, of the disturbed and disturbing planet, and are of very great importance in the planetary theory. Were it not for these, the theory of the planetary perturbations would be very simple, as the same treatment, or nearly so, could be applied in every case, and the magnitudes of the several inequalities would go on diminishing with more or less rapidity, as the arguments contained higher multiples of the mean motions. But these co-efficients prevent this regular progression of magnitude from taking place, and render it difficult to foresee without computation whether any assigned inequality may be neglected or not, and impossible to argue from its known minuteness in the case of one pair of planets to its probable smalness in that of another. Thus an inequality, whose maximum we know to be small in the case of Venus disturbed by the earth, may have a considerable magnitude in that of Jupiter disturbed by Saturn, merely from a relation subsisting between the periodic times in the latter case which does not in the former. In fact, it is evident that if the periods should happen to be so related as to render the denominator of any of the foregoing or similar fractions very small compared to the numerator, the inequality into which it enters as a co-efficient will, on this account alone, acquire a very great preponderance. Thus, if the period of the disturbed planet were very nearly half or double that of the disturbing; the terms, multiplied by $\frac{2\,n}{2\,n-n'}$ or by $\frac{n}{n-2\,n'}$ would become very large, and the length of the period of the inequality represented by it would be proportionally increased, and in the case of exact commensurability would actually become infinite; that is to say, the disturbance would go on to such an extent, as to make a total subversion of the original conditions of the problem. The physical reason of this is equally obvious. In the case just instanced, the two planets, at every revolution of the exterior, would be placed in exactly the same circumstances—the same disturbing forces would act in the same manner for a series of ages, and their effects, instead of compensating each other by mutual opposition, would go on accumulating in the same direction, till their orbits at length became entirely changed, and the commensurability of their periods at length ceased to subsist. In fact, an equation thus limited by no period, and affecting both the longitude and radius vector of each orbit always the same way, is equivalent to an alteration of the *mean* motion and *mean* distance; and as this would take place in opposite directions on the two planets, shortening the period of one, and lengthening that of the other, their periods would continually recede from commensurability; the magnitude of the inequality in question, as well as the length of its period, would both acquire finite values, which even then would continually diminish, till reduced within limits consistent with the stability of the system. It is probably by some such gradations (if we may hazard a conjecture on a part of the planetary theory so far beyond the reach of analysis or exact reasoning,) that our system has attained, in the course of almost indefinite ages, its present admirable state of equilibrium, in which no inequality of enormous magnitude exists; and those which have any notable value, can be proved to be confined within comparatively narrow bounds.

Let us next consider the part of the developement of Q, which arises from the term $\Delta\left(r\,\frac{d\,\Omega}{d\,r}\right)$. Now, if we regard only the first powers of the eccentricities, and consequently neglect the squares of $\Delta\,r$, &c. we have

$$\Delta\left(r\,\frac{d\,\Omega}{d\,r}\right)=\frac{d\,\Omega}{d\,r}\,.\,\Delta\,r+r\,\Delta\,\frac{d\,\Omega}{d\,r}$$

$$=\frac{d\,\Omega}{d\,r}\,\Delta\,r+r\,.\,\frac{d^2\,\Omega}{d\,r^2}\,\Delta\,r+r\,\frac{d^2\,\Omega}{d\,r\,d\,r'}\,\Delta\,r'+r\,\frac{d^2\,\Omega}{d\,r\,d\,w}\,\Delta\,w$$

in which the differential co-efficients are to have their circular values, which we may represent by $\frac{d\,\Omega}{d\,a}$, &c. Now these are,

$$\left.\begin{aligned}
\frac{d\,\Omega}{d\,a} &= \frac{d\,A}{d\,a} + \frac{d\,A'}{d\,a}\cos W + \frac{d\,A''}{d\,a}\,.\cos 2\,W + \&c.\\
a\,\frac{d^2\,\Omega}{d\,a^2} &= a\,\frac{d^2\,A}{d\,a^2} + a\,\frac{d^2\,A'}{d\,a^2}\,.\cos W + a\,\frac{d^2\,A''}{d\,a^2}\,.\cos 2\,W + \&c.\\
a\,\frac{d^2\,\Omega}{d\,a\,d\,a'} &= a\,\frac{d^2\,A}{d\,a\,d\,a'} + a\,\frac{d^2\,A'}{d\,a\,d\,a'}\,.\cos W + a\,\frac{d^2\,A''}{d\,a\,d\,a'}\,.\cos 2\,W + \&c.\\
a\,\frac{d^2\,\Omega}{d\,a\,d\,W} &= \qquad 1\,.\,a\,\frac{d\,A'}{d\,a}\,.\sin W + 2\,.\,a\,\frac{d\,A''}{d\,a}\,.\sin 2\,W + \&c.
\end{aligned}\right\}\ldots\ldots\ldots\ldots(150)$$

Let these be substituted in the above expression for $\Delta\left(r\,\frac{d\,\Omega}{d\,r}\right)$ and the values of $\Delta\,r, \Delta\,r'$ and $\Delta\,w$, given in (145) be put for them, and we shall have, by a process exactly similar to that gone through for $\Delta\,\Omega$,

$$\begin{aligned}
\Delta\left(r\,\frac{d\,\Omega}{d\,r}\right) = &- a\,e\,.\cos V\,.\left\{\left(\frac{d\,A}{d\,a} + a\,\frac{d^2\,A}{d\,a^2}\right) + \left(\frac{d\,A'}{d\,a} + a\,\frac{d^2\,A'}{d\,a^2}\right)\cos W + \&c.\right\}\\
&- a'\,e'\,.\cos V'\,.\left\{a\,\frac{d^2\,A}{d\,a\,d\,a'} + a\,\frac{d^2\,A'}{d\,a\,d\,a'}\,.\cos W + \&c.\right\} \qquad (151)\\
&- (2\,e\,.\sin V - 2\,e'\,.\sin V')\left\{1\,a\,\frac{d\,A'}{d\,a}\,.\sin W + 2\,.\,\&c.\right\}
\end{aligned}$$

which, by the use of the same trigonometrical formulæ, and by properly arranging the terms, becomes

$$\begin{aligned}
\Delta\left(r\,\frac{d\,\Omega}{d\,r}\right) = &- e\left\{a^2\,\frac{d^2\,A}{d\,a^2} + a\,\frac{d\,A}{d\,a}\right\}\cos V &&- e'\,.\,a\,a'\,\frac{d^2\,A}{d\,a\,d\,a'}\cos V'\\
&- \frac{e}{2}\left\{a^2\,\frac{d^2\,A'}{d\,a^2} - a\,\frac{d\,A'}{d\,a}\right\}\cos(W + V) &&- \frac{e'}{2}\left\{a\,a'\,\frac{d^2\,A}{d\,a\,d\,a'} + 2\,a\,\frac{d\,A'}{d\,a}\right\}\cos(W + V')\\
&- \frac{e}{2}\left\{a^2\,\frac{d^2\,A'}{d\,a^2} + 3\,a\,\frac{d\,A'}{d\,a}\right\}\cos(W - V) &&- \frac{e'}{2}\left\{a\,a'\,\frac{d^2\,A'}{d\,a\,d\,a'} - 2\,a\,\frac{d\,A'}{d\,a}\right\}\cos(W + V')\\
&- \frac{e}{2}\left\{a^2\,\frac{d^2\,A''}{d\,a^2} - 3\,a\,\frac{d\,A''}{d\,a}\right\}\cos(2\,W + V) &&- \frac{e'}{2}\left\{a\,a'\,\frac{d^2\,A''}{d\,a\,d\,a'} + 4\,a\,\frac{d\,A''}{d\,a}\right\}\cos(2\,W + V')\\
&- \frac{e}{2}\left\{a^2\,\frac{d^2\,A''}{d\,a^2} + 5\,a\,\frac{d\,A''}{d\,a}\right\}\cos(2\,W - V) &&- \frac{e'}{2}\left\{a\,a'\,\frac{d^2\,A''}{d\,a\,d\,a'} - 4\,a\,\frac{d\,A''}{d\,a}\right\}\cos(2\,W - V')\\
&- \&c. &&- \&c.\,; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(152)
\end{aligned}$$

We are now in a condition to express the whole value of $\Delta\,Q$, by combining this with the expressions (147) and (149) and it is evident that our result will be of the following form :—

$$\begin{aligned}
\Delta\,Q = (N\,e\cos V + N'\,e'\cos V') + O\,e\cos(W + V) + O'\,e'\cos(W + V') \qquad (153)\\
+ P\,e\cos(W - V) + P'\,e'\cos(W - V') + Q\,e\cos(2\,W + V) + \&c.
\end{aligned}$$

and the co-efficients of the several arguments will be

$$\left.\begin{aligned}
N &= -\left\{a^2\,\frac{d^2\,A}{d\,a^2} + 3\,a\,\frac{d\,A}{d\,a}\right\}\\
O &= -\frac{1}{2}\left\{a^2\,\frac{d^2\,A'}{d\,a^2} + \frac{2\,n + n'}{2\,n - n'}\,a\,\frac{d\,A'}{d\,a} - \frac{8\,n}{2\,n - n'}\,A'\right\}\\
P &= -\frac{1}{2}\left\{a^2\,\frac{d^2\,A'}{d\,a^2} + 3\,a\,\frac{d\,A'}{d\,a}\right\};\\
N' &= -\,a\,a'\,.\,\frac{d^2\,A}{d\,a\,d\,a'}\\
O' &= -\frac{1}{2}\left\{a\,a'\,.\,\frac{d^2\,A'}{d\,a\,d\,a'} + 2\,a\,\frac{d\,A'}{d\,a} + 2\,a'\,\frac{d\,A'}{d\,a'} + 4\,A'\right\}\\
P' &= -\frac{1}{2}\left\{a\,a'\,.\,\frac{d^2\,A'}{d\,a\,d\,a'} - 2\,a\,\frac{d\,A'}{d\,a} + \frac{2\,n}{n - 2\,n'}\,a'\,\frac{d\,A'}{d\,a'} - \frac{4\,n}{n - 2\,n'}\,A'\right\}
\end{aligned}\right\}\ldots\ldots\ldots(154)$$

The value of $\Delta\,Q$ being thus obtained, that of Π is had by mere substitution. The part of Π, independent of the eccentricities, will remain as in the preceding section, changing only in the several arguments w into W; and if we denote this by H, and by $\Delta\,\Pi$ the part of Π which depends on the eccentricities, the equation (128) will give

$$\Delta\,\Pi = \frac{e}{a^2}\left\{Q\,.\cos V - 2\int Q\,n\,d\,t\,.\sin V\right\} - \frac{\Delta\,Q}{a^2}$$

in which Q and Δ Q denote, as we have all along supposed, the parts of Q respectively independent, and dependent, on the eccentricities.

The value of Δ Π therefore consists of two parts; the latter $-\dfrac{\Delta\,Q}{a^2}$ is immediately obtained from the expression of Δ Q above found, and is equal to

$$-\frac{e}{a^2}\left\{N\;.\cos V + O\,.\cos(W+V) + P\;.\cos(W-V) + \&c.\right\}$$
$$-\frac{e'}{a^2}\left\{N'.\cos V' + O'.\cos(W+V') + P'.\cos(W-V') + \&c.\right\}$$

The former depends on Q, and must be determined by substituting for Q its value

$$Q = M + M'.\cos W + M''.\cos 2\,W + \&c.$$

where M, M′, &c. are co-efficients, whose values are assigned in equation (138, 1). This substitution made, we find

$$Q\cos V = M\,.\cos V + \frac{M'}{2}\,.\cos(W+V) + \frac{M'}{2}\,.\cos(W-V) + \&c.$$

$$\int Q\,n\,d\,t\,.\sin V = -M\,.\cos V - \frac{M'\,n}{2\,(2\,n-n')}\cos(W+V) - \frac{M'\,n}{2\,n'}\cos(W-V) - \&c.$$

So that the part of Δ Π now in question becomes

$$\frac{e}{a^2}\left\{3\,M\,.\cos V + \frac{4\,n-n'}{4\,n-2\,n'}\,M'.\cos(W+V) + \frac{2\,n+n'}{2\,n'}\,M'.\cos(W-V) + \&c.\right\}$$

and the whole value of Δ Π will be as follows :—

$$\Delta\,\Pi = \frac{e}{a^2}\left\{(3\,M-N)\cos V + \left(\frac{4\,n-n'}{4\,n-2\,n'}\,M'-O\right)\cos(W+V) + \left(\frac{2n+n'}{2\,n'}M'-P\right)\cos(W-V) + \&c.\right\}$$
$$-\frac{e'}{a^2}\left\{N'.\cos V' + O'.\cos(W+V') + P'.\cos(W-V') + \&c.\right\}; \qquad (155)$$

This found, the integral of the equation $\dfrac{d^2\,\delta\,u}{d\,t^2} + n^2\,u + \Pi = o$ will be obtained by the expressions 106, 107, and 108. The parts of $\delta\,u$, $\delta\,r$, and $\delta\,\theta$, independent of the eccentricities, have already been found; and calling therefore $\Delta\,\delta\,u$, $\Delta\,\delta\,r$, and $\Delta\,\delta\,\theta$, those parts of these respective quantities which depend on the eccentricities, we shall have (since $V = n\,t + \epsilon - \pi$, $W = (n-n')\,t + (\epsilon - \epsilon')$ and $V' = n'\,t + \epsilon' - \pi'$)

$$\left.\begin{aligned}\Delta\,\delta\,u = \frac{e}{a^2}\Big\{\,(3\,M-N)\,.\,\frac{t}{2}\cos V + \left(\frac{4\,n-n'}{4n-2\,n'}M'-O\right)\frac{\cos(W+V)}{(2\,n-n')^2-n^2} + \\ + \left(\frac{2\,n+n'}{2\,n'}M'-P\right)\frac{\cos(W-V)}{n'^2-n^2} + \;\&c.\end{aligned}\right\}$$
$$-\frac{e'}{a^2}\left\{\frac{N'}{n'^2-n^2}\cos V' + \frac{O'}{2}\,.\,\frac{t}{2}\cos(W+V') + \frac{P'}{(n-2\,n')^2-n^2}\cos(W-V') + \&c.\right\} \qquad (156)$$

The perturbation of the radius vector is now easily found; for, as we have by equation 124,

$$\delta\,r = -\,a\,\delta\,u\,\{1 + 2\,e\,.\cos V + \&c.\}$$

this gives

$$\Delta\,\delta\,r = -\,a\,\{\Delta\,\delta\,u + 2\,e\,\delta\,u\,.\cos V\}$$

whence $\Delta\,\delta\,r$ is readily obtained. With regard to the perturbation in longitude, or $m'\,\delta\,\theta$, no further difficulty than the length of the substitutions remains to be encountered; for the part not depending on the eccentricities being already obtained, that which depends on them will be had by merely substituting for $\delta\,r$, $r\dfrac{d\,\Omega}{d\,r}$ and Ω, the values of $\Delta\,\delta\,r$, $\Delta\left(r\dfrac{d\,\Omega}{d\,r}\right)$ and Δ Ω already obtained in the general expression (121).

But as this process of substitution and reduction presents no difficulties in principle, requiring only patience and exactness in performing the numerous combinations of the terms which occur in it, we shall not pursue it, but content ourselves with observing, that the part of it which depends on the first powers of the eccentricities will, on examination, be found to assume the form

$$\left.\begin{aligned} e\,.\,a\,t\,.\cos V + e'\,.\,b\,t\,.\cos(W+V') \\ + e\,\{A\,.\sin(W+V) + B\;.\sin(W-V) + C\,.\sin(2\,W+V) + \&c.\} \\ + e'\,\{A'.\sin V' \qquad + B'.\sin(W-V') + C'.\sin(2\,W+V') + \&c.\}\end{aligned}\right\};\ldots\ldots\ldots(157)$$

is here that we first encounter the secular equations of the planetary motions, in the form of two terms ining the time t disengaged from the signs sin and cos, and therefore capable of indefinite increase and nution. They are multiplied by the eccentricities, and therefore originate from the ellipticity of the planetary orbits; and in the case of strictly circular orbits, would not exist. Similar terms occur of course in the value of $\delta\, r$, being introduced by the integration of the equation for $\delta\, u$. But as the discussion of these terms is one of the most delicate and difficult points of the planetary theory, we shall not enter upon it till we have pointed out the method of taking into account the higher powers of the eccentricities.

SECTION V.

Of the inequalities depending on the squares and higher powers of the eccentricities.

It is not our intention to enter into any detailed account of this very complicated part of the planetary theory. Any such attempt would lead us far beyond our proper limits; and the reader, who is desirous to follow it into its minutiæ, must consult the original memoirs of Laplace, Lagrange, &c., the *Mécanique Céleste*, and other works of a similar nature. In the foregoing sections we have however followed, as nearly as possible, the course pursued in the last named immortal work, supplying only such steps in the analysis as cannot be expected to be discovered by the ordinary student, (and they are numerous) and endeavouring throughout to place the principles of the several processes in as strong a light as possible. In the present section, the explanation of the principles on which the process of approximation is to be pursued will be almost our sole object.

Let us resume the consideration of the function Ω.

$$\Omega = \mathrm{R} + \mathrm{R}' \,.\, \cos w + \mathrm{R}'' .\, \cos 2\, w + \mathrm{R}''' \,.\, \cos 3\, w + \&\text{c.}$$

When $r + \Delta\, r$, $r' + \Delta\, r'$, and $w + \Delta\, w$, are substituted for r, r', w, in this, Ω becomes $\Omega + \Delta\, \Omega$, and we have

$$\left.\begin{aligned}\Delta\,\Omega = {} & \frac{d\,\Omega}{d\,a} \,.\, \frac{\Delta\, r}{1} + \frac{d\,\Omega}{d\,a'}\, \frac{\Delta\, r'}{1} + \frac{d\,\Omega}{d\,\mathrm{W}}\, \frac{\Delta\, w}{1} \\ & + \frac{d^2\,\Omega}{d\,a^2}\, \frac{(\Delta\, r)^2}{1\,.\,2} + \frac{d^2\,\Omega}{d\,a\,d\,a'}\, \frac{\Delta\, r \,.\, \Delta\, r'}{1\,.\,1} + \frac{d^2\,\Omega}{d\,a'^2}\, \frac{(\Delta\, r')^2}{1\,.\,2} + \&\text{c.} \\ & + \frac{d^3\,\Omega}{d\,a^3}\, \frac{(\Delta\, r)^3}{1\,.\,2\,.\,3} + \&\text{c.}\end{aligned}\right\}, \ldots\ldots\ldots\ldots\ldots \quad (158)$$

The differential co-efficients of Ω are here supposed to have their circular values denoted by $\frac{d\,\Omega}{d\,a}$, $\frac{d\,\Omega}{d\,a'}$, $\frac{d\,\Omega}{d\,\mathrm{W}}$, &c.

In like manner, if we consider the value of $\Delta\left(r\,\frac{d\,\Omega}{d\,r}\right)$, or the augmentation of $r\,\frac{d\,\Omega}{d\,r}$ produced by the eccentricities, we have only to substitute for Ω in the expression (158) the circular value of the function in question, or $a\,\frac{d\,\Omega}{d\,a}$, and we get

$$\begin{aligned}\Delta\left(r\,\frac{d\,\Omega}{d\,r}\right) = {} & \frac{\Delta\, r}{1} \times \frac{d}{da}\left(a\,\frac{d\,\Omega}{d\,a}\right) + \frac{\Delta\, r'}{1} \,.\, \frac{d}{da'}\left(a\,\frac{d\,\Omega}{d\,a}\right) + \frac{\Delta\, w}{1} \,.\, \frac{d}{d\,\mathrm{W}}\left(a\,\frac{d\,\Omega}{d\,a}\right) \\ & + \frac{(\Delta\, r)^2}{1\,.\,2} \,.\, \frac{d^2}{d\,a^2}\left(a\,\frac{d\,\Omega}{d\,a}\right) + \frac{\Delta\, r\, \Delta\, r'}{1\,.\,1} \,.\, \frac{d^2}{d\,a\,d\,a'}\left(a\,\frac{d\,\Omega}{d\,a}\right) + \&\text{c.} \\ & + \&\text{c.} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (159)\end{aligned}$$

in which $\frac{d}{da}$, $\frac{d}{da'}$, &c. denote the differentiation relative to a, a', &c. respectively of the function to which they are prefixed, and the division of the resulting differential by $d\,a$, $d\,a'$, &c. according to the very convenient system of notation explained in (Lacroix, *Differential and Integral Calculus*, 8vo. English translation, Appendix.)

Since $\Omega = \mathrm{A} + \mathrm{A}' \,.\, \cos \mathrm{W} + \mathrm{A}'' .\, \cos 2\, \mathrm{W} + \&\text{c.}$ we must have

$$\left.\begin{aligned} a\,\frac{d\,\Omega}{d\,a} &= a\,\frac{d\,\mathrm{A}}{d\,a} + a\,\frac{d\,\mathrm{A}'}{d\,a} \,.\, \cos \mathrm{W} + a\,\frac{d\,\mathrm{A}''}{d\,a} \,.\, \cos 2\,\mathrm{W} + \&\text{c.} \\ \frac{d}{d\,a}\left(a\,\frac{d\,\Omega}{d\,a}\right) &= \frac{d}{da}\left(a\,\frac{d\,\mathrm{A}}{d\,a}\right) + \frac{d}{da}\left(a\,\frac{d\,\mathrm{A}'}{d\,a}\right) \cos \mathrm{W} + \frac{d}{da}\left(a\,\frac{d\,\mathrm{A}''}{d\,a}\right) .\, \cos 2\,\mathrm{W} + \&\text{c.} \\ \frac{d}{d\,a'}\left(a\,\frac{d\,\Omega}{d\,a}\right) &= \frac{d}{da'}\left(a\,\frac{d\,\mathrm{A}}{d\,a}\right) + \frac{d}{da'}\left(a\,\frac{d\,\mathrm{A}'}{d\,a}\right) \cos \mathrm{W} + \frac{d}{da'}\left(a\,\frac{d\,\mathrm{A}''}{d\,a}\right) .\, \cos 2\,\mathrm{W} + \&\text{c.} \\ \frac{d}{d\,\mathrm{W}}\left(a\,\frac{d\,\Omega}{d\,a}\right) &= -\, a\,\frac{d\,\mathrm{A}'}{d\,a} \,.\, \sin \mathrm{W} - 2\, a\,\frac{d\,\mathrm{A}''}{d\,a} \sin 2\,\mathrm{W} - \&\text{c.} \\ \frac{d^2}{d\,a^2}\left(a\,\frac{d\,\Omega}{d\,a}\right) &= \frac{d^2}{da^2}\left(a\,\frac{d\,\mathrm{A}}{d\,a}\right) + \frac{d^2}{da^2}\left(a\,\frac{d\,\mathrm{A}'}{d\,a}\right) \cos \mathrm{W} + \frac{d^2}{da^2}\left(a\,\frac{d\,\mathrm{A}''}{d\,a}\right) \cos 2\,\mathrm{W} + \&\text{c.} \\ \frac{d^2}{d\,a\,d\,\mathrm{W}}\left(a\,\frac{d\,\Omega}{d\,a}\right) &= -\,\frac{d}{da}\left(a\,\frac{d\,\mathrm{A}'}{d\,a}\right) \sin \mathrm{W} - 2\,\frac{d}{da}\left(a\,\frac{d\,\mathrm{A}'}{d\,a}\right) \sin 2\,\mathrm{W} - \&\text{c.}\end{aligned}\right\}; \quad (160)$$

and so on. Now these series may be regarded as completely developed; for the co-efficients of their several terms cos W, sin W, &c. are quantities completely given in numbers, when a, a', A, A′, A″, &c. and their differential co-efficients are known, by the equations

$$\left.\begin{array}{l}
\dfrac{d}{d\,a}\left(a\dfrac{d\,A}{d\,a}\right)=\dfrac{d\,A}{d\,a}+a\,\dfrac{d^2\,A}{d\,a^2};\;\dfrac{d}{d\,a}\left(a\,\dfrac{d\,A'}{d\,a}\right)=\dfrac{d\,A'}{d\,a}+a\,\dfrac{d^2\,A'}{d\,a^2};\;\&c.\\
\dfrac{d}{d\,a'}\left(a\dfrac{d\,A}{d\,a}\right)=\dfrac{d^2\,A}{d\,a\,d\,a'};\;\dfrac{d}{d\,a'}\left(a\dfrac{d\,A'}{d\,a}\right)=\dfrac{d^2\,A'}{d\,a\,d\,a'};\;\&c.\\
\dfrac{d^2}{d\,a^2}\left(a\dfrac{d\,A}{d\,a}\right)=2\,\dfrac{d^2\,A}{d\,a^2}+a\,\dfrac{d^3\,A}{d\,a^3};\\
\qquad\&c.
\end{array}\right\};\;\ldots\ldots\ldots(161)$$

If, instead of $r\,\dfrac{d\,\Omega}{d\,r}$, we had any other function to develope $\left(\text{such as for instance } r^2\,.\,\dfrac{d^3\,\Omega}{d^2\,r\,d\,r'}\right)$, we might treat it exactly in the same way, and should arrive at corresponding series, in which the sines and cosines of W and its multiples would be combined with co-efficients absolutely constant, and reducible to numbers.

In the values of $\Delta\,\Omega$ and $\Delta\left(r\,\dfrac{d\,\Omega}{d\,r}\right)$ in (158) and (159) we see therefore that the differential co-efficients of Ω introduce the sines and cosines of W, and its multiples, combined only with given quantities, and not involving the eccentricities. These latter arise from the factors $\Delta\,r$, $\Delta\,r'$, $\Delta\,w$, and their powers. Let us now examine these more nearly. Supposing then, as we have done all along,

$$V=n\,t+\epsilon-\varpi;\quad V'=n'\,t+\epsilon'-\varpi';\quad W=n\,t-n'\,t+\epsilon-\epsilon';$$

let us take α, β, &c. as follows:—

$$\left.\begin{array}{ll}
\alpha=-\,a\,.\cos V & \alpha'=-\,a'\,.\cos V'\\
\beta=\dfrac{a}{2}\,(1-\cos 2\,V) & \beta'=\dfrac{a'}{2}\,(1-\cos 2\,V')\\
\gamma=-\,\dfrac{a}{8}\,(3\cos 3\,V-3\cos V) & \gamma'=-\,\dfrac{a'}{8}\,(3\cos 3\,V'-3\cos V')\\
\quad\&c. & \quad\&c.\\
p=2\sin V & p'=2\sin V'\\
q=\dfrac{5}{4}\sin 2\,V & q'=\dfrac{5}{4}\sin 2\,V'\\
r=-\,\dfrac{1}{4}\sin V+\dfrac{13}{12}\sin 3\,V & r'=-\,\dfrac{1}{4}\sin V'+\dfrac{13}{12}\sin 3\,V'\\
\quad\&c. & \quad\&c.
\end{array}\right\};\;\ldots\ldots\ldots(162)$$

Then we shall have, by the equations (28) and (30)

$$\left.\begin{array}{l}
\Delta\,r=\alpha\,e+\beta\,e^2+\gamma\,e^3+\&c.\\
\Delta\,r'=\alpha'\,e'+\beta'\,e'^2+\gamma'\,e'^3+\&c.\\
\Delta\,w=(p\,e+\;q\,e^2+r\,e^3+\&c.)-(p'\,e'+q'\,e'^2+r'\,e'^3+\&c.).
\end{array}\right\};\ldots\ldots\ldots\ldots\ldots\ldots(163)$$

whence we obtain

$$\begin{array}{l}
(\Delta\,r)^2=\alpha^2\,e^2+2\,\alpha\,\beta\,e^3+(\beta^2+2\,\alpha\,\gamma)\,e^4+\&c.\\
\Delta\,r\,\Delta\,r'=\alpha\,\alpha'\,e\,e'+\alpha\,\beta'\,e\,e'^2+\beta\,\alpha'\,e^2\,e'+\&c.\\
\Delta\,r\,\Delta\,w=p\,\alpha\,e^2+(p\,\beta+q\,\alpha)\,e^3+\&c.-p'\,\alpha\,e\,e'-\&c.\\
(\Delta\,w^2)=p^2\,e^2+2\,p\,q\,e^3+\&c.+(p'^2\,e'^2+2\,p'\,q'\,e'^3+\&c.)-2\,p\,p'\,e\,e'-\&c.
\end{array}$$

and so on; and it only remains to substitute these values in the expressions for $\Delta\,\Omega$ and $\Delta\left(r\,\dfrac{d\,\Omega}{d\,r}\right)$. Confining ourselves to $\Delta\,\Omega$, since it is obvious that the process is exactly similar for the other function, we have, as before, for the part depending on the first powers of the eccentricities,

$$e\left\{\alpha\,\frac{d\,\Omega}{d\,a}+p\,\frac{d\,\Omega}{d\,W}\right\}+e'\left\{\alpha'\,\frac{d\,\Omega}{d\,a'}-p'\,\frac{d\,\Omega}{d\,W}\right\}$$

which, developed into series of sines and cosines, gives the result obtained in the last section.

The part of $\Delta\,\Omega$ depending on the squares of the eccentricities, consists of three terms multiplied respectively by e^2, $e\,e'$, and e'^2, which originate, 1st. From the terms $\beta\,e^2$, $\beta'\,e'^2$, $q\,e^2$, and $-\,q'\,e'^2$, in the simple powers of $\Delta\,r$, $\Delta\,r'$, $\Delta\,w$, and which are therefore affected with the differential co-efficients of the first order only: 2dly. With the terms $\alpha^2\,e^2$, $\alpha'^2\,e'^2$, $p^2\,e^2$, $p'^2\,e'^2$, and $-\,2\,p\,p'\,e\,e'$, in $(\Delta\,r)^2$, $(\Delta\,r')^2$, and $(\Delta\,w)^2$, and which are consequently affected with differential co-efficients of Ω of the second order.

3dly. With the terms $a\,p\,e^2$, $a'\,p'\,e'^2$, $a\,a'\,e\,a'$, $a\,p'\,e\,e'$, $a'\,p\,e\,e'$, which arise from combinations of $\Delta\sigma$ with $\Delta\,\varpi$ and with $\Delta\,r'$.

The aggregate of all these terms, with their proper co-efficients, is

$$\left.\begin{aligned}&\frac{e^2}{2}\left\{\left(2\,\beta\,\frac{d\,\Omega}{d\,a}+2\,q\,\frac{d\,\Omega}{d\,W}\right)+\left(a^2\,\frac{d^2\,\Omega}{d\,a^2}+2\,a\,p\,\frac{d^2\,\Omega}{d\,a\,d\,W}+p^2\,\frac{d^2\,\Omega}{d\,W^2}\right)\right\}\\ &+e.e'\left\{a\,a'\,\frac{d^2\,\Omega}{d\,a\,d\,a'}-a\,p'\,\frac{d^2\,\Omega}{d\,a\,d\,W}+a'\,p\,\frac{d^2\,\Omega}{d\,a'\,d\,W}-p\,p'\,\frac{d^2\,\Omega}{d\,W^2}\right\}\\ &+\frac{e'^2}{2}\left\{\left(2\,\beta'\,\frac{d\,\Omega}{d\,a'}-2\,q'\,\frac{d\,\Omega}{d\,W}\right)+\left(a'^2\,\frac{d^2\,\Omega}{d\,a'^2}-2\,a'\,p'\,\frac{d^2\,\Omega}{d\,a'\,d\,W}+p'^2\,\frac{d^2\,\Omega}{d\,W^2}\right)\right\}\end{aligned}\right\}\;;\;\ldots\ldots\ldots\ldots(164)$$

The co-efficients of e^3, $e^2\,e'$, $e\,e'^2$, e'^3, and of the higher powers and combinations, may in like manner be easily obtained; but the number of terms of which they consist, goes on increasing so rapidly, that they at length become of extreme complexity.

Let us now consider the nature of the terms into which the expressions for Ω and $r\,\frac{d\,\Omega}{d\,r}$ resolve themselves by the process of developement, and the manner in which they become modified by the several processes of substitution and integration they have to undergo in obtaining the values of Q, Π, $\delta\,u$, $\delta\,r$, and $\delta\,\theta$.

It is evident, then, since a, β, γ, a', β', &c. are all composed of cosines, and p, q, p', q', &c. of sines of V, V', and their multiples, without W, that any product or combination of these letters, (such as a^2, $a\,p$, $a\,a'$, &c.) is reducible by the trigonometrical formulæ so often employed in the foregoing pages into the simple sines or cosines of arcs, of the form $k\,V+l\,V'$. Thus, a^2 or $a^2\,.\cos V^2$ becomes $a^2\,.\,\frac{1+\cos 2\,V}{2}$, $a\,p$, or $-\,2\,a\,.\sin V\,.\cos V$ becomes $-\,a\,.\sin 2\,V$, $a\,a'$ or $a\,a'\,.\cos V\,.\cos V'$ is reduced to $\frac{a\,a'}{2}\,(\cos(V+V')+\cos(V-V'))$, $a\,p'$ into $a\,.\sin(V-V')-a\,.\sin(V+V')$, and so on. Moreover, it is evident, that whenever the combination in question consists only of the letters a, β, a', β', &c. or of these combined with any product of an *even* dimension, in p, q, p', q', &c., that the terms into which it is resolved will consist entirely of cosines; but when a product of an *odd* dimension in p, q, p', q', &c. occurs, then of sines. Now, the differential co-efficient of Ω combined with any such product, will, in the former case, evidently be differentiated an even, and in the latter an odd number of times relatively to W; so that in the former case it will represent a series of *cosines*, and in the latter, *of sines* of W.

Every term therefore formed by such combination, must be of one or other of the forms

$$\cos i\,W\,.\cos(k\,V+l\,V')\ \text{and}\ \sin i\,W\,.\sin(k\,V\pm l\,V')$$

both which, being further resolved, produce terms comprehended in the form

$$\cos(i\,W\pm k\,V\pm l\,V')$$

It is thus demonstrated, that the co-efficients of all the powers of e, e', in the developements of Ω and $r\,\frac{d\,\Omega}{d\,r}$ are generally reducible to series of cosines of arguments, of the form $i\,W\pm k\,V\pm l\,V'$; but there is a connection between the multiples of V and V' contained in any argument, and the dimension of the power or product of the eccentricities to which it belongs that we must now explain. In fact, it is obvious from the process above pursued, that if we regard a, a', p, p', as quantities of one dimension; β, β', q, q', as of two; γ, γ', r, r' of three, and so on, the dimension of every term multiplied by e, or e', will be one; that of the terms multiplied by e^2, $e\,e'$, e'^2, will be two, and so on. Now, the expressions of these quantities in V and V' involve, each, the sines or cosines of multiples of V, V', as far as the number expressing its own dimension; and when these come to be combined by multiplication, and then resolved by the usual formula, it is obvious that the resulting terms of the form $\cos(k\,V\pm l\,V')$ and $\sin(k\,V\pm l\,V')$ can only contain such multiples $k\,V$ and $l\,V'$ of V, V', as together (without regard to their signs) do not exceed the dimension of the combination from which they arose.

Moreover, since the alternate multiples of V, V' are absent in the expressions of a, β, &c. the same law will hold good in any combination of them when developed. Hence we may state it as a general law, that

The co-efficient of any power or product of the eccentricities of the dimension n, *in the developement of* Ω *or* $r\,\frac{d\,\Omega}{d\,r}$ *will consist of a series of cosines, the form of whose argument is*

$$i\,W\pm k\,V\pm l\,V'$$

in which i *may have every possible value from* 0 *to infinity, but* k *and* l *are restricted to certain particular values, viz. those which satisfy one or other of the equations*

$$k+l=n,\qquad k+l=n-2,\qquad k+l=n-4,\ \text{\&c.}$$

down to $k+l=1$, *or* $k+l=0$, *according as* n *is odd or even.*

Thus, as we have already seen, the parts independent of the eccentricities consist of terms of the form cos

$i\,\mathrm{W}$ only, and those depending on the first powers involve the arguments

$$i\,\mathrm{W}+\mathrm{V},\quad i\,\mathrm{W}-\mathrm{V},\quad i\,\mathrm{W}+\mathrm{V}',\quad i\,\mathrm{W}-\mathrm{V}'$$

and no other. Similarly, in the part dependent on the squares and product of the eccentricities, the arguments which can occur, are only

$i\,\mathrm{W}$, $i\,\mathrm{W}+2\,\mathrm{V}$, $i\,\mathrm{W}-2\,\mathrm{V}$, $i\,\mathrm{W}+\mathrm{V}+\mathrm{V}'$, $i\,\mathrm{W}+\mathrm{V}-\mathrm{V}'$, $i\,\mathrm{W}-\mathrm{V}+\mathrm{V}'$, $i\,\mathrm{W}-\mathrm{V}-\mathrm{V}'$, $i\,\mathrm{W}+2\,\mathrm{V}'$, and $i\,\mathrm{W}-2\,\mathrm{V}'$, and so on.

Here, it will be observed, we have again the argument $i\,\mathrm{W}$, which occurred in the part independent on the eccentricities, and it is easily seen to be a general law, that *any particular argument which first occurs combined with a power or product of the eccentricities of the dimension* n *will occur again, combined with products of the dimensions* $n+2$, $n+4$, &c. *to infinity, but not with* $n+1$, $n+3$, &c. For instance, the argument $i\,\mathrm{W}+3\,\mathrm{V}$ cannot occur combined with any dimension of the eccentricities less than the third, and will occur again in the terms multiplied by the 5th, 7th, &c. dimensions, but not by the 4th, 6th, or any even dimensions.

Since $\mathrm{W}=n\,t-n'\,t+\epsilon-\epsilon'$, $\mathrm{V}=n\,t+\epsilon-\varpi$, and $\mathrm{V}'=n'\,t+\epsilon'-\varpi'$, the argument $i\,\mathrm{W}+k\,\mathrm{V}+l\,\mathrm{V}'$ is equal to

$$(i+k)\,.\,n\,t-(i-l)\,n'\,t+(i+k)\,\epsilon-(i-l)\,\epsilon'-k\,\varpi-l\,\varpi'$$

If then we would inquire in what terms any proposed combination of n and n', such, for instance, as $(f\,n-g\,n')\,t$ can originate, we have only to put

$$i+k=f,\qquad i-l=g$$

which give

$$i=f-k,\qquad l=i-g=(f-g)-k$$

taking then in succession $k=o$, $\quad k=\pm 1$, $\quad k=\pm 2$, &c. we get

$$\begin{array}{lll} i=f, & i=f\mp 1, & i=f\mp 2,\ \&c. \\ l=f-g, & l=f-g\mp 1, & l=f-g\mp 2,\ \&c. \end{array}$$

For instance, if we would know from what terms the combination $(2\,n-n')\,t$ can originate, the corresponding values of i, k, l, are

1st. $i=2,\quad k=o,\quad l=-3$; $\quad$ 2dly. $\left\{\begin{array}{lll} i=1, & k=1, & l=-4 \\ i=3, & k=-1, & l=-2 \end{array}\right.$ $\quad$ 3dly. $\left\{\begin{array}{lll} i=o, & k=2, & l=-5 \\ i=4, & k=-2, & l=-1 \end{array}\right.$

So that any of the arguments comprised in the following series,

$$2\,\mathrm{W}-3\,\mathrm{V}',\ \left\{\begin{array}{l} \mathrm{W}+\mathrm{V}-4\,\mathrm{V}';\quad 2\,\mathrm{V}-5\,\mathrm{V}'; \\ 3\,\mathrm{W}-\mathrm{V}-2\,\mathrm{V}';\quad 4\,\mathrm{W}-2\,\mathrm{V}-\mathrm{V}';\quad 5\,\mathrm{W}-3\,\mathrm{V},\ \&c. \end{array}\right.$$

will produce the combination in question. Now, the lowest sum of the co-efficients of V, V′ in these arguments taken without regard to their signs, is 3: consequently, the combination $2\,n\,t-5\,n'\,t$ will first occur among the inequalities multiplied by the cubes or products of three dimensions of the eccentricities, and among them only in such terms as produce the arguments

$$2\,\mathrm{W}-3\,\mathrm{V}',\quad 3\,\mathrm{W}-\mathrm{V}-2\,\mathrm{V}',\quad 4\,\mathrm{W}-2\,\mathrm{V}-\mathrm{V}',\quad 5\,\mathrm{W}-3\,\mathrm{V}.$$

Let us now examine the co-efficients of the several arguments as they occur in the values of Q, Π, &c.

The co-efficient of any argument, such as $i\,\mathrm{W}\pm k\,\mathrm{V}\pm l\,\mathrm{V}'$ in the developement of Ω or $r\,\dfrac{d\,\Omega}{d\,r}$ will obviously consist only of combinations of a, a', A, A′, A″, &c. and their differential co-efficients with a power or product of e, e', and may therefore be regarded as a given quantity, and its value, with more or less trouble, numerically computed. Taking A for the general representative of such a combination, M will be a function of a, a', e, e', and of these only, and

$$\mathrm{M}\cos\,(i\,\mathrm{W}\pm k\,\mathrm{V}\pm l\,\mathrm{V}')$$

will be the general form of any term of Ω or $r\,\dfrac{d\,\Omega}{d\,r}$.

In the value of Q, the terms of $r\,\dfrac{d\,\Omega}{d\,r}$ enter unchanged; but since

$$d\,\mathrm{W}=n\,d\,t,\quad d\,\mathrm{V}=n\,d\,t,\quad d\,\mathrm{V}'=o$$

the term under consideration will produce in $d\,\Omega$ the term $-\,\mathrm{M}\,.\,(i\pm k)\,n\,d\,t\,.\sin\,(i\,\mathrm{W}\pm k\,\mathrm{V}\pm l\,\mathrm{V}')$ and in Q, the term

$$\frac{2\,(i\pm k)\,n}{(i\pm k)\,n\pm l\,n'}\,\mathrm{M}\cos\,(i\,\mathrm{W}\pm k\,\mathrm{V}\pm l\,\mathrm{V}')$$

so that Q will contain two species of terms, those whose co-efficients are of the form M, and $\mathrm{M}\,\dfrac{(i\pm k)\,n}{(i\pm k)\,n\pm l\,n'}$

The value of Q substituted in Π (equation 128) will produce terms comprised in one or other of the forms

$$\mathrm{M}\,.\cos\,(i\,\mathrm{W}\pm k\,\mathrm{V}\pm l\,\mathrm{V}')$$

$$\mathrm{M}\,.\frac{(i\pm k)\,n}{(i\pm k)\,n\pm l\,n'}\,.\cos\,(i\,\mathrm{W}\pm k'\,\mathrm{V}\pm l\,\mathrm{V}')$$

$$\mathrm{M}\,.\frac{n}{(i\pm k)\,n\pm l\,n'}\,.\cos\,(i\,\mathrm{W}\pm k\,\mathrm{V}\pm l\,\mathrm{V}')$$

$$\mathrm{M}\,.\frac{(i\pm k)\,n^2}{\{(i\pm k)\,n\pm l\,n'\}\,\{(i\pm k')\,n\pm l\,n'\}}\,.\cos\,(i\,\mathrm{W}\pm k'\,\mathrm{V}\pm l\,\mathrm{V}')$$

In the process of integration by which δu is derived from Π (or in the integration of the equation (129)) these terms again acquire factors of the form $\frac{1}{\{(i \pm k) n \pm l n'\}^2 - n^2}$ or $\frac{1}{\{(i \pm k') n \pm l n'\}^2 - n^2}$ according as k or k' occurs in the argument; and as these forms are obviously not altered in the transition from δu to δr, the terms of δr will necessarily be included in one of the forms

$$\text{M} . \frac{1}{\{(i \pm k) n \pm l n'\}^2 - n^2} \cos (i \text{W} \pm k \text{V} \pm l \text{V}')$$

$$\text{M} . \frac{(i \pm k') n}{\{(i \pm k) n \pm l n'\} \{(i \pm k) n \pm l n')^2 - n^2\}} \cos (i \text{W} \pm k \text{V} \pm l \text{V}')$$

$$\text{M} . \frac{n}{\{(i \pm k) n \pm l n'\} \{(i \pm k) n \pm l n')^2 - n^2\}} \cos (i \text{W} \pm k \text{V} \pm l \text{V}')$$

$$\text{M} . \frac{(i \pm k') n^2}{\{(i \pm k') n \pm l n'\} \{(i \pm k) n \pm l n'\} \{(i \pm k) n \pm l n')^2 - n^2\}} \cos (i \text{W} \pm k \text{V} \pm l \text{V}')$$

Let the several functions of n, n', in the co-efficients of these terms be represented indiscriminately by N, then will the general form of the terms of δr be $\text{M} . \text{N} . \cos (i \text{W} \pm k \text{V} \pm l \text{V}')$.

It remains only to consider the nature of the terms of which $\delta \theta$ consists. Now these will be,

1st. Those arising from the term $\frac{d r}{d t} \delta r$, which are of the form

$$\frac{\text{M} . k'' n . \text{N}.}{n a^2 \sqrt{1 - e^2}} \sin (i \text{W} \pm (k \pm k'') \text{V} \pm l \text{V}')$$

2dly. Those arising from $r \frac{d \delta r}{d t}$, whose form is

$$\frac{2 \text{M} \{(i \pm k) n \pm l n'\} \text{N}}{n a^2 \sqrt{1 - e^2}} \sin (i \text{W} \pm (k \pm k'') \text{V} \pm l \text{V}')$$

3dly. Those arising from $\int r \frac{d \Omega}{d r} n \, d t$ and $\iint d \Omega . n \, d t$ whose forms are respectively

$$\frac{2 a \text{M}}{\mu \sqrt{1 - e^2}} . \frac{n}{(i \pm k) n \pm l n'} \sin (i \text{W} \pm k \text{V} \pm l \text{V}'), \text{ and } \frac{3 a \text{M}}{\mu \sqrt{1 - e^2}} . \frac{(i \pm k) n^2}{\{(i \pm k) n \pm l n'\}^2} \sin (i \text{W} \pm k \text{V} \pm l \text{V})$$

and lastly those peculiar terms containing t out of the signs sin and cos which we have already noticed as giving rise to the secular equations.

The complete enumeration of all the possible varieties of terms which $\delta \theta$ may contain, will therefore be had by putting for N each of the four forms above assigned to it; but as those only really differ importantly in which the denominators of the fractions differ, we need only enumerate the latter quantities; which are,

$$(i \pm k) n \pm l n' ; \quad \{(i \pm k) n \pm l n'\}^2 ;$$

$$\{(i \pm k) n \pm l n'\}^2 - n^2 ; \quad ((i \pm k') n \pm l n') \{((i \pm k) n \pm l n')^2 - n^2\}$$

$$\{(i \pm k') n \pm l n'\} \{(i \pm k) n \pm l n'\} \{((i \pm k) n \pm l n')^2 - n^2\}$$

These then are the various forms of the divisors with which the processes of integration, &c., affect the inequalities in longitude. They are, as we have already remarked, of the highest importance in the theory of the planets, by reason of their effect on the numerical values of the maxima of the perturbations to which they belong. Such is the immense number of terms, or rather of series of terms, branching out in all directions, of which the perturbations consist, that it is manifestly in vain to attempt to take account of them all. It is therefore of the highest consequence to have some guiding principle to direct us in our choice of the terms to be retained or neglected. Were it not for these divisors, we might safely rely on the rapid convergency of the powers and products of the eccentricities; and reject, without further examination, all in which their dimension exceeded a certain limit; but should there be an approach to commensurability in the periodic times of the two planets, (as, for instance, should five times the mean motion of the disturbed planet ($5 n' t$) be very nearly equal to twice that of the disturbing, ($2 n t$)) this circumstance will render some one of their factors ($5 n' - 2 n$) very small. In consequence, all the divisors into which this factor enters will become very small, and the inequalities affected by them will, in consequence, acquire from this cause an unnatural magnitude (if we may use such an expression) and must be retained, even though of such an order as would otherwise authorize their rejection.—The terms so affected too, will originate in a great variety of manners from the developements, and may be affected with various powers of the eccentricities; so that their number will necessarily be infinite, and for the purpose of approximation only the most prominent can be selected.

The equations of the motions of Jupiter and Saturn, known by the name of the great inequalities of these planets, were long a difficulty in the way of the theoretical astronomer, and even a stumbling block in the way of the Newtonian philosophy. It was observed, on comparing very ancient observations of the oppositions of these planets with more modern ones, that their *mean motions* had undergone an apparent alteration; that of Saturn appearing to have been retarded, and that of Jupiter accelerated. In other words, that Saturn perpetually lagged behind, and Jupiter as constantly surpassed the places, when they ought to have been, on the hypothesis of the mean motion, or periodic time, remaining invariable.

We have seen that every inequality of very long period will appear, while on the increase, to affect the mean motion; it is obvious it must, if the latter, as determined by observations comprised within the periods of its increase, be compared with the result of similar observations made while its value is on the diminution. Now, the length of the period of any inequality depends on the multiples of the mean motions found in its argument; and it was not difficult for geometers to shew, that, so far as the first powers or squares of the eccentricities were concerned, no inequalities of such very long periods as the case required, could be found in the motion of either planet. The cubes and higher powers had all along been neglected without fear of error; but Laplace having, from other considerations, ascertained that an acceleration in Jupiter's motion *being supposed*, a retardation in Saturn's *must follow* of course, and that in the very proportion observed; and that therefore the phenomenon was not altogether inconsistent with the laws of gravity, set himself to examine the terms multiplied by the cubes of the eccentricities. Here he immediately encountered the argument $(5nt - 2n't + \text{const.})$; and the mean motion of Jupiter being to that of Saturn nearly in the proportion of 5 to 2, if we suppose n' to correspond to Jupiter's and n to Saturn's motion the co-efficient $5n-2n'$ is very small, and the corresponding period is found on calculation to amount to 918 years. The resulting inequality has also $5n-2n'$ for its divisor, and its magnitude is thus increased as well as its period lengthened. On executing the calculation, the inequalities of both planets were found to be such as would completely account for the apparent accelerations and retardations observed.

SECTION VI.

Of the variations of the elements of the planetary orbits, and the secular equations of their motions. Theory of the major axes, inclinations, nodes, eccentricities, and aphelia.

We have already taken occasion to observe, that the motions of the planets may be regarded as performed in ellipses, whose positions and magnitudes are continually, but very slowly, changing, by the effects of the disturbing forces. These forces are so small, that in a moderate period of time, as for instance, in a single revolution of a planet, the change is insensible; and, if we allow for those inequalities which depend on the configurations of the disturbed and disturbing planets, the theory of which has been exposed in the foregoing sections, the motion, equated by the application of these corrections, will coincide with almost rigorous exactness with the elliptic theory. But after the expiration of many revolutions of the planet, this exact coincidence will cease to take place, even when its place is corrected for such periodical inequalities. The place so corrected is found, it is true, in the circumference of an ellipse with the sun in its focus, but it is not an ellipse of precisely the same form and position as before. Its elements have undergone a change, and this change, though imperceptible in a single revolution, becomes gradually more and more evident, till at length it is too remarkable to be overlooked.

Such slow changes are what we understand by the secular variations of the elements of the planet's orbits. But there is another point of view in which we may consider the subject, which presents peculiar facilities to the application of mathematical investigation. It consists in referring *all* the inequalities resulting from perturbation, to the variation of the elliptic elements, not merely those of long periods, but those which pass rapidly from their maxima to their minima, and depend on the configurations of the bodies. We are indebted to Lagrange for this view of the subject, and shall endeavour to give an idea in this section of the luminous analysis of that great geometer.

If, at the expiration of any instant, the disturbing forces were to cease acting, the planet would go on describing an exact ellipse, of which the infinitesimal arc described in the last instant would be an elementary portion. The plane of the ellipse would be that in which this portion and the sun's centre lie; its eccentricity, position, and magnitude, would all be determined from the *position*, *magnitude*, and *curvature* of this element, and the laws of elliptic motion. In a word, it would be a real *ellipse of curvature* to the actual curve described by the planet, at that particular instant, subjected to the conditions of having its focus in the sun, and satisfying the other laws of elliptic motion during that moment. The elements of the planet's orbit then, at any moment, are no other than the elements of this ellipse, and are determined from three consecutive places of the planet, infinitely near each other. Thus every inequality in its motion will produce a corresponding fluctuation in the elements, which will thus be subject to as many equations, periodical or otherwise, as the planet's motion itself is affected with.

The periodical terms thus originating in the elements, will, in the course of many revolutions, compensate each other; but if it should happen that terms not periodical should find their way into their values, these will express *secular* changes, which it becomes of the utmost importance to investigate.

To determine the ellipse of curvature at any instant, is a matter of no difficulty; we have only to call to mind that any one of the constants in its equation may be insulated, and expressed in terms of the co-ordinates and their differential co-efficients, by the mere operations of differentiating and eliminating; so that, supposing the co-ordinates and their differential co-efficients given at any instant, any one of the constants may have its value ascertained by simple substitution. But it is not necessary to go through these processes—we may avoid that trouble by recurring to the origin of the ellipse itself. Now all we know of it is, that it satisfies the dynamical relations of the problem, on the supposition of the disturbing forces ceasing to act at the instant $d t$. Consequently its equations, however transformed, must be such as to satisfy the differential equations

$$\frac{d^2 x}{d t^2} + \frac{\mu x}{r^3} = 0$$

$$\frac{d^2 y}{d t^2} + \frac{\mu y}{r^3} = 0$$

$$\frac{d^2 z}{d t^2} + \frac{\mu z}{r^3} = 0$$

and its elements will be the constants introduced by the integration of these, or known functions of them.

For instance, let us consider its major semiaxis. If we pursue with these the same process of integration by which equation (109) was obtained, viz. multiply the first by $d x$, the second by $d y$, and the third by $d z$ and add, and integrate, we find

$$\frac{\mu}{a} = \frac{2 \mu}{r} - \frac{d x^2 + d y^2 + d z^2}{d t^2}$$

a being the arbitrary constant introduced by integration; and if we compare this with (33), we shall see that a is the semiaxis of the ellipse. Thus we know, that all that is necessary to obtain the semiaxis of the ellipse of curvature at any instant (let the body describe what curve it will) is merely to substitute for r, $\frac{d x}{d t}$, $\frac{d y}{d t}$, $\frac{d z}{d t}$, in the expression

$$\frac{\mu}{\frac{2 \mu}{r} - \left\{\left(\frac{d x}{d t}\right)^2 + \left(\frac{d y}{d t}\right)^2 + \left(\frac{d z}{d t}\right)^2\right\}}$$

those values which, in the case proposed, they *actually have* in virtue of the real motion of the body, such as the forces in action make it.

Now, in the case of disturbed motion, $d x$, $d y$, $d z$, are given by the equation (94); for if we integrate these after multiplying them respectively by $2 d x$, $2 d y$, $2 d z$, we find

$$\left(\frac{d x}{d t}\right)^2 = -\mu \int \frac{2 x d x}{r^3} - 2 m' \int \frac{d \Omega}{d x} d x$$

$$\left(\frac{d y}{d t}\right)^2 = -\mu \int \frac{2 y d y}{r^3} - 2 m' \int \frac{d \Omega}{d y} d y$$

$$\left(\frac{d z}{d t}\right)^2 = -\mu \int \frac{2 z d z}{r^3} - 2 m' \int \frac{d \Omega}{d z} d z$$

consequently adding all together,

$$\left(\frac{d x}{d t}\right)^2 + \left(\frac{d y}{d t}\right)^2 + \left(\frac{d z}{d t}\right)^2 = -\mu \int \frac{2 x d x + 2 y d y + 2 z d z}{r^3} - 2 m' \int \left(\frac{d \Omega}{d x} d x + \frac{d \Omega}{d y} d y + \frac{d \Omega}{d z} d z\right)$$

$$= \frac{2 \mu}{r} - 2 m' \int d \Omega$$

and substituting, we get

$$a = \frac{\mu}{2 m' \int d \Omega}, \text{ or } \frac{\mu}{a} = 2 m' \int d \Omega ; \qquad (165, I)$$

The same result will be obtained as follows:—If we integrate, as in (109), and instead of adding explicitly the arbitrary quantity $\frac{\mu}{a}$ to complete the integral, regard it as included under the sign $\int$ we have

$$\frac{2 \mu}{r} - \frac{d x^2 + d y^2 + d z^2}{d t^2} = 2 m' \int d \Omega$$

but if a be the major semiaxis of the ellipse of curvature,

$$\frac{\mu}{a} = \frac{2 \mu}{r} - \frac{d x^2 + d y^2 + d z^2}{d t^2}$$

Hence we get

$$\frac{\mu}{a} = 2 m' \int d \Omega$$

If (a) be the major semiaxis at the commencement of the time t, and $\int d \Omega$ be taken, so as to vanish when omy. Physical Astronomy

$t = o$, we have

$$\frac{\mu}{a} = \frac{\mu}{(a)} + 2\,m' \int \mathrm{d}\,\Omega \qquad (165,\ 2)$$

In this instance we have had no difficulty in arriving at once at the finite expression for the varied element. But it is in other cases more commodious to express its *momentary* variation. Let us therefore denote by the characteristic δ, that peculiar variation by which the ellipse of curvature passes from the form and position it had during $d\,t$ to that which it has in the consecutive instant, then $\delta\,a$, $\delta\,e$, $\delta\,\pi$, &c. will be the momentary variations of its semiaxis, eccentricity, perihelion, &c. Moreover, $\delta\,x$, $\delta\,y$, $\delta\,z$, and $\delta\,r$, will represent the excesses of the values of x, y, z, r, in the varied ellipse; not over what they *were* in the former instant, but over what they *would have been* had that ellipse remained unaltered. But, both in the one case and the other, the point in the ellipse to which they correspond, coincides with the real place of the planet. Hence the lines x, y, z, r, are the same in the varied ellipse, in the unvaried, and in the actual curve described; so that $\delta\,x = o$, $\delta\,y = o$, $\delta\,z = o$, and $\delta\,r = o$. Again, $\delta\,\frac{d\,x}{d\,t}$, $\delta\,\frac{d\,y}{d\,t}$, $\delta\,\frac{d\,z}{d\,t}$, are the excesses of the values of $\frac{d\,x}{d\,t}$, $\frac{d\,y}{d\,t}$, $\frac{d\,z}{d\,t}$, in the varied ellipse over what they would have been had the ellipse not varied—that is, had the disturbing force not acted,—in the instant consecutive to $d\,t$. Their values therefore would vanish, had the body remained in its former ellipse, and, in general, will be obtained by subtracting from the values actually assumed by $\frac{d\,x}{d\,t}$, &c. in the consecutive instant in the curve, what *would have been* assumed by them had the body continued in the same ellipse, or had the disturbing forces ceased to act at the end of $d\,t$. Now, in the *curve* the consecutive value of $\frac{d\,x}{d\,t}$ is

$$\frac{d\,x}{d\,t} + d\,\frac{d\,x}{d\,t} = \frac{d\,x}{d\,t} - \left\{\frac{\mu\,x}{r^3} + m'\,\frac{d\,\Omega}{d\,x}\right\} d\,t$$

because $\frac{\mu\,x}{r^3} + m'\,\frac{d\,\Omega}{d\,x}$ being the force in the curve, we must have $d\,\frac{d\,x}{d\,t} = -\left\{\frac{\mu\,x}{r^3} + m'\,\frac{d\,\Omega}{d\,x}\right\} d\,t$. On the other hand, had the disturbing force ceased acting, we should have had simply $d\,\frac{d\,x}{d\,t} = -\,\frac{\mu\,x}{r^3}\,d\,t$, so that the consecutive value of $\frac{d\,x}{d\,t}$ *would have been* merely $\frac{d\,x}{d\,t} - \frac{\mu\,x}{r^3}\,d\,t$.

Hence we have

$$\delta\,\frac{d\,x}{d\,t} = -\,m'\,\frac{d\,\Omega}{d\,x}\,d\,t$$

and similarly

$$\left.\begin{aligned} \delta\,\frac{d\,y}{d\,t} &= -\,m'\,\frac{d\,\Omega}{d\,y}\,d\,t \\ \delta\,\frac{d\,z}{d\,t} &= -\,m'\,\frac{d\,\Omega}{d\,z}\,d\,t \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (166)$$

To explain how the variations so obtained may be employed, let us take again the case already treated,

$$\frac{\mu}{a} = \frac{2\,\mu}{r} - \frac{d\,x^2 + d\,y^2 + d\,z^2}{d\,t^2}$$

If we differentiate this relative to the characteristic δ, we get

$$-\,\frac{\mu\,\delta\,a}{a^2} = -\,\frac{2\,\mu\,\delta\,r}{r^2} - 2\left\{\frac{d\,x}{d\,t}\,\delta\,\frac{d\,x}{d\,t} + \frac{d\,y}{d\,t}\,\delta\,\frac{d\,y}{d\,t} + \frac{d\,z}{d\,t}\,\delta\,\frac{d\,z}{d\,t}\right\}$$

in which, putting $\delta\,r = o$, and for $\delta\,\frac{d\,x}{d\,t}$ &c., their values above found, we find

$$-\,\frac{\mu\,\delta\,a}{a^2} = +\,2\,m'\left\{\frac{d\,\Omega}{d\,x}\,d\,x + \frac{d\,\Omega}{d\,y}\,d\,y + \frac{d\,\Omega}{d\,z}\,d\,z\right\} = 2\,m'\,\mathrm{d}\,\Omega$$

$\delta\,a$ is the *momentary* variation of a in the instant $d\,t$, so that this equation may be integrated relative to t, and we get

$$\frac{\mu}{a} = 2\,m' \int \mathrm{d}\,\Omega$$

the same result as before.

Before we proceed farther, we will stop to draw from this expression of the reciprocal axis, a most important conclusion. It is this—that all the variations to which the major axes of the planetary orbits are subjected by their mutual attraction are periodical—and that the mean distances, and consequently the mean motions of the planets are subject to no *secular* variations. In fact, when we consider only the first power of the disturbing forces, we have already proved that the developement of Ω is entirely composed of terms of the form

$$\mathrm{A}\,.\,\cos\,(i\,\mathrm{W} + k\,\mathrm{V} + l\,\mathrm{V}')$$

hence

$$\begin{aligned} \mathrm{d}\,\Omega &= \mathrm{A}\,.\,\left(i\,\frac{d\,\mathrm{W}}{n\,d\,t} + k\,\frac{d\,\mathrm{V}}{n\,d\,t}\right).\,\sin\,(i\,\mathrm{W} + k\,\mathrm{V} + l\,\mathrm{V}') \\ &= \mathrm{A}\,.\,(i + k)\,.\,\sin\left\{(i + k)\,n\,t - (i - l)\,n'\,t + (i + k)\,\epsilon - (i - l)\,\epsilon' - k\,\pi - l\,\pi'\right\} \end{aligned}$$

which is always periodic unless $i + k = o$ and $i - l = o$, when the argument becomes simply $i\,(\pi - \pi')$ a constant quantity; but in this case the whole term vanishes by the disappearance of its co-efficient. Thus $d\,\Omega$ and, of course, $\int d\,\Omega$ contains no term multiplied by t, and none but what is periodic; consequently, $\frac{\mu}{a}$ is periodic also.

This beautiful result, the demonstration of which is of almost elementary simplicity, assures us of the impossibility of any of the bodies of our system ever leaving it in consequence of the disturbances it may experience, and secures the general permanence of the whole, by keeping the mean distances and periodic times perpetually fluctuating between certain limits (very restricted ones) which they can neither exceed nor fall short of.

Let us next consider the variation in the position of the plane of the disturbed orbit. The equations (4), (5), and (6), give

$$h = y\frac{d\,x}{d\,t} - x\frac{d\,y}{d\,t}, \quad h' = z\frac{d\,x}{d\,t} - x\frac{d\,z}{d\,t}, \quad h'' = z\frac{d\,y}{d\,t} - y\frac{d\,z}{d\,t}$$

These quantities in the case of elliptic motion are constant, but in that of disturbed motion they will equally hold good, if h, h', h'', be regarded as variable. Now, either on the one or the other supposition, if we multiply the first by z, the second by $-\,y$, and the third by x, and add, we get

$$h\,z - h'\,y + h''\,x = o$$

which is the same with equation (7). But if we mutiply the first by $d\,z$, the second by $-\,d\,y$, and the third by $d\,x$, we shall also obtain by addition,

$$h\,d\,z - h'\,d\,y + h''\,d\,x = o$$

Consequently, even when we regard $h\;h'\;h''$ as variable, still the equation $h\,z - h'\,y + h''\,x = o$ and its differential relative to x, y, z, subsist together, just as if h, h', h'' were constant. Hence, it appears, that the body at the end of the instant $d\,t$ is still found in the plane represented by $h\,z - h'\,y + h''\,x = o$, or, that this plane is the plane in which the elementary arc described in the instant $d\,t$, lies. If therefore we call ϕ its inclination to that of the x and y taken as a fixed plane, and ω the longitude of its ascending node, we have

$$\tan\phi = \frac{\sqrt{h'^2 + h''^2}}{h}; \quad \tan\omega = \frac{h''}{h'}; \quad \mu\,a\,(1 - e^2) = h^2 + h'^2 + h''^2; \tag{167}$$

whence these elements (viz. the inclination, the longitude of the node, and the semiparameter,) are expressed in terms of h, h', h''.

That the equation $h\,d\,z - h'\,d\,y + h''\,d\,x = o$ must hold good at the same time with $h\,z - h'\,y + h''\,x = o$ is also evident from this consideration, that although the ellipse, it is true, varies from one instant to another, yet it must be regarded as invariable, while the body describes each of its elementary portions; because, by hypothesis, it is so adjusted that the body shall remain in its circumference during the whole of the instant $d\,t$, and it is not till the consecutive instant that it is necessitated to change its form, &c. to accommodate itself to the new course taken by the body. The same reasoning holds for any other finite equation of elliptic motion. Its first differential may be taken, as if the arbitrary constants it involves were rigorously such. The disturbing forces make no change on x, y, z, r, *their* consecutive values remain $x + d\,x$, $y + d\,y$, $z + d\,z$, $r + d\,r$, as before, and are common both to the curve and the ellipse, it is for the consecutive values of $d\,x$, $d\,y$, $d\,z$, that they differ, these becoming $d\,x + d^2\,x + \delta\,d\,x$ for the curve, and $d\,x + d^2\,x$ for the ellipse.

This premised, we have only to inquire the variations of h, h', h''; and to this end, by the equations (4, 5, 6,) we have

$$\delta\,h = y\,\delta\frac{d\,x}{d\,t} - x\,\delta\frac{d\,y}{d\,t}; \quad \delta\,h' = z\,\delta\frac{d\,x}{d\,t} - x\,\delta\frac{d\,z}{d\,z}; \quad \delta\,h'' = z\,\delta\frac{d\,y}{d\,t} - y\,\delta\frac{d\,z}{d\,t}.$$

In which, substituting for $\delta\frac{d\,x}{d\,t}$, $\delta\frac{d\,y}{d\,t}$, $\delta\frac{d\,z}{d\,t}$, their values $-\,m'\frac{d\,\Omega}{d\,x}\,d\,t$, $-\,m'\frac{d\,\Omega}{d\,y}\,d\,t$, and $m'.\frac{d\,\Omega}{d\,z}\,d\,t$, we find

$$\left.\begin{aligned}
\delta\,h &= m'\left\{x\frac{d\,\Omega}{d\,y} - y\frac{d\,\Omega}{d\,x}\right\}d\,t;\\
\delta\,h' &= m'\left\{x\frac{d\,\Omega}{d\,z} - z\frac{d\,\Omega}{d\,x}\right\}d\,t;\\
\delta\,h'' &= m'\left\{y\frac{d\,\Omega}{d\,z} - z\frac{d\,\Omega}{d\,y}\right\}d\,t;
\end{aligned}\right\}\cdots\cdots\cdots\cdots \tag{168}$$

and integrating relative to t,

$$h = (h) + m'\int\left\{x\frac{d\,\Omega}{d\,y} - y\frac{d\,\Omega}{d\,x}\right\}d\,t; \tag{169, 1}$$

$$h' = (h') + m'\int\left\{x\frac{d\,\Omega}{d\,z} - z\frac{d\,\Omega}{d\,x}\right\}d\,t; \tag{169, 2}$$

$$h'' = (h'') + m'\int\left\{y\frac{d\,\Omega}{d\,z} - z\frac{d\,\Omega}{d\,y}\right\}d\,t; \tag{169, 3}$$

where (h), (h'), and (h''), represent the primitive values of these respective quantities at the commencement of the time t.

As these expressions are rigorous, we may derive from them all the laws which regulate the motion of the nodes and the inclinations of the planes of the orbits; but as it is only the secular variations which concern us at present, we shall not regard the periodical parts of the expressions within the brackets under the integral signs. To develope them, we must consider the orbits as inclined to the plane of the x, y; but if we take the undisturbed orbit of m for this plane, the value of z at any time t will be of the order of the disturbing forces, and z' will be a very small quantity; so that z^2, $z\,z'$, and z'^2, may be neglected. Hence r $(= \sqrt{x^2 + y^2 + z^2})$ and r', will represent with this degree of approximation, their projections on the plane of the x, y, and we have (calling s the tangent of m's latitude,)

$$x = r \,.\, \cos\theta,\ y = r \,.\, \sin\theta,\ \ z = r\,s;\quad x' = r' \,.\, \cos\theta', y' = r' \,.\, \sin\theta',\ z' = r'\,s'$$

If then we recur to the expressions for the disturbing forces $m'\dfrac{d\,\Omega}{d\,x}$, &c. in Section I. Part II. we shall find

$$y\,\frac{d\,\Omega}{d\,x} - x\,\frac{d\,\Omega}{d\,y} = (y\,x' - x\,y')\left(\frac{1}{r'^3} - \frac{1}{\lambda^3}\right); \qquad (170,\,1)$$

$$z\,\frac{d\,\Omega}{d\,x} - x\,\frac{d\,\Omega}{d\,z} = (z\,x' - x\,z')\left(\frac{1}{r'^3} - \frac{1}{\lambda^3}\right); \qquad (170,\,2)$$

$$z\,\frac{d\,\Omega}{d\,y} - y\,\frac{d\,\Omega}{d\,z} = (z\,y' - y\,z')\left(\frac{1}{r'^3} - \frac{1}{\lambda^3}\right); \qquad (170,\,3)$$

Now, since $h\,z - h'\,y + h''\,x = o$, we have $z = -\dfrac{h''}{h}\,x + \dfrac{h'}{h}\,y$

Suppose then, $\dfrac{h''}{h} = p$ and $\dfrac{h'}{h} = q$, and let the quantities corresponding to p and q in the orbit of m' be p' and q'; then, if ϕ, ϕ', be the inclinations of the two orbits to the fixed plane, and ω, ω', the longitudes of their ascending nodes, we shall have $\tan\phi = \sqrt{p^2 + q^2}$, $\tan\omega = \dfrac{p}{q}$, $\tan\phi' = \sqrt{p'^2 + q'^2}$, $\tan\omega' = \dfrac{p'}{q'}$, and thus, when p and q are determined, the inclinations and places of the nodes are easily found. We have, moreover,

$$z = q\,y - p\,x,\quad z' = q'\,y' - p'\,x'$$
$$z\,x' - x\,z' = (p' - p)\,x\,x' + q\,y\,x' - q'\,x\,y'$$
$$z\,y' - y\,z' = -\,(q' - q)\,y\,y' - p\,x\,y' + p'\,y\,x'$$

and if we therefore suppose for a moment $\mathrm{M} = \dfrac{1}{r'^3} - \dfrac{1}{\lambda^3}$,

$$\frac{d\,h}{d\,t} = m'\,\mathrm{M} \,.\, (y\,x' - x\,y')$$
$$\frac{d\,h'}{d\,t} = m'\,\mathrm{M} \,.\, \{(p' - p)\,x\,x' + q\,y\,x' - q'\,x\,y'\}$$
$$\frac{d\,h''}{d\,t} = m'\,\mathrm{M} \,.\, \{-\,(q' - q)\,y\,y' - p\,x\,y' + p'\,y\,x'\}$$

Now, our design being to eliminate h, h', h'', from the formulæ, and obtain expressions involving only p and q, from which h, h', h'', may be deduced if wanted, we differentiate the equations $p = \dfrac{h''}{h}$ and $q = \dfrac{h'}{h}$, when we find

$$\frac{d\,p}{d\,t} = -\,\frac{1}{h}\left(\frac{d\,h''}{d\,t} - p\,\frac{d\,h}{d\,t}\right) \qquad \frac{d\,q}{d\,t} = \frac{1}{h}\left(\frac{d\,h'}{d\,t} - q\,\frac{d\,h}{d\,t}\right)$$

and if we substitute in these, the values of $\dfrac{d\,h}{d\,t}$, $\dfrac{d\,h'}{d\,t}$, and $\dfrac{d\,h''}{d\,t}$, as above found, we shall get

$$\frac{d\,p}{d\,t} = m'\,\mathrm{M} \,.\, \frac{(p' - p)\,y\,x' - (q' - q)\,y\,y'}{h} = \frac{m'\mathrm{M}}{h}\,y\,\{(p' - p)\,x' - (q' - q)\,y'\}$$
$$\frac{d\,q}{d\,t} = m'\,\mathrm{M} \,.\, \frac{(p' - p)\,x\,x' - (q' - q)\,x\,y'}{h} = \frac{m'\mathrm{M}}{h}\,x\,\{(p' - p)\,x' - (q' - q)\,y'\}$$

But if we neglect the squares of the disturbing forces, and the eccentricities and inclinations of the orbits, we have $\dfrac{m'}{h} = \dfrac{m'}{\sqrt{(1 + m)\,(1 - e^2)\,.\,a}} = \dfrac{m'}{\sqrt{a}}$, so that these expressions become

$$\frac{d\,p}{d\,t} = m'\,\mathrm{M} \,.\, \frac{(p' - p)\,x' - (q' - q)\,y'}{\sqrt{a}}\,y \qquad (171)$$

$$\frac{d\,q}{d\,t} = m'\,\mathrm{M}\,.\,\frac{(p'-p)\,x' - (q'-q)\,y'}{\sqrt{a}}\,x \qquad (172)$$

Let us suppose the function $(a^2 - 2\,a\,a'\,.\cos w + a'^2)^{-\frac{3}{2}}$ developed in a series of cosines of w, and its multiples $\mathrm{S} + \mathrm{S}'\,.\cos w + \mathrm{S}''\,.\cos 2\,w + \&c.$
then we shall have, by (135)

$$\left.\begin{aligned} \mathrm{S} &= \frac{1}{a^3}\left\{1 + \left(\frac{3}{2}\right)^2 a^2 + \left(\frac{3\,.\,5}{2\,.\,4}\right)^2 a^4 + \left(\frac{3\,.\,5\,.\,7}{2\,.\,4\,.\,6}\right)^2 a^6 + \&c.\right\} \\ \mathrm{S}' &= \frac{2}{a^3}\left\{\frac{3}{2}\,a + \frac{3^2\,.\,5}{2^2\,.\,4}\,a^3 + \frac{3^2\,.\,5^2\,.\,7}{2^2\,.\,4^2\,.\,6}\,a^5 + \&c.\right\} \\ \mathrm{S}'' &= \frac{2}{a^3}\left\{\frac{3\,.\,5}{2\,.\,4}\,a^2 + \frac{3^2\,.\,5\,.\,7}{2^2\,.\,4\,.\,6}\,a^4 + \&c.\right\} \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots \quad (173)$$

and if we neglect in the present research, as is allowable, the eccentricities and inclinations, or suppose the orbits circular and in one plane,

$$\mathrm{M} = \frac{1}{a'^3} - \frac{1}{\lambda^3} = \left(\frac{1}{a'^3} - \mathrm{S}\right) - \mathrm{S}'\,.\cos(\theta' - \theta) - \mathrm{S}''\,.\cos 2\,(\theta' - \theta) - \&c. \qquad (174)$$

$$r = a,\ r' = a',\ \theta = n\,t + \epsilon,\ \theta' = n'\,t + \epsilon',\ \theta - \theta' = n\,t - n'\,t + \epsilon - \epsilon' = \mathrm{W}$$

$$\begin{aligned} y\{(p'-p)\,x' - (q'-q)\,y'\} &= a\,a'\,.\,\frac{q'-q}{2}\{\cos(\theta'+\theta) - \cos(\theta'-\theta)\}; \qquad (175) \\ &+ a\,a'\,.\,\frac{p'-p}{2}\{\sin(\theta'+\theta) - \sin(\theta'-\theta)\} \end{aligned}$$

$$\begin{aligned} x\{(p'-p)\,x' - (q'-q)\,y'\} &= a\,a'\,.\,\frac{p'-p}{2}\{\cos(\theta'+\theta) + \cos(\theta'-\theta)\}; \qquad (176) \\ &- a\,a'\,.\,\frac{q'-q}{2}\{\sin(\theta'+\theta) + \sin(\theta'-\theta)\} \end{aligned}$$

Each of these latter quantities is to be combined with M by multiplication, and in resolving each of the products of sines and cosines so originating into simple sines and cosines of sums and differences, it is obvious that constant terms will arise whenever similar terms are combined, by reason of the property $\cos \mathrm{A} \times \cos \mathrm{A} = \frac{1}{2} \cos 2\,\mathrm{A} + \frac{1}{2}$. Now the only argument common to both factors is $\theta' - \theta$; and, of course, the only terms in (174) and (175) from whose combination a constant term can originate, are $-\,\mathrm{S}'\,.\cos(\theta - \theta')$ and $-\,a\,a'\,.\,\frac{q'-q}{2}\,.\cos(\theta' - \theta)$. Consequently, if we reject all the periodical terms, and put $I = \frac{a\,a'}{4}\,\mathrm{S}'$, we have

$$\frac{d\,p}{d\,t} = \frac{m'}{\sqrt{a}}\,.\,I\,.\,(q' - q); \qquad (177,\ 1)$$

and similarly

$$-\frac{d\,q}{d\,t} = \frac{m'}{\sqrt{a}}\,.\,I\,(p' - p); \qquad (177,\ 2)$$

If we go through a process exactly analogous, so as to obtain differential equations for determining p' and q' relative to the orbit of m', we shall find them to be

$$\frac{d\,p'}{d\,t} = \frac{m}{\sqrt{a'}}\,I\,(q - q'); \qquad (177,\ 3)$$

$$-\frac{d\,q'}{d\,t} = \frac{m}{\sqrt{a'}}\,I\,(p - p'); \qquad (177,\ 4)$$

These four equations, being of the first order, and with constant co-efficients (for the secular variations of a, a', and therefore of S′ and of $\frac{a\,a'}{4} \times \mathrm{S}'$ or I, which are symmetrical functions of a, a', vanish,) are easily integrated. As they subsist simultaneously among the four variables p, p', q, q', we may integrate them all together, if we assume

$$p = \mathrm{A}\,.\sin(g\,t + k) \qquad p' = \mathrm{A}'\,.\sin(g\,t + k)$$
$$q = \mathrm{A}\,.\cos(g\,t + k) \qquad q' = \mathrm{A}'\,.\cos(g\,t + k)$$

for if we substitute these values, we find that the variable part divides off, and there remain the following equations of condition between the constants A, A′, and g,

$$g\,\mathrm{A} = \frac{m'}{\sqrt{a}}\,I\,(\mathrm{A}' \quad \mathrm{A}); \qquad (178,\ 1)$$

$$g\,A' = \frac{m}{\sqrt{a'}}\,I\,(A - A')\,; \tag{178, 2}$$

In these one of the constants A, A′, g, remains indeterminate. Let this be A, then eliminating A′, we get for determining g,

$$\left.\begin{aligned} g^2 + \left(\frac{m'}{\sqrt{a}} + \frac{m}{\sqrt{a'}}\right) I\,g = o \\ g = o, \text{ or } g = -\frac{m\sqrt{a} + m'\sqrt{a'}}{\sqrt{a\,a'}}\,.\,I \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{179, 1, 2}$$

Now it is evident, that if g and g' be two values of g which satisfy the equations of condition, then A . sin $(g\,t + k)$ and B . sin $(g'\,t + k')$ will each be satisfactory values of p, and so for the rest; consequently (the equations being linear) their sum will be so, and we have

$$\begin{cases} p = A\,.\,\sin\,(g\,t + k) + B\,.\,\sin k' & (180, 1) \\ q = A\,.\,\cos\,(g\,t + k) + B\,.\,\cos k' & (180, 2) \end{cases}$$

$$\begin{cases} p' = A'\,.\,\sin\,(g\,t + k) + B\,.\,\sin k' & (180, 3) \\ q' = A'\,.\,\cos\,(g\,t + k) + B\,.\,\cos k' & (180, 4) \end{cases}$$

where A, B, k, k', are four arbitrary constants, and

$$g = -I\,.\,\frac{m\sqrt{a} + m'\sqrt{a'}}{\sqrt{a\,a'}}\,;\quad A' = -\frac{m\sqrt{a}}{m'\sqrt{a'}}\,.\,A\,; \tag{181, 1, 2}$$

From these values of p and q, p' and q' it is easy to eliminate sin $(g\,t + k)$ and cos $(g\,t + k)$; for if we multiply (180, 1) by $m\sqrt{a}$, and (180, 3) by $m'\sqrt{a'}$, and add, noticing that $A\,.\,m\sqrt{a} + A'\,.\,m'\sqrt{a'} = o$ by reason of the relation between A and A′ (181, 2) we get

$$m\sqrt{a}\,.\,p + m'\sqrt{a'}\,.\,p' = (m\sqrt{a} + m'\sqrt{a'})\,B\,.\,\sin k' = \text{constant}\,; \tag{182}$$

and similarly,

$$m\sqrt{a}\,.\,q + m'\sqrt{a'}\,.\,q' = (m\sqrt{a} + m'\sqrt{a'})\,B\,.\,\cos k' = \text{const.}\,; \tag{183}$$

Moreover, we have

$$\tan\phi^2 = p^2 + q^2 = (A^2 + B^2) + 2\,AB\,.\,\cos\,(g\,t + k - k')\,; \tag{184, 1}$$

$$\tan\phi'^2 = p'^2 + q'^2 = (A'^2 + B^2) + 2\,A'B\,.\,\cos\,(g\,t + k - k')\,; \tag{184, 2}$$

Consequently,

$$m\sqrt{a}\,.\,\tan\phi^2 + m'\sqrt{a'}\,.\,\tan\phi'^2 = m\sqrt{a}\,(A^2 + B^2) + m'\sqrt{a'}\,(A'^2 + B^2) = \text{constant}\,; \tag{185}$$

The arbitrary constants A, B, k, k', may be determined in any particular case, either by comparing the general expressions for p, p', q, q', with their actual values at any assigned instant as derived from observation, or from these last derived equations; for since $p^2 + q^2 = \tan\phi^2$ and $\frac{p}{q} = \tan\omega$, we have

$$p = \tan\phi\,.\,\sin\omega\,;\quad q = \tan\phi\,.\,\cos\omega\,;\quad p' = \tan\phi'\,.\,\sin\omega'\,;\quad q' = \tan\phi'\,.\,\cos\omega'\,; \tag{186}$$

Now the equation (180) gives

$$(p' - p) = (A' - A)\,.\,\sin\,(g\,t + k)\,;\quad q' - q = (A' - A)\,.\,\cos\,(g\,t + k)$$

consequently, if we take $t = o$, or if we assume for our data, the elements ϕ, ϕ', ω, ω', as they were observed at the epoch or origin of the time t, we find

$$\tan k = \frac{p' - p}{q' - q} = \frac{\tan\phi'\,.\,\sin\omega' - \tan\phi\,.\,\sin\omega}{\tan\phi'\,.\,\cos\omega' - \tan\phi\,.\,\cos\omega}\,; \tag{187}$$

Hence A and A′ are found; for, the values of S, S′, &c. being known from equations (173), $I = \frac{a\,a'}{4}\,S'$ is also known; and since by (181, 2)

$$A' - A = -\frac{m\sqrt{a} + m'\sqrt{a'}}{m'\sqrt{a'}}\,.\,A \text{ we get } A = -\frac{m'\sqrt{a'}\,(p' - p)}{(m\sqrt{a} + m'\sqrt{a'})\,\sin k} \tag{188}$$

Again, if we divide (182) by (183), we find

$$\tan k' = \frac{m\sqrt{a}\,.\,\tan\phi\,.\,\sin\omega + m'\sqrt{a'}\,.\,\tan\phi'\,.\,\sin\omega'}{m\sqrt{a}\,.\,\tan\phi\,.\,\cos\omega + m'\sqrt{a'}\,.\,\tan\phi'\,.\,\cos\omega'}\,; \tag{189}$$

$$B = \frac{m\sqrt{a}\,.\,\tan\phi\,.\,\sin\omega + m'\sqrt{a'}\,.\,\tan\phi'\,.\,\sin\omega'}{(m\sqrt{a} + m'\sqrt{a})\,.\,\sin k'}\,; \tag{190}$$

These constants once computed, the laws, periods, and limits of the motions of the planes of both orbits are known. The period in which the inequalities recur is deducible at once from the value of g. If we express the time t in Julian years, n, n' represent the mean motions of the planets m, m', in one such year, in parts of a whole circumference, and $n^2 a^3 = 1$, that is, $\sqrt{a} = \frac{1}{n a}$, hence $g \left(= - I . \left(\frac{m}{\sqrt{a'}} + \frac{m'}{\sqrt{a'}} \right) \right)$ will be found as follows :—

$$g = - (m n' a' + m' n a) . I$$

and if we call T the whole period, $- g \mathrm{T} = 1$ circumference $= 1$; and

$$\mathrm{T} = - \frac{1}{g} = \frac{1}{(m n' a' + m' n a) . I}; \qquad (191)$$

where

$$I = \frac{a a'}{4} \mathrm{S}'$$

If n, n' be expressed in seconds, the numerator of T instead of being unity, must be $360 \times 60 \times 60'' = 2196000''$.

The limits of the variations of the inclinations are readily found; for it appears from the equations (184, 1, 2) that their maxima and minima occur when $g t + k - k' = o$ and 180°, and have for their corresponding values

$$\left.\begin{matrix} \mathrm{A} + \mathrm{B} \\ \mathrm{A}' + \mathrm{B} \end{matrix}\right\} \text{ and } \left.\begin{matrix} \mathrm{A} - \mathrm{B} \\ \mathrm{A}' - \mathrm{B} \end{matrix}\right\}; \dots\dots\dots\dots \qquad (192)$$

Now, it is obvious from the values of p and q, (180, 1, 2) that A and B are small quantities of the same order as p, q; so that the inclinations can never increase or diminish beyond certain very narrow limits. This follows too from the equation (185); for in the present state of our system, $\tan \phi$ and $\tan \phi'$ being extremely small, the sum $m \sqrt{a} . \tan \phi^2 + m' \sqrt{a'} . \tan \phi'^2$ is always a very minute quantity; and since $\sqrt{a}$ and $\sqrt{a'}$ must both be taken positively, (for $\sqrt{a} = \frac{1}{n a}$ and n is positive for both planets, the motions being both in the same direction) neither term separately can exceed the value of the constant; so that ϕ or ϕ' must remain for ever confined to a value not greatly different from what it now has, and the planes of the planetary orbits must keep for ever oscillating within very confined limits about their mean positions.

With regard to the nodes it is different. These are liable to great changes of place, and may even circulate for ever in one direction without returning. In fact, if we would determine the maxima and minima of their longitudes, we have only to put $d \omega = o$, $d \omega' = o$; the roots of which equations, if real, will indicate the stationary points; and, if imaginary, will shew that such points do not exist, or that the nodes circulate. Now we have

$$d \omega = \frac{d . \tan \omega}{1 + \tan \omega^2} = o, \text{ or } d . \tan \omega = o, \text{ or } \frac{d p}{p} = \frac{d q}{q}, \text{ or } q \frac{d p}{d t} - p \frac{d q}{d t} = o$$

in which equation, substituting for $\frac{d p}{d t}$ and $\frac{d q}{d t}$ their values in equations (177, 1, 2) we find

$$q q' - q^2 + p p' - p^2 = o, \text{ or } p p' + q q' = p^2 + q^2; \qquad (193)$$

in which, substituting for p, p', q, q', their values (180) we get ultimately

$$\mathrm{A} + \mathrm{B} . \cos (g t + k - k') = o \text{ which gives } \cos (g t + k - k') = - \frac{\mathrm{A}}{\mathrm{B}}; \qquad (194)$$

Hence, if $\mathrm{B} \mathrel{7} \mathrm{A}$ (no regard being had to the signs) this will correspond to a real value of t, and the node will then merely have a libratory motion, advancing and receding alternately; if $\mathrm{B} \angle \mathrm{A}$, they will circulate always in one direction. In the former case, if we substitute in the value of $\tan \phi^2$ (equation 184, 1) this value of $\cos (g t + k - k')$ we shall find $\tan \phi^2 = \mathrm{B}^2 - \mathrm{A}^2$; $\tan \phi = \sqrt{\mathrm{B}^2 - \mathrm{A}^2}$, which gives the inclination corresponding to the stationary points of the node. These points are attained when $\cos (g t + k - k') = - \frac{\mathrm{A}}{\mathrm{B}}$ while the maxima and minima of the inclinations happen when $\cos (g t + k - k') = \pm 1$. The stationary positions of the node therefore do not correspond either to the maxima and minima of the inclinations, or to the semi-intervals between them.

If we had considered more than two bodies the results would have been analogous, and we should have arrived at similar expressions for p and q only containing more terms, and analogous equations to those in 182, 183, and 185, viz.

$$\left.\begin{matrix} m \sqrt{a} . p + m' \sqrt{a'} . p' + m'' \sqrt{a''} . p'' + \&\text{c.} = \text{const.} \\ m \sqrt{a} . q + m' \sqrt{a'} . q' + m'' \sqrt{a''} . q'' + \&\text{c.} = \text{const.} \\ m \sqrt{a} . \tan \phi^2 + m' \sqrt{a'} . \tan \phi'^2 + m'' \sqrt{a''} . \tan \phi''^2 + \&\text{c.} = \text{const.} \end{matrix}\right\}, \dots\dots\dots \qquad (195)$$

Let us apply this theory to an example, and we will take that of the orbits of Jupiter and Saturn, the two principal planets of our system. If we take for our epoch the year 1700, we have, by Halley's tables,

$$\omega = 101° \ 5' \ 6''; \quad \phi = 2° \ 30' \ 10''; \quad a = 9{\cdot}54007$$

$$\omega' = 97° \ 34' \ 9''; \quad \phi' = 1° \ 19' \ 10'; \quad a' = 5{\cdot}20098$$

Hence we find the values of p, p', q, q', for that epoch, by the equations (186) as follows :—

$$p = 0{\cdot}04078, \quad q = - 0{\cdot}01573; \quad p' = 0{\cdot}02283, \quad q' = - 0{\cdot}00303$$

whence, having computed I, and assuming $m' = \frac{1}{1067}$ and $m = \frac{1}{3358}$, we shall find A, B, k, k' and g as follows:

$$B = 0{\cdot}02905\,; \quad A = 0{\cdot}01537\,; \quad k = 125^\circ\, 15'\, 40'', \quad k' = 103^\circ\, 38'\, 40''$$

and $A' = -\,0{\cdot}00661$, and finally $g = -\,25''{\cdot}5756$

Hence we obtain, in the case of Saturn,

$$\tan \phi = 0{\cdot}03287\,.\,\sqrt{1 + 0{\cdot}82665\,.\cos \left\{21^\circ\, 37' - t \times 25''{\cdot}5756\right\}}$$

and for Jupiter,

$$\tan \phi' = 0{\cdot}02980\,.\,\sqrt{1 - 0{\cdot}43290\,.\cos \left\{21^\circ\, 37' - t \times 25''{\cdot}5756\right\}}$$

Also we have $B + A = 0{\cdot}04442$; $B - A = 0{\cdot}01368$, so that the maxima and minima of the inclinations of Saturn's orbit are 2° 32′ 40″ and 0° 47′, and its greatest deviation from the mean state will not exceed 52′ 50″. In Jupiter's orbit the maximum is 2° 2′ 30″, and the minimum 1° 17′ 10″, and the greatest deviation from a mean state 0° 22′ 40″.

The longitude of the node, ω, has a maximum and a minimum in both orbits, because B 7 A; and the extent of its librations will be, in the case of Jupiter's orbit, 13° 9′ 40″, and in that of Saturn's, 31° 56′ 20″ on either side of its mean station, on the plane of the ecliptic supposed immoveable.

The period in which these changes take place, or the whole time in which the inclinations vary from their greatest to their least values, and the nodes from their greatest to their least longitudes, and back again, is equal to $\dfrac{360^\circ}{g} = \dfrac{360^\circ}{25''{\cdot}5756} = 50673$ Julian years; an immense period, and which may serve to give some idea of the extent to which the Newtonian theory, assisted by the refined methods of the modern analysis, enables us to carry our views of the past and future condition of our system; as this, though subject of course to some subordinate corrections, is perhaps one of the least uncertain of the results of perturbation.

Let us now consider the secular variations of the eccentricities and aphelia. Our first object, agreeable to the theory of the variation of the arbitrary constants already exposed, must be to obtain such an equation of the ellipse of curvature as shall be adapted to our purpose, by containing these elements (or convenient functions of them) in a state sufficiently disengaged from the variables x, y, z, and their differentials. Now if we differentiate the quantities $\frac{x}{r}$, $\frac{y}{r}$, $\frac{z}{r}$, noticing that $r^2 = x^2 + y^2 + z^2$, we get

$$d\,\frac{x}{r} = \frac{r\,d\,x - x\,d\,r}{r^2} = \frac{r^2\,d\,x - x\,.\,r\,d\,r}{r^3} = \frac{r^2\,d\,x - \frac{1}{2}\,x\,d\,(r^2)}{r^3}$$

$$= \frac{(x^2 + y^2 + z^2)\,d\,x - (x\,d\,x + y\,d\,y + z\,d\,z)\,.\,x}{r^3}$$

$$= \frac{y\,(y\,d\,x - x\,d\,y) + z\,(z\,d\,x - x\,d\,z)}{r^3}$$

That is, substituting for $y\,d\,x - x\,d\,y$ and $z\,d\,x - x\,d\,z$ their values $h\,d\,t$, $h'\,d\,t$,

$$\left.\begin{aligned} d\left(\frac{x}{r}\right) &= \frac{h\,y + h'\,z}{r^3}\,d\,t \\ \text{similarly,}\quad d\left(\frac{y}{r}\right) &= \frac{-\,h\,x + h''\,z}{r^3}\,d\,t \\ d\left(\frac{z}{r}\right) &= \frac{-\,h'\,x - h''\,y}{r^3}\,d\,t \end{aligned}\right\}\ \text{or}\ \left\{\begin{aligned} \mu\,d\left(\frac{x}{r}\right) &= \left(h\,\frac{\mu\,y}{r^3} + h'\,\frac{\mu\,z}{r^3}\right)d\,t \\ \mu\,d\left(\frac{y}{r}\right) &= \left(-\,h\,\frac{\mu\,x}{r^3} + h''\,\frac{\mu\,z}{r^3}\right)d\,t \\ \mu\,d\left(\frac{z}{r}\right) &= \left(-h'\,\frac{\mu\,x}{r^3} - h''\,\frac{\mu\,y}{r^3}\right)d\,t \end{aligned}\right.$$

Now, in the case of elliptic motion, $\frac{\mu\,x}{r^3} = -\,\frac{d^2\,x}{d\,t^2}$; $\frac{\mu\,y}{r^3} = -\,\frac{d^2\,y}{d\,t^2}$ and $\frac{\mu\,z}{r^3} = -\,\frac{d^2\,z}{d\,t^2}$, consequently substituting these, we obtain

$$\mu\,d\left(\frac{x}{r}\right) = d\left\{-\,h\,\frac{d\,y}{d\,t} - h'\,\frac{d\,z}{d\,t}\right\};\ \mu\,d\left(\frac{y}{r}\right) = d\left\{h\,\frac{d\,x}{d\,t} - h''\,\frac{d\,z}{d\,t}\right\}$$

$$\mu\,d\left(\frac{z}{r}\right) = d\left\{h'\,\frac{d\,x}{d\,t} + h''\,\frac{d\,y}{d\,t}\right\}$$

and integrating,

$$f = \mu\,\frac{x}{r} + \left(h\,\frac{d\,y}{d\,t} + h'\,\frac{d\,z}{d\,t}\right); \tag{196, 1}$$

$$f' = \mu\,\frac{y}{r} + \left(-\,h\,\frac{d\,x}{d\,t} + h''\,\frac{d\,z}{d\,t}\right); \tag{196, 2}$$

$$f'' = \mu\,\frac{z}{r} + \left(-\,h'\,\frac{d\,x}{d\,t} - h''\,\frac{d\,y}{d\,t}\right); \tag{196, 3}$$

These equations will serve our purpose, as the arbitrary constants f, f', f'', are completely disengaged; but before we proceed to employ them, we must determine the values of f, f', f'', in terms of the elements, and *vice versa*; and for this purpose must first eliminate the differential co-efficients, which (from the peculiar form of the equations) is practicable, by merely multiplying the first by x, the second by y, and the third by

z; and adding the results, when we get

$$\mu r - h\,\frac{y\,dx - x\,dy}{dt} - h'\,.\,\frac{z\,dx - x\,dz}{dt} - h''\,.\,\frac{z\,dy - y\,dz}{dt} = f x + f' y + f'' z$$

or, by reason of the equations (4) (5) (6),

$$\mu r = f x + f' y + f'' z + (h^2 + h'^2 + h''^2)\,; \tag{197}$$

This equation expresses the general property of the conic sections, in virtue of which a line drawn from the focus to the circumference is always in a given ratio to a perpendicular let fall from that point on the directrix.

If we multiply (196, 1) by h'', (196, 2) by $-h'$, and (196, 3) by h, and take the sum of the results, (observing that $h'' x - h' y + h z = o$) it will be found that all the variable terms will destroy each other, leaving simply the equation of condition,

$$h'' f - h' f' + h f'' = o \tag{198}$$

Suppose X, Y, Z, the co-ordinates of the perihelion. At this point $d r = o$, or $x\,dx + y\,dy + z\,dz = o$, but if for h, h', h'', we write their values in (4) (5) (6) we have

$$h\,\frac{dy}{dt} + h'\,\frac{dz}{dt} = -x\,.\,\frac{dy^2 + dz^2}{dt^2} + dx\,.\,\frac{y\,dy + z\,dz}{dt^2} \tag{199, 1}$$

$$-h\,\frac{dx}{dt} + h''\frac{dz}{dt} = -y\,\frac{dx^2 + dz^2}{dt^2} + dy\,.\,\frac{x\,dx + z\,dz}{dt^2} \tag{199, 2}$$

$$-h'\frac{dx}{dt} - h''\frac{dy}{dt} = -z\,\frac{dx^2 + dy^2}{dt^2} + z\,\frac{x\,dx + y\,dy}{dt^2} \tag{199, 3}$$

At the perihelion therefore, substituting for $y\,dy + z\,dz$, $x\,dx + z\,dz$, $x\,dx + y\,dy$, their equals $-\,\mathrm{X}\,d\mathrm{X}$, $-\,\mathrm{Y}\,d\mathrm{Y}$, and $-\,\mathrm{Z}\,d\mathrm{Z}$, respectively, these quantities become

$$-\mathrm{X}\,.\,\frac{d\mathrm{X}^2 + d\mathrm{Y}^2 + d\mathrm{Z}^2}{dt^2}$$

$$-\mathrm{Y}\,.\,\frac{d\mathrm{X}^2 + d\mathrm{Y}^2 + d\mathrm{Z}^2}{dt^2}$$

$$-\mathrm{Z}\,.\,\frac{d\mathrm{X}^2 + d\mathrm{Y}^2 + d\mathrm{Z}^2}{dt^2}$$

Consequently we have, putting $\dfrac{d\mathrm{X}^2 + d\mathrm{Y}^2 + d\mathrm{Z}^2}{dt^2} = \mathrm{V}^2$, $\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2 = \mathrm{R}^2$

$$f = \mathrm{X}\left(\frac{\mu}{\mathrm{R}} - \mathrm{V}^2\right);\quad f' = \mathrm{Y}\,.\left(\frac{\mu}{\mathrm{R}} - \mathrm{V}^2\right);\quad f'' = \mathrm{Z}\left(\frac{\mu}{\mathrm{R}} - \mathrm{V}^2\right); \tag{200}$$

Hence it is easy to obtain the following equation,

$$f^2 + f'^2 + f''^2 = \mathrm{R}^2\left(\frac{\mu}{\mathrm{R}} - \mathrm{V}^2\right)^2; \tag{201}$$

Now R is the perihelion distance, $\mathrm{R} = a\,(1 - e)$ and V being the velocity at the perihelion, we have, by (33) $\mathrm{V}^2 = \mu\left(\dfrac{2}{\mathrm{R}} - \dfrac{1}{a}\right)$ Hence we get

$$f^2 + f'^2 + f''^2 = \mathrm{R}^2\left(\frac{\mu}{a} - \frac{\mu}{\mathrm{R}}\right)^2 = \frac{\mu^2\,(\mathrm{R} - a)^2}{a^2} = \mu^2 e^2$$

$$\mu e = \sqrt{f^2 + f'^2 + f''^2}\,; \tag{202}$$

Moreover we have

$$\frac{f'}{f} = \frac{\mathrm{Y}}{\mathrm{X}},\quad \frac{f''}{f} = \frac{\mathrm{Z}}{\mathrm{X}};\quad \frac{f''}{\sqrt{f^2 + f'^2}} = \frac{\mathrm{Z}}{\sqrt{\mathrm{X}^2 + \mathrm{Y}^2}}; \tag{203}$$

But $\dfrac{\mathrm{Y}}{\mathrm{X}}$ is the tangent of the longitude of the perihelion, or of the angle which the projection of the perihelion distance makes with the axis of the x: also, π is the longitude of the perihelion reckoned on the orbit; and if its plane is but very little inclined to that of the x, y, this angle differs from its projection on that plane only by a very small quantity of the second order; so that if we disregard, as we have hitherto done in this research, the squares of the eccentricities, inclinations, and disturbing forces, we have

$$\tan \pi = \frac{f'}{f}\,; \tag{204}$$

These three equations, viz. $\tan \pi = \dfrac{f'}{f}$, $\mu e = \sqrt{f^2 + f'^2 + f''^2}$, and $h f'' - h' f' + h'' f = o$, determine f, f', f'', in terms of the elements of the orbit. The variations of the eccentricities and longitudes of the perihelia, e and π, may be immediately determined from those of f and f', so that it is to these we shall now turn our attention. Resuming then the equations (196, 1, 2, 3) let us differentiate them on the supposition of f, f', f'', being variable quantities, their variations being such as to express the effect of the disturbing forces only. To express this we have used the characteristic δ, and we shall continue to do so, to keep the principle on which the process is founded distinctly in view. No inconvenience can arise from the confusion of two symbols, δ and d, in the same investigation, when we bear in mind that any expression such as δf denotes, strictly, the whole amount of the momentary change which the quantity f undergoes while the planet passes from *one elementary portion* of its unknown curve to *that immediately consecutive to it*, while such expressions as $d\,x, d\,y$, &c., denote the changes which x, y, &c. undergo while it passes from one end to the other of *the same elementary portion*. In both points of view, the accumulated effects during a finite time are obtained by the same rules of integration; the whole variation of f is legitimately expressed by $\int \delta f$ just as that of x is by $\int d\,x$, and when δf is expressed in terms of $d\,t$, the integration must be performed in the usual manner, as on a function of t.

We have therefore, since $\delta\,\frac{x}{r} = o$, $\delta\,\frac{y}{r} = o$, $\delta\,\frac{z}{r} = o$

$$\delta f = \frac{d\,y}{d\,t}\,\delta\,h + \frac{d\,z}{d\,t}\,\delta\,h' + h\,\delta\,\frac{d\,y}{d\,t} + h'\,\delta\,\frac{d\,z}{d\,t}$$

Substituting therefore for $\delta\,h$, $\delta\,h'$ their values in the equations (168) for $\delta\,\frac{d\,y}{d\,t}$, $\delta\,\frac{d\,z}{d\,t}$, $-\,m'\,\frac{d\,\Omega}{d\,y}$ and $-\,m'\,\frac{d\,\Omega}{d\,z}$ respectively; and for h, h', their values $\frac{y\,d\,x - x\,d\,y}{d\,t}$ and $\frac{z\,d\,x - x\,d\,z}{d\,t}$, we get

$$\delta f = -\left\{\begin{array}{l} d\,y\left\{y\,\frac{d\,\Omega}{d\,x} - x\,\frac{d\,\Omega}{d\,y}\right\} + d\,z\left\{z\,\frac{d\,\Omega}{d\,x} - x\,\frac{d\,\Omega}{d\,z}\right\} \\ + (y\,d\,x - x\,d\,y)\,\frac{d\,\Omega}{d\,y} + (z\,d\,x - x\,d\,z)\,\frac{d\,\Omega}{d\,z} \end{array}\right\} \times m' \ldots\ldots\ldots\ldots (205, 1)$$

and similarly,

$$\delta f' = -\left\{\begin{array}{l} d\,x\left\{x\,\frac{d\,\Omega}{d\,y} - y\,\frac{d\,\Omega}{d\,x}\right\} + d\,z\left\{z\,\frac{d\,\Omega}{d\,y} - y\,\frac{d\,\Omega}{d\,z}\right\} \\ + (x\,d\,y - y\,d\,x)\,\frac{d\,\Omega}{d\,x} + (z\,d\,y - y\,d\,z)\,\frac{d\,\Omega}{d\,z} \end{array}\right\} \times m' \ldots\ldots\ldots\ldots (205, 2)$$

These are the rigorous values of δf, $\delta f'$, and their integrals regarded as functions of t, express the total variations of these elements produced in that time. But, as we have done in the less complicated theory of the nodes and inclinations we shall neglect, in developing them, all their periodical terms, at least such as depend on the configurations of the planets, as well as all terms containing the squares of the disturbing forces, eccentricities, and inclinations, and take the primitive orbit of m for our fixed plane. This will simplify them greatly; for in this case z, and $\frac{d\,\Omega}{d\,z}$ are quantities of the order of the disturbing forces, and therefore, when multiplied by m', may be rejected. Moreover, on this supposition, it is indifferent whether we reckon the longitudes on the orbit or on the fixed plane, since the difference is of the order z^2; consequently, representing that of m by θ, we have

$$x = r\,.\cos\theta,\; y = r\,.\sin\theta;$$

whence we find

$$x\,\frac{d\,\Omega}{d\,y} - y\,\frac{d\,\Omega}{d\,x} = (y'\,x - x'\,y)\,.\left(\frac{1}{r'^3} - \frac{1}{\lambda^3}\right)$$

$$= r\,r'\,.\sin\,(\theta' - \theta)\,.\left(\frac{1}{r'^3} - \frac{1}{\lambda^3}\right) = \frac{d\,\Omega}{d\,\theta}; \qquad (206)$$

$$\delta f = -\,d\,y\,\frac{d\,\Omega}{d\,\theta} - h\,\frac{d\,\Omega}{d\,y}\,d\,t$$

Now we have $\frac{d\,\Omega}{d\,y} = \frac{y'}{r'^3} - \frac{y' - y}{\lambda^3} = \frac{\sin\theta'}{r'^2} - \frac{r'\,.\sin\theta' - r\,.\sin\theta}{\lambda^3}$

But since $\lambda = \sqrt{r^2 - 2r\,r'\,.\cos\,(\theta - \theta') + r'^2}$

$$\Omega = \frac{r\,.\cos\,(\theta - \theta')}{r'^2} - \frac{1}{\lambda}$$

$$\frac{d\,\Omega}{d\,r} = \frac{\cos\,(\theta - \theta')}{r'^2} + \frac{r - r'\,.\cos\,(\theta - \theta')}{\lambda^3} \qquad (207)$$

$$\frac{d\,\Omega}{d\,\theta} = -\,\frac{r\,.\sin(\theta-\theta')}{r'^2} + \frac{r\,r'.\sin(\theta-\theta')}{\lambda^3} \qquad (208)$$

Consequently

$$\frac{d\,\Omega}{d\,y} = \frac{1}{r}\left\{r\,\frac{d\,\Omega}{d\,r}\,.\sin\theta + \frac{d\,\Omega}{d\,t}\,.\cos\theta\right\} \qquad (209)$$

and as we have $d\,y = d\,r\,.\sin\theta + r\,d\,\theta\,.\cos\theta$, and $h\,d\,t = r^2\,d\,\theta$,

$$\delta f = -\,\{d\,r\,.\sin\theta + 2\,r\,d\,\theta\,.\cos\theta\}\frac{d\,\Omega}{d\,\theta} - r^2\,d\,\theta\,\frac{d\,\Omega}{d\,r}\,.\sin\theta; \qquad (210)$$

In like manner we should have found

$$\delta f' = +\,\{d\,r\,.\cos\theta - 2\,r\,d\,\theta\,.\sin\theta\}\frac{d\,\Omega}{d\,\theta} + r^2\,d\,\theta\,\frac{d\,\Omega}{d\,r}\,.\cos\theta; \qquad (211)$$

Now, if we neglect the squares of the eccentricities, we have

$$r = a\{1 - e\,.\cos(\theta-\pi)\},\ d\,r = a\,e\,.\sin(\theta-\pi)\,d\,\theta,\ r^2\,d\,\theta = n\,a^2\,d\,t,$$

whence we have

$$d\,\theta = n\,d\,t\,(1 + 2\,e\,.\cos(\theta-\pi) \text{ and } d\,r = n\,a\,e\,d\,t\,.\sin(\theta-\pi)$$

and consequently,

$$\delta f = -\,n\,a\,d\,t\,.\frac{d\,\Omega}{d\,\theta}\{(2 - 2\,e\,.\cos\overline{\theta-\pi})(1 + 2\,e\,.\cos\overline{\theta-\pi})\,.\cos\theta + e\,.\sin\theta\,.\sin\overline{\theta-\pi}\} - n\,a^2\,d\,t\,.\frac{d\,\Omega}{d\,r};$$

which reduced, rejecting e^2, becomes

$$\delta f = -\,m'\,n\,d\,t\left\{a\,\frac{d\,\Omega}{d\,\theta}\left(2\cos\theta + \frac{3}{2}\,e\,.\cos\pi + \frac{e}{2}\,.\cos(2\,\theta-\pi)\right) + a^2\,\frac{d\,\Omega}{d\,r}\,.\sin\theta\right\} \qquad (212)$$

In order to find the secular part of the variation of f, we must develope this expression, retaining only such terms as are not periodical. Now, recurring to the notation of Sections 3, 4, 5, Part II. we have

$$\theta = n\,t + \epsilon + 2\,e\,.\sin V = V + \pi + 2\,e\,.\sin V$$

therefore

$$\sin\theta = \sin(V+\pi) + 2\,e\,.\sin V\,.\cos(V+\pi)$$
$$= \sin(V+\pi) + e\,.\sin(2\,V+\pi) - e\,.\sin\pi$$
$$2\cos\theta = 2\cos(V+\pi) - 4\,e\,.\sin V.\sin(V+\pi)$$
$$= 2\cos(V+\pi) + 2\,e\,\cos(2\,V+\pi) - 2\,e\,.\cos\pi$$
$$\frac{e}{2}\,.\cos(2\,\theta-\pi) = \frac{e}{2}\,.\cos(2\,V+\pi);\quad \frac{3\,e}{2}\,.\cos\pi = \frac{3\,e}{2}\,.\cos\pi$$

neglecting e^2 &c. Again we have

$$a^2\,\frac{d\,\Omega}{d\,r} = a^2\,\frac{d\,\Omega}{d\,a} + a^2\,\Delta\,\frac{d\,\Omega}{d\,r}$$

When this therefore is multiplied by sin θ, the term $-\,e\,.\sin\pi$ will combine with the constant term $a^2\,\frac{d\,A}{d\,a}$ of $a^2\,\frac{d\,\Omega}{d\,a}$, and produce $-\,a^2\,e\,.\frac{d\,A}{d\,a}\,.\sin\pi$, and this is the only constant term which can arise from $a^2\,\frac{d\,\Omega}{d\,a}$, because every other term contains W, and therefore both V and V′; while the value of sin θ contains only V, and, of course, V′ cannot be destroyed by their combination. Again, with respect to $\Delta\,\frac{d\,\Omega}{d\,r}$, it has been proved, that (regarding only the first powers of the eccentricities,) this is resolvable into a series of terms of the form $A\,.\cos(i\,W \pm V)$, $A\,.\cos(i\,W \pm V')$. Now, since $\Delta\,\frac{d\,\Omega}{d\,r}$ is of the order of the eccentricities, we may disregard the term of sin θ multiplied by e, and take it simply equal to $\sin(V+\pi)$. If we multiply this into the series $\Sigma\,A\,.\cos(i\,W \pm V)$ since $W = V - V' + \pi - \pi'$, only one term can produce a constant argument, viz. that into which V′ does not enter, or in which $i = o$; that is, the term multiplied by cos V. The co-efficient in this term in $\Delta\,\frac{d\,\Omega}{d\,r}$ is $-\,a\,e\,\frac{d^2\,A}{d\,a^2}$, (as appears from (147), writing $\frac{d\,\Omega}{d\,r}$ for Ω, and of course $\frac{d\,A}{d\,a}$, &c. for A, &c.) Hence the term so originating in $a^2\,\frac{d\,\Omega}{d\,r}\,.\sin\theta$ will be equal to the constant part of $-\,a^3\,e\,\frac{d^2\,A}{d\,a^2}\,.\cos V\,.\sin(V+\pi)$, that is, to $-\,\frac{a^3\,e}{2}\,.\,\frac{d^2\,A}{d\,a^2}\,.\sin\pi$. Again the multiplication of $\sin(V+\pi)$ with the series $\Sigma\,A\,.\cos(i\,W \pm V')$ will produce a constant term by the combination of $\sin(V+\pi)$ with $\cos(W+V') = \cos(V+\pi-\pi')$. The co-efficient of $\cos(W+V')$ in the developement of $\Delta\,\frac{d\,\Omega}{d\,r}$ is $-\,\frac{e'}{2}\left(a\,\frac{d^2\,A'}{d\,a\,d\,a'} + 2\,\frac{d\,A'}{d\,a}\right)$. Hence the constant part of $a^2\,\frac{d\,\Omega}{d\,r}\,.\sin\theta$ so originating, will be $-\frac{a^2 e'}{2}\left(a'\frac{d^2 A'}{d a\, d a'} + 2\frac{d A'}{d a}\right) \times$ (the constant part of $\sin (V + \pi) . \cos (V + \pi - \pi')$

$= - a^2 . \frac{e'}{4} \sin \pi' \left\{ 2\frac{d A'}{d a} + a'\frac{d^2 A'}{d a\, d a'}\right\}$

Let us next consider the constant terms which can originate from the part multiplied by $a\frac{d \Omega}{d \theta}$. This quantity is equal to $a\frac{d \Omega}{d w} . \frac{d w}{d \theta} = a\frac{d \Omega}{d w} = a\frac{d \Omega}{d W} + a\, \Delta\frac{d \Omega}{d w}$. But we have

$$\frac{d \Omega}{d W} = - A . \sin W - 2 A' . \sin 2 W - \&c.$$

and as all these contain V′, and this arc does not enter into any of the terms multiplying $\frac{d \Omega}{d \theta}$ in the expression of δf, it is obvious that no combination not containing V′, can arise in this way, *i. e.* no constant term can occur in the part of the developement depending on $\frac{d \Omega}{d W}$. With regard to $\Delta\frac{d \Omega}{d w}$ being already of the order of the eccentricities, it need only be combined with the first term, $2 \cos \theta$, and this term may be regarded as equal simply to $2 . \cos (V + \pi)$. We have only then to investigate the constant part of

$$2 a . \cos (V + \pi) \left\{ \Delta\frac{d \Omega}{d w} = \frac{d^2 \Omega}{d a\, d W}\Delta r + \frac{d^2 \Omega}{d a'\, d W}\Delta r' + \frac{d^2 \Omega}{d W^2}\Delta w \right\}$$

Now $\Delta r = - a \epsilon . \sin V$, $\Delta r' = - a' e' . \sin V'$, $\Delta w = 2 e . \sin V - 2 e' . \sin V'$

1st. The combination $\quad - 2 a^2 e . \cos V . \cos (V + \pi) . \Sigma - i\frac{d A^i}{d a'} \sin i W$

can contain no constant term, because the terms sin V and cos (V + π) cannot eliminate V′ from sin *i* W.

2dly. $\quad - 2 a a' e' . \cos (V + \pi) . \cos V' . \Sigma - i\frac{d A^i}{d a'} \sin i W$

will contain a constant term, for taking $i = 1$, $\cos V' \times \sin W = \frac{1}{2} \sin (W + V') + \frac{1}{2} \sin (W - V')$, the last term of which does not contain V′, being equal to $+ \frac{1}{2} \sin (V + \pi - \pi')$. The combination now under consideration, will therefore contain the term

$$+ a a' e' . 1 . \frac{d A'}{d a'} : \cos (V + \pi) . \sin (V + \pi - \pi')$$

which, developed, will produce a constant term,

$$- a a' . \frac{e}{2} . \frac{d A'}{d a'} . \sin \pi'$$

Lastly, the combination

$$4 a . \cos (V + \pi) \{e . \sin V - e' . \sin V'\} . \Sigma \quad i^2 . A^i . \cos i W$$

will produce a constant term; viz. that arising from

$$- 4 a e' . \cos (V + \pi) . \sin V' \times - 1^2 A' . \cos W$$

for sin V′ . cos W produces a term free from V′, viz. $\frac{1}{2} \sin (V + \pi - \pi')$ which again combined with cos (V + π) produces $\frac{-1}{4} \sin \pi'$; so that the complete result of this combination will be

$$- a e' . A' . \sin \pi'$$

Assembling therefore all the constant terms so found, we get

$$\delta f = m' . a n d t . e \sin \pi \left\{ a\frac{d A}{d a} + \frac{a^2}{2}\frac{d^2 A}{d a^2}\right\} \tag{213}$$
$$+ m' . a n d t . e' . \sin \pi' \left\{ A' + \frac{a}{2}\frac{d A'}{d a} + \frac{a'}{2}\frac{d A'}{d a'} + \frac{a a'}{4} . \frac{d^2 A'}{d a\, d a'}\right\}$$

The value of $\delta f'$ may be derived by a similar process, but without executing the whole process, we may obtain it more readily by considering the expressions (210) and (211) we then see that δf changes to $\delta f'$ simply by changing θ to $\theta - 90°$, but leaving r, $\frac{d \Omega}{d r}$ and $\frac{d \Omega}{d \theta}$ unchanged. Now Ω, $\frac{d \Omega}{d r}$ and $\frac{d \Omega}{d \theta}$ remain manifestly unaltered, whatever direction we assume as that from which we reckon the longitudes. Suppose then that we shift this direction 90°, then will each of the angles θ, θ', ϵ, ϵ', π, π', be diminished 90°, and this will produce no alteration in r, because $\theta - \pi$ will remain unchanged. Hence, the effect produced by this change on δf will transform it into $\delta f'$. The value of $\delta f'$ therefore will be had at once by changing sin π and sin π′ into − cos π, and − cos π′ in that of δf; and this will be found to be confirmed by the direct process.

Suppose now we denote the constant co-efficients within the brackets for brevity by $- I$ and K, so as to have

$$-I = a\frac{d\,A}{d\,a} + \frac{a^2}{2}\cdot\frac{d^2\,A}{d\,a^2} \qquad (214)$$

$$K = A' + \frac{a}{2}\;\frac{d\,A'}{d\,a} + \frac{a'}{2}\cdot\frac{d\,A'}{d\,a'} + \frac{a\,a'}{4}\cdot\frac{d^2\,A'}{d\,a\,d\,a'} \qquad (215)$$

Then it will be observed, that K is obviously symmetrical with respect to a, a', because the co-efficient A is so; and the same will be found to hold good with I; for if we write it as follows,—

$$I = -\frac{1}{2}\,\frac{d}{d\,a}\left\{a^2\,\frac{d\,A}{d\,a}\right\} \qquad (216)$$

and for A substitute its value given in (137, 1) it will be found that the execution of the operations here indicated will lead to precisely the same result as those denoted by $\frac{d}{d\,a'}\left\{a'^2\,\frac{d\,A}{d\,a'}\right\}$: consequently I and K are both symmetrical functions of a, and a', and do not alter when these elements are mutually transposed. With regard to I, it may be easily shewn to be identical with the function $\frac{a\,a'}{4}$ S′ which we have already denoted by the same symbol. To see this, we need only write for A and S′ their values in (136, 1) and (173, 2), and executing on the former series the operations indicated by $-\frac{1}{2}\cdot\frac{d}{d\,a}\left(a^2\frac{d}{d\,a}\right)$, (remembering that $\alpha = \frac{a'}{a}$, $\frac{d\,\alpha}{d\,a} = -\frac{\alpha}{a}$; $\frac{d^2\,\alpha}{d\,a^2} = +\frac{2\,\alpha}{a^2}$) and multiplying the latter by $\frac{a\,a'}{4}$, the result will be identical. We have then

$$\left.\begin{aligned}\delta f &= -m'.n\,a.I\,d\,t \times e.\sin\varpi + m'.n\,a.K\,d\,t \times e'\sin\varpi\\ \delta f' &= +m'.n\,a.I\,d\,t \times e\cos\varpi - m'.n\,a.K\,d\,t \times e'\cos\varpi'\end{aligned}\right\};\ldots\ldots\ldots\ldots(217)$$

Now, if we recur to the values of f, f', f'', we find

$$\frac{f'}{f} = \tan\varpi;\quad \frac{f'}{\sqrt{f^2 + f'^2}} = \sin\varpi,\quad f'' = \frac{h'f' - h''f}{h}$$

but h' and h'' are of the order of the disturbing forces when we take the plane of m's undisturbed orbit for our fixed plane; consequently, f'' is so, and therefore $\sqrt{f^2 + f'^2} = \sqrt{f^2 + f'^2 + f''^2} = \mu\,e$, neglecting the squares of the disturbing forces. Hence we have $f = \mu\,e.\cos\varpi$ and $f' = \mu\,e.\sin\varpi$.

In imitation of what we did in the theory of the nodes and inclinations, let q and p represent these quantities; then $d\,q$ and $d\,p$ being the momentary variations of q and p, will replace $\delta\,f$ and $\delta\,f'$; and if in like manner we put q' and p' for $\mu'\,e'.\cos\varpi'$ and $\mu'\,e'.\sin\varpi'$, the corresponding quantities in the orbit of m', we shall have (putting $\frac{1}{\sqrt{a}}$, $\frac{1}{\sqrt{a'}}$, for $n\,a$, $n'\,a'$; and supposing $\mu = 1$, and $\mu' = 1$, which is allowable since the squares of the disturbing forces are neglected, and the quantities under consideration are all of the first order of these forces,)

$$\left.\begin{aligned}+\frac{d\,p}{d\,t} &= \frac{m'}{\sqrt{a}}\left\{I\,q - K\,q'\right\}\\ -\frac{d\,q}{d\,t} &= \frac{m'}{\sqrt{a}}\left\{I\,p - K\,p'\right\}\\ +\frac{d\,p'}{d\,t} &= \frac{m}{\sqrt{a'}}\left\{I\,q' - K\,q\right\}\\ -\frac{d\,q'}{d\,t} &= \frac{m}{\sqrt{a'}}\left\{I\,p' - K\,p\right\}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (218,\ 1,\ 2,\ 3,\ 4)$$

Thus we see that the problem of the secular variations of the eccentricities and aphelia, depends, exactly as in the case of the nodes and inclinations, on the simultaneous integration of four differential equations of the first order with constant co-efficients. If we compare these, however, with those of (177), we shall observe that the former research is rather more complicated than the latter, by reason of the two co-efficients, I and K, which it involves, while the system of equations on which the nodes and inclinations depend, involves only the first. However, this circumstance does not render the integration more difficult; the same substitution succeeds, and the integrals have a form almost exactly similar; we have only to take

$$\left.\begin{aligned}p &= A.\sin(g\,t + k) + B.\sin(h\,t + l)\\ q &= A.\cos(g\,t + k) + B.\cos(h\,t + l)\\ p' &= A'.\sin(g\,t + k) + B'.\sin(h\,t + l)\\ q' &= A'.\cos(g\,t + k) + B'.\cos(h\,t + l)\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(219)$$

If these be substituted, the equations will be satisfied, provided the following equations hold good between the constants,

$$\left.\begin{aligned} g\,\mathrm{A} &= \frac{m'}{\sqrt{a'}}\,(I\mathrm{A} - K\mathrm{A}') \\ g\,\mathrm{A}' &= \frac{m}{\sqrt{a'}}\,(I\mathrm{A}' - K\mathrm{A}) \end{aligned}\right\} \text{ and } \left\{\begin{aligned} h\,\mathrm{B} &= \frac{m'}{\sqrt{a'}}\,(I\mathrm{B} - K\mathrm{B}') \\ h\,\mathrm{B}' &= \frac{m}{\sqrt{a'}}\,(I\mathrm{B}' - K\mathrm{B}) \end{aligned}\right.$$

It is obvious that the elimination of A′ from the first pair, and of B′ from the second pair, will lead to equations of exactly the same form for g and h, which are therefore the two roots of the quadratic,

$$g^2 - I\,g\,.\,\frac{m\sqrt{a} + m'\sqrt{a'}}{\sqrt{a\,a'}} + \frac{m\,m'}{\sqrt{a\,a'}}(I^2 - K^2) = o \qquad (220)$$

and g and h being found, and A, B, k, l, remaining arbitrary, A′ and B′ are easily found.

To adapt these values to any particular case, the general values of p, q, p', q', at any assigned epoch, must be made to coincide with the observed values of $e\,.\,\sin\pi$, $e\,.\,\cos\pi$, $e'\,.\,\sin\pi'$, and $e'.\,\cos\pi'$; which condition will furnish equations to determine all the arbitrary constants.

For example, in the case of Jupiter and Saturn, we shall find on computation $g = 21''{\cdot}9905$, $h = 3''{\cdot}5851$.

$\mathrm{A} = 0{\cdot}04877$; $\mathrm{B} = 0{\cdot}03532$; $\mathrm{A}' = -\,0{\cdot}01715$; $\mathrm{B}' = 0{\cdot}04321$; $k = 306^\circ\,34'\,40''$; $l = 210^\circ\,16'\,40''$

Now, since $p = e\,.\,\sin\pi$, and $q = e\,.\,\cos\pi$, we have

$$\tan\pi = \frac{p}{q}, \text{ and } e = \sqrt{p^2 + q^2} = \sqrt{\mathrm{A}^2 + \mathrm{B}^2 + 2\,\mathrm{AB}\cos\left\{(g-h)\,t + k - l\right\}}$$

which gives, by substituting the numbers already found,

$$e = 0{\cdot}06021\,.\,\sqrt{1 - 0{\cdot}95009 \times \cos\,(83^\circ\,42' - 18''{\cdot}4054 \times t)}$$

which is the eccentricity of Saturn's orbit; and similarly,

$$e' = 0{\cdot}04649\,.\,\sqrt{1 + 0{\cdot}68592\,.\,\cos\,(83^\circ\,42' - 18''{\cdot}4054 \times t)}$$

for that of Jupiter, after any number t of years since 1700.

The longitudes of their respective aphelia, will also be known by the formulæ

$$\tan\pi = \frac{\mathrm{A}\,.\,\sin\,(g\,t + k) + \mathrm{B}\,.\,\sin\,(h\,t + l)}{\mathrm{A}\,.\,\cos\,(g\,t + k) + \mathrm{B}\,.\,\cos\,(h\,t + l)}\,; \qquad (221)$$

$$\tan\pi' = \frac{\mathrm{A}'.\,\sin\,(g\,t + k) + \mathrm{B}'.\,\sin\,(h\,t + l)}{\mathrm{A}'.\,\cos\,(g\,t + k) + \mathrm{B}'.\,\cos\,(h\,t + l)}\,; \qquad (222)$$

and the maxima and minima of these, or their greatest deviations from their mean places will take place, when

$$g\,\mathrm{A}^2 + h\,\mathrm{B}^2 + \mathrm{AB}\,(g + h)\,.\cos\left\{(g-h)\,t + (k - l)\right\} = o$$

that is, when

$$\cos\left\{(g-h)\,t + (k-l)\right\} = -\,\frac{g\,\mathrm{A}^2 + h\,\mathrm{B}^2}{(g+h)\,\mathrm{AB}}\,; \qquad (223)$$

If this fraction be less than unity, the aphelia will librate, as in the case of the nodes about a mean position, if not, they move in one direction continually. In the case of Jupiter and Saturn now before us, $g\,\mathrm{A}^2 + h\,\mathrm{B}^2$ 7 $(g + h)$ AB, so that the aphelia go on for ever in one direction.

The period in which the eccentricities go through all their evolutions, and return to the same state, is represented by $\dfrac{360^\circ}{g-h} = \dfrac{360^\circ}{18''{\cdot}4054} = 70414$ Julian years.

The greatest and least values of the eccentricities are respectively A ± B and A′ ± B′, for the two planets; and in the case before us

$$\left.\begin{array}{l} \text{for Saturn, } \ldots\ldots\; 0{\cdot}08409 \text{ and } 0{\cdot}01345 \\ \text{and for Jupiter, } \ldots\ldots\; 0{\cdot}06036 \text{ and } 0{\cdot}02606 \end{array}\right\} \begin{array}{l} \text{the maximum of the one corresponding to the minimum of} \\ \text{the other planet.} \end{array}$$

Finally, we may derive relations between the eccentricities, masses, and semiaxes, similar to those obtained between the inclinations, &c. in equations (182, 183, &c.) for since $p^2 + q^2 = e^2$ it is easy to see that we must have

$$\begin{aligned} m\,.\,e^2\sqrt{a} + m'.\,e'^2\sqrt{a'} &= m\sqrt{a}\,(\mathrm{A}^2 + \mathrm{B}^2) + m'\sqrt{a'}\,(\mathrm{A}'^2 + \mathrm{B}'^2) \\ &\quad + 2\,(m\sqrt{a}\,.\,\mathrm{AB} + m'\sqrt{a'}\,.\,\mathrm{A'B'})\,.\cos\left\{(g-h)\,t + (k-l)\right\} \end{aligned} \qquad (224)$$

Now we have, from the equations of condition,

$$\left.\begin{aligned} \mathrm{A}\left(g - \frac{m'}{\sqrt{a}}\,.\,I\right) &= -\,\mathrm{A}'\,.\,\frac{m'}{\sqrt{a}}\,K \\ \mathrm{B}\left(h - \frac{m'}{\sqrt{a}}\,.\,I\right) &= -\,\mathrm{B}'\,.\,\frac{m'}{\sqrt{a}}\,K \end{aligned}\right\}; \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(225)$$

so that $$\mathrm{AB}\left(g - \frac{m'}{\sqrt{a}}\,I\right)\left(h \quad \frac{m'}{\sqrt{a}}\,I\right) = \mathrm{A'B'}\,.\,\frac{m'^2}{a}\,K^2$$

but g and h being the roots of (220), we have

$$g\,h = \frac{m\,m'}{\sqrt{a\,a'}}\,(I^2 - K^2) \qquad g + h = I\,.\,\frac{m\,\sqrt{a} + m'\,\sqrt{a'}}{\sqrt{a\,a'}}$$

Consequently, our equation, on executing the multiplications indicated, and substituting, reduces itself to

$$m\,\sqrt{a}\,.\,\mathrm{AB} + m'\,\sqrt{a'}\,.\,\mathrm{A'B'} = o$$

so that the equation (224) becomes simply

$$m\,e^2\,.\,\sqrt{a} + m'\,e'^2\,\sqrt{a'} = m\,\sqrt{a}\,(\mathrm{A}^2 + \mathrm{B}^2) + m'\,\sqrt{a'}\,(\mathrm{A}'^2 + \mathrm{B}'^2) = \text{constant}, \tag{226}$$

because the major axes are constant; and had we considered a greater number of planets than two, we should, in like manner, have arrived at the equation

$$m\,e^2\,\sqrt{a} + m'\,e'^2\,\sqrt{a'} + m''\,e''^2\,\sqrt{a''} + \&\text{c.} = \text{constant}; \tag{227}$$

PART III.

Of the Theory of the Moon.

Section I.

Rigorous investigation of the differential equations of the moon's motion, and general expression of the disturbing forces.

The extreme minuteness of the masses of the planets, as well as their great distance, renders it allowable in the theory of their perturbations to neglect altogether the squares of the disturbing forces, and affords such facilities to the whole investigation, as to permit us to express at once the true longitude and radius vector in explicit functions of the mean longitude or the time. This is not the case in the lunar theory, in which one of the most remarkable of the inequalities depends for nearly half its value on the square of the disturbing force, and in which the whole investigation is so delicate, as to render it necessary to abandon the direct process followed in the planetary theory and adopt a route, apparently more circuitous, but possessing advantages of a peculiar kind. It consists in expressing the radius vector and the time or *mean* longitude in functions of the *true*, and thence by the reversion of series deducing the true longitude in terms of the mean. The advantage of this is, that we are thus enabled to set out from differential equations in which nothing has been neglected, and consequently have it in our power fully to appreciate the influence of all the terms we may afterwards neglect in their integration on the final result.

Our directing principle in this investigation will be to follow out, step by step, with the proper modifications, the same system of transformations, by which the differential equations 1, 2, 3, of undisturbed motion were converted into (9) and (11) expressing the radius vector and time in terms of θ.

Our first step will consist in the investigation of an equation corresponding to (9): to this end (as we have supposed $d\,t$ constant) multiply the first of the equations (94) by y, and the second by $-\,x$, and add, and we get

$$o = \frac{y\,d^2x - x\,d^2y}{d\,t^2} + m'\,.\left\{y\,\frac{d\,\Omega}{d\,x} - x\,\frac{d\,\Omega}{d\,y}\right\}$$

and integrating,

$$\frac{y\,d\,x - x\,d\,y}{d\,t} = h + m'\!\int\!\left\{x\,\frac{d\,\Omega}{d\,y} - y\,\frac{d\,\Omega}{d\,x}\right\}d\,t; \tag{228}$$

but $y\,d\,x - x\,d\,y = \rho^2\,d\,\theta$, so that this equation becomes

$$\rho^2\,\frac{d\,\theta}{d\,t} = h + m'\!\int T\,\rho\,d\,t \tag{229}$$

if we assume

$$T = \frac{x\,\dfrac{d\,\Omega}{d\,y} - y\,\dfrac{d\,\Omega}{d\,x}}{\rho}. \tag{230}$$

The function T is the measure, and $m'\,.\,T$ the quantity, of what, in the lunar theory, is called the tangential force, or that part of the disturbing force acting on m, which is perpendicular to the direction of the radius vector. This is evident, if we consider that $m'\,\dfrac{d\,\Omega}{d\,y}$ and $m'\,\dfrac{d\,\Omega}{d\,x}$ representing the disturbing forces in the directions QP and PM (fig. A) if we resolve these each into the direction $K\,$Q perpendicular to QM they will become respectively Fig. A.

$$-\,m'\,\frac{x}{\rho}\,.\,\frac{d\,\Omega}{d\,y} \text{ and } +\,m'\,\frac{y}{\rho}\,.\,\frac{d\,\Omega}{d\,x}$$

so that their aggregate, or the whole *tangential force,* will have for its expression the quantity above represented by T in equation (230).

The equation (229), if m' were zero, would coincide with (9), and would express the proportionality of the areas to the times. The term $m' \int T \rho \, d t$ then expresses the momentary effect of the disturbing forces in deranging this proportionality; and since

$$\int \rho^2 \, d \theta = h \, (t + \text{const.}) + m' \int d t \int T \rho \, d t \qquad (231)$$

the term $m' \int d t \int T \rho \, d t$ expresses their total effect after the lapse of the time t; and we see that this effect takes place solely in virtue of the tangential force, agreeably to what Newton has advanced in the 11th section of the *Principia.*

Our next step is to investigate an equation for the disturbed motion, corresponding to (11) in the undisturbed. The process we followed in the deduction of (11), it will be remembered, consisted

1st. In making θ the independent variable.

2dly. In eliminating t and its differentials.

Now, if in the equations (94) instead of supposing $d t$ constant, we regard it as variable, we must change $\frac{d^2 x}{d t^2}$ into $\frac{1}{d t} d \frac{d x}{d t}$ or $\frac{d^2 x}{d t^2} - \frac{d x}{d t} \frac{d^2 t}{d t^2}$, and so for $\frac{d^2 y}{d t^2}$, &c. This done, let the first of the equations (94) be multiplied by x, and the second by y; their sum will be

$$o = \left\{ (x \, d^2 x + y \, d^2 y) - (x \, d x + y \, d y) \, . \frac{d^2 t}{d t} \right\} \frac{1}{d t^2}$$
$$+ \mu \, . \frac{x^2 + y^2}{r^3} + m' . \left\{ x \frac{d \Omega}{d x} + y \frac{d \Omega}{d y} \right\}$$

but since $x = \rho \, . \cos \theta$, and $y = \rho \, . \sin \theta$, $r = \sqrt{x^2 + y^2 + z^2}$
$= \sqrt{\rho^2 + z^2}$

this equation will become (supposing $d \theta$ constant),

$$o = \left\{ \rho \, d^2 \rho - \rho^2 \, d \theta^2 - \rho \, d \rho . \frac{d^2 t}{d t} \right\} \frac{1}{d t^2}$$
$$+ \mu \, . \frac{\rho^2}{(\rho^2 + z^2)^{\frac{3}{2}}} + m' \left\{ x \frac{d \Omega}{d x} + y \frac{d \Omega}{d y} \right\} \qquad (232)$$

We have only now to find the value of $\frac{d^2 t}{d t}$ on the supposition of $d \theta$ being invariable, and substitute it in this equation; and for this purpose we must employ our equation (229) which gives (multiplying it by $T \rho \, d t$)

$$T \rho^3 \, d \theta = h \, . \, T \rho \, d t + m' . \, T \rho \, d t \, . \int T \rho \, d t$$

and integrating,

$$\int T \rho^3 \, d \theta = h \int T \rho \, d t + \frac{m'}{2} \left(\int T \rho \, d t \right)^2$$

from which we get, by the solution of a quadratic equation,

$$m' . \int T \rho \, d t = - h + \sqrt{h^2 + 2 \, m' \int T \rho^3 \, d \theta}$$

and differentiating, and dividing by $m' \, T \rho$

$$d t = \frac{\rho^2 \, d \theta}{\sqrt{h^2 + 2 \, m' \int T \rho^3 \, d \theta}} \, ; \qquad (233)$$

If we take the logarithmic differential on both sides of this, supposing $d \theta$ constant, we get

$$\frac{d^2 t}{d t} = 2 \frac{d \rho}{\rho} \quad m' . \frac{T \rho^3 \, d \theta}{h^2 + 2 \, m' \int T \rho^3 \, d \theta} \, ; \qquad (234)$$

Substituting this then in equation (232), and for $\frac{1}{d t^2}$, writing its value $\frac{h^2}{\rho^4 \, d \theta^2} + 2 \, m' . \frac{\int T \rho^3 \, d \theta}{\rho^4 \, d \theta^2}$ given by (233), it becomes,

$$o = \frac{1}{\rho \, d \theta^2} \left\{ \frac{d^2 \rho}{\rho^2} - 2 \frac{d \rho^2}{\rho^3} - \frac{d \theta^2}{\rho} \right\} \left(h^2 + 2 \, m' \int T \rho^3 \, d \theta \right)$$
$$+ \frac{\mu}{\rho} . \frac{1}{\left(1 + \frac{z^2}{\rho^2} \right)^{\frac{3}{2}}} + m' . \left\{ \left(x \frac{d \Omega}{d x} + y \frac{d \Omega}{d y} \right) + 7 \, . \frac{d \rho}{d \theta} \right\}, \qquad (235)$$

In this equation let $\frac{z}{\rho} = s$, or $z = \rho\, s$, so that $s =$ tan heliocentric elevation of m above the plane of the x and y, $=$ tan m MQ fig. (A). Moreover, let

$$\mathrm{V} = \frac{x\,\frac{d\,\Omega}{d\,x} + y\,\frac{d\,\Omega}{d\,y}}{\rho}; \tag{236}$$

then will V be the measure, and m' . V the quantity, of that part of the disturbing force which operates to increase the gravity of m towards the central body. For if we resolve the forces $m' . \frac{d\,\Omega}{d\,x}$ and $m'\,\frac{d\,\Omega}{d\,y}$ which act in the directions of the co-ordinates x and y into the direction of the distance ρ, they will become respectively

$$+\,m' . \frac{x}{\rho} . \frac{d\,\Omega}{d\,x} \text{ and } +\,m' . \frac{y}{\rho} . \frac{d\,\Omega}{d\,y}$$

whose sum is m' . V.

These substitutions made, and ρ being put (as in the elliptic theory) equal to $\frac{1}{\mu}$, the equation will become when multiplied by $-\frac{\rho}{h^2}$,

$$o = \frac{d^2\,u}{d\,\theta^2} + u - \frac{\mu}{h^2} \times \frac{1}{(1+s^2)^{\frac{3}{2}}} \tag{237}$$

$$-\frac{m'}{h^2}\left\{\frac{\mathrm{V}}{u^2} - \frac{T}{u^3} . \frac{d\,u}{d\,\theta} - 2\left(\frac{d^2\,u}{d\,\theta^2} + u\right)\int\frac{T\,d\theta}{u^3}\right\};$$

which equation is rigorous, nothing having been neglected in the previous process. But if the squares of the disturbing forces be neglected, we have $\frac{1}{(1+s^2)^{\frac{3}{2}}} = 1$ because s or $\frac{z}{\rho}$ is of the order of the disturbing forces; and since, in virtue of this very equation,

$$m'\left(\frac{d^2\,u}{d\,\theta^2} + u\right) = m'\left\{\frac{\mu}{h^2} . \frac{1}{(1+s^2)^{\frac{3}{2}}} + m' \times \text{\&c.}\right\}$$

$$= m' . \frac{\mu}{h^2} + m'^2 \times \text{\&c.}$$

the equation becomes

$$o = \frac{d^2\,u}{d\,\theta^2} + u - \frac{\mu}{h^2} - \frac{m'}{h^2}\left\{\frac{\mathrm{V}}{u^2} - \frac{T}{u^3} . \frac{d\,u}{d\,\theta} - \frac{2\,\mu}{h^2}\int\frac{T\,d\,\theta}{u^3}\right\}; \tag{238}$$

Now $\mu = 1 + m$, and if we neglect the powers and products of the disturbing forces and the mass m, the equation becomes

$$o = \frac{d^2\,u}{d\,\theta^2} + u - \frac{\mu}{h^2} - \frac{m'}{h^2}\left\{\frac{\mathrm{V}}{u^2} - \frac{T}{u^3} . \frac{d\,u}{d\,\theta} - \frac{2}{h^2}\int\frac{T\,d\,\theta}{u^3}\right\}; \tag{239}$$

It is on this equation, (or the rigorous equation (237)) that the perturbation of the radius vector is made to depend in the lunar theory. It remains to derive an equation from which the perturbation in latitude can be obtained. For this purpose, all that is necessary will be to regard x and y as known, and to employ the third of our general equations (94) to obtain a value of z, or, which comes to the same, of s, since $z = \rho\, s$, and ρ is given by the previous theory.

Now, if in our equation (94) we change the independent variable from t to θ, it becomes

$$o = \frac{d^2\,z}{d\,t^2} - d\,z . \frac{d^2\,t}{d\,t^3} + \mu . \frac{z}{r^3} + m' . \frac{d\,\Omega}{d\,z}$$

but since

$$\left.\begin{aligned} \frac{1}{d\,t^2} &= \frac{h^2 + 2\,m'\int T\,\rho^3\,d\,\theta}{\rho^4\,d\,\theta^2} \\ \frac{d^2\,t}{d\,t^3} &= \frac{1}{d\,\theta}\left\{\frac{2\,d\,\rho}{\rho^5\,d\,\theta}\left(h^2 + 2\,m'\int T\,\rho^3\,d\,\theta\right) - m' . \frac{T}{\rho}\right\} \end{aligned}\right\}; \quad \ldots\ldots\ldots\ldots\ldots (240)$$

as appears from equations (233) and (234); if we substitute these, we shall get

$$o = \left\{h^2 + 2\,m'\int T\,\rho^3\,d\,\theta\right\}\left(\frac{1}{\rho} . \frac{d^2\,z}{d\,\theta^2} - \frac{2\,d\,\rho}{\rho^2\,d\,\theta}\,\frac{d\,z}{d\,\theta}\right)$$

$$+\,\rho^3\left\{\frac{\mu\,z}{r^3} + m' . \frac{d\,\Omega}{d\,z} + m' . \frac{T}{\rho} . \frac{d\,z}{d\,\theta}\right\}$$

Now we have $z = \rho\, s$, $d\,z = \rho\,d\,s + s\,d\,\rho$, $d^2\,z = \rho\,d^2 s + 2\,d\,\rho\,d\,s + s\,d^2\,\rho$, whence we find

$$\frac{1}{\rho} \cdot \frac{d^2 z}{d\,\theta^2} - \frac{2\,d\,\rho}{\rho^2\,d\,\theta} \cdot \frac{d\,z}{d\,\theta} = \frac{d^2 s}{d\,\theta^2} + \frac{\rho^2\,s}{\rho\,d\,\theta^2} \left\{ \frac{d^2 \rho}{\rho^2} - \frac{2\,d\,\rho^2}{\rho^3} \right\}$$

But, by (235) we have

$$(h^2 + 2\,m' \int T \rho^3\,d\,\theta) \times \frac{1}{\rho\,d\,\theta^2} \left\{ \frac{d^2 \rho}{\rho^2} - \frac{2\,d\,\rho^2}{\rho^3} \right\} =$$

$$= \frac{1}{\rho^2} (h^2 + 2 m' \int T \rho^3\,d\,\theta) - \frac{\mu}{\rho} \cdot \frac{\rho^3}{r^3} - m' \left\{ \rho\,\mathrm{V} + T . \frac{d\,\rho}{d\,\theta} \right\}$$

so that, by substitution we obtain,

$$o = (h^2 + 2\,m' \int T \rho^3 d\,\theta) \left(\frac{d^2 s}{d\,\theta^2} + s \right)$$

$$+ m'\,\rho^3 \left\{ \frac{d\,\Omega}{d\,z} - s\,\mathrm{V} \right\} + m'\,T\,\rho^2 \left\{ \frac{d\,z}{d\,\theta} - \frac{s\,d\,\rho}{d\,\theta} \right\}$$

but $d\,z = \rho\,d\,s + s\,d\,\rho$; so that $\frac{d\,z}{d\,\theta} - \frac{s\,d\,\rho}{d\,\theta} = \frac{\rho\,d\,s}{d\,\theta}$, and we get

$$o = \left\{ h^2 + 2\,m' \int T \rho^3\,d\,\theta \right\} \left(\frac{d^2 s}{d\,\theta^2} + s \right) + m'\,\rho^3 \left\{ T \frac{d\,s}{d\,\theta} - s\,\mathrm{V} + \frac{d\,\Omega}{d\,z} \right\}$$

this equation $\left(\text{writing } u \text{ for } \frac{1}{\rho} \right)$ becomes

$$o = \frac{d^2 s}{d\,\theta^2} + s + m' . \frac{T \frac{d\,s}{d\,\theta} - s\,\mathrm{V} + \frac{d\,\Omega}{d\,z}}{u^3 \left\{ h^2 + 2\,m' \int T \frac{d\,\theta}{u^3} \right\}} ; \ldots\ldots\ldots\ldots\ldots\ldots\ldots (250)$$

and when the square of the disturbing force is neglected,

$$o = \frac{d^2 s}{d\,\theta^2} + s + \frac{m'}{h^2\,u^3} \left\{ T \frac{d\,s}{d\,\theta} - s\,\mathrm{V} + \frac{d\,\Omega}{d\,z} \right\} \qquad (241)$$

Such are the fundamental equations on which the problem of the moon's motion depends. They were first investigated by Clairaut, in a piece which gained the prize of the Petersburgh Academy for 1750, and have been deduced by almost every writer since his time, as the groundwork of the lunar theory. The method we have followed shews how the expressions of the tangential and centripetal disturbing forces peculiar to this theory, originate in the general equations of the problem of these bodies; and thus connects the modern with the more ancient methods followed by Newton, Clairaut, &c.

Hitherto we have only considered the rigorous equations of the moon's motion. It remains to apply to them the methods of approximation appropriate to the case, and deduce in succession the equations of the disturbed motion. The great length to which the details of this complicated process would lead, will preclude our entering minutely into it. We must content ourselves with leading results and general principles. One great peculiarity of the lunar theory consists in this—that the mass of the disturbing body, instead of being, as in that of the planets, small in comparison with the disturbed and central bodies, exceeds them both several hundred thousand times. Its vast distance, however, makes up for this, and renders its disturbing effect small in comparison with the attraction of the central body. In the foregoing equations then the quantities m, and m', are not to be our guides in regulating the orders of the corrections. The very small quantities, whose powers determine the convergency of our approximations are T and V, or rather $m'\,T$, and $m'\,\mathrm{V}$, for m' never occurs unmultiplied by one or other of these quantities. Let us therefore, first of all, consider the nature and magnitude of these forces.

Expression of the centripetal disturbing force, $m'\,\mathrm{V}$.

$$m'\,\mathrm{V} = m' \left(\frac{x}{\rho} \frac{d\,\Omega}{d\,x} + \frac{y}{\rho} \frac{d\,\Omega}{d\,y} \right) = \frac{m'}{\rho} (x\,x' + y\,y') \left(\frac{1}{r'^3} - \frac{1}{\lambda^3} \right) + \frac{m'\,\rho}{\lambda^3}$$

$$= m'\,\rho' . \cos{(\theta - \theta')} \left\{ \frac{1}{r'^3} - \frac{1}{\lambda^3} \right\} + \frac{m'\,\rho}{\lambda^3}$$

Now, if we neglect the inclination of the moon's orbit, and take the plane of the ecliptic for our fixed plane, we have

$$z = o, \quad z' = o, \quad r = \rho, \quad r' = \rho', \quad s = o,$$

and

$$\frac{1}{\lambda^3} = \left\{ r'^2 \quad 2\,r\,r' . \cos{(\theta - \theta')} + r^2 \right\}^{-\frac{3}{2}}$$

$$= \frac{1}{r'^3}\left\{1 - \left(2\frac{r}{r'}\cos(\theta - \theta') - \left(\frac{r}{r'}\right)^2\right)\right\}^{-\frac{3}{2}}$$

$$= \frac{1}{r'^3}\left\{1 - \left(2\frac{r}{r'}\cos w - \left(\frac{r}{r'}\right)^2\right)\right\}^{-\frac{3}{2}}$$

$$= \frac{1}{r'^3}\left\{1 + \left(3\frac{r}{r'}\cos w - \frac{3}{2}\left(\frac{r}{r'}\right)^2\right) + \frac{3\,.\,5}{2\,.\,4}\left(2\frac{r}{r'}\,.\cos w - \left(\frac{r}{r'}\right)^2\right)^2 + \&\text{c.}\right\}$$

$$= \frac{1}{r'^3}\left\{1 + 3\frac{r}{r'}\cos w + \frac{9 + 15\,.\cos 2w}{4}\,.\left(\frac{r}{r'}\right)^2 + \&\text{c.}\right\}$$

putting $w = \theta - \theta'$, and developing in powers of $\frac{r}{r'}$. Hence, we obtain (writing r for ρ in the expression for V) after all reductions,

$$m'\,\mathrm{V} = -\frac{m'}{r'^2}\left\{\frac{1 + 3\cos 2w}{2}\,.\frac{r}{r'} + \frac{9\cos w + 15\cos 3w}{8}\left(\frac{r}{r'}\right)^2 + \&\text{c.}\right\} \qquad (242)$$

Now $\frac{m'}{r'^2}$ is the accelerating force exerted by the sun on the earth, whose intensity, as we have shewn in Section 1. Part I., is represented by 2·17399 (in its mean state) on the supposition of the earth's attraction on the moon being unity. Also, $\frac{r}{r'} = \frac{60\cdot23799}{23405} = \frac{1}{388}$ nearly, and $\left(\frac{r}{r'}\right)^2 = \frac{1}{119920}$; so that we may safely neglect $\left(\frac{r}{r'}\right)^2$ &c. The constant, or non-periodical parts of $m'\,\mathrm{V}$ is therefore equal to $-\frac{1}{2}\,.\frac{m'\,r}{r'^2} = -\frac{1}{2}\,m'\left(\frac{r}{r'}\right)^3 \times \frac{1}{r^2} = \text{nearly} - \frac{1}{357\cdot5} \times \frac{1}{r^2}$.

Let a and a' be the actual mean distances of the moon and sun from the earth, and supposing

$$\alpha = m'\left(\frac{a}{a'}\right)^3$$

α will be nearly $\frac{1}{178\cdot7}$, and the mean effect of the centripetal disturbing force will be $-\frac{\frac{1}{2}\alpha}{r^2}$ indicating a diminution of the moon's mean gravity, amounting to $\frac{1}{357}$th of its whole quantity.

The moon's mean gravity being diminished by the action of the sun, and the velocity remaining nearly the same, the centrifugal force which would be balanced by the centripetal in a circular orbit without the sun's action, will obtain the preponderance when the sun acts and carry the moon farther out; thus the disturbing force has the effect of increasing the distance and periodic time above what they would be in the undisturbed orbit.

Let us next consider the tangential disturbing force $m'\,T$. Now we have

$$m'\,T = m'\left(\frac{x}{\rho}\,\frac{d\,\Omega}{d\,y} - \frac{y}{\rho}\,\frac{d\,\Omega}{d\,x}\right)$$

$$= m'.\frac{x\,y' - y\,x'}{\rho}\left(\frac{1}{r'^3} - \frac{1}{\lambda^3}\right) = -\,m'\,r'\,.\sin w\left(\frac{1}{r'^3} - \frac{1}{\lambda^3}\right)$$

$$= +\frac{m'}{r'^2}\left(3\frac{r}{r'}\cos w + \left(\frac{r}{r'}\right)^2.\,\&\text{c.}\right).\sin w$$

that is

$$m'\,T = \frac{3}{2}\,\frac{m'}{r'^2}\,.\,\frac{r}{r'}\,.\sin 2w\,; \qquad (243)$$

The tangential force therefore vanishes both in syzigies and quadratures, and is a maximum in octants, or at 45° of elongation. Now we have seen, that the description of areas would be equable, were it not for this tangential force. Hence, at the former points, the equable description of areas still holds good; while, at the latter, an acceleration or retardation takes place. This, of course, produces an equation in the moon's longitude, whose period is semimenstrual, as during the first quarter of its synodic revolution, the moon is continually retarded in its orbit (supposed circular): at the first quadrature the momentary retardation ceases and changes to an acceleration—and here the effect accumulated through the whole of the quadrant has reached its maximum. In the next quadrant, the moon is gradually accelerated; and at the full moon, its motion has regained all it had lost, and is restored to its mean state, so that here the equation is nothing, and so on. This is the origin of that equation of the moon's motion to which astronomers have attached the name of the variation.

SECTION II.

Approximate integration of the equation of the moon's orbit.

In this inquiry we will commence with the equation (238), in which the square of the disturbing force is neglected. Now, the obvious mode of beginning the approximation, would be (in analogy with what we did in the theory of the planets) to substitute for u its undisturbed, or elliptic value, $\frac{1 + e \,.\, \cos \theta}{a\,(1 - e^2)}$. But, as it is obvious that the more nearly our first approximation approaches the truth, the more rapid will be the convergency of all the succeeding ones, we may take advantage of what observation has taught us respecting the form of the lunar orbit, and assume, for our first value of u, an expression representing that form, more nearly than the ellipse. Now, we learn from observation, that during a single revolution, it is true, the orbit does not deviate materially from an ellipse, but that in a very few revolutions the elliptic radius vector differs very sensibly from the real one, by reason of the rapid motion of the lunar apsides, which perform a whole circuit in little more than nine years. Instead then of representing by u the inverse radius vector of a *fixed* ellipse, we will give it such a form, in terms of θ, as shall express that of an ellipse in motion, and revolving in its own plane in the direction of the moon's motion, as observation informs us it does.

Let us then take c, such that $1 : 1 - c ::$ the moon's motion in longitude to the motion of the apsides, then will $\theta\,(1 - c)$ be the angle described by the apsis, while the moon describes θ; and, consequently, supposing (as we may) that the origin of the time t is fixed at the epoch when the apsis coincided with the axis of the x, we shall have $\theta - (1 - c)\,\theta = c\,\theta$ for the moon's true anomaly, and in the place of taking $u = \frac{1 + e \,.\, \cos \theta}{a\,(1 - e^2)}$, we may take $u = \frac{1 + e \,.\, \cos c\,\theta}{a\,(1 - e^2)}$ for our initial value of u.

Although the idea of commencing the approximation with this value of u is, it is true, taken from observation, yet, when once suggested, no matter from what source, we may look on it as a mere analytical artifice; and the truth or falsehood, convenience or inconvenience of the assumption will be tried by actual substitution; when, if we find that it reproduces the original expression with a train of small corrections multiplied by the disturbing force, or can be made to do so by a proper assumption of the constant c, we may rest assured that the assumption is (mathematically speaking) a legitimate one, and concern ourselves no farther with the way in which we arrived at it.

Now, neglecting e^2, we have

$$u = \frac{1}{a}\,(1 + e \,.\, \cos c\,\theta)$$

whence we get

$$\frac{1}{u^3} = a^3\,(1 - 3\,e \,.\, \cos c\,\theta)\,;\quad \frac{1}{u^4} = a^4\,(1 - 4\,e \,.\, \cos c\,\theta)$$

$$\frac{d\,u}{d\,\theta} = -\frac{c\,e}{a} \,.\, \sin c\,\theta$$

and, by substituting this in (238) which is equivalent to

$$o = \frac{d^2\,u}{d\,\theta^2} + u - \frac{1}{h^2} + m'\,\frac{u'^3}{h^2\,u^3} \,.\, \frac{1 + 3\cos 2\,w}{2} - \frac{3\,m'\,u'^3}{2\,h^2\,u^4} \,.\, \frac{c\,e}{a}\sin c\,\theta \,.\, \sin 2\,w$$
$$+ \frac{3}{h^4}\int m'\,\frac{u'^3}{u^4}\sin 2\,w \,.\, d\,\theta$$

Now, if we neglect the eccentricity of the earth's orbit, $u' = \frac{1}{a'}$, so that $\left(\text{recollecting that } m' \,.\, \left(\frac{a}{a'}\right)^3 = a = \frac{1}{179}\right)$

$$\begin{aligned} o = \frac{d^2\,u}{d\,\theta^2} + u - \frac{1}{h^2} &+ \frac{a}{2\,h^2}\,(1 + 3\cos 2\,w)\,(1 - 3\,e \,.\, \cos c\,\theta) \\ &- \frac{3\,c\,e}{2\,h^2}\,a \,.\, \sin c\,\theta \,.\, \sin 2\,w \\ &+ \frac{3\,a}{h^4} \,.\int (1 - 4\,e\cos c\,\theta)\sin 2\,w \,.\, d\,\theta \end{aligned} \qquad (244)$$

The only difficulty now in the way of integrating this, is to express $\cos 2\,w$ and $\sin 2\,w$ in terms of θ. Now we have $w = \theta - \theta'$; and since we neglect the eccentricity of the earth's orbit, θ' is proportional to the time, or $\theta' = n'\,t$; also we have θ by the equation, $h\,d\,t = r^2\,d\,\theta$, or

$$t = \frac{1}{h}\int\frac{d\,\theta}{u^2} = \frac{1}{h}\int d\,\theta\,(1 - 2\,e \,.\, \cos c\,\theta)\,a^2 = \frac{a^2}{h}\left(\theta - \frac{2\,e}{c} \,.\, \sin c\,\theta\right)$$

Hence, since $h = \sqrt{a}$ very nearly, and the motion of the apsides being slow in comparison with that of the moon itself, c is nearly unity, at least sufficiently so for a first approximation; consequently,

$$t = a^{\frac{3}{2}} (\theta - 2e . \sin c\,\theta)$$

or
$$n\,t = \theta - 2e . \sin c\,\theta$$

and eliminating t between this and the equation $\theta' = n'\,t$, we have

$$\frac{n}{n'}\,\theta' = \theta - 2e . \sin c\,\theta$$

and therefore,

$$\theta - \theta' = \theta\left(1 - \frac{n'}{n}\right) + 2e . \frac{n'}{n} . \sin c\,\theta$$

putting n and n' for the mean motions of the moon and earth. Hence we find, putting $\frac{n'}{n} = k$

$$\cos 2w = \cos 2(1-k)\,\theta - \sin 2(1-k)\,\theta . \sin\left\{4e . \frac{n'}{n} . \sin e\,\theta\right\}$$

$$= \cos 2(1-k)\,\theta + 2ke\left\{\cos(2-2k+c)\,\theta - \cos(2-2k-c)\,\theta\right\}; \qquad (245)$$

and similarly,

$$\sin 2w = \sin 2(1-k)\,\theta + 2ke\left\{\sin(2-2k+c)\,\theta - \sin(2-2k-e)\,\theta\right\} \qquad (246)$$

It only remains to substitute these values in (244) where it becomes, after reducing all the products of sines and cosines to simple multiples, and rejecting e^2,

$$o = \frac{d^2u}{d\theta^2} + u - \frac{1}{h^2}\left(1 - \frac{a}{2}\right) - \frac{3e}{2h^2}\,a . \cos c\,\theta \qquad (247)$$

$$+ a\left\{A . \cos(2-2k)\,\theta + Be . \cos(2-2k-e)\,\theta + Ce\cos(2-2k+c)\,\theta\right\}$$

where A, B, C, are co-efficients easily expressed in functions of k and c. If we integrate this, we get

$$u = \frac{1}{h^2}\left(1 - \frac{a}{2}\right) - \frac{3e}{2h^2} . \frac{a}{c^2 - 1} . \cos c\,\theta \qquad (248)$$

$$+ a\left\{A' . \cos(2-2k)\,\theta + B'e . \cos(2-2k-c)\,\theta + C' . e . \cos(2-2k+c)\,\theta\right\}$$

Let us now consider the nature of this result. We set out with $u = \frac{1}{a} + \frac{e}{a} . \cos c\,\theta$ for a first assumed value of u; and having substituted this, have deduced another value, which ought to be a nearer approximation, and which ought to coincide with the former, if $a = o$.

Now, the terms within the brackets being all multiplied by a, vanish as they should do when $a = o$, and the two first terms of this expression will coincide with the uncorrected value of u, provided we take

$$\frac{1}{h^2}\left(1 - \frac{a}{2}\right) = \frac{1}{a} \quad \text{or } h = \sqrt{a} . \sqrt{1 - \frac{a}{2}} = \sqrt{a}\left(1 - \frac{a}{4}\right)$$

and
$$\frac{3e}{2h^2} . \frac{a}{1 - c^2} = \frac{e}{a}, \quad \text{or } 1 - c^2 = \frac{3}{2}\,a . \frac{a}{h^2}$$

or
$$1 - c^2 = \frac{3}{2}\,a . \frac{a}{h^2} = \frac{3}{2}\,\frac{a}{1 - \frac{a}{2}} = \frac{3}{2}\,a + \frac{3}{4}\,a^2$$

$$1 - c = \frac{3}{4}\,a + \frac{21}{32}\,a^2, \quad \text{or } 1 - c = \frac{3}{4}\,a$$

neglecting a^2; that is, since $a = \frac{1}{178 \cdot 7}$, $c = 0{\cdot}99580$, and $1 - c = 0{\cdot}0042$. Hence, according to this view of the subject, the progression of the apsides in one revolution of the moon is $0{\cdot}0042 \times 360^\circ$, or about 1° 31′.

Now, it is remarkable that this progression of the lunar apsides, as determined by a first approximation, is only half the quantity actually observed; and this is the conclusion Newton had arrived at, where, after going through a process capable of being identified with that now under consideration, he remarks (in the 9th section of the *Principia,*) "Apsis lunæ est duplo velocior circiter." Every attempt to obtain a nearer coincidence, by taking into account the higher powers of the eccentricities, the inclination of the moon's orbit (which is pretty considerable, and materially modifies some of its inequalities) &c, failed; and this difficulty, which Newton evidently felt, though he had passed it in this apparently negligent manner, as being at that time beyond his reach, or deferred for farther consideration, became so great a stumbling-block in the way of succeeding geometers, as to shake their faith in the theory of gravity, till Clairaut shewed that it would vanish on pushing the process to a second approximation, and that, in fact, the value of c depends, for nearly half its quantity, on a term containing the square of the disturbing force.

To shew this, we have only to substitute the approximate value of u (248) back into the original equation

$$o = \frac{d^2u}{d\theta^2} + u - \frac{1}{h^2} + \left\{\frac{m'\,T}{h^2u^3}\,\frac{du}{d\theta} - \frac{m'\,V}{h^2u^2} + 2\left(\frac{d^2u}{d\theta^2} + u\right)\int\frac{m'\,T}{h^2u^3}\,d\theta\right\}$$

Supposing now we represent the first assumed value, or $\frac{1}{a}$ $(1 + e \,.\, \cos c\,\theta)$ by u and by $\delta\, u$ all the terms added by the first approximation, then will the value resulting from a second approximation be had by substituting $u + \delta\, u$ for u in all the part of the above equation within the brackets, and integrating. As we only require at present to know the terms introduced by this second approximation, let us call them $\delta^2 u$, and the value of $\delta^2 u$ will be had at once by integrating the equation

$$o = \frac{d^2\,\delta^2 u}{d\,\theta^2} + \delta^2 u + \frac{m'}{h^2}\left\{\begin{array}{l}\delta\,.\,\frac{T}{u^3}\,\frac{d\,u}{d\,\theta} - \delta\,\frac{V}{u^2} + 2\left(\frac{d^2\,\delta\, u}{d\,\theta^2} + \delta\, u\right)\int\frac{T}{u^3}\,d\,\theta \\ \qquad + 2\left(\frac{d^2 u}{d\,\theta^2} + u\right)\int\delta\,.\,\frac{T}{u^3}\,d\,\theta\end{array}\right\};\ldots\ldots..(249)$$

Now, since $h^2 = a$ very nearly, and $\alpha = m'\,.\frac{a^3}{a'^3}$, we have

$$\frac{m'}{h^2}\,\delta\left(\frac{T}{u^3}\,\frac{d\,u}{d\,\theta}\right) = \frac{3}{2}\,\alpha\,\delta\left\{\frac{\sin 2\,w}{(a\,u)^4}\,\frac{d\,u}{d\,\theta}\right\}$$

$$= \frac{3}{2}\,\alpha\left\{\frac{2\cos 2\,w}{(a\,u)^4}\,\frac{d\,u}{d\,\theta}\,\delta\, w + \frac{\sin 2\,w}{(a\,u)^4}\,.\,\frac{d\,\delta\, u}{d\,\theta} - 4\,a\,\frac{\sin 2\,w}{(a\,u)^5}\,\frac{d\,u}{d\,\theta}\,\delta\, u\right\}$$

$$-\frac{m'}{h^2}\,\delta\,\frac{V}{u^2} = -\frac{\alpha}{2\,a}\,\delta\,\frac{1 + 3\cos 3\,w}{(a\,u)^3} = \frac{3\,\alpha}{2\,a}\left\{\frac{2\sin 2\,w}{(a\,u)^3}\,\delta\, w + a\,.\,\frac{1 + 3\cos 2\,w}{(a\,u)^4}\,\delta\, u\right\}$$

$$\frac{m'\,T}{h^2\,u^3} = \frac{3\,\alpha}{2}\,.\,\frac{\sin 2\,w}{(a\,u)^4}$$

$$\int\delta\,\frac{m'\,T}{h^2\,u^3}\,d\,\theta = 3\,\alpha\int\left\{\frac{\cos 2\,w}{(a\,u)^4}\,\delta\, w - 2\,a\,\frac{\sin 2\,w}{(a\,u)^5}\,\delta\, u\right\}d\,\theta$$

Let these values be substituted in the general equation (249), and it becomes

$$\begin{aligned} o = \frac{d^2\,\delta^2 u}{d\,\theta^2} + \delta^2 u &+ \frac{3}{2}\,\alpha\left\{\frac{2\cos 2\,w}{(a\,u)^4}\,\frac{d\,u}{d\,\theta}\,\delta\, w + \frac{\sin 2\,w}{(a\,u)^4}\,\frac{d\,\delta\, u}{d\,\theta} - 4\,a\,\frac{\sin 2\,w}{(a\,u)^5}\,\frac{d\,u}{d\,\theta}\,\delta\, u\right\} \\ &+ \frac{3}{2}\,\alpha\left\{\frac{2\sin 2\,w}{a\,(a\,u)^3}\,\delta\, w + \frac{1 + 3\cos 2\,w}{(a\,u)^4}\,\delta\, u\right\} \qquad (250) \\ &+ 3\,\alpha\left(\frac{d^2\,\delta\, u}{d\,\theta^2} + \delta\, u\right)\int\frac{\sin 2\,w}{(a\,u)^4}\,d\,\theta \\ &+ 6\,\alpha\left(\frac{d^2 u}{d\,\theta^4} + u\right)\int\left\{\frac{\cos 2\,w}{(a\,u)^4}\,\delta\, w - 2\,a\,\frac{\sin 2\,w}{(a\,u)^5}\,\delta\, u\right\}d\,\theta \end{aligned}$$

To reduce this equation to the degree of approximation required, let us agree to neglect all terms which contain α^2 multiplied by any power of the eccentricity; or in that part of the approximation which depends on the square of the disturbing force, to regard the orbits as circular. We shall have then $u = \frac{1}{a}$ $a\,u = 1$, and supposing $\delta\, u = \frac{\alpha}{a}\,.\,\phi$, where ϕ is the sum of any number of terms of the form $A\,.\cos i\,\theta$, we have

$$\delta\, t = \frac{1}{h}\int\delta\,\frac{d\,\theta}{u^2} = \frac{-1}{h}\int\frac{2\,\delta\, u}{u^3}\,d\,\theta = -\,\frac{2\,a^2}{h}\,\alpha\int\phi\,d\,\theta$$

or since $n = \frac{1}{a^{\frac{3}{2}}} = \frac{h}{a^2}$

$$\delta\,.\,n\,t = -\,2\,\alpha\int\phi\,d\,\theta$$

consequently, since $w = \theta - \theta' = \theta - \frac{n'}{n}\,t = \theta - k\,.\,n\,t$, we get

$$w = \theta\,(1 - k), \quad \text{and } \delta\, w = -\,k\,.\,\delta\, n\,t = 2\,k\,\alpha\int\phi\,d\,\theta$$

We have also $\frac{d\,u}{d\,\theta} = o$ and $\frac{1}{(a\,u)^n} = 1$; so that the equation (250) reduces itself to

$$\begin{aligned} o = \frac{d^2\,\delta^2 u}{d\,\theta^2} + \delta^2 u &+ \frac{3\,\alpha^2}{2\,a}\sin 2\,w\,.\,\frac{d\,\phi}{d\,\theta} + \frac{6\,k\,\alpha^2}{a}\sin 2\,w\,.\int\phi\,d\,\theta + \frac{3\,\alpha^2}{2\,a}\,(1 + 3\cos 2\,w)\,.\,\phi \\ &+ \frac{3\,\alpha^2}{a}\left(\frac{d^2\,\phi}{d\,\theta^2} + \phi\right)\int d\,\theta\,.\sin 2\,w \qquad (251) \\ &+ \frac{12\,\alpha^2}{a}\int\left\{k\cos 2\,w\int\phi\,d\,\theta - \sin 2\,w\,.\,\phi\right\}d\,\theta \end{aligned}$$

Let us now, more particularly, consider any one of the terms to which ϕ consists; as, for instance, A . cos B θ. This gives

$$\phi = \mathrm{A} . \cos \mathrm{B}\,\theta, \quad \frac{d\,\phi}{d\,\theta} = -\,\mathrm{AB} . \sin \mathrm{B}\,\theta, \quad \frac{d^2\,\phi}{d\,\theta^2} = -\,\mathrm{AB}^2 . \cos \mathrm{B}\,\theta;\ \int \phi\, d\,\theta = \frac{\mathrm{A}}{\mathrm{B}} . \sin \mathrm{B}\,\theta, \ \&c.$$

and the equation will give, on substitution and reduction,

$$\begin{aligned} o = \frac{d^2\,\delta^2\,u}{d\,\theta^2} + \delta^2\,u &+ \left\{\frac{6\,k\,a^2}{a} . \frac{\mathrm{A}}{\mathrm{B}} - \frac{3\,a^2}{2\,a} . \mathrm{A\,B}\right\} \sin 2\,w . \sin \mathrm{B}\,\theta \\ &+ \left\{\frac{3\,a^2}{2\,a}\,(1 + 3 . \cos 2\,w) + (\mathrm{B}^2 - 1) . \frac{3\,a^2}{a} . \frac{\cos 2\,w}{2\,(1-k)}\right\} \mathrm{A} . \cos \mathrm{B}\,\theta \\ &- \frac{6\,\mathrm{A}\,a^2}{a}\left\{\frac{k - \mathrm{B}}{\mathrm{B}\,(\mathrm{B} + 2 - 2\,k)} \cos (\mathrm{B} + 2 - 2\,k)\,\theta + \frac{k + \mathrm{B}}{\mathrm{B}\,(\mathrm{B} - 2 + 2\,k)} \cos (\mathrm{B} - 2 + 2\,k)\,\theta\right\} \end{aligned}$$

or resolving the products of sines and cosines by the formula (A), page 690,

$$\begin{aligned} o = \frac{d^2\,\delta^2\,u}{d\,\theta^2} + \delta^2 u &+ \frac{3\,\mathrm{A}\,a^2}{2\,a}\left(\frac{2k}{\mathrm{B}} - \frac{\mathrm{B}}{2}\right)\left\{\cos (\mathrm{B} - 2 + 2\,k)\,\theta - \cos (\mathrm{B} + 2 - 2\,k)\,\theta\right\} \\ &+ \frac{3\,\mathrm{A}\,a^2}{2\,a} . \cos \mathrm{B}\,\theta \\ &+ \frac{3\,\mathrm{A}\,a^2}{4\,a}\left\{3 + \frac{\mathrm{B}^2 - 1}{1 - k}\right\}\left\{\cos (\mathrm{B} - 2 + 2\,k)\,\theta + \cos (\mathrm{B} + 2 - 2\,k)\,\theta\right\} \\ &- \frac{6\,\mathrm{A}\,a^2}{a}\left\{\frac{k + \mathrm{B}}{\mathrm{B}\,(\mathrm{B} + 2 - 2\,k)} . \cos (\mathrm{B} + 2 - 2\,k)\,\theta + \frac{k + \mathrm{B}}{\mathrm{B}\,(\mathrm{B} - 2 + 2\,k)} \cos (\mathrm{B} - 2 + 2\,k)\,\theta\right\} \end{aligned}$$

That is, assembling together the co-efficients of like terms,

$$\begin{aligned} o = \frac{d^2\,\delta^2\,u}{d\,\theta^2} + \delta^2\,u &+ \frac{3\,\mathrm{A}\,a^2}{a}\left\{\frac{k}{\mathrm{B}} - \frac{\mathrm{B}}{4} + \frac{3}{4} + \frac{\mathrm{B}^2 - 1}{4\,(1-k)} - \frac{2\,(k + \mathrm{B})}{\mathrm{B}\,(\mathrm{B} - 2 + 2\,k)}\right\} \cos (\mathrm{B} - 2 + 2\,k)\,\theta \\ &+ \frac{3\,\mathrm{A}\,a^2}{a}\left\{-\frac{k}{\mathrm{B}} + \frac{\mathrm{B}}{4} + \frac{3}{4} + \frac{\mathrm{B}^2 - 1}{4\,(1-k)} - \frac{2\,(k - \mathrm{B})}{\mathrm{B}\,(\mathrm{B} + 2 - 2\,k)}\right\} \cos (\mathrm{B} + 2 - 2\,k)\,\theta \\ &+ \frac{3\,\mathrm{A}\,a^2}{2\,a} . \cos \mathrm{B}\,\theta \end{aligned}$$

Which equation, integrated, gives

$$\delta^2\,u = \frac{3\,\mathrm{A}\,a^2}{2a} . \left\{\begin{aligned} &\frac{\cos \mathrm{B}\,\theta}{\mathrm{B}^2 - 1} + \left(\frac{2k}{\mathrm{B}} + \frac{3 - \mathrm{B}}{2} + \frac{\mathrm{B}^2 - 1}{2\,(1-k)} - \frac{4\,(k + \mathrm{B})}{\mathrm{B}\,(\mathrm{B} - 2 + 2\,k)}\right)\frac{\cos (\mathrm{B} - 2 + 2k)\,\theta}{(\mathrm{B} - 2 + 2\,k)^2 - 1} \\ &+ \left(-\frac{2k}{\mathrm{B}} + \frac{3 + \mathrm{B}}{2} + \frac{\mathrm{B}^2 - 1}{2\,(1-k)} - \frac{4\,(k - \mathrm{B})}{\mathrm{B}\,(\mathrm{B} + 2 - 2\,k)}\right)\frac{\cos (\mathrm{B} + 2 - 2k)\,\theta}{(\mathrm{B} + 2 - 2\,k)^2 - 1} \end{aligned}\right\} \quad (252)$$

Such is the value of that part of $\delta^2\,u$, which originates in the term A . cos B θ in the value of ϕ, or in the term $\frac{a}{a}$. A cos B θ in u itself. A similar set of terms, having for their arguments respectively,

$$\mathrm{B}'\,\theta, \quad (\mathrm{B}' - 2 + 2\,k)\,\theta. \quad (\mathrm{B}' + 2 - 2\,k)\,\theta$$

and

$$\mathrm{B}''\,\theta, \quad (\mathrm{B}'' - 2 + 2\,k)\,\theta, \quad (\mathrm{B}'' + 2 - 2\,k)\,\theta; \ \&c.$$

will arise from the other terms $\frac{a}{a}\,\mathrm{A}' . \cos \mathrm{B}'\,\theta$, $\frac{a}{a}\,\mathrm{A}'' . \cos \mathrm{B}''\,\theta$, &c. Consequently, by substituting for B, B′, B″, &c. the several values which they actually have in the first approximate value of u (equation 248), or $2 - 2\,k$, $2 - 2\,k - c$, $2 - 2\,k + c$, we shall obtain the arguments of the several terms in the second approximation, and so on. These are then as follows :—

From $c\,\theta$ arise the arguments $c\,\theta$, $2\,c\,\theta$

From $(2 - 2\,k)\,\theta$ arise the arguments $(2 - 2\,k)\,\theta$; $c\,\theta$; $(4 - 4\,k)\,\theta$

From $(2 - 2\,k - c)\,\theta$ arise $(2 - 2\,k - c)\,\theta$; $c\,\theta$; $(4 - 4\,k - c)\,\theta$

From $(2 - 2\,k + c)\,\theta$ arise $(2 - 2\,k + c)\,\theta$; $c\,\theta$; $(4 - 4\,k + c)\,\theta$

Thus we see, 1st. that each term which in the expression of u is multiplied by the first power of the disturbing force a, reproduces itself on a second approximation, multiplied by the square of a, and thus the co-efficient of every term in the general expression of u, consists in fact of an infinite series of powers of a; for what takes place at the second approximation will, of course, do so at every succeeding step; and, in fact, the very same analysis, and the same resulting formula, may be applied to deduce the terms depending on a^3, a^4, &c., in all of which the same arguments will occur.

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Hence, it follows, that if we set out, as a first hypothesis, with $u = \frac{1}{a}(1 + e \,.\, \cos c\,\theta)$ the process of approximation leads us to a value of u, of the form

$$\begin{aligned} u = {} & \frac{1}{a}\{A + B\,a + C\,a^2 + D\,a^3 + \&c.\} - \frac{1}{h^2} \\ & + \cos c\,\theta\,\{A'\,a + B'\,a^2 + C'\,a^3 + \&c.\} \\ & + \cos(2 - 2\,k)\,\theta\,.\,\{A'\,a + B''\,a^2 + \&c.\} \\ & + \cos(4 - 4\,k)\,\theta\,.\,\{B'''\,a^2 + \&c.\} \\ & + \cos(2 - 2\,k + c)\,\{A^{\text{iv}}\,a + B^{\text{iv}}\,a^2 + \&c.\} \\ & + \cos(2 - 2\,k - c)\,\{A^{\text{v}}\,a + B^{\text{v}}\,a^2 + \&c.\} \\ & + \&c. \end{aligned}$$

We have therefore only to take care that this shall not contradict the original hypothesis, and therefore we must have

$$1 = A + B\,a + C\,a^2 + \&c. - \frac{a}{h^2} \tag{253}$$

$$\frac{e}{a} = A'\,a + B'\,a^2 + C'\,a^3 + \&c. \tag{254}$$

These equations established, the two first terms of u agree with the original value, and the remaining ones are the equations due to the effect of perturbation, not capable of being expressed by the hypothesis of a revolving ellipse.

The equation (253) expresses a relation between a and h, which we shall consider presently; the other (254) gives the value of $1 - c$, or the velocity of the motion of the lunar apogee. We have already seen (248) that the value of A' is $\frac{3\,e}{2\,h^2\,(1 - c^2)}$, and if we break off the equation (254) at the first power of a we have also seen that the resulting value of $1 - c$ is $\frac{3}{4}\,a$, amounting to only half the value given by observation.

If we make B successively equal to $2 - 2\,k - c$ and $2 - 2\,k + c$ in (252), we find for the co-efficients of the argument $c\,\theta$, respectively,

$$\frac{1}{c^2 - 1}\,.\,\frac{3\,a^2\,e}{2\,a}B'\left\{\frac{2 + 6\,k + c - 4\,k^2 - 4\,k\,c - c^2}{2\,(2 - 2\,k - c)} + \frac{(2 - 2\,k - c)^2 - 1}{2\,(1 - k)} + \frac{4\,(2 - k - c)}{c\,(2 - 2\,k - c)}\right\}$$

and

$$\frac{1}{c^2 - 1}\,.\,\frac{3\,a^2\,e}{2\,a}C'\left\{\frac{2 + 6\,k - c - 4\,k^2 + 4\,k\,c - c^2}{2\,(2 - 2\,k + c)} + \frac{(2 - 2\,k + c)^2 - 1}{2\,(1 - k)} - \frac{4\,(2 - k + c)}{c\,(2 - 2\,k + c)}\right\}$$

when B' and C' are the co-efficients of $e\,.\,\cos(2 - 2\,k - c)\,\theta$ and $e\,.\,\cos(2 - 2\,k + c)\,\theta$ within the parentheses of equation (248).

Thus our equation (254) carried as far as a^2, will become

$$1 - c^2 = \frac{3}{2}a + \frac{3}{2}a^2.\left\{\begin{aligned} & B'\left(\frac{2 - 6\,k + c - 4\,k^2 - 4\,k\,c - c^2}{2\,(2 - 2\,k - c)} + \frac{(2 - 2\,k - c)^2 - 1}{2\,(1 - k)} + \frac{4\,(2 - k - e)}{c\,(2 - 2\,k - c)}\right) \\ + {} & C'\left(\frac{2 + 6\,k - c - 4\,k^2 + 4\,k\,c - c^2}{2\,(2 - 2\,k + c)} + \frac{(2 - 2\,k + c)^2 - 1}{2\,(1 - k)} - \frac{4\,(2 - k + c)}{c\,(2 - 2\,k + c)}\right) \end{aligned}\right\} \tag{255}$$

This equation is to be resolved by approximation, so as to find c; and as a is a very small quantity, the approximation may be performed easily; for we have only to rescind a^2, and take $c = 1 - \frac{3}{4}a$ for a first approximation $= 0{\cdot}9957$, and then substitute this for c in the co-efficient of a^2, which will then become a given number (B', C', being previously computed, and k being known,) after which, if this co-efficient be called β, we have

$$1 - c^2 = \frac{3}{2}(a + \beta\,a^2) \quad c = \sqrt{1 - \frac{3}{2}(a + \beta\,a^2)} = 1 - \frac{3}{4}a - \frac{6\,\beta + 1}{8}a^2$$

and, again substituting this for c, we find a nearer approximation, and so on.

Now, it is a remarkable circumstance, that when this process is executed in numbers, the term $-\frac{6\,\beta + 1}{8}a^2$ thus added to the value of c is found to be very nearly equal to the term $-\frac{3}{4}a$ introduced by the first step of the approximation; and on pushing it still farther, the subsequent corrections are found to be inconsiderable. Now the velocity of the apogee of the lunar orbit is expressed by $1 - c$, or by $\frac{3}{4}a + \frac{6\,\beta + 1}{8}a^2 + \&c.$ We have already seen that the value of this, as deduced from a first approximation (or the part $\frac{3}{4}a$) came out only half what observation has assigned as its real amount, and that this was regarded as a great difficulty in the way of the full admission of the Newtonian law of gravity. We now see the reason of this; the remaining half is accounted for by the term $\frac{6\beta+1}{8}a^2$ arising from a second approximation; and the fact, so far from being an objection against gravity, is thus converted into a most cogent argument in its favour—its effects being thus shewn to correspond not merely to general results and first approximations, but to the refinements and niceties of theory.

SECTION III.

Expression of the moon's mean longitude in terms of the true, and vice versâ; and of the variation and evection of the moon.

THE purposes of astronomy require us to know the moon's true longitude and latitude at any assigned instant, or to know the angle θ in terms of the time t. For this we must have recourse to the equation.

$$t=\int\frac{\rho^2\,d\theta}{\sqrt{h^2+2m'\int T\rho^3\,d\theta}}=\int\frac{d\theta}{u^2\sqrt{h^2+2m'\int\frac{T}{u^3}\,d\theta}}$$

If we develope this, and retain only the first power of the disturbing force, we get

$$t=\frac{1}{h}\int\frac{d\theta}{u^2}\left\{1-\frac{m'}{h^2}\int\frac{T}{u^3}\,d\theta\right\}=\frac{1}{h}\int\frac{d\theta}{u^2}\left\{1-\frac{m'}{a}\int\frac{T}{u^3}\,d\theta\right\}$$

because $h^2=a$, if the disturbing force and square of the eccentricity be neglected. Now, in this for u let us substitute

$$u=\frac{1}{a}.(1+e.\cos c\theta)+a\{A'.\cos(2-2k)\theta+B'e.\cos(2-2k-c)\theta+C'e.\cos(2-2k+c)\theta\}$$

and for $\frac{m'}{h^2}\int\frac{T}{u^3}\,d\theta$ its value $\frac{3a}{2}\int\frac{\sin 2w}{(au)^4}\,d\theta$

$$=\frac{3a}{2}\int\sin 2w\,(1-4e.\cos c\theta)\,d\theta$$

$$=\frac{3a}{2}\int\{\sin(2-2k)\theta-(2-2k)e\sin(2-2k+c)\theta-(2+2k)e\sin(2-2k-c)\theta\}\,d\theta$$

$$=\frac{3a}{2}\left\{\frac{\cos(2-2k)\theta}{2(k-1)}-\frac{2(k-1)e}{2-2k+c}\cos(2-2k+c)\theta+\frac{2(k+1)e}{2-2k-c}\cos(2-2k-c)\theta\right\}$$

whence we find

$$\frac{ht}{a^2}=\int d\theta\left\{\begin{array}{l}1-2e.\cos c\theta-2aa\{A'.\cos(2-2k)\theta+B'e.\cos(2-2k-c)\theta+C'e\cos(2-2k+c)\theta\}\\ \qquad +3A'a.ae\{\cos(2-2k+c)\theta+\cos(2-2k-c)\theta\}\\ -\frac{3a}{2}(1-2e\cos c\theta)\left(\frac{\cos 2(1-k)\theta}{2(k-1)}-\frac{2(k-1)e}{2-2k+c}\cos(2-2k+c)\theta+\frac{2(k+1)e}{2-2k-c}\cos(2-2k-c)\theta\right)\end{array}\right\}\tag{256}$$

or putting $\frac{h^2}{a^2}=n$

$$nt=\theta-\frac{2e}{c}.\sin c\theta+Pa.\sin(2-2k)\theta+Qae.\sin(2-2k+c)\theta+Rae.\sin(2-2k-c)\theta\Big\}\ldots\tag{257}$$

A′, B′, C′, being as in (248)
where

$$\left.\begin{array}{l}P=-\frac{aA'}{1-k}+\frac{3}{8(1-k)^2}\\ Q=\frac{a(A'-2C')}{2-2k+c}+\frac{3}{4(k-1)(2-2k+c)}+\frac{3(k-1)}{(2-2k+c)^2}\\ R=\frac{a(A'-2B')}{2-2k-c}+\frac{3}{4(k-1)(2-2k-c)}-\frac{3(k+1)}{(2-2k-c)^2}\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\tag{258}$$

In order to have the value of θ in terms of nt, we must revert this series, and the simplest mode of accomplishing this will be to follow the steps by which we resolved the equation $nt=v-e.\sin v$ in the elliptic theory (vide page 656, col. 1, equation 26). Putting it therefore into the form

$$\theta = n\,t + \frac{2\,e}{c} . \sin c\,\theta - \mathrm{P}\,a \sin (2 - 2\,k)\,\theta \qquad (259)$$
$$- \mathrm{Q}\,a\,e \,.\, \sin (2 - 2\,k + c)\,\theta$$
$$- \mathrm{R}\,a\,e \,.\, \sin (2 - 2\,k - c)\,\theta$$

$n\,t$ appears to be the first approximate value of θ, neglecting e and a. Let this be substituted in the whole of the second member, and we get

$$\theta = n\,t + \frac{2\,e}{c} . \sin c\,n\,t - \mathrm{P}\,a \,.\, \sin (2 - 2\,k)\,n\,t \qquad (260)$$
$$- \mathrm{Q}\,a\,e \,.\, \sin (2 - 2\,k + c)\,n\,t$$
$$- \mathrm{R}\,a\,e \,.\, \sin (2 - 2\,k - c)\,n\,t$$

for a second approximation carried as far as the first powers of e and a. If we now substitute this value in (259) the terms multiplied by e and a only will of course remain; but in addition to those multiplied by $a\,e$, others will be introduced; and if we reject e^3 and a^2, and retain only $a\,e$, these terms will be, (putting $c = 1$)

$$- 2\,\mathrm{P}\,a\,e \,.\, \sin (2 - 2\,k)\,n\,t \,.\, \cos c\,n\,t$$
$$- 2\,\mathrm{P}\,a\,e\,(1 - k) \,.\, \cos (2 - 2\,k)\,n\,t \,.\, \sin c\,n\,t$$

whose sum, after the usual reductions, is

$$- \mathrm{P}\,(2 - k)\,a\,e \,.\, \sin (2 - 2\,k + c)\,n\,t - \mathrm{P}\,k\,a\,e \,.\, \sin (2 - 2\,k - c)\,n\,t$$

Consequently, if we push the approximation only to such terms, we get

$$\theta = n\,t + \frac{2\,e}{c} . \sin c\,n\,t - \mathrm{P}\,a \,.\, \sin (2 - 2\,k)\,n\,t \qquad (261)$$
$$- \left\{\mathrm{Q} + (2 - k)\,\mathrm{P}\right\} a\,e \,.\, \sin (2 - 2\,k + c)\,n\,t$$
$$- \left\{\mathrm{R} + k\,\mathrm{P}\right\} a\,e \,.\, \sin (2 - 2\,k - c)\,n\,t$$

The *variation* of the moon is that inequality which is represented by the term $- \mathrm{P}\,a \,.\, \sin (2 - 2\,k)\,n\,t$. The numerical magnitude of the co-efficient being considerable, (2146″) it, (as well as the inequality represented by the term, whose argument is $(2 - 2\,k - c)\,n\,t$, which is called the *evection*, and whose co-efficient is still greater (4830″) was long observed before its cause was known. If we observe that $n\,t$ being the mean longitude of the moon $k\,n\,t$ will be that of the sun, and if we put ☽ for the former, and $\odot$ for the latter, the argument of the variation will be 2 ☽ $- 2\,\odot$, or simply ☽ $- \odot$, and its *period*, half a synodic revolution of the moon $= 14^{\mathrm{d}}{\cdot}765$. The argument of the evection is in like manner

$$2\,☽ - 2\,\odot - c$$

where c is the longitude of the moon's perigee, and its period is nearly that of the variation, a little longer, however, on account of the progression of the perigee during a revolution of the moon. The term depending on the argument $(2 - 2\,k + c)\,n\,t$, or 2 ☽ $- 2\,\odot + c$, is found on calculation to have its co-efficient very small, and has no particular name.

In the theory we have now investigated we have been only anxious to simplify and curtail as much as possible the developements, it being far beyond our design to enter into minutiæ on so complicated a problem. It must suffice to have shewn in rather more than a general way the origin of the principal inequalities, and traced them from the differential equations to their final expression in the value of the longitude. In so doing, we have purposely neglected the inclination of the moon's orbit, and its effect both in modifying the disturbing forces which act in the plane of the orbit, and in altering the longitudes by reducing them from their actual values to their projections on the plane of the ecliptic, or by what is called in astronomy, the "Reduction to the ecliptic." The effect of the inclination, however, will be briefly touched on in the next section.

SECTION IV

Of the effect of the inclination of the plane of the moon's orbit.—Of the motion of the nodes, and the precession of the equinoxes.

IN determining the effect of the inclination of the lunar orbit to the plane of the ecliptic and the motion of its nodes, we shall suppose the orbit circular, and neglect the square of the disturbing force, the cube and higher power of the inclination, and the product of the disturbing force and inclination. Thus our equation (237) will become simply,

$$o = \frac{d^2 u}{d\,\theta^2} + u - \frac{1}{h^2\,(1 + s^2)^{\frac{3}{2}}} - \frac{m'}{h^2} \left\{ \frac{\mathrm{V}}{u^2} - \frac{2}{h^2} \int \frac{T\,d\,\theta}{u^3} \right\}; \qquad (262)$$

The value of the part depending on the disturbing forces $m'\,\mathrm{V}$ and $m'\,T$ in this equation remains unaltered, if we neglect $a\,s^2$; and therefore the part of u arising from perturbation is not altered when we only push the approximation to the extent now proposed. The only new terms arising in the value of u from the introduction of the inclination, are those introduced by the term $\frac{1}{h^2(1+s^2)^{\frac{3}{2}}}$; and to obtain them, we have only to integrate the equation

$$\frac{d^2 u}{d\,\theta^2} + u - \frac{1}{h^2}\left(1 - \frac{3}{2}s^2\right) = o$$

neglecting s^4. Now, in this we must put for s its undisturbed value, which is given by the equation $\frac{d^2 s}{d\,\theta^2} + s = o$ (making $m' = o$, in equation 240) and which is

$$s = \gamma\,.\sin(\theta - \mathrm{N}) \tag{263}$$

where γ is the tangent of the inclination, and N the longitude of the ascending node. This gives $s^2 = \gamma^2 . \frac{1 - \cos 2(\theta - \mathrm{N})}{2}$

$$\frac{d^2 u}{d\,\theta^2} + u - \frac{1}{h^2}\left(1 - \frac{3\,\gamma^2}{4}\right) - \frac{3\,\gamma^2}{4}.\cos 2(\theta - \mathrm{N}) = o$$

which integrated, gives

$$u = \frac{1}{h^2} - \frac{3\,\gamma^2}{4\,h^2} - \frac{\gamma^2}{4\,h^2}\cos 2(\theta - \mathrm{N}) \tag{264}$$

Thus we see that the effect of the inclination is to add to

$$u = \frac{1}{h^2} \text{ or } u = \frac{1}{a} \text{ the terms } -\frac{\gamma^2}{4\,h^2}(3 + \cos 2(\theta - \mathrm{N}))$$

Now, in approximating to u, we assumed in the foregoing pages an expression for a first value derived not *a priori* from theory, but from the *observed* motion of the lunar apsis. It is equally a matter of observation, that the node of the lunar orbit does not remain fixed, but is continually retreating on the ecliptic, in a direction contrary to the motion of the moon in its orbit. Let us then introduce this modification into our expression for u above found, and taking for the origin of the time, the moment when the node coincided with the axis of the x, we have only to write $g\,\theta$ in place of $\theta - \mathrm{N}$, when we get

$$u = \frac{1}{h^2}\left\{1 - \frac{3}{4}\gamma^2 - \frac{\gamma^2}{4}.\cos 2\,g\,\theta\right\} \tag{265}$$

Let us next investigate the effect of the inclination on that part of u which arises from perturbation. Now this may be done at once, by merely taking in 250, 251,

$$\delta\,u = \frac{a}{a}\phi = -\frac{1}{a}\left(\frac{3}{4}\gamma^2 + \frac{1}{4}\gamma^2\cos 2\,g\,\theta\right)$$

or

$$\phi = -\frac{3}{4}\frac{\gamma^2}{a}.\cos o\,.\,\theta \qquad \frac{1}{4}\frac{\gamma^2}{a}\cos 2\,g\,\theta$$

that is, supposing in 252, 1st. $\mathrm{A} = -\frac{3\,\gamma^2}{4\,a}$, $\mathrm{B} = o$, and 2dly, $\mathrm{A} = -\frac{\gamma^2}{4\,a}$ and $\mathrm{B} = 2\,g$

The first supposition gives

$$\delta^2\,u = \frac{9}{8\,a}\,a\,\gamma^2\left\{1 - \left(3 - \frac{1}{1-k}\right)\frac{\cos(2 - 2\,k)\,\theta}{(2 - 2\,k)^2 - 1}\right\} \tag{266}$$

and the second in like manner introduces into the value of u the new arguments $\cos(2\,g - 2 + 2\,k)\,\theta$ and $\cos(2\,g + 2 - 2\,k)\,\theta$ with co-efficients affected with $a\,\gamma^2$.

These arguments existing in the expression of u will in like manner be introduced into the value of t or $\int\frac{d\,\theta}{h\,u^2}$ and thence, by the process of reversion before explained, will arise terms of the form $a\,\gamma^2.\sin(2\,g - 2 + 2\,k)\,n\,t$ and $a\,\gamma^2.\sin(2\,g + 2 - 2\,k)\,n\,t$ in the longitude, of which this mention must suffice, as their co-efficients may easily be calculated from the principles above laid down, and nothing very remarkable in the lunar theory depends on them. Other terms also will arise from the part of (250) multiplied by $\frac{d\,u}{d\,\theta}$ which in this case is not $= o$ as was supposed in (251) but is equal to $\frac{g\,\gamma^2}{2\,h^2}\sin 2\,g\,\theta$.

Let us now examine the manner in which the disturbing forces affect the moon's latitude. For this purpose we must take the equation (241) in which writing for s the expression $\gamma\,.\sin(g\,v - \mathrm{N})$, for $\frac{d\,s}{d\,\theta}$ its value $g\,\gamma\,.\cos(g\,v - \mathrm{N})$ for $m'\,T$ and $m'\,\mathrm{V}$ their values $\frac{3}{2}\frac{a}{a^2}\sin 2(1 - k)\,\theta$ and $-\frac{1}{2}\frac{a}{a^2}(1 + 3\cos 2(1 - k)\,\theta)$

and for $m' \frac{d\,\Omega}{d\,z}$ its value $\frac{m'\,z}{\lambda^3} = \frac{m'\,\rho\,s}{\lambda^3} = \frac{m'\,a\,s}{\lambda^3} = \frac{m'\,a\,s}{a'^3}\left(1 + 3\,\frac{a}{a'}\cos w\right)$ (see page 718, line 4,) or, neglecting the product $s \times \frac{a}{a'}$, simply

$$m'\,\frac{d\,\Omega}{d\,z} = \frac{m'\,a\,s}{a'^3} = \frac{a}{a^2}\,s$$

we shall find

$$o = \frac{d^2\,s}{d\,\theta^2} + s + \frac{3}{2}\,a\,\gamma\,.\,g\,\sin 2\,(1-k)\cos\theta\,(g\,\theta - \mathrm{N})$$

$$+ \frac{a}{2}\,s\,(1 + 3\cos 2\,(1-k)\,\theta) + a\,s$$

and resolving the products of sines and cosines into simple sines,

$$o = \frac{d^2\,s}{d\,\theta^2} + s + \frac{3\,a}{4}\,a\,\gamma\,.\left\{\sin\left(\overline{2 - 2\,k + g}\,.\,\theta - \mathrm{N}\right) + \sin\left(\overline{2 - 2\,k - g}\,.\,\theta + \mathrm{N}\right)\right\}$$

$$+ \frac{3}{2}\,a\,.\,\gamma\sin(g\,\theta - \mathrm{N}) + \frac{3}{4}\,a\,\gamma\left\{\sin\left(\overline{2 - 2\,k + g}\,.\,\theta - \mathrm{N}\right) - \sin\left(\overline{2 - 2\,k - g}\,.\,\theta + \mathrm{N}\right)\right\}$$

or

$$o = \frac{d^2\,s}{d\,\theta^2} + s + \frac{3}{2}\,a\,\gamma\,.\sin(g\,\theta - \mathrm{N}) \tag{267}$$

$$+ \frac{3}{4}\,a\,\gamma\,(1+g)\,.\sin\left(\overline{2 - 2\,k + g}\,.\,\theta - \mathrm{N}\right)$$

$$- \frac{3}{4}\,a\,\gamma\,(1-g)\,.\sin\left(\overline{2 - 2\,k - g}\,.\,\theta + \mathrm{N}\right)$$

The integration of this gives

$$s = \frac{3\,a\,\gamma}{2\,(g^2-1)}\sin(g\,\theta - \mathrm{N}) + a\,\gamma\sin\left(\overline{2 - 2\,k + g}\,.\,\theta - \mathrm{N}\right) \times \text{\&c.}$$

$$+ a\,\gamma\sin\left(\overline{2 - 2\,k - g}\,.\,\theta + \mathrm{N}\right) \times \text{\&c.}$$

Now the process of approximation ought necessarily to reproduce the original value $s = \gamma\,.\sin(g\,\theta - \mathrm{N})$. Hence, exactly as in the case of the apogee, we ought to have

$$\frac{3\,a\,\gamma}{2\,(g^2-1)}\,.\sin(g\,\theta - \mathrm{N}) = \gamma\,.\sin(g\,\theta - \mathrm{N})$$

and

$$g^2 = 1 + \frac{3a}{2},\quad g = 1 + \frac{3}{4}\,a$$

Thus we obtain the motion of the node, for $g - 1$ or $\frac{3}{4}\,a$ expresses the ratio of the retrograde velocity of the node to that of the moon in its orbit. If we execute the calculation, we find this ratio to be that of 0·0042 : 1 (see page 720, line 15 from bottom,) and it is remarkable, that it is exactly the same as a first approximation gives for the direct motion of the apogee; but there is one remarkable difference, that in the present case, this first approximated value is very near the truth as given by observation, and is little altered by a second approximation; whereas, in the other, the repetition of the process doubles the value.

The regression of the moon's nodes is a circumstance in their theory which admits of a very easy and familiar illustration.

The part of the disturbing force which acts in the direction of the z, or which tends to draw the moon out of the plane of its orbit is $m'\,\frac{d\,\Omega}{d\,z} = m'\,.\left(\frac{z'}{r'^3} - \frac{z'-z}{\lambda^3}\right)$. If we wish to know the whole force which tends to draw the moon in a direction at right angles to the plane of its own orbit, we must assume that plane for the plane of the x, y; then will $z = o$, and

$$m'\,\frac{d\,\Omega}{d\,z} = m'\,z'\left(\frac{1}{r'^3} - \frac{1}{\lambda^3}\right)$$

Now $m'\left(\frac{1}{r'^3} - \frac{1}{\lambda^3}\right) = -\,\frac{3\,m'\,r}{r'^4}\cos w =$ nearly $-\,\frac{3\,a}{a^2\,a'}\cos w$, and z' being a perpendicular let fall from the sun on the plane of the moon's orbit is equal to r' or a' multiplied by the sine of the sun's angular distance from the node $(\sin(\theta' - \mathrm{N}))$ and by that of the inclination or by γ. Hence, the expression of this force is

$$-\,\frac{3\,a}{a^2} \times \cos w\,.\sin(\theta' - \mathrm{N})\,.\sin I$$

w is the moon's angular distance from the sun as seen from the earth, and therefore $\cos w$ is equal to sine of

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$$-\frac{3\,a}{a^2} \times \sin \text{☾'s dist. from quadrat.} \times \sin \text{☉'s dist. from ☊} \times \text{sin. inclin.}; \qquad (268)$$

for the value of the force in question. Let us now examine how this force tends to produce a motion in the node. To this end, suppose the moon to set off from its ascending node, and at any point in its orbit (supposed circular) to describe an infinitely small arc which will be a portion of a great circle seen from the earth's centre. If the disturbing force at this moment ceased to act, this arc would be a portion of the undisturbed orbit, and being prolonged backwards would cut the ecliptic in the ascending node, ☊, whose position therefore would be the same at the beginning and end of this infinitesimal instant. To limit our ideas, let us conceive the moon to be within 90° from the ☊, and therefore its motion is from the plane of the ecliptic. Now, let the disturbing force act, and suppose its direction to be such as to draw the moon out of its orbit towards the plane of the ecliptic, then will the elementary arc actually described by the moon in virtue of its own inertia combined with the new impulse given by this force, be *less inclined* to the ecliptic than the last described portion; and therefore being produced backwards, will cut the ecliptic in a point *behind* the former place of the node. This point is the new or consecutive place of the node, which therefore has retreated on the plane of the ecliptic by the action of the disturbing planet.

On the other hand, had the disturbing force been directed from the plane of the ecliptic, the new path of the moon would be more inclined than in the preceding instant; and therefore, produced backwards, would cut the ecliptic in a point more *forward* than the previous place of the node, so that under these circumstances the node advances.

Thus we see that when the moon moves from the ecliptic and the force acts *to* it, the node retreats; but when the force acts from it, the node advances.

Again, let the moon be approaching the ecliptic; then, if the force act *to* that plane, it will approach it more rapidly, and will cut it in a point nearer than the node to which it is approaching. This node (and of course both) will then move to meet the moon, or in a direction contrary to its motion, *i. e.* will retreat; and *vice versâ*, if the force act from the plane, the node advances.

From this analysis of all the cases, we see that whenever the disturbing force tends to elevate the moon *from* the plane of the ecliptic, the node advances, and in every other case retreats. Now, it is easily seen, that the former condition never holds good unless the moon is between the node and the quadratures; and as the extent of the angle in which this can happen during a whole revolution of the moon in its orbit, is necessarily less than two right angles, the preponderant tendency of the node on the average of a whole revolution is always in favour of its retreat. In fact, when the node is in quadratures, it retreats at every instant of the lunation; and in the most unfavourable case, when the node is in syzigies, its retreat is barely counterbalanced by its advance, and the node only rests for an instant; the sun being then for a moment in the plane of the lunar orbit.

The general tendency of the node to recede on the ecliptic is thus clearly made out; but we may go further on these principles, and make the quantity of its recess a matter of calculation. For, let us denote by $\frac{K}{a^2}$ the force expressed in (268). Then, since the lunar gravity $\left(\frac{1}{a^2}\right)$ draws the moon in the instant of time $d\,t$ through the versed sine of an arc $= a\,d\,\theta$ or through a space equal to $\frac{(a\,d\,\theta)^2}{2a}$, the force $\frac{K}{a^2}$ will draw it *in the same time* through the space $K \cdot \frac{(a\,d\,\theta)^2}{2a}$. The inclination therefore of its new path to its old will be represented by the infinitely small angle

$$2\,\frac{K \cdot \dfrac{(a\,d\,\theta)^2}{2a}}{a\,d\,\theta} = K\,d\,\theta$$

Let the new elementary portion be prolonged till it meets the ecliptic in ☊′, ☊ being the former place of the node, then we shall have ☊ ☊′ for the momentary change of the node's place. Now this is the side of a spherical triangle opposite to the infinitely small angle $K\,d\,\theta$. The included side is the arc of the moon's orbit between the moon and node, or the moon's distance from its node, while the included angle is I the inclination. Hence, if we call L the longitude of the node, we shall have, by spherical trigonometry,

$$-\,d\,L = K\,d\,\theta \times \frac{\sin\,(\theta - L)}{\sin I}$$

but $K = -\,3\,\alpha \,.\, \sin\,(\text{☾} - \text{☉}) \,.\, \sin\,(\text{☉} - \text{☊}) \,.\, \sin I$. Hence, we have

$$d\,L = 3\,\alpha \,.\, \sin\,(\theta - k\,\theta) \,.\, \sin\,(k\,\theta - L) \,.\, \sin\,(\theta - L)\,d\,\theta$$

from which differential equation the relation between L and θ may be deduced.

If we assume L as constant during one lunation in the second member, we get by integration the whole change of L in that interval approximately, or the mean motion of the node in a lunation, which we will call Δ L

$$\triangle L = 3a . \int d\theta . \sin(1-k)\theta . \sin(k\theta - L) . \sin(\theta - L)$$

This is in principle the method followed by Newton in that part of the third book of the *Principia*, where he treats of the motion of the moon's node; the most elegant and satisfactory instance of the application of his geometry to the lunar theory.

The precession of the equinoxes is explicable on the same principles as the motion of the moon's nodes. The centrifugal force at the earth's equation throws out a portion of the matter of which it consists into the form of an oblate or flattened spheriod; and we may conceive this redundant matter, as a spheriodical shell investing an inscribed sphere. Suppose now every particle of this shell at liberty to obey any impulse, unfettered by the others, or conceive it to consist of an infinite number of infinitely small moons: each of these will describe an orbit, whose nodes have a tendency to recede on the plane of the ecliptic, and though some (those which happen to lie between their quadratures with the sine and their nodes,) will have their nodes in a state of advance; all the rest, which constitute the greater number, will have theirs in a state of recess. Conceive now the particles to cohere and form a solid ring, *unconnected* with the central globe, the motion of this ring will be a mean among all the motions of its parts, and the nodes of the ring will in consequence continually recede, with a certain velocity. Now, let the ring adhere to the sphere, then must all its motion be divided between itself and the whole mass of the earth, which *alone* has no such tendency. The redundant matter at the equator, however, bears a very small ratio to the whole mass of the earth; so that owing to this cause, the retrogradation is exceedingly diminished in rapidity; and owing to the enormous distance of the sun and the smallness of the earth (whose radius being only $\frac{1}{23405}$th part of the distance; so that here, $a = m' . \left(\frac{a}{a'}\right)^3 = \frac{m'}{(23405)^3}$ a quantity quite insensible,) is rendered too small to be distinctly perceived.

But the moon also exerts a disturbing force. That luminary is to our imaginary moons, or terrestrial molecules, what the sun is to the moon itself in the theory of the lunar perturbation. If we reduce these principles to calculation, assuming such a mass and distance of the moon as we know to be near the truth, we shall find that a retrograde motion of the earth's equator on the ecliptic of about 50″ per annum will actually result. Now this is the very phenomenon known by the name of the precession of the equinoxes; and though its strict theory is much more complicated and difficult than the general view here taken, its accordance with observation is perfect, and affords one of the most refined verifications of that admirable law which holds the frame of nature in the harmony we are now so well able to appreciate.

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MAGNETISM.

PART I.

etism.

Magnetism is that science which treats of the phenomena of magnets, whether natural or artificial; their relation and reciprocal action on each other; the direction which they assume when freely suspended, &c. &c.

The term magnet is differently derived; some deduce it from one *Magnes*, a shepherd, who is said to have first discovered its attractive power by the iron of his crook on Mount Ida. Others assert that it was first discovered in Heraclea, a city in Magnesia, and that it hence derived its denomination magnet, as well as that of *Lapis Heracleus*, as it was sometimes called by ancient authors. At all events the Greeks called those ferruginous ores which possessed this power of attracting iron, μάγνης, and from them we derive our terms magnet and magnetism.

It is but very lately that magnetism has assumed that scientific character which enables us to reduce its phenomena to the dominion of analysis, whereby to calculate and to predict with certainty the effect that will be produced, by any proposed combination, or given position, of different bodies of this kind acting on each other; but it is now advanced to a high rank in the physical sciences, and it is under this character we shall consider it in the present treatise. We propose to divide the subject into three distinct heads; Historical, Experimental, and Theoretical. In the former we shall take a succinct view of the various steps that have been made from time to time towards advancing this doctrine, from its first rude beginning to its present scientific form. In the second we shall confine ourselves to subjects wholly experimental, or at least we shall abstain from entering into any but the most general and obvious theoretical principles. We shall in this part explain the method of making artificial magnets, shewing the effect of temper, form, and proportion: we shall then describe the various magnetical instruments, and the relative magnetic powers of different natural and artificial substances.

In the theoretical part we shall examine the mathematical principles of the laws of magnetic action: the phenomena of terrestrial magnetism, the annual and diurnal motion of the terrestrial polarizing axis, annexing to it tables of the most authentic observations in all parts of the earth, from the first record of them to the present time. We propose also to give, in continuation of the above, a complete treatise of Electro Magnetism.

History.

History of ancient and modern discoveries of magnetical phenomena.

Ancient authors.

1. We have already seen that the ancients were acquainted with some of the properties of the magnet, its attractive powers being mentioned by Homer, Pythagoras, and Aristotle. Plato, Euripides, and Pliny, have also such references to the subject as would lead us to imagine that they not only were acquainted with its common attractive properties, but also were aware that it might be communicated; for Plato has described a chain of iron rings suspended by one another, the first being sustained by the loadstone.

The most important property of magnets, that of direction, appears however to have been altogether unknown to the philosophers of those times, and there seems to be much uncertainty as to the date of this invaluable discovery. By most writers it is represented, that a certain Neapolitan, named Flavio, or John de Gioja, or Giova, or Gira, who lived in the thirteenth century, has the best title to this honour; yet his claim is by no means indisputable. Dr. Gilbert, an English writer of the sixteenth century, in his work "*de Magnete,*" affirms, that Paulus Venetus brought the invention of the compass to Italy in the year 1260, having learned it of the Chinese. Ludi Vertomanus asserts, that when he was in the East Indies, about the year 1500, he saw a pilot direct his course by a magnetic needle, fastened and formed like those now in use; and Mr. Barlow, in his *Navigators' Supply*, anno 1597, relates, that in a personal conference with two East Indians, they affirmed, that instead of our compasses, they used a magnetic needle of about six inches in length, suspended upon a pin in a dish of white china earth, filled with water, in the bottom of which there were marked two cross lines to indicate the principal winds; the rest of the divisions being left to the skill of their pilots. But these two last observations, being of a date much posterior to the use of the magnetic needle in Europe, lead to nothing conclusive with respect to its original discovery; since the use of that magnetic property might have been introduced into Asia by some European.

Claims of the Chinese.

2. P. Duhalde, in his *General History of China*, (vol. 1.) in the annals of the Chinese monarchy, speaking of the Emperor *Hoangti*, when he gave battle to *Tchi Yeou*, says, "He, perceiving that thick fogs saved the enemy from his pursuit, and that the soldiers rambled out of the way, and lost the course of the wind, made a carr, which shewed them the four cardinal points: by this method he overtook *Tchi Yeou*, made him prisoner, and put him to death. Some say there were engraved in his carr, on a plate, the characters of a rat and a horse, and underneath was placed a needle, to determine the four parts of the world." In another part of the same book, speaking of certain ambassadors, he says, "After they had their audience of leave, in order to return to their own country, *Tcheou Kong* gave them an instrument, which on one side pointed towards the north, and on the opposite side towards the south, to direct them better on their way home, than they had been directed in coming to China. The instrument was called *Tchi Nan*, which is the same name as the Chinese now give to the sea compass: and this circumstance has given occasion to think, that *Tcheou Kong* was the inventor of the compass. This happened in the twenty-second cycle, more than 1040 years before Christ."

Notwithstanding these assertions, however, Renaudot adduces strong arguments against the supposed knowledge of the mariner's compass amongst the ancient people of China, and of Arabia. See Kircher *de Magnete*. We must, therefore, leave these claims upon the very uncertain basis on which they stand, and proceed to about the twelfth and thirteenth century of the Christian era, when it appears at least that the compass was known in Europe, however rudely and imperfectly constructed.

Early use of the compass in Europe

3. Cavallo, in the first edition of his *Treatise on Magnetism*, and in the supplement to the two subsequent editions, has collected together several curious and interesting statements connected with the early history of this science in Europe, amongst which perhaps the following is the most interesting. In the works of Claude Fauchet, entitled, *Recueil de l'Origine de la Langue et Poësie Françoise*, there is a quotation from an old French poem, called *la Bible Guiot*, in which the mariner's compass is obviously mentioned, and the same work is likewise quoted by Mushenbroeck, in his *Dissertatio de Magnete*. The passage in which the compass is quoted forms part of the above mentioned poem, contained in a curious quarto MS. of the thirteenth century, on vellum, belonging to the Royal Library at Paris, which was never published. The poem entitled *la Bible Guiot* forms the first article of the volume; the author of which, Guiot de Provence, as mentioned in the poem itself, was at the court of the Emperor Frederic Barbarossa, held at Mentz in the year 1181, when the Emperor's two sons were knighted.

The following is a verbatim copy of this part of the poem, with its literal translation:—

" Icelle estoile ne se muet
Une arts font qui mentir ne puet
Par la vertu de la Manete
Une piere laide et brunete
Ou il fers volenters se joint
Ont regardent lor droit point
Puez c'une aguile lont touchie
Et en un festu lont fishie
En longue-la mette sens plus
Et il festui la tient desus
Puis se torne la point toute
Contre lestoile sans doute
Quant il nuis est tenebre et brune
Con ne voit estoile ne lune
Lor font a laguille alumer.
Puiz ne puent ils assorer
Contre lestoile vers le pointe
Par se sont il mariner cointe
De la droite voie tems
C'est uns ars qui ne puet mentir."

TRANSLATION.

" This same (the pole) star does not move, and
They (the mariners) have an art which cannot deceive,
By the virtue of the magnet,
An ugly brownish stone,
To which iron adheres of its own accord.
Then they look for the right point,
And when they have touched a needle (on it)
And fixed it on a bit of straw,
Lengthwise in the middle, without more,
And the straw keeps it above;
Then the point turns just
Against the star undoubtedly.
When the night is dark and gloomy,
That you can see neither star nor moon,
Then they bring a light to the needle;
Can they not then assure themselves
Of the situation of the star towards the point? (of the needle)
By this the mariner is enabled
To keep the proper course;
This is an art which cannot deceive."

There can be no question, from the whole bearing of this very curious passage, that the mariner's compass, however rude in its construction, was known prior to the year 1200: at the same time, the very rough form of the instrument would seem to imply that it had not been then very long employed, although it does not appear to be described as a recent discovery.

Professor Hansteen, in his *Inquiries concerning the Magnetism of the Earth*, quotes some expressions of an Icelandic historian, which seems to prove that the directive quality of the natural loadstone, if not the artificial needle, was known nearly a century before the date of this poem. "Are Frode," says Mr. Hansteen, "by whom the *Landnamabok* of Iceland, or the account of the discovery of that island was written, mentions, (part I. chap. ii. p. 7.) that Floke Vilgerderson, a renowned *viking* or pirate, the third discoverer of the island, departed from Rogaland, in Norway, to seek *Gadersholm*, (Iceland,) some time in the year 868. He took with him three ravens to serve as guides; and in order to consecrate them to his purpose, he offered up a great sacrifice in Smörsund where his ship lay ready to sail. FOR IN THOSE TIMES SEAMEN HAD NO LOADSTONE IN THE NORTHERN COUNTRIES. (*Paiat pa hofdo hafsiglingarmenn enger Leidarstein i pan pinna á nordorlondum.*) In Icelandic, *Leid* signifies region, and on this account the pole star is named *Leidstjerna*, consequently *Leiderstein* signifies guiding-stone. According to the testimony of Snorro Sturleson, Are Frode was born in the year 1068; this account was therefore probably written about the end of the eleventh century, at which time it appears unquestionable that the directive property of the magnet or loadstone was known: although there is nothing in the passage that can lead to imagine that the form of the compass had at that time been suggested."

Our preceding quotations seem to refer the first use of the mariner's compass to a period antecedent to the thirteenth century; and before the fourteenth we have proof of its being known to the northern nations; for in 1266, on the death of the Swedish Count Byrgeris, whom the Swedes denominate Duke Pontana Byrges Jarl, Sturla celebrated his memory in a poem consisting of twelve verses; and when he attended the King to the boundaries of the kingdom, he produced this poem in an assembly of the states; and as a reward for his genius, he was presented with a mariner's compass. See Torfaeus, (Thormodus) *Historia Rerum Norvegicarum*, 4 tom. p. 345.

Another claim of this kind is made by Francis Cabeus, a jesuit of Ferrara, who says that the first thing he knows professedly written respecting the directive property of the magnet, was an epistle of Petrus Peregrinus Gallus, about the latter end of the thirteenth century. A few years after, this epistle was disguised by one John Tasnier, who published it in his own name, under the title of *Opusculum perpetua memoria dignissimum de natura et effectibus magnetis.* Some years ago, Mr. Senebier, of Geneva, sent the following memorandum concerning this letter to Dr. Lorimer, in London:—

"*Epistola Petri Peregrini de Marcourt.*" "*Ad Sigerium de Foueancourt Militem de Magnete.*"

"The work contains a description of that stone, the means of finding the poles, its property of attracting iron, and proves that the part of the magnet which is turned to the north attracts that which is turned to the south." It then teaches the manner of employing the magnet in astronomy, and of playing tricks like those of Comus. It deserves to be remarked, that the author knew not that the magnet could be employed in navigation; for though he frequently speaks *de stella nautica*, he never mentions the use that might be made of the magnetic needle in sea voyages. Vide *Bibliotheca Bibliothecarum*, p. 1400; *Catalogue of the Manuscripts in the Library of Geneva*, by Senebier, p. 207.

Among the manuscripts of the University of Leyden, there is a volume containing several scientific tracts, one of which is a Latin letter of Peter Adsiger, on the properties of the magnet. It is, in fact, a little methodical treatise, divided into two parts; the first of which is subdivided into ten, and the second into three chapters. This letter, which seems intended for the instruction of some particular friend, is dated in the year 1269. A few years ago, Mr. Cavallo obtained an exact copy of this curious letter, of which he inserted very ample extracts, both in the original Latin and in English, in the supplement to the second and third edition of his *Treatise on Magnetism;* and from which it appears that the writer, at that early time, was acquainted with all the principal properties of the magnet. The following is the translation of the most remarkable part of the above mentioned epistle, which describes the compass, and mentions the declination of the magnetic needle.

"Part II. chap. ii. *On the construction of a better instrument to answer the purpose; viz. to find out the azimuth of the sun, the moon, or any star upon the horizon.*

"In the present chapter, you will be informed of the construction of another instrument of more certain effect. A vessel must be made of wood, copper, or any other material, and let it be turned like a box, of small depth, and competently wide. Let a cover of some transparent substance, as glass or crystal, be adapted to it; and if the whole were made of some transparent matter, it would be still better. A slender axis of copper or silver must be adapted to the middle of this vessel, applying its extremities to the upper and lower parts of the box, viz. to the cover and to the box; the axis, however, must not be so firmly fastened, as not to be capable of moving very freely. Two holes must be perforated in the middle of the axis, at right angles to each other; and an iron style, like a needle, must pass through one of those holes, whilst another style of silver or copper passes through the other hole in a direction crossing the iron one. The cover must first be divided into four parts, and each of these into ninety parts, agreeably to the instructions given concerning the other instrument in the preceding chapter; and upon it mark the north and the south, the east and the west points; and let a ruler of some transparent substance, with sights on its extremities, be adapted to it. Then place whichever part of the magnet you please, viz. the north or the south, near the glass, until the needle be moved towards the said magnet, and acquires the virtue from it; after which, the magnet being removed, the extremity of the needle will turn itself towards the pole. This being done, turn the box until one extremity of the needle remains directed towards the north part of the instrument, or the north part of the heavens; then turn the ruler towards the sun in the day-time, and towards the stars in the night-time, after the manner mentioned in the preceding chapter. By means of this instrument, you may direct your course towards cities, and islands, and all other parts of the world, either on land or at sea, provided you are acquainted with the longitudes and latitudes of those places; for, if the town or island to which I intend to go be in a lower latitude than the place in which I am, I shall go straight before me, towards that end of the ruler which is directed to the sun or stars; but if the latitude of the place be greater, I shall proceed in the opposite way, viz. in the direction of the other extremity of the ruler: observe, that the south part of the ruler, which is to be used as a guide, must be made to decline towards the west by one point; and this must be done by the declination of the north part towards the east, because the south part of the instrument is destitute of divisions.

First mention of declination.

"Take notice, that the magnet, as well as the needle that has been touched by it, does not point exactly to the poles, but that part of it which is reckoned to point to the south, declines a little to the west: and that part which looks towards the north, inclines as much to the east. The exact quantity of this declination I have found, after numerous experiments, to be five degrees. However, this declination is no obstacle to our guidance, because we make the needle itself decline from the true south by nearly one point and a half towards the west."*

To this letter Mr. Cavallo, in his treatise, subjoins the following observations. It appears that the suspension of the needle in the above description, is a very clumsy one; and that the beautiful suspension by a cap upon a pin, which is now universally used, was unknown to the author. It is likely, therefore, that for want of this suspension, the needles at that time not moving sufficiently easy, were not actually used in navigation, at least in Europe, though the suspension by means of a cap upon a pin, seems not to have remained long unknown after the date of this letter. Cavallo's *Treatise of Magnetism.*

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* There is some ambiguity in these different estimates of the variation, but we have no means of correcting it.

Such are the very imperfect historical documents relative to the first discovery of this most valuable instrument; from which can be drawn very little to be entirely depended upon, although there can be no doubt, that whether the mariner's compass was known as such, prior to the date of the French poem we have quoted or not, the directive power of the magnet was not unknown in Europe so lately as some authors have stated; but as Montucla very justly observes in his *Histoire de Mathematiques*, tom. i. p. 527, inventions such as that of the mariner's compass, proceed step by step with such imperceptible degrees of advancement, that it is very difficult to do justice to the respective claims of different nations; and it is not astonishing, considering the high importance of the compass to navigation, that we should find various claimants to the honour of the discovery.

We have seen that the French claim this honour on the authority of the poem which we have quoted; they also assert that the usual mode of ornamenting the card with the *fleur de lis*, which is to this day practised by every nation in Europe, is a farther indication of their prior claim. Others urge still further its French origin from the term *calamita*, by which the Italians still denote the compass, and which is said to be derived from an ancient French word *calamite*, signifying a little frog, to which the compass in the first instance was assimilated, when it was made to float on water. Dr. Wallis, an English mathematician of celebrity, claims the honour of the invention for this country, from the general denomination *compass* or an analogous term, used by most European nations. We must however acknowledge that this learned philosopher has displayed more partiality than judgment in making this claim, although the English are allowed to have been the first who suspended the box in gimbals; but the suspension of the needle by a cap and point, as now universally adopted, is certainly of unknown origin.

4. It appears however to be unquestionably established, that the mariner's compass in a rude form was at least partially known, and some of the more obvious magnetic phenomena, before the year 1300, and even the variation is positively if not clearly stated in the letter quoted in a preceding article; the commonly received opinion therefore, that it was Columbus who first discovered this declination is not strictly correct. The truth perhaps is, that the variation of the compass from the true north was known before the date of the voyage of this celebrated navigator, and that he was merely the first to discover that the variation was not the same in all places. His son Ferdinand states, that on the 14th of September 1492, his father first discovered the variation; and that in consequence his crew mutinied, supposing that the needle had lost its polarity, and that they would not be able to find their way back to Europe. If this means any thing more than that Columbus then discovered the variation to be different in different latitudes, it is erroneous; at least if it was then for the first time discovered by Columbus, it is obvious from what has been stated, that the fact itself was known nearly 200 years before that time.

Columbus said to have discovered the declination or variation.

Dip of needle covered.

5. The next important discovery in magnetism is due to our countryman Robert Norman, a mathematical instrument maker in, or near, London, who in 1576 found, that besides the known directive power of the needle in azimuth, it had also a certain direction of inclination commonly called the dip. He tells us in his book, *The new Attractive*, that, "having made many compasses, and always balancing the needles before he touched them with the magnet, he found continually, that, after he had touched them, the north point would incline downward under the horizon, inasmuch that the fly (or card) of the compass, which was before level, he was still constrained to put some small piece of wire on the end thereof to counterpoise it." He then communicated his discovery to some of his literary friends, who advised him to frame an instrument and make an exact trial of what would be the greatest angle it would make with the horizon, and he found the inclination, or dip, when the discovery was made, to be 71° 50′ or thereabouts. Churchman's *Magnetic Atlas.*

Variation first ob observed.

6. In the above work, and in *A Discourse on the Variation of the Compass* by Borough, the friend and patron of Norman, and Comptroller of the Navy of Queen Elizabeth, both published in 1581 and bound together, they say, "the variation is 11° 15′ east," this being deduced from actual observation made by Borough in 1580 at Lime-house. The altitude of the sun was taken by him to each degree from 17° to 25°, being the mean of nine several observations, and at every altitude, turning his compass to the sun, he observed the degree and minutes cut thereby: then comparing the magnetical azimuth made in the afternoon, with those in the forenoon, the half difference was taken for the variation. These two works were corrected, amended, and reprinted in the year 1585; but the variation is still given as being 11° 15′ E. and nothing is said relative to any change in it, or as it is termed, of the variation of the variation.

Variation of the variation discovered.

7. In the following century, professor Gellibrand, who was the next English writer on the subject, clearly ascertained this important fact; stating the variation as only 4° 5′ in 1633, and at the same time correcting Borough's observations, or rather his calculations, for certain omissions, which reduced the former variation from 11° 15′ to 11°. Bond, however, in his *Longitude Found*, says that the declination was discovered to vary and to decrease first by Mr. John Mair, secondly by Mr. Gunter, thirdly by Professor Gellibrand, and then by himself in 1640. Whether these several authors and observers all notice this fact independently of each other, is now very uncertain; but it is obvious upon the whole, that about this time, the annual change in the variation was become a subject of inquiry; and soon after this a tolerably regular set of observations are recorded, as will be seen by referring to our general table I.

Variation charts.

8. The first idea of the formation of variation charts, had its origin a short time before the period to which we have now arrived; for in the year 1599, a work was published by Prince Maurice of Nassau, lord high Admiral of the United Provinces of the Low Countries, and which was translated into English by Edward

Wright, a well known British mathematician. In this work all seamen are strongly urged to keep registers of the variation in all places they may visit, in order to facilitate the practice of navigation, and which was thought of so much the more importance, as the variation in the variation was not then suspected, although, as we have seen, it followed soon after.

9. Hitherto no attempt had been made that we are aware of, to advance any theory relative to the nature of terrestrial magnetism; this however now began to engage the attention of philosophers. Dr. William Gilbert, physician in ordinary to Queen Elizabeth, took the lead in this pursuit: for in 1600 he published his work *de Magnete*, and offered certain theoretical ideas on the nature of this mysterious power. "He supposed that the earth itself, being in all its parts magnetical, and the water not, wherever the land was there would the needle turn, as to the greater quantity of magnetical matter." This work contains a number of curious experiments and many ingenious suppositions, but it affords no hint relative to the variation of the variation. We shall again refer to this treatise when we give an abstract of the different theories of magnetism that have been advanced: we shall confine our remarks in this place to the discovery of phenomena.

10. We have seen that early in the seventeenth century, it was discovered that the variation was not always the same, and that it was at that time in London decreasing, or the needle approaching nearer in direction to the true pole of the world. Bond, to whom we have already referred, published during this century two different works on the subject of magnetism, in which he not only clearly points out this variation of the variation, but he also attempts to compute the annual change: his rules however were merely empirical, and were soon found to be erroneous.

11. In 1683, Dr. Edmund Halley published a theory of magnetism, and a few years after, in consequence of an application made to the government of William and Mary, the command of one of the ships in the royal navy was given to him, with orders "to seek by observation the discovery of the rule for the variation of the compass. This voyage was made in 1698; he returned in 1699; set out again the same year, and traversed various parts of the Atlantic and Pacific oceans, and lastly formed a chart of the variation, which is highly valuable as a source of authentic observations on the variation of the needle at that time. An account of these voyages was published in 1701. See also various papers by Dr. Halley on the subject of magnetism in the *Abridgement of the Phil. Trans.* from vol. ii. to vol. vi. inclusive. Since this time different magnetical charts have been published and corrected to particular periods, the last of which is by Mr. Yeates in 1817.

12. In the year 1722 another curious fact was discovered in magnetism by Mr. Graham, an ingenious philosopher and instrument maker in London. This was, that the needle, besides its annual change in its declination, which carries it constantly to the westward, is subject to a daily variation, by which the north end of the needle moves westward during the early part of the day, and returns again, during the latter part, to the bearing it had in the morning; remaining stationary or nearly so during the night.

When Mr. Graham first noticed this change, he attributed it to certain defects in the form of his needles; these he therefore varied, and made some light and some heavy. "But after many trials Mr. G. found all the needles would not only vary in their direction on different days, but frequently at different times of the same day; and this difference would sometimes amount to upwards of half a degree in the same day, sometimes in the same hour. Mr. G. found it of no consequence whether the needle was drawn aside or let alone, the shaking of the floor by walking upon it, or the trembling of the house by the coaches in the street was sufficient to overcome the small friction on the point.

"The box was placed in a room more than six feet from the nearest wall, and above thirteen feet from the grate in the chimney, and no iron was at any time brought near it. The needles were all touched by that excellent loadstone presented to the Society by the Right Hon. the Lord Paisley. It may not be improper to take notice, that the needles were not touched on the naked stone, but with its armour on, generally on that part of the capping nearest the poles; but there was no difference in the direction by touching them on another part. When the needle was observed increasing or decreasing in its variation, Mr. G. very frequently with a key, drew it the contrary way several degrees, from the place it stood at immediately before, it stopped there for some time by holding the key at a proper distance, and withdrawing his hand gradually, he tried to make it stand short of its former place, but could never succeed. By this method, and several others made use of, he was well assured these changes in the direction were owing to some other cause than the friction of the needle on the pin; but as to what the cause was he did not attempt to offer a conjecture; it seemed neither to depend on heat nor cold, a dry or moist air, clear, cloudy, windy nor calm weather, nor on the state of the barometer. The only thing that had any appearance of regularity was, that the variation was generally the greatest between ten o'clock and four in the day, and the least about six or seven in the evening." With these needles and their apparatus, the author made numerous observations on many days, from February 5th to May 10th, 1722, and for many times each day, under all variety of circumstances: indeed from February 6th, 1772, to the 10th of May following, he made 1000 observations in the same place; and the greatest variation westward was found 14° 45′ and the least 13° 50′; but it was seldom more than 14° 35′ nor less than 14°. *Abridg. Phil. Trans.* vol. vii. p. 29.

More recent observations.

13. More modern observations have shown us that the limits of the daily variation are stated too widely in the above paper, as it seldom exceeds 15′; but still the account is important as fixing decidedly the date of this discovery, and the amount of the variation in London at that time; which unquestionably must have been between the limits last stated. We are not aware of any farther attention being paid to the diurnal variation till the subject was taken up by Wargentin, secretary to the Swedish Academy of Sciences in 1750, whose observation we have recorded in our tables at the end of this treatise; and about the same time, viz. 1756, Mr. John Canton, an English philosopher, made about 4000 observations on the same subject, and various experiments, with a view, not only of discovering the actual amount of this change, but of explaining its cause. From these it was found that the daily variation is greater in the summer than in the winter months,

and all observations that have since been made have tended to establish this fact. According to the general mean of all Canton's results, the daily variation in each of the twelve months is as below.

January	7′	8″
February	8	58
March	11	17
April	12	26
May	13	0
June	13	21
July	13	14
August	12	19
September	11	43
October	10	36
November	8	9
December	6	58

The greatest deviation was always found to take place every day about two hours after noon, commencing from about seven or eight o'clock in the morning, and returning to its morning bearing or nearly so, about the same hour in the evening. Similar observations were more recently made by Van Swinden, with nearly the same results, except that he found sometimes prior to the hour in the morning above stated, and after that hour in the evening, a motion of the needle both to the eastward and westward; viz. that the morning westerly variation of the needle is sometimes preceded by a small easterly variation, and that the principal easterly variation in the evening is followed by a light westerly variation. So that the course of the several daily changes may be represented as follows: W. E.; *e.* W. E.; *e.* W. E. *w.* the roman letters indicating the greater change, and the italic the less. See *Recueil des Mémoires sur l'Analogie de l'Electricité et du Magnétisme.*

Observations of the same kind have also now been made for several years by Colonel Beaufoy, and published monthly in the *Annals of Philosophy*. These however do not point out the minor variations in the morning and evening, being registered only three times every day; viz. about eight o'clock in the morning, at half past one o'clock in the afternoon, and at six o'clock in the evening. The most remarkable fact deduced from this series of observations is, that the daily variation appears to be the greatest in June and August, and somewhat less in July; so that there seem to be two maxima and two minima of variation in the course of the year; the two maxima in June and August, and the two minima in December and July; the latter minima however is much greater than the former.

The daily change for the several months in the year, taking a mean for 1817, 1818, 1819, is as below.

April	11′	48″
May	9	53
June	11	15
July	10	43
August	11	26
September	9	44
October	8	46
November	7	10
December	4	7
January	5	3
February	6	3
March	8	22

Professor Hansteen, the distinguished Norwegian professor, has also made numerous observations connected with this inquiry, to which we shall have occasion to refer, when we have to treat of the daily variation in the following treatise. Some very curious experiments on the daily variation have also been lately made at Woolwich by Messrs. Barlow and Christie. The account of them has been read before the Royal Society, but they have not yet been published. We shall speak of them more at length in the course of the following treatise.

Compa needle fected b the nort ern ligh

14. Another curious phenomenon should also be mentioned in this place, and which was we believe first noticed by Professor Wargentin in 1750, that is to say, the effect which is produced on a needle by the northern lights. At the time of this meteor the needle is found to be in a perpetual state of agitation, varying as much as 3°, 4° or 5°. Some observations of this kind will also be recorded as we proceed; the fact is merely noticed in this place to preserve a certain connection in point of date, between the discoveries of the several class of phenomena. Mr. Dalton, so well known by his various ingenious works, has entered minutely upon this subject in his *Meteorological Observations and Essays*, where he has proved in the most satisfactory manner that such an action actually takes place, and has also offered an hypothesis by way of explanation, to which we shall have occasion to refer in a subsequent article.

The above are the principal experimental facts that have been established, relative to the natural phenomena of the magnetic needle; we must now retrace our steps for about a century, to take a general view of the several theories that have been advanced, and theoretical deductions that have been made during that period: we shall also have to lay before our readers certain experimental facts elicited during the same period, and connected with the laws of magnetic action, although not immediately dependent on the magnetism of the terrestrial sphere.

Dr Hal theory of magneti

We have already stated that Dr. Gilbert was the first to attempt any hypothesis with a view to reducing the laws of magnetic action to something like a theoretical form, but there were too few facts yet known to enable him to approach to a correct theory. To him, however, we owe the first idea that the magnetic action of iron bodies is produced by induction from the terrestrial magnetism; an hypothesis which is still admitted in the modern theories. After this time Dr. Halley's hypothesis is the first that is deserving of particular notice, or we should have rather said hypotheses; for after the first had been promulgated for some years, during which he had made various observations, and collected numerous facts, he found himself under the necessity of remodelling his first theory, and replacing it with a new one.

According to Dr. Halley's first ideas, the magnetism exhibited by the earth, was caused "by four poles or points of attraction, two of them near each pole of the equator; and that, in those parts of the world which lie nearly adjacent to any one of those magnetic poles, the needle is governed thereby, the nearest pole being always predominant over that more remote. The parts of the earth wherein these magnetic poles lie," he says, "cannot as yet be exactly determined for want of sufficient data; but as near as conjecture can reach, I reckon that the pole which is at present nearest to us, lies in or near the meridian of the Land's End, and not above 7° from the north pole." The other north magnetic pole was supposed to be on the meridian of California, and about 15° from the pole of the earth. The two south poles he conceived to be situated, the one about 16° from the southern terrestrial pole, and in a meridian about 20° to the westward of Magellan's straits, or 95° W. of London. The other, which he supposed the greatest or most powerful of the four, was conceived to be 20° from the south pole of the earth, and in about 120° E. longitude of London. By means of these, the author endeavoured to explain the various bearing of the needle in different parts of the world: but, as he had thus made no provision for that change of variation to which we have alluded, he proposed, as we have stated, some years afterwards, his new theory, in which the exterior surface of the earth is considered merely as that of a hollow shell; and that within this shell, there is a solid globe performing its rotations in nearly the same time as the earth, about the same centre of gravity. He further supposed that this globe has two magnetic poles, and the shell two others; and that the variation in the variation is caused by the want of coincidence in the periods of the times of rotation of the globe and shell. On this subject the author observes, "So then the external parts of the globe may well be considered as the shell, and the internal as the nucleus, or inner globe, included within ours, with a fluid medium between, which having the same common centre and axis of diurnal rotation, may turn about with our earth, each in twenty-four hours; only this outer sphere, having its turbinating motion some small matter either swifter or slower than the internal ball. And a very minute difference in length of time, by many repetitions becoming sensible, the internal parts will by degrees recede from the external, and not keeping pace with each other, will appear gradually to move either to east or west by the difference of their motion. Now supposing such an external sphere, having such a motion, we may solve the two great difficulties in my former hypothesis; for if this exterior shell of earth be a magnet, having its poles at a distance from the poles of diurnal rotation, and if the internal nucleus be likewise a magnet, having its poles in two other places distant also from its axis; and these latter, by a gradual and slow motion, change their places in respect of the external, we may then give a reasonable account of the four magnetical poles, as also of the changes of the needle's variation. The period of this motion being wonderfully great, and there being hardly a century since these variations have been duly observed, it will be very hard to bring this hypothesis to a calculus, especially since, though the variation increases and decreases regularly in the same place, yet in different places, at no great distance, there are found such casual changes of it, as can nowise be accounted for by a regular hypothesis; as depending on the unequal and irregular distribution of the magnetical matter within the substance of the external shell, or coat of earth, which deflects the needle from the position it would acquire from the effect of the general magnetism of the whole. Of this, the variation of London and Paris affords a notable instance; for the needle has been certainly about $1\frac{1}{2}$° more easterly at Paris than at London, though it be certain that, according to the general effect, the difference ought to be the contrary way. Notwithstanding which, the variation in both places changes alike.

"Hence, and from some other reasons of the like nature, I conclude, that the two poles of the external globe are fixed in the earth, and if the needle were wholly governed by them, the variation would be always the same, with some little irregularities on the account just now mentioned; but the internal sphere, having such a gradual translation of its poles, influences the needle, and directs it variously, according to the result of the attractive or directive power of each pole; and consequently there must be a period of revolution of this internal ball, after which the variation will return as before." *Abridgement Phil. Trans.* vol. iv. p. 470.

Although perhaps no philosopher ever admitted the whole of Dr. Halley's machinery, yet the last result, viz. of two fixed and two movable poles, or of four movable poles, has found many able defenders, amongst whom we may mention Professor Hansteen, whose elaborate work on this subject we have already, and shall hereafter, have frequent occasion to quote. Others, as Euler, thought that it was possible to explain all the phenomena by the admission of two poles only. Others again have supposed a greater or less number of poles according to their particular fancies; but very few have attempted to submit their hypotheses to the test of observation and experiment.

Conjectures concerning the cause of terrestrial magnetism.

15. The cause of this motion in the polarizing axis or axes of the earth, is another subject on which conjectures have been more abundant than rational. One has attributed it to the oxidation of the metals in the bowels of the earth, which was supposed to proceed with order and regularity; another to a sort of rotation in a principal focus of cold, which is supposed to proceed from east to west; others again attribute it to the solar action; others to electricity and so on; but the most recent conjecture is, that it is due to the newly discovered electro-magnetic action which has place between the galvanic and magnetic fluids.

Magnetic poles not at the surface.

16. In all the theories to which we have referred, the idea has been, that the points or poles to which the needle is directed, are found either on the surface of the earth, or at no very considerable depth below it. At length, after the scientific travels of M. Humboldt, who made numerous accurate observations on the dip and variation of the needle, in various parts of the earth, sufficient data appeared to have been collected, to determine to what points in the supposed terrestrial magnetic axis, the needle was attracted and repelled. M. Biot undertook this determination; he assumed two points at an indeterminate distance, and having obtained his general law in terms of that distance, and other quantities which were given, he compared his actual experimental results with his formula, by taking different numbers for his indeterminate distance; and he thus found, contrary to what had been before supposed, that the nearer his two poles were taken to each other, the nearer the computed and observed results were found to coincide: and lastly, by taking them indefinitely near in the centre of the earth, the two agreed with each other as nearly as could reasonably have been expected. His resulting expression, however, was very complicated; but in the mean time Professor Kraft, of St. Petersburgh, in 1809, undertook a comparison of the same experimental results, with each other, and with their respective situations on the globe, and hence arrived

at this very simple law, i. e. "supposing a circle circumscribed about the earth, having the two extremities of the magnetic axis for its poles, and considering it as a magnetic equator;" he found, that the tangent of the dip of the needle in any magnetic latitude, was equal to double the tangent of that latitude; that is $\tan \Delta = 2 \tan \lambda$, where Δ is the dip of the needle and λ the magnetic latitude. M. Biot afterwards, on a reexamination of his formula, found that it reduced itself to the same simple expression, and thus the same result was obtained by two different philosophers, independent of each other, and on principles wholly distinct: this law has since been verified by our recent Northern expeditions, and may perhaps be admitted as the first established principle of terrestrial magnetism.

As we shall have occasion to enter upon this subject at length in the following treatise, what we have now advanced may be considered sufficient for our present purpose, which is altogether historical.

Action of one magnet on another.

17. We have hitherto referred only to the discovery of facts, and the invention of hypotheses relating to terrestrial magnetism. We must now bestow a few pages to the detail of some other interesting inquiries which were entered into during the last century, in order to discover the cause, and to develope the laws of the action of magnets on each other.

It was not till about the year 1750, that the method of communicating magnetism to steel bars was well understood; indeed we are not aware that it had been before attempted on bars of any considerable magnitude, and consequently it was not till this date that the great advantage which these bars possessed over natural magnets was properly appreciated. This discovery we owe to Dr. Knight an English physician, and a fellow of the Royal Society. Improvements were afterwards made on his methods by Du Hamel, Michel, Canton, and others, which will be found described at length in our second part; and which we shall therefore pass over in this place.

Another important investigation also now began to be undertaken, which was, to determine the laws according to which a single insulated particle of magnetic fluid acts on another insulated particle. Hitherto very different deductions had been made by different authors; some had found it to vary inversely as the fifth power of the distance, others as the fourth, some again as the third, and some inversely as the square of the distance, as in the case of universal gravitation.

One amongst the first that elicited the true law was Lambert of Berlin, who in detailing his experiments observes, that the great difficulty met with in these researches is, that when we wish to determine the fundamental laws, probably in themselves very simple, it is necessary to find means of simplifying the experiments to such a degree, that, being disengaged from all foreign circumstances, and all mechanism of too complex a nature, they may present nothing but the mere effect of the law which we wish to discover. But this presupposes an acquaintance with those circumstances, and that mechanism, and a knowledge more or less accurate of the influence they may have in altering the issue of the proposed experiment. The magnet presents all these obstructions; one of its poles attracts, while the other repels; and the effect which results is always mixed. The number of poles may be augmented, but no art has yet succeeded in reducing them to one, which would be the best means of discovering its individual effect, unmingled with the alteration proceeding from the directly opposite effect of the other pole. Besides, it is not at the pole alone that the magnet attracts; it attracts more or less at every point of its surface, with very unequal degrees of force. Hence, the mean direction and the compound force vary in a manner extremely complicated. To succeed in finding the elementary laws, we should have it in our power to make the experiment for each isolated point; but experiment can show us nothing more than the sum that results from the combined action of all the points.

To these difficulties still more must be added. In order to determine by experiment the forces of the magnet, we have nothing to present to it but ferruginous matters or another magnet. Now, how short soever the experiment may be, the iron which we bring to be attracted by the magnet, begins itself to share in the magnetic force, and the effect which results, begins of course to be more or less disturbed. The same thing happens, if, instead of iron, we employ another magnet. And farther, if the experiments require a considerable space of time, it is not certain that the magnet we use will preserve its force invariably till the progress is concluded; the magnetic force, and its mean direction, being known to change from one moment to another.

Obscured by these uncertainties, the true law of magnetic attraction was long unknown or mistaken Newton, and his commentators, La Jacquier, and La Sueur, imagined the force to be inversely as the cube of the distance. Muschenbroeck's experiments led him to adopt the same opinion; some even contended that it was not proportional to any power of the distance. And though Mayer, in an unpublished paper read before the Society of Gottingen in 1760; and Lambert, a few years later, asserted the inverse square of the distance to be the true expression of this law, their statements seem rather to have proceeded on general and somewhat vague deduction, than on decisive facts. Coulomb (1786) was the first who established this principle, on grounds at all satisfactory. By means of the simple instrument which he constructed after Michel, and named the Balance of Torsion, he was enabled to measure, with great precision, the force necessary to make a magnetic bar, horizontally suspended, deviate by any number of degrees from the position assigned to it. Having by this means verified the discovery of Lambert, that the effect of the terrestrial magnet is proportional to the sine of the angle which its meridian makes with that of the magnet acted upon by it; and having farther ascertained, that a magnetised steel wire, 24 inches long, and $1\frac{1}{4}$ line in diameter, requires 35° of torsion to force it one degree from the magnetic meridian of the earth, he took another wire, of equal dimensions, of the same degree of magnetism, and placing it *vertically* in the magnetic meridian, (so that its extremities or poles would have intersected the *homologous* pole of the horizontal wire, had no influence existed between them, or their poles not been homologous) at the distance of an inch from the end of each, he observed the force of torsion necessary to overcome certain quantities of their mutual repulsion. It was found that the vertical wire at first repelled the horizontal one 24°; that three circles of torsion, or three complete turns of the micrometer, brought the latter within 17° of the former; and lastly, that eight circles brought the wires within 12° of each other; and adding to each of these results, the effect of the terrestrial magnet, already estimated at 35° of torsion for each degree of deviation, Coulomb calculated, that to keep the magnetised wires within 24° of each other, $24^\circ \times 24 \times 35^\circ = 864^\circ$ of torsion were necessary: within 17° of each other, $3 \times 360^\circ \times 17^\circ \times 17 \times 35^\circ = 1692^\circ$ of torsion were necessary; and within 12° of each other, $8 \times 360^\circ \times 12^\circ \times 12 \times 35^\circ = 3312^\circ$ were necessary. At the distance, 12°, 17°, and 24°, therefore, it appears, that the forces of repulsion are respectively proportional to the numbers 3312, 1692, and 864: now had these numbers been 3312, 1650, and 828, the repulsion found by the experiment would have been accurately in the inverse duplicate ratio of the distances. The errors 42 and 36 then might be supposed to result from the unavoidable incorrectness of such an experiment. Their existence, however, is sufficiently explained; for although the portion of each wire submitted to the other's influence was very small, it did not consist of a single particle as the calculation demands, but of a perceptible space, comprehending other particles besides the one whose action was required; so that the force of these other particles being less oblique, and therefore stronger at greater distances, it ought actually to produce an excess similar to that found in the experiments. A result entirely analogous was found when the *contrary* poles of the same were exposed to each other; and as these facts were confirmed by others, and contradicted by no established principle, Coulomb was warranted in concluding, with considerable confidence, that the attraction and repulsion existing between two magnetic particles is always *inversely as the square of the distance;* agreeing thus with the law observed in electricity, the force of gravity, and with every other force or principle emanating from a centre in right lines.

The same result was afterwards further verified by the experiments of Professor Hansteen, to whose work we have already referred; and no doubt any longer remained, of the truth of the assertion first propagated by Mayer and Lambert, of the inverse law of the square of the distance.

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18. In this state of the science, experiments were undertaken by Mr. Barlow, of the Royal Military Academy; and from these he first deduced certain empirical laws, which possessed an extraordinary degree of simplicity, when compared with the various and apparently anomalous action which is exhibited between one magnet and another, or between a magnet and simple iron. These experiments were made by observing the deviations which an iron ball or shell produces on a compass needle freely suspended, in different situations; and in the course of which a most remarkable fact was discovered; namely, that the magnetic power of iron resides wholly on its surface, and is therefore independent of the mass; this fact, since confirmed by Captain Kater, (*Phil. Trans.* 1821,) threw a new light on the nature of magnetic action; and which, together with the other laws elicited by the same experiments, have been the means of placing magnetism in the first rank of physical sciences, and of reducing all its varied phenomena to the dominion of analysis. The following is Mr. Barlow's summary of his laws depending on experiment, as they were given by him in the first edition of his *Essay on Magnetic Attractions.*

1. "There exists a plane of *no attraction* in every ball or shell of iron, which plane, in this latitude, inclines from north to south, and forms, with the horizon, an angle equal to the complement of the dip.

2. "Considering any circle in this plane as an equator to an ideal sphere concentric with the ball or shell, and imagining circles of latitude and longitude to be drawn thereon, the first meridian being supposed to pass through the east and west points, we shall have (while the diameter of the ball and the distance are the same,) the tangent of the angle of deviation proportional to the rectangle of the sine of the double latitude, and the cosine of the longitude of the place of the compass as referred to the above sphere.

3. "Instead of conceiving the imaginary sphere to surround the ball, we may imagine a similar sphere concentric with the pivot of the needle, then it it obvious that the centre of the ball will have the same relative position on the latter sphere as the pivot of the compass has with respect to the former, so that the reference may be made to either at pleasure; but when the mass of iron is irregular, it will then be preferable to refer the common centre of attraction of the iron to the imaginary sphere circumscribing the compass.

4. "The distance only being variable, the tangent of deviation is reciprocally proportional to the cube of the distance.

5. "All other things being the same, the tangents of deviation are proportional to the cubes of the diameters of the balls or shells, whatever may be their masses, provided only that the thickness exceed a certain quantity.

6. "All these laws may be expressed by the general formula

$$\tan \Delta = \frac{D^3}{A\,d^3} (\sin 2\lambda \cos l) \text{ or}$$

$$\tan \Delta = \frac{r^3}{A\,d^3} (\sin 2\lambda \cos l) \text{ or}$$

where Δ is the angle of deviation, λ the latitude, and l the longitude on the ideal sphere, D the diameter, and r the radius of the ball or shell, d the distance of the centre of the ball from the pivot of the needle, and A a constant coefficient, to be determined by experiment.

7. "Since the force has been shown to vary with the surface or square of the diameter, independently of the mass, while the tangents of deviation are as the cubes of the diameters, it follows that the squares of the tangents of deviation are directly proportional to the cubes of the forces.

8. "The same inference may also be drawn from the law of the distances, by assuming, that the force varies inversely as the square of the distance; for then, the tangents being inversely as the cubes of the distances, while the force varies as the squares of the same, it follows, as above, that the squares of the tangents of deviation are directly as the cubes of the forces."

9. Mr. Barlow also, in the second edition of the same

work, reports the results of several curious and interesting experiments on the relative magnetic power of different descriptions of iron and steel; and on the effect of temperature in changing both the quality and quantity of the attractive power of iron; but as we shall have occasion to refer more particularly to these experiments as we proceed, the above notice of the results will be sufficient for our present purpose, which is wholly historical.

The above laws, as we have stated, were merely empirical, having been drawn from a comparison of different results obtained under different circumstances; but in the second edition of the same work above referred to, under the title of *An Essay on Magnetic Attractions, and on the Laws of Terrestrial and Electro Magnetism*, the author shows that the same results, (at least with certain inconsiderable corrections,) are deducible from an hypothesis relative to the nature of magnetic action, extremely simple in itself, and of easy application.

His hypothesis differs very little from that previously laid down by Coulomb and Biot, with the exception of that particular modification arising out of the experimental fact to which we have alluded; viz. that the magnetic power all resides on the surface of iron bodies, and is independent on its mass. It was this remarkable circumstance that enabled Mr. Barlow to get a general analytical expression for the disturbing power of an iron ball at its surface, as compared with that of the earth, and whereby he deduced theoretically all those laws he had before found by experiment. The power of magnetic attraction varies, as we have seen, like all other emanating forces, inversely as the square of the distance from the centre; and since the surfaces of balls and shells vary also as the squares of the diameter, but in a direct proportion, it followed from what has been already stated, that the magnetic power of iron balls and shells at the surface would always be the same, whatever may be their diameter; and to this constant value Mr. Barlow was indebted for the full success of his investigation. It should, however, be stated, that notwithstanding the author succeeded in this ultimate determination, and performed the experiments on which it was founded, yet the first idea of the theoretical investigation is due to Mr. Charles Bonnycastle. It was this ingenious mathematician who first undertook the solution, and who succeeded in deducing the greater part of the laws; but in consequence of having adopted the theory of Æpinus, which supposes an accumulation of the two magnetic fluids in the poles of the body, he was not able to deduce the whole of them. See Mr. Bonnycastle's *Investigation*, in vol. lx. of the *Philosophical Magazine*.

Theory of Æpinus.

19. Having alluded above to the theory of Æpinus, and as we shall have no occasion to refer to it in the following treatise, it may not be improper to conclude this part of our historical sketch by a short outline of its principal features. According to this philosopher, iron and all ferruginous matter contain a quantity of magnetic fluid, which is equably dispersed through their substance, when those bodies are not magnetic; in which state they shew no attraction or repulsion against each other, because the repulsion between the particles of the magnetic fluid is balanced by the attraction between the matters of those bodies and the said fluid, in which case those bodies are said to be in a natural state; but, when in a ferruginous body, the quantity of magnetic fluid that belongs to it is driven to one end, then the body becomes magnetic, one extremity of it being now overcharged with magnetic fluid, and the other extremity undercharged. Bodies thus constituted, viz. rendered magnetic, exert a repulsion between their overcharged extremities, in virtue of the repulsion between the particles of that excess of magnetic fluid; which is more than overbalanced by the attraction of their matter. There is an attraction exerted between the overcharged extremities of one magnetic body, and the undercharged extremity of the other, on account of the attraction between that fluid and the matter of the body; but to explain the repulsion, which takes place between their undercharged extremities, we must either imagine that the matter of ferruginous bodies, which, deprived of its magnetic fluid, must be repulsive of its own particles, or that the undercharged extremities appear to repel each other, only because either of them attract the opposite overcharged extremities; both which suppositions are embarrassed with difficulties.

A ferruginous body, therefore, is rendered magnetic by having the equable diffusion of magnetic fluid throughout its substance disturbed, so as to have an overplus of it in one or more parts, and a deficiency of it in one or more other parts; and it remains magnetic as long as it impermeability prevents the restoration of the balance between the overcharged and undercharged parts. Moreover, the piece of iron is rendered magnetic by the vicinity of a magnet; because, when the overcharged part or pole of the magnet is presented to it, the overplus of magnetic fluid in that pole repels the magnetic fluid away from the nearest extremity of the iron, which, therefore, becomes undercharged, or possessed of the contrary polarity to the most remote part of the iron, which consequently becomes overcharged, or possessed of the same polarity as the presented pole of the magnet. When the piece of iron is rendered magnetic by presenting to it the undercharged extremity or pole of the magnet, then the part of the iron which is nearest to it, becomes overcharged, &c. because that part of the magnet, being deprived of its magnetic fluid, attracts the magnetic fluid of the iron to that extremity of the iron which lies nearest to itself.

In consequence of which it appears, that, in order to give magnetism to a body, as a piece of steel, the strength of the magnet employed must be such as to overcome the resistance, which the substance of the steel makes against the free passage of the magnetic fluid; hence, a piece of soft steel is rendered magnetic more easily than a hard one; hence a stronger magnet will render magnetic such ferruginous bodies, as other smaller magnets have no power upon. The actions of two magnets upon each other is likewise explained by this hypothesis; for when two equal magnets oppose their contrary poles to each other, they preserve and strengthen the power; but when the homologous poles of two magnets are placed near, then, if the strength and quality of those magnets be equal, they will only diminish each other's magnetic power; but, if they be unequal in power or other quality, as in their hardness, shape, &c. then the weakest will have its power diminished, destroyed, or changed, in proportion to its softness, weakness of magnetism, and other circumstances.

There is undoubtedly a great degree of ingenuity in the suggestions here advanced, but more recent experiments have proved that it only approximates to the true state of the case. It will be seen as we proceed, that the magnetic fluids in simple iron bodies are not, as this author imagines, accumulated in the distinct parts of the body, but that the displacement which produces the phenomena takes place only in every individual particle; which are each in precisely the same state as Æpinus attributes to the whole body.

al at- on. 20. We believe that we have now enumerated most of the principal facts connected with magnetic discoveries; many of them, it is true, have been touched upon very slightly, and some perhaps may have altogether escaped us; some also have been purposely deferred till we arrive at the proper places to speak of them in the following treatise. There is, however, still one circumstance to be mentioned, which can scarcely be called a discovery, because it was the necessary consequence of facts long established; we allude to the *local attraction* of vessels on the compass. As it has been known from the earliest times that iron attracts the magnet, it was a necessary inference that the great quantity of iron, particularly in ships of war, must have a tendency to disturb the natural direction of the compass: but so little was known of the laws of magnetism, that it was supposed, as the iron, guns, &c. were pretty equally distributed on each side of the ship, that the attractions would act with equal power on the needle, and thereby leave it under the same circumstances as if there had been no iron on board. It is only within a few years that this source of error has been detected by navigators; but since it has been known, it has attracted considerable attention, and all confidence has been withdrawn from those magnetic charts which attempt to lay down the variation of the compass for different parts of the world; these having been all constructed from observations made under circumstances liable to this source of error.

The first notice of such an effect, that we are aware of, occurs in the voyages of Captain Cook, it having been first observed by Mr. Wales, his astronomer; the cause of the deviation of the needle, however, seems not to have been suspected, and the subject at that time attracted little attention.

The next reference to the local attraction of vessels, and in which the cause is clearly pointed out, is found in Walker's *Treatise on Magnetism*, published in 1794: it is contained in a report from Mr. Downie, master of his Majesty's ship Glory, where he says, "I am convinced that the quantity and vicinity of iron in most ships, have an effect in attracting the needle; for it is found by experience that the needle will not always point in the same direction, when placed in different parts of the ship, also it is rarely found that two ships steering the same course, by their respective compasses, will go exactly parallel to each other; yet these compasses, when compared on board the same ship, will agree exactly."

A few years after this, the action of the iron of the vessel was more minutely noticed by Captain Flinders, who was the first to trace its connection with the dip of the needle, and through whose perseverance some attention was paid by government to the subject, and several experiments were made, by order of the Admiralty, on various ships at the Nore, by which the general fact was established. The subject, however, seems to have been again lost sight of, till Mr. Bain published his valuable little *Treatise on the Variation of the Compass*; in which the fatal consequences attending this source of error are put in so clear a point of view, as to strike the most indifferent reader. And as at this time our Arctic expeditions were in contemplation, the local attraction of the vessels was one of the objects to which the attention of the officers was particularly directed. The results of the experiments made in these instances are given by Captains Ross and Parry, in the accounts of their respective voyages, as also by Captain Sabine, in the *Philosophical Transactions*, part i. 1819; another memoir on this subject is also published in the same volume, by Captain Scoresby. These statements all agree in establishing the existence of this source of error, and in showing the necessity of some method of correcting it, particularly in high latitudes.

This subject we shall therefore examine as we proceed in a subsequent part of our treatise; and it would consequently be useless to pursue it farther in this historical sketch. The works that have been published on magnetism, as a distinct science, are very few: what has been written, has been generally given in the Transactions of the learned Societies of Europe, particularly in those of the Royal Society of London. But, in general, these communications are merely the details of some isolated facts, unconnected with, and frequently contradicting each other. A *Treatise of Magnetism*, by Cavallo, 1784; another by Walker, (1794;) and *The Magnetic Atlas*, by Churchman, 1804, are, we believe, the only modern English works on this subject. To which we may add, the *Essay on Magnetic Attractions*, by Mr. Barlow, although it does not profess to be a complete treatise. A small work has also lately been published by Mr. Lecount, a midshipman in the Royal Navy, *On the Polarization of Iron*.

PART II.

§ I. *General definitions; and properties peculiar to magnetic bodies.*

Definitions. 1. As it is the property of attracting and being attracted by iron that has given the denomination of magnet to the mineral possessing that quality, so we may say conversely, that any mineral in which this property exists is a *natural magnet*; for no other condition is requisite to constitute it such a body. The mineralogical characteristic of the *native magnet* is given as follows in Professor Kidd's *Outlines of Mineralogy*:—

"*Native Magnet*. Applicable peculiarly to the amorphous granular varieties.

"It is not malleable.

"Of a grey colour, and dull metallic lustre.

"The colour of its powder is black.

"All varieties are attracted by the magnet; and some have the attractive power of that substance: the first, according to Mr. Kirwan's suggestion, may be called passively, and the latter actively, magnetic.

"Specific gravity varies from 4·24 to 4·93.

"Primitive form, the regular octohedron.

Iron	85	to	76
Oxygen......	15	to	24
	100		100"

What is stated above, of passively magnetic ore, has the property of common iron; viz. it is attracted by the magnet, and will of course attract the magnet; but it has no perceptible action on simple iron.

2. Artificial magnets are those which have the magnetic property communicated to them; they are generally bars of steel rendered hard for the purpose.

3. If a piece of natural magnet be immersed or rolled in iron filings, it will be found that these latter are attracted to its surface in various places, and in different quantities; but in general there are two points only in which the accumulation is very considerable. These points are called the *poles of the magnet.*

4. Let now a small steel bar, or needle which has been hardened, be brought in contact with these poles; viz. first one of its extremities with one of the poles, and then its other extremity with the other pole; and it will be found to have acquired a magnetic property similar to that of the natural magnet, and more or less, according to the power of the latter, and the time it was allowed to remain in contact; at least this will be the case within certain limits; but its power can in no way be made to surpass a certain quantity. This will be explained as we proceed; we merely mention the experiment in this place, in order to show how the communication of the magnetic property may be made, and that we may be able, as soon as possible, to proceed in our investigation with artificial instead of natural magnets; the former having all the properties of the latter, with the advantages of a greater uniformity of figure, texture, and distribution of the magnetic power.

First principle of magnetic action.

5. Let us conceive now a second equal steel bar to be submitted to the same process as the first; viz. to have its extremities brought respectively in contact with the two poles of the natural magnet, and this likewise will have acquired the same properties: we shall thus have a pair of equal magnets, and which, in the subsequent part of this chapter, will be those we shall suppose to be employed in the experiments we shall have occasion to refer to.

Figs. 1 and 2, plate I.

Let us now suspend each of these magnetic bars, at a distance from each other, by a piece of untwisted silk, so that they are perfectly horizontal, and free to take any direction; and it will be found that both bars will arrange themselves according to a certain direction; viz. nearly north and south, and perfectly parallel to each other: the ends of each, which were touched with the same poles of the natural magnet, being both directed towards the same point. Let that end of the one magnetic bar which is turned to the north, be supposed to be marked N′; and the corresponding end of the other N; also let the other two ends be marked S and S′, as in figures 1 and 2, plate I, Magnetism. If now one of the magnets N S be left suspended, and the other removed, and if the ends of this last be applied successively, at a small distance from the ends of the other, the following phenomena will be observed: viz. the extremity N′ of the magnet N′ S′ will attract the extremity S of the other magnet, and will repel the end N; and in like manner the end S′, will attract N, and repel S: that is, the ends which have a contrary direction will attract each other, and those which have the same direction, will repel. This may be considered one of the fundamental principles of magnetic action.

Of terres- trial direc- tion.

6. The magnetic bar N S being supposed still suspended, let it be drawn out of its present position by the action of the magnet N′ S′, or in any other way, and then left to itself, and it will be found to oscillate backwards and forwards on each side of the line of its natural direction; the arcs of vibration diminishing more and more, till at length it will come to rest in its first position.

As this is the case in all parts of the earth, we must attribute this tendency of the bar to a certain point or direction, to some magnetic power in the earth itself, and which operates upon a magnet in particular, in something like the same manner as gravity acts upon any pendulum in general, when the latter is disturbed from its natural vertical position; and this power, whatever it may be, we are accustomed to speak of under the denomination of *terrestrial magnetism;* or the *magnetic power of the earth.*

Definition of poles.

We have seen that one end in particular of the magnet will direct itself towards the north, and the other towards the south; we must therefore admit a north and south polarity in the earth; we call that north, which is towards the north terrestrial pole; and the south, that which is towards the south pole, but then, in order to preserve a certain analogy in our language, it must be observed, that the end of our magnet which has been marked N, to indicate the direction in which it points, is in reality its south pole; for it must be of a contrary name or character to that pole of the terrestrial magnet by which it is attracted; and the same with the other extremity that points to the south, which must be considered as the north pole, being attracted towards the south pole of the earth. This nomenclature appears to be necessary, for the sake of analogy in certain cases; but as it might lead to some ambiguity if generally used, we shall, except in those instances where it is actually necessary, speak of the *north end* and *south end* of the magnet, meaning thereby the ends which point to the north and south, and which, if spoken of as poles, must be called the south and north pole respectively.

Dip of the needle.

7. If instead of balancing the bar N S after being magnetised, we suppose it to be balanced first, and then touched on the magnet, it will be found, that the needle will still point towards the north and south as before, except that it will be no longer horizontal, but will have its north end (if the experiment is made any where in these latitudes) inclined below the horizon several degrees, but more or less according to the place in which the experiment is performed. This angle is called the *dip,* or *inclination* of the magnetic needle, and the angle which it forms with the true north is called the *declination* or *variation;* where the deviation of the north end is to the eastward, it is called *east* variation, and where it is to the westward, *west* variation: that is, the estimation is always made in reference to the north end of the needle.

Fig. 3.

8. The vertical plane which passes through the needle when freely suspended, as supposed in the last experiment, is called the *magnetic meridian,* as S N E C; and the plane which passes at right angles to this, and to the needle, through its centre, is called the *magnetic equator* of that needle, as E H C W, (see fig. 3.) and which therefore inclines to the horizon at an angle equal to the complement of the dip. These definitions are all that appear necessary for our present purpose, and we shall therefore proceed to

indicate, in the most concise terms possible, the theoretical principles now generally admitted, for explaining the more common and obvious magnetic phenomena; the complete developement of the theory, however, will not be attempted in this division of our treatise.

In the preceding articles we have shewn, that a small steel bar, or needle, may be rendered magnetic by bringing its extremities successively in contact with the poles of a strong magnet; this experiment, however, was simply referred to for the purpose of furnishing us with two magnets, in order the better to illustrate our definitions, and to establish the first principles of the science; for although small pieces of steel wire may be by the method there described rendered permanently magnetic, yet the power thus developed will be very inconsiderable. To produce magnetic bars of great power, other modes of proceeding must be had recourse to, and which it will be our object to illustrate in the following section.

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9 It is our intention to treat this part of the subject merely as experimental; yet, in order to see somewhat more intelligibly the intention of the processes recommended to be followed, we propose to explain concisely the theoretical view which is generally taken of the subject.

First, as there are obviously two distinct species of action in magnetic bodies, the one repelling where the other attracts, philosophers have been in the habit of attributing them to the existence of two fluids, whose properties are such, that the particles of the one will attract the particles of the other, but each of them repelling the particles of the same kind as itself. This at least is supposed to be the case as soon as they are separated from each other; but in their combined state they are conceived to neutralize each other, and thus to produce that state which we have defined by the term *passively magnetic*.

This is precisely the case in simple iron and soft steel bars; that is, they possess a degree of magnetism in a latent state, which may be immediately called into action by a body actively magnetic; but they will produce no action, nor receive any impression from another passively magnetic body.

10. If the iron or steel bars on which we are experimenting be perfectly soft and malleable, the magnetism excited in them by the presence of a natural or artificial magnet, will remain only while the magnet and iron retain their relative situations; for which reason *soft iron* can never be rendered permanently magnetic; but if a piece of steel be made hard, and then rendered magnetic, it will retain its state for a long time; in fact, it will become permanently magnetic if it be rendered sufficiently hard; viz. so as not to admit of being cut by a file; but otherwise it will, after a greater or less time, return entirely, or very nearly, to its natural passively magnetic state.

11. With reference to the separation of the two magnetic fluids from each other, it is proper to observe, that they are not imagined to be so far separated, as that one half of the bar, for example, shall contain one kind of magnetism, and the other half the contrary, as some authors have supposed: on the contrary, there are strong reasons for assuming that the displacement takes place only in each individual particle of the iron or steel. For if a bar rendered permanently magnetic be parted in the middle, it will still be found that each half of it is a complete magnet, having a north and south pole as when it was whole; and that the two faces, which were immediately before in contact, exhibit now opposite magnetic powers, that which belongs to the half bar, or needle where the north pole is, becoming a south pole, and the other face a north pole; and this will be found to be the case, however minute the part is which is detached from the original bar. This is, we conceive, a decisive demonstration, that the fluids are only displaced in each particle, and not wholly separated, as is the case in electricity.

Fig. 4.

12. To illustrate this hypothesis, let S N (fig. 4.) represent a line of magnetic particles in any passively magnetic body; then, if we conceive two magnets to be applied, with contrary poles, at N and S, these, by the hypothesis, will separate the two latent fluids from each other, and produce that kind of arrangement which we have endeavoured to indicate in fig. 5; where the shaded parts are supposed to represent one species of the magnetic fluids, and the light parts the other. This is at least perfectly consistent with the facts stated above, that every detached part of a magnetic bar is similar to the whole, which would not be the case if we were to suppose the accumulation of each fluid to be made in the opposite branches.

13. In order to explain the well-known fact, that soft iron and steel will only admit of a temporary state of magnetism, while steel, properly hardened, receives it permanently, it has been assumed, that the cohesive power of the hardened steel resists the recombination of the fluids after separation; but that in its soft state, the resistance of the metal is inferior in its power to that with which the fluids have a tendency to combine; and thus it is, that in one case we produce a permanent, and in the other only a temporary magnetism. This hypothesis is rendered the more probable from the known fact, that the harder the steel is, the more difficult it is to produce in it the full degree of magnetic developement; the separation being, even with a strong exciting power, produced with difficulty, and the recombination seems to be impossible without the application of a similar process.

On the other hand, perfectly soft iron or steel receives its entire magnetic developement almost instantaneously, and loses it with equal facility; and in all intermediate states of hardness, the time and labour requisite to produce the developement, seems to be proportional to those with which the recombination is effected.

This being premised, it follows that the formation of artificial magnets consists only in the separation of the two fluids from each other; and this will serve as an illustration of the processes recommended in the following experiments.

§ II. *Different methods of producing artificial magnets.*

On the method of making artificial magnets.

14. The best proportional dimensions of steel bars for the formation of artificial magnets, appear to be, to have the length about ten or twelve times the breadth, and the thickness about one-third of the breadth; and in order that they may retain the magnetism permanently after it has been excited, the bars should be previously rendered as hard as possible. We shall, in a subsequent section, examine the effects

of temper on the excitement and retention of magnetism; but what is stated above appears to be sufficient for our present purpose; that is, we shall suppose the bars prepared for the experiment, and that they only require to be rendered magnetic by means of another magnet.

1. *Method of magnetising practised by the philosophers of the seventeenth century.*

15. This consisted simply in rubbing the steel bar to be magnetised, at right angles upon one of the poles of either a natural or artificial magnet, as shewn in fig. 6; where A B is intended to represent the given magnet, and *a b* the bar required to be magnetised, which is supposed to be passed over the pole A of the magnet from *b* to *a*, and which by this means will be found slightly magnetic; but the developement can never be complete unless the bar be extremely small, or the magnet very powerful, as will appear from the following considerations. Fig. 6.

For the sake of simplifying our remarks, we shall distinguish the two species of magnetism, or magnetic fluids, by the terms *austral*, or southern, and *boreal*, or northern, it being understood that these are to be employed under the same conditions as the northern and southern poles, (art. 7;) so that, that branch, or half, of a magnetic bar which is directed to the north, is supposed to have the southern or austral magnetism predominant, and the other the boreal or northern. This being premised, let us consider the bar *a b*, when its extremity *b* begins to be first applied on the pole A of the magnet A B, and let us suppose that A is the south pole of this loadstone. In this case the austral action of the part C A predominating over the boreal magnetism of the part C B, it will produce in *b* a decomposition of the latent fluids; the austral magnetism of each particle will be repelled towards a, and the boreal attracted towards *b*, and will thus render *b* a north pole.

But when the pole A quits the extremity *b*, and comes in contact with the succeeding points in *a b*, it will produce upon each of them the very same effect; that is, it will attract the boreal magnetism of each particle towards the actual point of contact, and will repel from it the austral magnetism. But the continuance of this repulsion will at last be found to have destroyed entirely the first decomposition which it had excited by immediate contact at *b*. In proportion as it advances on the bar, the pole will continue to produce the same effect, and to destroy successively by its influence at a distance, the decomposition it had produced by actual contact upon the points which it had previously touched. At the last extremity, however, this destruction no longer takes place, so that the extremity *a* of the bar *a b*, is ultimately left either precisely or very nearly in the state in which *b* was at the commencement of the operation; little or nothing being produced in this way beyond what would have been effected by simple contact.

Another method of procedure, much more effectual, and very simple, which we can scarcely conceive not to have been practised, although it has not been described by the early writers, is, to rub first from the middle *c* of the bar upon the pole A the half *c b*, and then on the pole B, the half *c a*, which it is obvious must have produced a more powerful developement, because in this case, each extremity would have been left in that state of magnetic action which it was intended it should possess.

2. *To magnetise steel bars according to the method of Dr. Knight.*

16. It was only about the middle of the last century that any important advance was made in the formation of artificial magnets: this was effected by Dr. Gowan Knight, a physician in London, whose mode of procedure was afterwards made known by Mr. Wilson; for the Doctor himself enveloped his discovery in great mystery, and refused to discover the secret; it appears, however, to have been as follows.

Having joined by their contrary poles two strong magnetic bars placed in the same line, he laid the middle of the bar he intended to magnetise exactly over their point of junction, (fig. 7.) the bar or needle having been tempered at what the workmen call the *cherry red heat*. He then separated the bars, drawing them straight from each other, and by this means each half was passed over the pole of the contrary name to that which was intended to be given to the corresponding ends of the needle or magnet, and which thus acquired a much more intense degree of magnetism than had hitherto been obtained; and perhaps there is no better method known at present when we have to operate upon a single bar or needle.

It is obvious, that in this process each magnet acts upon the half of the small bar, which it passes over as in the former method; but in that case the influence of the same magnet acted alone over all the length of the bar, in order to develope the two magnetisms, whereas in this way, the decomposition is favoured by the presence of the other magnet; for in all the points which lie between them, their influence is combined, and the kind of magnetism that is attracted by one to the one extremity of the bar, is at the same time repelled by the other to the same extremity. By employing this method, and making use of bars very large and strongly magnetised, it is still found that small bars, such for instance as compass needles, will acquire nearly all the magnetism they are susceptible of. Dr. Knight invented also a particular sort of ferruginous paste, with which he made bars which acquired a very high degree of magnetic power. This paste is described at length in a subsequent article; at present it will be more to our purpose to proceed with an account of the different processes for magnetising:

3. *Du Hamel's method of touching artificial magnets.*

17. This philosopher, to whom we are indebted for several valuable experiments in different branches of science, proceeded as follows.

Having placed parallel to one another two bars of steel of the same length, A B, A′ B′, (fig. 8.) he united their extremities by two small prisms of soft iron, A B′, A′ B, so as to form a right-angled parallelogram. He then took two bundles of bars, *a b*, *a′ b′*, previously magnetised, or two strong magnets, and united their poles of different names towards the middle of one of the bars of steel; after which, inclining the two magnets as represented in the figure, he made them pass towards each extremity of the bar; and by a successive repetition of this friction upon each of the bars, he obtained a considerable degree of magnetism. By this arrangement, each bundle or magnet acts upon the half of the bar which it runs over, as in the first method; but there is here this important difference, that the soft iron becomes as it were a connecting link in the succession of points or particles of the developed fluid, and thus holds them in that state, while the system or bundles of magnets are employed in exciting a farther developement. A system of small magnets combined have also a decided advantage in point of strength over a solid steel bar of the same size; because the magnetism in each bar in the bundle may be more completely excited, than in the corresponding interior particles of the solid bar. We may, however, improve this method, by adopting a part of that of Æpinus; that is, by connecting the ends of the steel bars required to be magnetised, by two other strong magnets, instead of using soft iron, as above described. See a subsequent article.

4. *Michel's method of double touch.*

18. Mr. Michel employed two bundles of bars, A B, A′ B′, fig. 9, strongly magnetised, and parallel to each other, the poles of contrary name having been united at each extremity, in such a manner, however, that there remained an interval between them of about one-third of an inch. He then placed in the same straight line several bars, *a b*, *a′ b′*, *a″ b″*, which he wished to magnetise; and then he caused to pass over these bars, at right angles, and in the direction of the line formed by them, one of the extremities of the double bundle. By this method he found that the intermediate bars in the chain *a b*, *a′ b′*, &c. acquired a great magnetic force; which ought necessarily to happen, viewing the subject in the light we have given of it in (article 12.) fig. 4 and 5. For the different bars, *a b*, *a′ b′*, &c. placed in contact by their extremities, have here the same effect as the small bars of iron employed by Du Hamel; but with this difference, that they are not rendered continuous, and which is the reason that the developement in the extreme bars is inferior to that in the intermediate ones.

5. *Method proposed by Æpinus for touching artificial magnets.*

19. This method is exemplified in fig. 10; it differs from that of Du Hamel in this, that he allowed the poles of the two systems of magnets to move at a small distance from one another, without even separating them, while he inclined them in an opposite direction, as recommended by the former philosopher. He also united the ends of the bars with strong magnets, instead of the soft iron employed by Du Hamel; and thus produced a more powerful developement of magnetism, where the bars were of considerable dimensions; but in other cases this had no advantage over the former method.

6. *Dr. Robison's method of making artificial magnets.*

20. This consisted in raising the temperature of the bar at the moment of magnetising, whereby the resistance to magnetic developement is much diminished; but there is a danger of its leaving the steel too soft. This philosopher, also, when he wished to produce a very uniform developement, placed the bar heated to a proper degree for being tempered, between the poles of two powerful magnets, and then immersed the whole in cold water, in order to give the proper degree of temper to the bar.

7. *Coulomb's method of forming bundles or systems of magnets.*

Of the formation of systems of magnets.

21. For this purpose he employed steel bars, tempered at a cherry red heat, having a length of about twenty-one or twenty-two inches, breadth $\frac{6}{10}$ths of an inch, and a thickness of $\frac{1}{5}$th.

These being separately magnetised to saturation, they were united by their poles of the same name, and formed into beds of five bars each, separated by small parallelopipeds of soft iron, and which projected a little beyond their extremities, (see fig. 11.) These formed his fixed batteries, (fig. 12.)

Fig. 11 and 12.

The moving bundles he commonly formed of four bars of tempered steel, similar to the above; sixteen inches long, $\frac{3}{5}$ths of an inch broad, and $\frac{1}{5}$th thick. After magnetising them as strongly as possible, he united two of them by their widths, and two by their thicknesses, which gave to each bundle or system a width of 1·2 inch, and a thickness of $\frac{2}{5}$ths of an inch.

An improvement in this combination is recommended by the writer of the article Magnetism, in Brewster's *Encyclopædia*, which, from experience, he says he has found to be of considerable importance. This consists in making the centre magnets project a little beyond those nearer the sides, the centre one having the greatest projection, as shown in fig. 13. The whole being bound together with a collar of soft iron, held firmly by a screw. The soft iron at the extremities, called the armour, being terminated in a trapezoidal form, as represented in the drawing. With this arrangement the operation of magnetism is performed by placing the large bundles in the same straight line; so that their north and south poles are opposite to each other, (see fig. 13.) The bar to be magnetised is then placed upon one extremity of their armours, in such a manner that it goes very little over it; after which, two moving bundles are placed upon the centre of the bar, with the south pole of the one opposite the north pole of the corresponding fixed bundle, and the north pole of the other opposite the south pole of the other fixed bundle; and the two moveable ones inclined at an angle varying from 15° to 20°. This being done, we may proceed according to the methods either of Du Hamel or Æpinus, and it was in this way that Coulomb was enabled to make the comparison between these two processes, which we have detailed in the following section.

Fig. 13.

8. *Method of touching and forming horse-shoe magnets.*

Of horse-shoe magnets.

22. This denomination is derived from the form given to the bar, which is nearly that of a horse-shoe, except that experience has shewn, that the branches ought to be longer, and the two extremities brought nearer to each other than in the actual horse-shoe; see fig. 15.

Fig. 15.

In order to excite magnetism in a bar bent into this form, we must connect the two extremities with a piece of soft iron, and it will then strongly resemble the combination of Du Hamel explained above, and we may proceed according to the same principle to develope the magnetism in the two branches; or we may, after the manner of Æpinus, apply two strong magnets, one to each pole, and connect their extremities either with a piece of soft iron, or with another magnet; or again, we may apply two horse-shoe magnets to each other, uniting them so as to have the ends which are intended to be of contrary names in contact; we may then rub them with another strong horse-shoe magnet, placing the latter so that the north pole of it is next to the south pole of one of the horse-shoe formed bars, and consequently its south pole next to the north pole; carrying the moveable magnet round and round always in the same direction, it does not signify which, and it will soon be found that a very high degree of magnetic action will be thus excited; we believe (after numerous experiments) much more than can be done by any other means, see fig. 16, 17. Fig. 16 and 17.

9. *Compound horse-shoe magnets.*

Compound horse-shoe magnets.

23. These consist of a number of bars, or plates (all of the form last described,) combined together by screws, or otherwise, and which thus acquire an immense sustaining power, even to the amount of several hundred weight.

We shall describe here the process we have followed in the construction of a small compound magnet of this kind, see fig. 15. The bars before being bent were twelve inches each, or at least after being bent the length of each was six inches, they are an inch broad at the curved part, three-fourths of an inch at their extremities, or poles, and are each one-fourth of an inch thick: they were filed very nicely to correspond and lie flat one upon the other. They were then each drilled with three holes as shewn in the figure; through these holes screws passed, and nine such bars were bound firmly together; the hole of one of the outside bars being *tapped* to receive the thread of the screws, and the holes on the other outside one *countersunk* to receive the heads of the screws, in order to prevent any unsightly appearance. In this form the face or ends of the bars were very nicely filed, so as to render the surface of both very plain and smooth. The several bars were then disunited and each carefully hardened by a good workman, precautions having been taken to prevent warping. This being done, and the bars well cleaned and brightened, but not polished, they were separately magnetised in the manner above described; and lastly, they were reunited, and their extremities connected by a piece of soft iron, or *lifter*, furnished with a hook to suspend a weight upon. It should be stated, that notwithstanding the *face* be perfectly flat before hardening, it will always require to be properly rubbed down with putty, on a flat surface, after hardening. For much of the sustaining power depends upon the accuracy of the contact between the poles of the magnet and the lifter.

This combination suspends with ease forty pounds. We have found however since, by experiment, that more than a proportionate degree of power is obtained by having the bars longer, in comparison with their breadth, different views of this magnet are shewn in figure 18, 19, and 20. Fig. 18, 19, 20.

10. *To magnetise a number of steel bars, by means of a horse-shoe magnet.*

24. The method described below we have practised with very great facility and success in several instances; we shall select a particular case for describing it. Having occasion to magnetise, at one time, eighteen pair of bar magnets, each twelve inches in length, an inch and a quarter broad, and $\frac{1}{8}$ths thick, they were all placed on a large platform, or table, so as to form a square, nine feet on the side; the end of each bar, marked for the north end, having been brought in contact with the south end of the next bar, throughout. The interior edges of the bars at the angles were brought together, and the opening thus left externally, was filled up by four pieces of soft iron, an inch and a quarter square, and of the same thickness as the steel bars; as shown for one angle in fig. 21. The horse-shoe magnet, which was that above described, was now placed on any one of the bars, so that its north end was towards the south end, or, that intended to be the south end of the bar, and its south to the north. It was then carried or rubbed along the entire circuit, after which the bars had obviously acquired considerable magnetic developement, as shewn by their attraction towards each other; the same was repeated without removing the horse-shoe a dozen times, and then, but still without removing the horse-shoe magnet, each bar was turned, one by one, with its other side upwards, without breaking the connection; and then the process above described was continued, passing the horse-shoe round again another dozen times, after which the entire circuit of magnets was found so highly charged that considerable force was requisite to separate the bars, in order to combine them in pairs. The whole operation occupied not more than half an hour.

The processes last described, although they are capable of exciting almost instantaneously a high degree of magnetic developement, yet they require powerful magnets to operate with, and means which are not in the possession of every amateur in these experiments. To these it will be interesting to learn how, by more slow, but not less certain means, we may produce magnetism, where no such power is apparent, and hence by degrees to acquire that intense force we have been just describing: to such readers Mr. Canton's method cannot fail of being highly instructive.

11. *Canton's method of forming artificial magnets.*

25. The following is Mr. Canton's directions as they are given in *Phil. Trans. Abridged*, vol. x., p. 131. "Procure a dozen bars; six of soft steel, each three inches long, one quarter of an inch broad, and $\frac{1}{20}$th of an inch thick, with two pieces of iron, each half the length of one of the bars, but of the same breadth and thickness; and six of hard steel, each five inches and a half long, half an inch broad, and $\frac{3}{20}$ths of an inch thick, with two pieces of iron of half the length, but the whole breadth and thickness of one of the hard bars; and let all the bars be marked with a line quite round them at one end.

"Then take an iron poker and tongs, or two bars of iron, (fig. 21. plate II.) the larger they are, and the longer they have been used, the better; and fixing the poker upright between the knees, hold to it near the top one of the soft bars, having its marked end downwards, by a piece of sewing silk, which must be pulled tight with the left hand, that the bar may not slide; then grasping the tongs with the right hand, a little below the middle, and holding them nearly in a vertical position, let the bar be stroked by the lower end, from the bottom to the top, about ten times on each side, which will give it a magnetic power sufficient to lift a small key at the marked end; which end, if the bar was suspended on a point, would turn towards the north, and is therefore called the north end, and the unmarked end is, for the same reason, called the south end.

"Four of the soft bars being impregnated after this manner, lay the other two, (fig. 22.) parallel to each other, at the distance of about one-fourth of an inch, between the two pieces of iron belonging to them, a north and a south pole against each piece of iron; then take two of the four bars already magnetical, and place them together, so as to make a double bar in thickness, the north pole of one even with the south pole of the other, and the remaining two being put to these, one on each side, so as to have two north and two south poles together. Separate the north pole from the south pole at one end by a large pin, and place them perpendicularly with that end downwards, on the middle of one of the parallel bars, the two north poles towards the south, and the two south poles towards its north end; slide them backward and forward three or four times the whole length of the bar, and removing them from the middle of this, place them on the middle of the other bar, as before directed, and go over that in the same manner; then turn both the bars the other side upwards, and repeat the former operation: this being done, take the two from between the pieces of iron, and placing the two outermost of the touching bars in their stead, let the other two be the outermost of the four to touch these with: and this process being repeated till each pair of bars have been touched three or four times over, which will give them a considerable magnetic power. Then put the whole half dozen together, after the manner of the four, (fig. 23.) and touch with them two pair of the hard bars, placed between their irons at the distance of about half an inch from each other: then lay the soft bar aside, and with the four hard ones let the other two be impregnated, (fig. 24.) holding the bars apart at the lower end near $\frac{3}{10}$ths of an inch, to which distance let them be separated after they are set on the parallel bar, and brought together again before they are taken off: this being observed, proceed according to the method described above, till each pair has been touched two or three times over. But as this vertical way of touching a bar will not give it quite so much of the magnetic virtue as it will receive, let each pair be now touched once or twice over, in their parallel position between the irons, (fig. 25.) with two of the bars held horizontally, or nearly so, by drawing at the same time the north of one, from the middle over the south end, and the south of the other, from the middle over the north end of a parallel bar; then bringing them to the middle again without touching the parallel bar, give three or four of these horizontal strokes to each side. The horizontal touch, after the vertical, will make the bars as strong as they can possibly be made; as appears by their not receiving any additional strength, when the vertical touch is given by a greater number of bars, and the horizontal by those of a superior magnetic power. This whole process may be gone through in half an hour, and each of the larger bars, if well hardened, may be made to lift twenty-eight troy ounces, and sometimes more. And when these bars are thus impregnated, they will give to a hard bar of the same size its full virtue in less than two minutes, and therefore will answer all the purposes of magnetism in navigation and experimental philosophy, much better than the loadstone, which is well known not to have sufficient power to impregnate hard bars. The half dozen being put into a case (fig. 26.) in such a manner, as that two poles of the same denomination may not be together, and their irons with them, as one bar, they will retain the virtue they have received; but if their power should, by making experiments, be ever so far impaired, it may be restored without any foreign assistance in a few minutes. And if out of curiosity, a much larger set of bars should be required, these will communicate to them a sufficient power to proceed with, and they may in a short time, by the same method, be brought to their full strength." *Phil. Trans.* 1750.

Fig. 26.

12. *Captain Scoresby's method of making artificial magnets.*

Captain Scoresby's method.

26. Since the above was written, Captain Scoresby has published in the *Phil. Trans.* for 1822, part ii., a set of experiments relative to the making of magnets by percussion, of which the following extract will not be uninteresting; it is much more simple and more effectual than the preceding. The author observes, "The strong magnetising effects of percussion on soft steel, induced me to apply this property to the formation of magnets; for this purpose I procured two bars of soft steel, thirty inches long, and an inch broad, also six other flat bars of soft steel, eight inches long, and half an inch broad, and a large bar of soft iron. The large steel and iron bars were not, however, absolutely necessary, as common pokers answer the purpose very well; but I was desirous to accelerate the process by the use of substances capable of aiding the developement of the magnetical properties in steel: the large iron bar was first hammered in a vertical position, it was then laid on the ground with its acquired south pole towards the south, and upon this end of it the large steel bars were rested while they were hammered; they were also hammered upon each other. On the summit of one of the large steel bars, each of the small bars, held also vertically, was hammered in succession; and in a few minutes they had all acquired considerable lifting powers. Two of the smaller bars, connected by two short pieces of soft iron in the form of a parallelogram, were now rubbed with the other four bars in the manner of Canton. These were then changed for two others, and these again for the last two. After treating each pair of bars in this way for a number of times, and changing them whenever the manipulations had been continued for about a minute, the whole of the bars were at length found to be magnetised to saturation; each pair readily lifting above eight ounces.

"In accomplishing this object, I took particular care that no magnetic substance was used in the process. All the bars were freed of magnetism before the experiment, so that none of them, not even the largest, produced a deviation of five degrees on the compass at three inches distance. Any bars which had been strongly magnetised, and had had their magnetisms destroyed or neutralised, (either by hammering, heating, or by the simultaneous contact of the two poles of another magnet placed transversely,) I always found had a much greater facility for receiving polarity in the same direction as before, than the contrary. Hence it generally happened, that one blow with the original north end downwards, produced as much effect as two or three blows did with the original south end downward."

By following the above directions, any person may supply himself with magnets, by merely procuring steel bars proper for the purpose; and with these, any number may be afterwards produced by any of the methods described above.

13. *Dr. Knight's method of forming artificial magnetic paste.*

Dr.Knight's magnetic paste.

27. This process, as we have elsewhere observed, was kept a secret by the philosopher in question, but was at length published in the *Phil. Trans.* for 1779, by Mr. Benjamin Wilson.

"Having provided himself with a large quantity of clean filings of iron, Dr. K. put them into a large tub, that was more than one-third filled with clean water; he then, with great labour, worked the tub to and fro for many hours together, that the friction between the grains of iron by this treatment might break off such small parts as would remain suspended in the water for some time; the obtaining of which very small particles in sufficient quantity, seemed to him to be one of the principal desiderata in the experiment. The water being by this treatment rendered very muddy, he poured it into a clean earthen vessel, leaving the filings behind; and when the water had stood long enough to become clear, he poured it out carefully, without disturbing such of the iron sediment as still remained, which was now reduced to almost impalpable powder: this powder was afterwards removed into another vessel, in order to dry it, but as he had not obtained a proper quantity of it by this first step, he was obliged to repeat the process many times. Having at last procured enough of this very fine powder, the next thing to be done was to make a paste of it, and that with some vehicle which could contain a considerable quantity of the phlogistic principle; for this purpose he had recourse to linseed oil, in preference to all other fluids. With these two ingredients only he made a stiff paste, taking particular care to knead it well before he moulded it into convenient shapes. Sometimes, while the paste continued in its soft stâte, he would put the impression of a seal on several pieces; one of which is in the British Museum. This paste was then put upon wood, and sometimes on tiles, in order to bake or dry it before a moderate fire, at about a foot distance. The doctor found that a moderate fire was most proper, because a greater degree of heat made the composition frequently crack in many places.

"The time required for baking or drying this paste was, generally, five or six hours, before it attained a sufficient degree of hardness. When that was done, and the several baked pieces were become cold, he gave them their magnetic virtue in any direction he pleased, by placing them between the extreme ends of his magazine of artificial magnets for a few seconds or more, as he saw occasion. By this method the virtue they acquired was such, that when any one of those pieces was held between two of his best ten guinea bars, with its poles purposely inverted, it immediately of itself turned about to recover its natural direction, which the force of those very powerful bars was not sufficient to counteract."

§ III. *Experiments made with a view of comparing the power excited by the methods of magnetising above described; the effect of temper, figure, &c.*

Inten powe mate the n of vib tions,

28. Before we proceed farther in our inquiries, it may be proper to offer a few remarks upon the best criterion for judging of the advantages or disadvantages of the different modes of operating, which have been described above; the effect of temper, and of the different forms that we may give to our bars or needles. When a magnetic needle, or bar, freely suspended and in equilibrio, is placed in different situations, not very remote from each other, in comparison with the diameter of the terrestrial sphere, it will, as we have seen (art. 5.) from some invisible power or force, which we have called terrestrial magnetism, assume a certain direction, which under the limitation we have supposed, will always be parallel to a certain plane, which has been called the magnetic meridian of those places; if we go to greater distances, these directions are not parallel to each other, as we shall see when we come to treat of terrestrial magnetism; but at present, limiting ourselves to shorter distances, it is clear that the centre of force, wherever it may be, and to whatever it may be due, is so far distant, that we may consider its action as made in right lines, parallel to each other, on the several particles of the magnetic fluid in the needle; and consequently that, as in the case of gravity, we may estimate the amount of a magnetic force by the number of vibrations which a needle makes in a given time, the latter being under the influence of the magnetic force only; that is its centre of gravity being supported, as we have supposed above. The only difference in the two cases is, that in the instance of gravity, the matter in the pendulum remains constant, and the difference of power is caused by the different distances, or other circumstances relating to the power of the attracting body; whereas in those we are about to consider, the attracting power will be the same, and the change in acceleration will be due to the greater or less capacity of the vibrating body, for the matter on which that power is exerted. This view of the subject is all that is necessary for our present purpose, and from this, as in the case of pendulums, it will follow, that the time in which the same magnetic needle vibrates, under different degrees of magnetic power, will vary inversely as the square root of that power; or the force will be inversely as the square of the time of vibration, or directly as the square of the number of vibrations performed in a given time.

By th lance of torsi

Another method of estimating the directive intensity of bars and needles, is by the balance of torsion (see our article BALANCE.) Here the needle being suspended on its cradle, the quantity of torsion requisite to deflect it a certain number of degrees from its natural direction is noted, and the intensity of the needle will be proportional to the torsion.

Coulomb employed both these methods, but Captain Kater used only the latter; they are, however, now both sufficiently explained to enable us to proceed in the detail of the experiments in question

Abstract of Coulomb's experiments.

29. In order to appreciate with accuracy the different methods of making artificial magnets, this philosopher undertook to make the comparison, by means of the first of the two tests we have just been illustrating, viz. by causing them to vibrate, and counting the number of vibrations performed in a given time. He first employed wires of tempered steel of very small diameter, and then thin plates of steel untempered by annealing; from this he passed to equal plates of small thickness, tempered at a *cherry red* heat; and finally, preserving the same degree of tempering, he magnetised by different methods, bars of different thicknesses, forms, and lengths.

In order that his comparisons should be made with the greater accuracy, he began by procuring ten bars of tempered steel, having each a length of seventeen inches, one inch wide, and one-fifth of an inch in thickness, and he magnetised them one after another, as strongly as possible, by subjecting each of them to a series of operations, so that their magnetism might be equal.

His first experiment was made with a wire of tempered steel, twelve inches long, and one twenty-fifth of an inch in diameter; he first rubbed it at right angles on the pole of a single artificial magnet, he then made it oscillate in a horizontal plane, by suspending it in a paper dish, with an assemblage of untwisted fibres of silk, in order to observe the intensity of the action exerted upon it by the magnetic force of the earth.

The following are a few of his results:

1. He found that in virtue of this action, it performed

Ten oscillations in seventy-four seconds.

2. The same wire, when rubbed at right angles upon the pole of four of ten united bars, did not acquire stronger magnetism; for it likewise performed

Ten oscillations in seventy-four seconds.

3. When magnetised by the method of Du Hamel, and that of Æpinus, by submitting it to the action of a large apparatus it still performed

Ten oscillations in seventy-four seconds.

From which it was concluded, that steel wires of small diameters, acquire by all the different methods, the same degree of magnetism that they do by simple contact. This degree, therefore, has been denominated the state of *magnetic saturation*, that is to say, that the greatest possible degree of magnetic developement has been excited, of which the bar is susceptible.

Afterwards, instead of a wire, a plate of annealed steel was substituted, twelve inches long, one-third of an inch wide, and one forty-second of an inch thick; so that it differed only in its width from the preceding wire.

4. This plate being rubbed at right angles upon the pole of a single magnet, it performed

Ten oscillations in seventy-seven seconds.

5. Upon the pole of two united bars,

Ten oscillations in seventy-five seconds.

6. Upon the pole of ten united bars,

Ten oscillations in seventy-five seconds.

7. By the methods of Du Hamel and Æpinus, employing only a single moveable bar on each side,

Ten oscillations in seventy-five seconds.

In these experiments, the different effect of the methods began to be perceptible, but it was still very feeble. It, however, manifested itself in a much stronger degree as the temper was changed.

8. A plate of steel about 6.5 inches long, $\frac{2}{15}$ths of an inch wide, and $\frac{1}{42}$nd thick, tempered at *cherry red*, and rubbed at right angles upon the poles of two united bars, performed

Ten oscillations in fifty-one seconds.

9. Upon the poles of five bars united,

Ten oscillations in forty-nine seconds.

10. Upon the poles of eight and ten bars united,

Ten oscillations in forty-seven seconds and a half.

11. By uniting only two bars, and rubbing the plate upon the poles at an angle of 15° or 20°, in place of a right angle, the needle performed

Ten oscillations in forty-seven seconds and a half.

12. When magnetised by the method of Du Hamel and Æpinus, with only two moveable bars, it performed

Ten oscillations in forty-seven seconds and a half.

In these experiments, as well as in the subsequent ones, we can only draw a comparison between the duration of the oscillations performed by the same plate, successively magnetised by the same methods; for in different plates, the absolute results are altered by the change of dimensions, and by the different degrees of temper. As the thickness of the plate increases, the difference in the effect produced becomes more perceptible.

13. A plate eight inches long, $\frac{14}{15}$ths of an inch wide, and $\frac{1}{18}$th thick, after being rubbed several times at right angles upon the pole of a single bar performed

Ten oscillations in seventy-three seconds.

14. Upon the pole of four united bars,

Ten oscillations in sixty-two seconds.

15. Upon the pole of ten united bars,

Ten oscillations in fifty-nine seconds.

16. By uniting only two bars, and rubbing them upon the plate, at an inclination of 15° or 20°, the same bar performed

Ten oscillations in fifty-three seconds.

17. At the same inclination with four united bars,

Ten oscillations in forty-nine seconds.

18. At the same inclination with ten united bars,

Ten oscillations in forty-nine seconds.

19. By the method of Du Hamel and Æpinus, with one or more bars on each side, still

Ten oscillations in forty-nine seconds.

These experiments are sufficient to shew that the methods of Du Hamel and Æpinus are superior to the other, since they give the same degree of magnetism with a much smaller number of moving bars. It will be observed also, that these two methods are equally effective, when we apply them to plates which are $\frac{1}{15}$ or $\frac{2}{15}$ths of an inch thick, but when the bars or plates exceed this thickness, that of Æpinus has decidedly the advantage. Coulomb made this trial upon one of the large bars of his bundle, which was sixteen inches long, $\frac{5}{10}$ths of an inch wide, and one-fifth thick, tempered at *cherry red.*

20. By applying to one of them the method of Æpinus, with two moveable bars, rubbing upon its surface, it performed

Ten oscillations in one hundred and ten seconds.

21. By augmenting the number of moveable bars it acquired no greater power, and had therefore reached the state of saturation. But by the method

of Du Hamel, it could not be brought to the same state without employing moveable bundles composed of four bars each.

22. The same comparison was also repeated on still thicker bars, sixteen inches long, one inch broad, and $\frac{9}{16}$ths of an inch thick. By applying to it at first, the method of Æpinus, and forming moveable bundles of one, two, or three bars, it acquired a magnetism progressively increasing; but by employing bundles composed of four bars, or even of ten, or a greater number, it performed

Ten oscillations in one hundred and fifty-three seconds.

By employing the method of Du Hamel, it was impossible to attain the same degree of magnetism, even by using bundles of ten bars each. By this last operation the bar performed

Ten oscillations in one hundred and sixty-two seconds.

The former method therefore gave a stronger magnetic force than the latter, in the ratio of $(162)^2$ to $(153)^2$ or as nine to eight nearly.

Such were the principal results of Coulomb, as directed to this inquiry: the following are the experiments of Captain Kater, published in the *Phil. Trans.* for 1821, part i.

Captain Kater's experiments relative to the different effects of magnetising, &c.

Captain Kater's experiments.

30. In these experiments the directive force was ascertained by means of the balance of torsion. The author's statement of these comparisons is as follows: "Two needles were made of the same kind of steel, in the form of right-angled parallelograms, five inches long, the one seven inches wide, and the other half this width. The widest was reduced in thickness till it was of the same weight as the other, viz. 142 grains. They were in the same state of softness as was necessary to work them. The magnets were placed together perpendicularly on the centre of the needle, their opposite poles being joined; their lower extremities were then separated and kept asunder, by placing a piece of wood, a quarter of an inch thick, between them, their upper extremities remaining in contact. The magnets were then slid along the needle backwards and forwards, from end to end: this was repeated on both sides, till it was conceived the needle must be saturated."

Experiment I.

Small parallelogram directive force	655
Large parallelogram	674

The needles were again magnetised in the same manner as before, excepting that the magnets were separated at the top by a piece of wood of the same thickness as that at the bottom.

Experiment II.

Small parallelogram directive force	595
Large parallelogram	580

The magnets were placed perpendicularly together on the centre of the needle, and their lower extremities separated by a piece of wood to the distance of half the length of the needle, the upper extremities remaining in contact. They were then slid on the needle backwards and forwards, from end to end, as before.

Experiment III.

Small parallelogram directive force	760
Large parallelogram	780

The magnets joined, placed perpendicularly on the centre of the needle, as before, then moved in opposite directions, from the centre to the extremities, keeping each magnet perpendicular to the needle; afterwards joined at a distance from the needle, placed again on its centre, and the operation thus continued.

Experiment IV.

Small parallelogram directive force	993
Large parallelogram	1155

The author remarking that the surface of the small parallelogram was unequal, so as to be touched by the magnet in very few places, he filed it flat, and having reduced the large parallelogram to the same weight, they were magnetised by joining the magnets, placing them perpendicularly on the centre of the needle, separating their lower extremities, and carrying them to each end of the needle, the upper ends remaining in contact.

Experiment V.

Small parallelogram directive force	1025
Large parallelogram	1150

The needles were next magnetised according to the method of Du Hamel, the magnets being inclined to an angle of about forty-five degrees, and carried as before from the centre to the ends of the needle.

Experiment VI.

Small parallelogram directive force	1070
Large parallelogram	1170

The magnets forming with the needle an angle of about twenty degrees.

Experiment VII.

Small parallelogram directive force	1085
Large parallelogram	1195

Magnets forming an angle, with the needle of about two or three degrees.

Experiment VIII.

Small parallelogram directive force	1160
Large parallelogram	1275

Magnets laid flat on the surface of the needle, and drawn from the centre to the ends.

Experiment IX.

Small parallelogram directive force	1158
Large parallelogram	1261

Magnets forming with the needle an angle of two or three degrees, their other extremities being connected by a very soft iron wire.

Experiment X.

Small parallelogram directive force	1145
Large parallelogram	1261

The iron wire was now removed, and the needle magnetised as before, at an angle of about two or three degrees.

Experiment XI.

Small parallelogram directive force	1160
Large parallelogram	1273

Both the needles were now hardened throughout at a bright red, and then softened from the middle to within three quarters of an inch from the ends till the blue had disappeared. This was done by laying the large parallelogram on a red hot poker; but from the thickness of the small parallelogram this heat was found insufficient, and that of a lamp was employed. The needles were then magnetised as in the last experiment.

Experiment XII.

Small parallelogram directive force	1815
Large parallelogram........................	1660

It here occurred to the author, that the heat employed in tempering the large parallelogram might not have been sufficient, it was therefore exposed to the flame of the lamp; but in doing this, a small piece, which weighed ten grains, was broken off from its end. It was, however, remagnetised, and the directive force was now found to be increased to 1720.

From these last experiments, it appears that the greatest directive force was given to the needle when the magnets were inclined to it in an angle not exceeding two or three degrees, and that this force is little, if at all, influenced by extent of surface; as Captain Kater conceives the small difference in favour of the greater surface, may be due to some difference in the quality of the steel, or its temper, both of which appear to have very considerable influence on the directive force. Two needles, the one five, and the other eight inches long, were cut out of the same plate of steel; they were of equal weight, but the short one was of greater width than the other. Being magnetised to saturation their directive forces were as follows:

Experiment XIII.

Long parallelogram directive force	2275
Short parallelogram........................	1193

They were now hardened at a red heat, and tempered beyond the blue, from the middle to within an inch of the extremities.

Experiment XIV.

Long parallelogram directive force	2277
Short parallelogram........................	1865

If the mean of these experiments be taken, it will be found, as was observed by Coulomb, that the directive force of a needle of a greater length than five inches is probably as its length.

In a subsequent article we shall give Captain Kater's experiments on the best form for *compass needles*, but it will be proper first to describe that instrument. The experiments stated above, although in fact, directed to the same inquiry, were so intimately connected with the matter delivered in the preceding articles, that we considered it best given as the conclusion of this section.

§ IV. *On the magnetic action of simple iron bodies, and the relative power of different species of metal.*

31. We have defined those bodies which possess the magnetic fluids in combination, or in a latent state, as *passively magnetic*; and of this class is simple malleable iron, that is, a certain degree of magnetic energy may be temporarily impressed upon it, but it would be totally inactive without the presence of an exciting magnet.

In fact, we cannot examine the effect of iron in that latent state to which we have alluded, in consequence of the constant presence of the terrestrial magnetic action, which ought necessarily, according to the hypothesis we have advanced, (art. 12.) to produce a developement of the magnetic fluids in the iron, in the direction in which the former is most active, that is, in the natural direction of the dipping needle in any place; but it is clear that such a body can *never possess a directive quality*; for whatever may be its position, the arrangement of its magnetic fluids will always correspond in direction with those of the terrestrial magnetism.

If a bar of iron, for example, be held in a line, corresponding with the natural direction of the dip, its lower half being opposed to the boreal magnetism of the terrestrial sphere will become effectually a south pole, and its upper end a north pole, corresponding in quality to what we have called the north end and south end of a magnet. The lower extremity of such a bar ought, therefore, to attract the south end of a needle, and the upper extremity the north end. This effect may be thus exhibited.

Illustrative experiments.

Fig. 52, 53, plate VI.

32. Suspend, as in (fig. 52, plate VI.) a magnetised bar or needle at its centre by an assemblage of untwisted silk, and equilibrate it by a counter weight at its south end, so that it may maintain itself in the horizontal plane, as well as in that of the magnetic meridian. Now take a bar of soft iron, A B, about two and a half, or three feet in length, and half an inch square: then, inclining this bar very nearly or exactly in the line of the magnetic dip, bring its lower end A, near the north end of the needle, and the latter will be repelled: approach, on the contrary, the upper end B, as in fig. 53, plate VI. by gradually lowering the bar parallel to itself, and the same end will be attracted. We see, then, that in this state of inclination the bar is found suddenly animated with the magnetic influence of the terrestrial globe, as it would have been if presented to any other magnet; its inferior or lower half possessing a magnetism contrary to that which predominates in our hemisphere, viz. the austral magnetism; and the upper half acquiring the opposite species, or the boreal.

The two ends of the bar are, therefore, in the same state as the needle, *a b*, itself, and this is the reason that on bringing the end A of the bar to the extremity *a* of the needle, we observe repulsion, while on approaching the other end of the bar to the same end *a* of the needle, we find attraction.

In order to show that these phenomena depend on a sudden animation impressed upon the bar in virtue of its position, we have only to invert it end for end, its inclination remaining the same, when each of its extremities will produce again the same phenomena as we have described above; and, consequently, these phenomena will be directly the reverse of those which the same ends produced in the first instance. The magnetic poles of the bar have been therefore suddenly changed by this inversion; but, in order that this effect may be instantaneous, it is requisite to employ a bar of soft iron, and not one of hard iron or steel.

33. This experiment is given by Biot, and we have followed him in requiring soft iron for the purpose; this, however, we have ascertained not to be actually necessary. The effect is *greater* with soft iron than with hard iron or steel; but, provided the latter be not previously magnetised, the change of power, with a change of position, is equally instantaneous.

It appears, therefore, that unmagnetised iron or steel possesses a certain portion of the magnetic fluids, which are easily and instantaneously displaced by the action of any exciting magnet, without contact, and which may for distinction sake be called *free* magnetism, in contradistinction to that which after being excited in hard iron and steel, is rendered *fixed* by the resisting power of the metal to a recombination.

34. It is stated above, that soft malleable iron possesses the highest magnetic power, which is contrary to what was for a long time thought to be the case, and which, in fact, remained doubtful till Mr. Barlow undertook a course of experiments directed wholly to this inquiry; from which it will be seen, that soft iron has nearly double the magnetic power of cast iron, *when cold*, but that every description of iron and steel, when softened in a furnace, and while still soft from the effect of the heat, possesses the same, or very nearly the same quantity of free magnetism; a fact which goes far towards demonstrating the accuracy of the hypothesis which has been advanced in the preceding articles, for illustrating the action of the iron on the freely suspended needle.

Mr. Barlow's experiments on the relative magnetic power of different descriptions of iron and steel.

Comparative magnetic power of different species of iron.

35. In these experiments, the first trial was made on four different kinds of metal; viz. cast iron, soft or malleable iron, soft blistered steel, and hard blistered steel. Of each there were two bars formed, twenty-four inches in length, and one inch and a quarter square. The four bars of blistered steel were of the same quality, except that two of them were softened, and two hardened for the experiments, which latter were conducted in the following manner:

The needle was first placed, so as to read correctly north and south; a situation was then taken for the bars, so that the lower end of each, successively, was on a level with the pivot of the needle, and distant from it 10·6 inches, first to the east, and then to the west: the bar itself being made to incline in the direction of the dip.

For the purpose of recording the results, the ends of each bar were marked A and B, and the four faces were numbered 1, 2, 3, 4, each of which was successively turned towards the needle; but as little or no difference could thus be detected, the author simply gives the results of each end of the several bars, and the means of each different specimen.

36. *Table of the deviations produced on a magnetised needle, by different descriptions of iron.*

Description of metal.		East of needle.		West of needle.		Mean.
		End A.	End B.	End A.	End B.	
Cast Iron	No. 1.	7° 30′	7° 37′	8° 0′	7° 45′	7° 48′
	No. 2.	6 30	9 30	6 0	9 38	
Malleable Iron	No. 1.	15 30	16 22	15 30	16 22	15 54
	No. 2.	16 0	15 45	16 0	15 45	
Soft blistered steel	No. 1.	10 56	9 56	10 52	9 56	10 50
	No. 2.	14 22	8 7	14 22	8 7	
Hard blistered steel	No. 1.	9 56	8 0	10 0	8 0	8 37
	No. 2.	9 30	7 0	9 30	7 0	

A similar series of experiments was made at the distance of 6·7 inches, which gave angles of deviation very nearly proportional to the above.

37. The first thing which calls for particular remark in these experiments is, the near agreement in the mean results of each of the two bars of the same kind, as for example in the two cast iron bars, which differ from each other only a few minutes, and the same with the two of soft iron. With the steel bars there is rather a greater difference, but even here the agreement is sufficiently close to show that the results obtained were not merely accidental, but that they exhibit the permanent difference in the magnetic quality of these different kinds of iron.

Another remarkable fact is, that although we find in some of the bars a considerable difference in the action of their two extremities, yet the mean of the two still agree with the mean of the other bar of the same kind. This is particularly exemplified in the two cast iron specimens.

Bars of shear steel of the same lateral dimensions as the above could not be obtained, at least not without the operation of forging, which it was apprehended might injure the texture, and alter the character of the metal. In order, therefore, to obtain a comparison with the above results, the author took four bars of shear steel from the rollers, and had two of them hardened, and two softened for the experiment; he had also two soft iron bars made of the same dimensions; viz. twenty-four inches long, one inch broad, and half an inch thick; these six bars were then tried precisely in the same manner as in the last experiments. The results were as below:

Preceding Table continued.

Description of metal.		East of needle.		West of needle.		Mean.
		End A.	End B.	End A.	End B.	
Soft iron	No. 1.	21° 30′	23° 10	22° 0′	22° 30′	22° 15′
	No. 2.	17 10	27 10	17 20	27 10	
Soft shear steel	No. 1.	13 20	17 0	not observed.	not observed.	15 0
	No. 2.	12 30	17 10			
Hard shear steel	No.1.	10 0	14 0	not observed.	not observed.	12 17
	No.2.	17 40	17 30			

In these experiments, the distance between the bottom of the bar and the centre of the compass was 5·2 inches.

The same remarks apply to these experiments as to the foregoing; viz. that each bar of the same kind of metal gives very nearly the same mean result, notwithstanding the difference in the action of their extremities.

Only one small specimen of cast steel could be procured, viz. a bar nine inches long, and seven-eights of an inch square: but in order to continue the comparison as above, an iron bar of the same dimensions was forged; and this latter was tried, first against the steel bar rendered soft, and then against the same when hardened; the following are the results:

Preceding Table continued.

Description of metal.	East of needle.		West of needle.		Mean.
	End A.	End B.	End A.	End B.	
Soft iron - -	16° 20′	17° 20′	not	obser.	16° 50′
Cast steel, soft	14 20	11 0	not	obser.	12 40
Cast steel, hard	8 0	8 45	not	obser.	8 22

By collecting the mean results from the three preceding tables, and comparing in each the particular deviations with the corresponding deviation produced by the iron bar; asuming also the tangents of the several angles as the measures of the deflecting power, and that due to the iron bar as unity; we shall have the relative power of the different kinds of metal, as in the last column of the following table:

Table of the proportional magnetic powers of different descriptions of iron and steel.

Description of metal.	Angles of deviation	Tangent.	Proportional magnetic power.
Cast iron - - -	7° 48′	·1369	·479
Malleable iron -	15 54	·2843	1·000
Blistered steel, soft	10 50	·1913	·673
Blistered steel, hard	8 37	·1515	·532
Malleable iron -	22 15	·4091	1·000
Shear steel, soft -	15 0	·2679	·655
Shear steel, hard -	12 17	·2177	·530
Malleable iron -	16 50	·3025	1·000
Cast steel, soft -	12 40	·2247	·743
Cast steel, hard -	8 22	·1470	·486

Or if we express these ratios by the nearest integral numbers, they will stand thus:

Malleable iron - -	100	Shear steel, hard -	53
Blistered steel, soft -	67	Cast iron - - - -	48
Blistered steel, hard -	53	Cast steel, soft - -	74
Shear steel, soft - -	66	Cast steel, hard - -	49

Barlow's *Essay on Magnetic Attraction*, second edition.

§ V. *On the effect of heat upon the developement of free magnetism.*

38. If we examine the results obtained by the experiments given in the last section, it will be seen that the harder the metal was, upon which the experiment was performed, the less it exhibited of a magnetic quality; a fact equally established by each species of iron and steel, and which is highly favourable to the hypothesis that has been advanced (art. 10 *et seq.*) relative to the resisting power of the steel preventing the entire developement, as it does also the recombination of the fluids, when these have been displaced by the contact of a powerful magnet. But, in order to establish it beyond every possibility of doubt, it was necessary to ascertain whether they would each exhibit the same powers when reduced to the same degree of softness, which could only be done by heating them in a furnace, and making the trial whilst in that state; and it was concluded, that if this should be found to be the case, the hypothesis in question must be considered as experimentally established. The following results will show that this is actually the case.

Comparison of the magnetic powers of cast and malleable iron, when heated in a furnace.

Experiments on heated iron.

39. For the purpose of these experiments there were procured a bar of soft iron, twenty-five inches in length, and one inch and a quarter square, and a cast iron bar of nearly the same dimension.

The needle was placed nearly level with the upper extremity of the bar, and at the distance of six inches, the latter being inclined as in the former experiments in the direction of the dip.

The following are the results:

Experiment I.

$$\begin{array}{l}\text{Cast iron,}\\ \quad\text{cold}\end{array}\left\{\begin{array}{lr}\text{End A, deviation} & 21^\circ\ 30'\\ \text{End B,} & 21\ \ 30\end{array}\right\}\text{mean } 21^\circ\ 30'.$$

Ditto, white heat, zero; blood-red - - 62 0.

Experiment II.

$$\begin{array}{l}\text{Malleable iron,}\\ \quad\text{cold}\end{array}\left\{\begin{array}{lr}\text{End A, deviation} & 37^\circ\ 0'\\ \text{End B,} & 43\ \ 0\end{array}\right\}\text{mean } 40^\circ\ 0'.$$

Ditto, white heat, zero; blood-red - - - 55 0.

These two experiments were repeated with exactly the same results.

The positions of the bars were now changed, viz. they were raised about two inches, but the dip was still observed.

Experiment III.

$$\begin{array}{l}\text{Cast iron,}\\ \quad\text{cold}\end{array}\left\{\begin{array}{lr}\text{End A, deviation} & 24^\circ\ 20'\\ \text{End B,} & 24\ \ 20\end{array}\right\}\text{mean } 24^\circ\ 20'.$$

Ditto, white heat, zero; blood-red - - 78 30.

Experiment IV.

$$\begin{array}{l}\text{Malleable iron,}\\ \quad\text{cold}\end{array}\left\{\begin{array}{l}\text{End A, deviation}\\ \text{End B, not observed.}\end{array}\right\}$$

Ditto, white heat, zero; blood-red - - - 70° 30′.

These experiments were repeated with the same results.

It should be observed here, that the great attraction produced by the heat did not subside with it, provided the bar remained in its place undisturbed; for after some days it was found that the power of the bar continued just the same as at the time of making the experiment, when it had not been displaced; but then the bar upon trial was always found to possess a certain degree of fixed magnetism, its other extremity producing an opposite effect upon the needle; but if the bar was inverted, while it retained any visible colour from the heat, both ends produced exactly the same deflection: as to the magnetic effect to which we have alluded above, it was lost, or at least a great part of it, after leaving the bar for some time horizontal, or, after its being thrown about with other pieces of iron.

It is also proper to observe that the needle always began to indicate the power of the iron upon it, as soon as it arrived at that state of temperature shown by a high blood-red heat, and its motion generally proceeded gradually till it had reached its maximum deviation, which it commonly attained in about a minute or two. It will be remarked as very singular, that cast iron, which is so decidedly inferior in its action, when cold, should possess an equal or superior power when hot, which happened uniformly in every experiment that was made, the two bars being placed under like circumstances.

Comparison of the magnetic power of soft iron and shear steel, when heated in a furnace.

Experiments on heated steel.

40. The bars employed in these experiments were those, whose effects are recorded in our second table of the preceding section. The results were as follow:

Experiment V.

Malleable iron, cold	End A, deviation	16°30′	mean 15° 10′.
	End B,	13 30	
Ditto, white heat, zero; blood-red	- - -	41 11.	

Experiment VI.

Soft shear steel, cold	End A, deviation	11° 30′	mean 11° 0′.
	End B,	10 30	
Ditto, white heat, zero; blood-red	- -	48 0	

Experiment VII.

Hard shear steel, cold	End A, deviation	15° 30′	mean 8° 0′.
	End B,	0 30	
Ditto, white heat, zero; blood-red	- - -	47 30.	

There can be no question after the above experiments, that every species of iron or steel possesses a greater capacity for the developement of its magnetism when rendered soft by fire, than when cold, and whence it is conceived we have a right to conclude, that the complete developement, when cold, is prevented only by the hardness or resisting power of the metal, agreeably to the hypothesis advanced, (§ I. part II.)

§ VI. *On the destruction of the magnetic quality of iron and steel at high temperature, and the anomalous effect observed during the bright red and red heats.*

Effect of iron at a white heat.

41. The effect of temperature in changing and destroying the magnetic power of iron bodies had been long imperfectly known, but it had never been satisfactorily established prior to Mr. Barlow's experiments. It is, for example, stated in Newton's *Optics*, that red hot iron has no magnetic property, while father Kircher asserts, that the magnet will attract red hot iron as well as cold; Mr. Cavallo again found, that although iron at a red heat had a greater power over the magnet than when cold, yet at the white heat it had less; but he was not aware that it was entirely lost at a white heat; this is, however, obviously the case from all the preceding experiments, and it is certainly a highly curious fact, of the cause of which we cannot venture to offer the slightest conjecture.

Of the anomalous effect at the red heat.

The following is a still more extraordinary circumstance, viz. after the iron loses its entire power of attraction at the white heat, it acquires, as that colour subsides into the bright red and red, an attractive power, the reverse of what it had when cold; so that if the bar and needle are so situated that the north end is attracted when the iron is cold, the south end will be attracted during the above interval, that is, while it is passing through those shades of colour mentioned above, after which the usual attractive power again takes place in the high degree stated in the preceding section.

This remarkable fact was communicated by Mr. Barlow to the Royal Society, and published in part i. of the *Phil. Trans.* for 1821; it was noticed first by him while carrying on the preceding series of experiments, and the following were wholly directed to this inquiry.

42. The first attempt that was made failed of detecting this effect, but it appeared that the position of the bar and needle was not the same as in those cases where it had been observed; the bar was therefore raised about four inches, it was inclined in the angle of the dip, and the needle placed to the west of it, about the same quantity (four inches) below its upper extremity, in which situation an obvious negative attraction was observed, amounting to 4½°.

Having gained this by raising the bar four inches, its base was now raised six inches; and on applying it in this place, a deviation of 10½° was obtained, which remained fixed about two minutes; when the needle suddenly yielded to the natural magnetic power of the iron, and obtained almost instantaneously a deviation of 81° the opposite way.

It was thus rendered obvious, that the quantity of negative attraction at the red heat depended upon the height or depth of the centre of the bar; and as the natural effect of the cold iron was changed by placing the needle below the centre of the bar, the next question was, will the character of the negative attraction change also? To decide this point, the needle was lowered to within six inches of the bottom of the bar; in which position the cold iron attracted the south end of the needle, and produced a deviation of 21°; and upon being heated, it was found as usual, that all its power upon the needle ceased at the white heat; but as this subsided into the bright red, the negative attraction began to manifest itself, and it soon amounted to 10½°, the north end of the needle being attracted towards the iron: it remained stationary for a short interval, and then returned gradually, first to due north, and ultimately to 70° 30′ on the opposite side.

Having made these preliminary experiments the author proceeded with his general series, and which we shall give in his own words.

43. "In the experiments detailed in the following table, I used four different bars, each twenty-five inches long, and one inch and a quarter square; two of them of cast iron, denoted in the first column by C B, No. 1, C B, No. 2; and two of malleable iron, denoted by M B, No. 1, M B, No. 2.

"I had also two other bars, one of cast and one of malleable iron, of the same dimensions, which were not heated, but kept as standards for determining the quantity of cold attraction, as this could not safely be done by the bars used in the experiments after being so repeatedly heated.

"The time employed in each experiment was about a quarter of an hour: the white heat commonly remained about three minutes, when the negative attraction commenced; this lasted about two minutes more, when the usual attraction took place: this sometimes arrived at its maximum very rapidly, but at others it proceeded increasing very gradually; and commonly within the time stated above, the needle had been found perfectly stationary.

"In the following table, to avoid confusion, that attraction which took place according to the known laws of iron, is marked *plus*, whichever end of the needle approached the iron, and the opposite attraction is marked *minus*. For example, when the needle is above the centre of the bar, the north end of it should be drawn towards the iron, but when the needle is below the centre, the south end should approach the iron; these therefore are both marked *plus*, and the contrary attraction at the red heat is marked *minus*."

TABLE,

Showing the effect of iron on the compass needle at different degrees of heat.

No.	Description of bar.	Height or depth of centre of bar from needle.	Distance of bar from needle.	Position of needle.	Effect Cold.	Effect White Heat.	Effect Red Heat.	Effect Blood Red Heat.	REMARKS.
		Inch.	Inch.	°	° ′	° ′	° ′	° ′	
1	C. B. No. 1.	0.0	6.0	S. 80 W.	+ 0 0	0 0	− 17 0	0 0	South end drawn to the bar at red heat.
2	M. B. No. 2.	4.5 below.	6.0	ditto.	+30 0	ditto.	0 0	+45 0	
3	C. B. No. 2.	ditto.	6.0	ditto.	+18 0	ditto.	0 0	+49 0	
4	M. B. No. 1.	ditto.	6.0	ditto.	+29 30	ditto.	− 12 0	+44 0	
5	ditto.	1.3 below.	6.0	ditto.	Not obs.	ditto.	0 0	+52 0	
6	ditto.	4.5 below.	6.0	N. 80 W.	ditto.	ditto.	− 12 30	+70 0	This bar being left standing, it attracted the same three days after.
7	ditto.	4.5 above.	6.0	S. 80 W.	ditto.	ditto.	− 12 30	+30 0	
8	ditto.	ditto.	6.0	ditto.	ditto.	ditto.	0 0	+25 0	The needle suspected to touch the box.
9	ditto.	ditto.	6.0	ditto.	ditto.	ditto.	− 19 0	+30 0	
10	ditto.	1.0 above.	6.0	ditto.	ditto.	ditto.	− 15 0	+ 4 0	
11	M. B. No. 2.	12.5 below.	8.5	N. 80 W.	+29 30	ditto.	0 0	+37 30	Observed at the same time with two compasses.
12	ditto.	ditto.	8.5	N. 80 E.	+30 0	ditto.	0 0	+41 0	
13	C. B. No. 1.	ditto.	8.5	N. 80 W.	+16 0	ditto.	0 0	+42 30	Ditto.
14	ditto.	ditto.	8.5	N. 80 E.	+15 30	ditto.	0 0	+47 30	
15	M. B. No. 2.	9.0 below.	8.5	N. 80 W.	+28 30	ditto.	− 1 0	+39 30	Ditto.
16	ditto.	ditto.	8.5	N. 80 E.	+29 30	ditto.	− 1 30	+42 0	
17	C. B. No. 1.	ditto.	8.5	N. 80 W.	+15 45	ditto.	− 1 30	+45 0	Ditto.
18	ditto.	ditto.	8.5	N. 80 E.	+16 0	ditto.	− 1 30	49 0	
19	M. B. No. 2.	6.0 below.	8.5	N. 80 W.	+25 0	ditto.	− 3 0	+32 30	Ditto.
20	ditto.	ditto.	8.5	N. 80 E.	+26 0	ditto.	− 3 30	+33 0	
21	C. B. No. 1.	ditto.	8.5	N. 80 W.	+11 30	ditto.	− 3 30	+36 30	Ditto.
22	ditto.	ditto.	8.5	N. 80 E.	+13 0	ditto.	Not obs.	+36 30	
23	M. B. No. 2.	3.0 below.	6.0	S. 80 E.	+ 8 0	ditto.	− 21 30	Not obs.	Ditto.
24	ditto.	ditto.	6.0	N. 45 W.	Not obs.	ditto.	− 25 30	+25 30	
25	M. B. No. 1.	0.0	6.0	ditto.	0 0	ditto.	− 40 0	0 0	North end drawn to the bar at red heat.
26	M. B. No. 2.	1.0 above.	5.3	N. 60 W.	+ 2 0	ditto.	− 4 30	+ 5 30	Both attractions very gradual.
27	M. B. No. 1.	ditto.	5.3	ditto.	Not obs.	ditto.	− 12 30	+ 5 30	Passed suddenly to 12½°, but returned immediately.
28	M. B. No. 2.	9.0 above.	6.0	N. 85 E.	+47 30	ditto.	− 2 30	+60 0	Attractions gradual.
29	M. B. No. 1.	ditto.	6.0	ditto.	+47 30	ditto.	− 2 30	+60 0	Ditto.
30	M. B. No. 2.	1.0 below.	5.5	N. 45 W.	Not obs.	ditto.	− 55 0	+ 5 45	Negative attraction rather sudden.
31	M. B. No. 1.	4.5 above.	7.0	N. 75 E.	ditto.	ditto.	− 2 30	+33 30	Motion of needle very slow.
32	M. B. No. 2.	1.7 below.	5.5	N. 45 W.	ditto.	ditto.	+100 0	+13 30	100° very sudden, returned immediately.
33	M. B. No. 1.	1.7 above.	5.5	ditto.	ditto.	ditto.	− 26 0	+13 30	Both attractions gradual.
34	M. B. No. 2.	ditto.	5.5	ditto.	ditto.	ditto.	+ 30 0	+13 30	The same as No. 32; both anomalous.
35	M. B. No. 1.	4.5 above.	6.0	N. 55 E.	ditto.	ditto.	− 5 30	+35 30	Attractions very gradual.
36	M. B. No. 2.	ditto.	6.0	ditto.	ditto.	ditto.	− 0 0	+35 30	
37	M. B. No. 1.	0.0	4.7	West.	+ 3 30	ditto.	− 50 0	+ 8 0	Motion regular, but quick,
38	M. B. No. 2.	0.0	4.7	North.	0 0	ditto.	0 0	0 0	No motion in the needle.

44. It should be observed, that all the above experiments were made with the bars inclined in the direction of the dipping needle, or nearly in that direction, and it will be seen that the negative attraction was the greatest where the natural attraction was the least; that is, opposite the middle of the bar. The author was led, therefore, to make a few experiments with the bar inclined at right angles to its former position, but the results were by no means so strongly marked as in the preceding experiments: he always found a certain quantity of negative attraction, but it was very inconsiderable, never amounting to more than 2½°.

He also made one experiment with an iron twenty-four pound ball, but the heat was too intense to make any very accurate observation. The numbers obtained were

Cold attraction + 13° 30′ White heat 0° 0′.
Red heat − 3° 30′. Blood-red heat + 19° 20′.

It may be proper also to state, that, being doubtful how far the heat itself, independent of the iron, might be the cause of the anomalous action above described, the author procured two copper bolts of rather larger dimensions than the iron bars, and had them heated to the greatest degree that metal would bear; but on applying them to the compass, no motion whatever could be discovered in the needle.

Effect produced in the decomposition of iron by acids.

It may not be improper to mention, in connection with the preceding experiments, the results obtained by Mr. Cavallo, by the decomposition of iron. This philosopher, after trying a few experiments of a similar kind to the foregoing, but on a smaller scale, was desirous to ascertain the effect produced by mixing sulphuric and other acids with iron dust or filings; with this view he placed a quantity of such filings in an earthen pot, and having brought the vessel near the south end of a needle, he observed the quantity of its attraction; he then poured some sulphuric acid to the iron, which produced an effervescence, and increased the power of the iron on the needle; the same effect was produced with nitric and nitrous acids, but it was nearly imperceptible when he employed the muriatic acid; this effect has probably some relation with electro magnetism, which will be treated of in its proper place.

§ VII. *Description of natural magnets, and the method of arming them.*

Description of natural magnets.

45. As it is difficult to explain and illustrate the peculiar properties of natural magnets without some previous knowledge of the nature of magnetic attractions and repulsions, and as this knowledge is better deduced from uniform artificial bars of steel, rendered magnetic, than from the natural magnet itself; we have deferred to this place any further description of the latter, than was sufficient for giving us two poles in an artificial bar, for the purpose of introducing the reader to the first principles of the science; we propose, however, now to enter a little more particularly into a description of this singular natural body.

We have already (art. 1.) described the mineralogical characteristic of the natural magnet; we have also stated, that if such a body be rolled in iron dust, or filings, it will attract a quantity of them to itself, and that the greatest accumulation will be found in two points on its surface, and which are accordingly denominated its poles. If the natural magnet were previously cut into a bar form, and these accumulations were found only at its extremities, there would be no difference between such a bar, and one of steel magnetised, at least as far as regards its magnetic properties; but in general we find such bodies in the most irregular forms, and their poles very irregularly posited, and sometimes several such poles in the same magnet; the power of them also is generally very inferior to those artificial bars formed for the purpose. It has been observed, that of natural magnets the smallest generally possess a greater attractive power than those which are larger. It has been said, that certain natural magnets not exceeding the weight of twenty or thirty grains, have been known to lift a piece of iron of forty or fifty times their own weight. Mention is made of a small magnet that Sir I. Newton wore in a ring, weighing only about three grains, that would lift 746 grains, or nearly 250 times its own weight; and Mr. Cavallo states that he has actually seen one, which he thinks could not weigh more than six or seven grains, that was capable of raising 300 grains. These, however, are very rare specimens, for in general natural magnets, *unarmed*, possess but a very inferior suspending power; and in no case, we believe, has a natural magnet which exceeds two pounds in weight, been known to possess a power sufficient to support a weight more than ten times its own.

In reference to the greater proportional power of the smaller magnets, it may be observed, that the magnetic ore is frequently much blended with other matter, which contains little or no magnetic property, and therefore, if such a body be fractured into various pieces, some of those pieces may be found to possess a great accumulation of the pure ore, and others little, or none at all; and consequently, the former will not only possess most of the power belonging to the entire body, but being now in a more condensed form, it may be, and indeed has been known to be, capable of suspending a greater weight than the whole body before the fracture.

Meth arming tural nets.

46. As both magnetic poles, when they can be simultaneously applied to a piece of soft iron, will support a much greater weight than the same two poles separately, and as the two poles of a natural magnet are generally in opposite parts of its surface, in which situation it is almost impossible to adapt the same piece of iron to them both at the same time; it is a common practice to apply two broad pieces of soft iron to the poles of such a magnet, and to make them both project on one side as shown in fig. 27. In that case, the two pieces of iron having become themselves magnetic by their contact with the body, and having thus their contiguous extremities impregnated with the opposite magnetic powers, a piece of iron applied so as to touch them both will be strongly attracted, and thereby the suspending power very considerably increased. The pieces of iron are generally held fast upon the magnet, by means of a brass or silver box; in this form the magnet is said to be *armed*, and the pieces of iron are called its *armour* or *armature*.

Fig. 27.

The process and construction above described are represented in fig. 27; the centre part of the figure is a section of the natural magnet; the two side pieces are the armour, which projecting below the magnet constitute its poles, to which latter is applied a lifter, as in the horse-shoe magnets. This lifter is furnished with a ring and hook, or a scale for carrying a weight, and by means of which the suspending power of the stone may be ascertained. On its upper part is also a ring attached to the brass or silver box, where the whole may be suspended in any convenient situation.

A natural magnet thus armed may be strengthened in the same way as we have described for magnetising horse-shoe magnets; or its suspending power may be, perhaps, excited to its highest pitch, by suspending from the hook above mentioned a small scale or bag, into which is to be inserted as many small shot, or as much sand as the magnet will support, every now and then, at intervals of about a day, adding a little more shot or sand; by which means the magnet will, at length, be found capable of supporting two or three times the weight it would have done, if it had been all applied at the same time. It will, however, be proper to observe, that if we at length pursue our additive process, until we overpower the attraction of the magnet, its power will be reduced to nearly the same as it was at first, and we must again proceed as before, if we are desirous of imposing upon the magnet its maximum load. But if before we have actualld overpowered the attraction of the magnet we unloay it, it will then take up, when the weight is reapplied, nearly the same load as when it was displaced. This is not, however, peculiar to natural magnets, for precisely the same applies to artificial horse-shoe magnets, as well as to those of the bar form, although the effect is not so obvious in the latter case as in the two former.

Terr or g

47. Sometimes a piece of natural magnet is brought down to the form of a perfect globe, in which case it is called a *terrella*, and was supposed by the early writers on magnetism to represent the earth; whence its denomination; but since the laws of magnetism

have been better understood, such forms of the magnet are very seldom had recourse to.

§ VIII. *Of the magnetism of certain substances not commonly considered as ferruginous.*

48. It was for a long time supposed that iron was the only substance naturally endowed with magnetic properties; but recent experiments have shown that certain metals and some other substances have a similar, although a very inferior, magnetic action: some doubts however have been, and are still entertained, whether the magnetism exhibited, in many of these cases at least, is due to the natural magnetic property of the matter itself, or to ferruginous particles which may be supposed to enter into the composition of the bodies. After numerous experiments, this matter, as we have said above, may still be considered as doubtful, although perhaps the general tendency of the results go rather to establish, that the magnetism belongs to the proper matter of the body, than that it is derived from foreign combination.

49. Next to iron nickel possesses, or will at least receive and retain the magnetic property in a stronger degree than any other metal; so that even needles have been formed of it which had an obvious directive quality, although in a degree very inferior to those formed of steel; and that it possesses this power *per se* has been assumed from the fact, that nickel has been known to retain its magnetism after being repeatedly purified. (Kirwan's *Mineralogy*.) This, however, has been denied by others, who have not been able to detect such a power in this metal, that is in certain specimens of it; and from which circumstance they conclude that the power is not necessary and inherent in the nickel, as it is in iron.

50. Cobalt is likewise attracted by the magnet, as are also zinc and bismuth, as well as their ores; antimony is attracted when exposed to a gentle heat, but arsenic is not in any perceptible degree affected by the strongest magnet. Cavallo's *Treatise of Magnetism*. Those minerals, which are not metallic, are nearly all attracted by the magnet, at least after having been exposed to the action of the fire. Of pure earths, the calcareous is the least or not at all, and silicious the most frequently attracted, especially the Black Sea sand, which evidently contains a good deal of martial earth, or rather half formed iron.

51. Amber and other combustible materials are generally attracted by the magnet, and particularly after burning. Of the precious stones, those which are pellucid, as the diamonds and crystals, are not attracted. The amethist, topaz, chalcedony, and generally those which are deprived of their colour by fire are not attracted. The other precious stones are all more or less magnetic; viz. the ruby, especially the oriental, the chrysolite, and tourmalins. The emerald, and particularly the garnet, are not only attracted but frequently they exhibit a determinate polarity, from the influence of a strong magnet, so that afterwards they are attracted from one side and repelled by the other.

52. Almost every part of animal and vegetable bodies, after combustion, is in some measure attracted by the magnet; the flesh, and particularly the blood, possesses this property in the strongest, and the bone in the weakest degree; vegetables after burning are almost generally attracted, but not all with equal force; but unburned and fresh vegetable and animal substances, very seldom if ever show any particular attraction to the magnet. It is also said that soot, and the dust which usually falls upon whatever is exposed to the atmosphere are sensibly magnetic.

Part II.

In most of the instances above cited, the magnetism is probably due to the combination of iron, but this does not appear to be the case in nickel; and the following experiments, made by Mr. Cavallo, seem also to show that the magnetism sometimes exhibited by brass proceeds from some cause independent of the presence of iron.

Cavallo's experiments on the magnetism of brass.

Magnetism of brass.

53. This philosopher, from various observations, had detected very commonly a magnetic power in brass, and he moreover noticed that it more commonly was exhibited in pieces that had been hammered than in those which had not been exposed to this operation; his object therefore in the following experiments was principally to examine this peculiarity, and the inquiry being of great importance, as connected with the construction of magnetical instruments, we shall not hesitate to give the experiments at length in the author's own words.

Experiment I.

Experiments.

"An oblong piece of brass, weighing somewhat less than half an ounce, being examined, by presenting every part of its surface to the suspended needle, showed no sign of magnetism whatever. It was then hammered for about two minutes; the consequence of which was, that it became magnetic so far as to attract either end of the needle from about a quarter of an inch distance. This same piece of brass being now put into the fire so as to become red hot, by which means it was softened, and when cold being presented to the suspended needle, its magnetism was found to be entirely gone. Hammering made it again magnetic; softening by fire took the magnetism away a second time: and thus the magnetism was repeatedly given it by hammering, and was destroyed by softening; sometimes shewing to have acquired a sensible degree of that power even after two or three strokes of the hammer."

Experiment II.

"The result of the first experiment would naturally induce one to suspect, that the hammer and anvil might have imparted some small quantity of steel to the brass, which rendered it magnetic; and that this magnetism was destroyed in softening the brass, insomuch as the fire calcined the small quantity of steel that had adhered to it. In consequence of which consideration, I took other pieces of brass besides that used before, and hammered them between card paper, changing the pieces of paper as often as was necessary, since they were easily broken by the hammer; but the pieces of brass became constantly magnetic by the hammering, and their magnetism was destroyed by fire.

"In this experiment I generally gave to the brass not above thirty strokes with the hammer."

Experiment III.

"Still suspecting that the hammer and anvil might have imparted some small quantity of iron to the brass, because the pieces of card paper sometimes were broken by the first or second stroke, in which case either the hammer or the anvil touched the brass, I hardened a piece of brass by beating it between two large flints; viz. using one for the hammer and the other for the anvil. The piece of brass became magnetic, though in this case it seemed to have acquired not so much power as when it had been hardened with the hammer; but it must be observed, that the flints being rough and irregular, the piece of brass could not be hardened by them so easily, or so equally, as by the other method.

"The flints, being examined both before and after the experiment, were found to have not the least degree of magnetism."

Experiment IV.

"A piece of brass, which by hammering had been rendered so strongly magnetic as to attract either pole of the needle from about a quarter of an inch distance, was put into a crucible, together with a considerable quantity of charcoal dust, which surrounded it every where. The crucible was covered with clay, and being placed into the fire, was kept red hot for about ten minutes. After cooling, the piece of brass was taken out of the crucible, and being examined, was found to have entirely lost its magnetism. The object of this experiment was to ascertain whether the loss of magnetism, in a piece of brass that was softened, was owing to the calcination of the ferruginous particles, which, notwithstanding the preceding experiment, might still be suspected to be imparted to it; because in this way of softening brass, the ferruginous particles being surrounded with charcoal dust could not have been calcined; hence the brass ought not to have lost its magnetism, which however was not the result of the experiment."

Experiment V.

"One of those pieces of brass, which had been used for the foregoing experiments, and which had been deprived of magnetism by fire, was hammered between two large and pretty thick pieces of copper, which had not the least magnetism; and after a few strokes of the hammer, it became sensibly magnetic."

54. The author next examined various kinds of brass, foreign and English, new and old. He found most of them subject to become magnetic by hammering, but some were not; these, however, exhibited no external peculiarity by which to judge whether or not they were liable to become magnetic by the process described: and although the precautions taken in the latter case seem to prove that the action which the brass exhibited after hammering, was entirely independent of any particles of iron that could be acquired in the experiment, yet it did not appear to be quite satisfactory whether, supposing iron to exist in small quantities in the body of the metal itself, these might not be rendered magnetic by the process, and that this property was again destroyed by the heat. In order to examine the subject under this point of view, the following experiment was made.

Experiment VI.

"Having chosen a piece of brass, which would acquire no magnetism by hammering, I placed it upon an anvil, together with a considerable quantity of *crocus martis*, which crocus had no action on the magnetic needle; then began hammering the brass, and turning it frequently, in order to let part of the crocus adhere to it; and, in fact, the crocus had, in several places, been fastened so well into the brass, that hard wiping, with a woollen cloth would not rub it off. The brass appeared red in those places; but, after having been hammered for a long time, it acquired no magnetism whatever."

The effect of the hammering was therefore not sufficient in this case to render the iron calx magnetic.

Experiment VII.

"In order to diversify the preceding experiment, I drilled a hole, about one-eighth of an inch long, and little more than one-fiftieth of an inch in diameter, into a piece of brass that was not rendered magnetic by hammering, and filled it with *crocus martis;* then I hammered the piece of brass, thus enclosing the calx of iron, and afterwards presented it to the suspended magnetic needle; but there was not the least sign of attraction: the martial earth, therefore, had not acquired any phlogiston from the brass by the action of hammering."

Experiment VIII.

"The same piece of brass, containing a small quantity of calx of iron, was put into the fire, and was made quite red hot, in which state it remained for about three minutes. Then, after cooling, it was presented to the magnetic needle, and this was attracted by the brass only in that place wherein the calx of iron was contained. The action, therefore, of the fire had rendered the martial earth so far phlogisticated as to attract the magnetic needle; hence, if the magnetism of brass were owing to any ferruginous matter contained in it, a piece of brass ought to become magnetic when softened; which is contrary to the foregoing observations."

Experiment IX.

"A hole, similar to that mentioned in the seventh experiment, was drilled in a piece of brass that would not become magnetic by hammering, and into which was put some black calx of iron, which was so far phlogisticated as to be attractable by the magnet, and the hole was closed by a few strokes of the hammer. In consequence of this, the piece of brass, when presented to the suspended needle, would attract it only about that place where the magnetic calx was contained. This attraction was very weak. The piece of brass thus prepared, was then put into the fire, and was kept for about six minutes in a heat very little short of that necessary to melt brass, and after cooling I presented it to the needle, expecting that the fire might have dephlogisticated the calx of iron so far as not to let it act any longer upon the needle; but the attraction appeared to be of the same degree it was before the heating.

"It seems, therefore, clear, that the magnetism acquired by brass, when hammered, is not owing to iron contained in it; and consequently, that *magnetism, or the power of being attracted by, and attracting, the magnet, may exist independent of iron.*"

55. From these and other similar experiments Mr. Cavallo draws the following conclusions. Summar of result

"First. That most brass becomes magnetic by hammering, and loses the magnetism by annealing or softening in the fire, or at least its magnetism is so far weakened by it as afterwards to be only discoverable when set to float on quicksilver.

"Secondly. That the acquired magnetism is not owing to particles of iron or steel imparted to the brass by the tools employed, or naturally mixed with the brass.

"Thirdly. Those pieces of brass which have that property, retain it without any diminution after a great number of repeated trials, viz. after having been repeatedly hardened and softened.

"Fourthly. A large piece of brass has generally a magnetic power somewhat stronger than a smaller piece; and the flat surface of the piece draws the needle more forcibly than the edge or corner of it.

"Fifthly. If only one end of a large piece of brass be hammered, then that end alone will disturb the magnetic needle, and not the rest.

"Sixthly. The magnetic power which brass acquires by hammering has a certain limit, beyond which it cannot be increased by further hammering. This limit is various in pieces of brass of different thickness, and likewise of different quality.

"Seventhly. Though there are some pieces of brass which have not the power of being rendered magnetic by hammering, yet all the pieces of magnetic brass, that I have tried, lose their magnetism, so as no longer to affect the needle, by being made red hot; excepting indeed when some pieces of iron are concealed in them, which sometimes occurs; but in this case, the piece of brass, after having been made red hot and cooled, will attract the needle more forcibly with one part of its surface than with the rest of it; and hence, by turning the piece of brass about, and presenting every part of it successively to the suspended magnetic needle, one may easily discover in what part of it the iron is lodged.

"Eighthly. In the course of my experiments on the magnetism of brass, I have twice observed the following remarkable circumstance:—a piece of brass, which had the property of becoming magnetic by hammering, and of losing the magnetism by softening, having been left in the fire till it was partially melted, I found, upon trial, that it had lost the property of becoming magnetic by hammering; but having been afterwards fairly fused in a crucible, it thereby acquired the property it had originally, viz. that of becoming magnetic by hammering.

"Ninthly. I have likewise often observed, that a long continuance in a fire so strong as to be little short of melting hot, generally diminishes, and sometimes quite destroys, the property of becoming magnetic in brass. At the same time, the texture of the metal is considerably altered, becoming what some workmen call *rotten*. From this it appears, that the property of becoming magnetic in brass by hammering, is rather owing to some particular configuration of its parts, than to the admixture of any iron; which is confirmed still farther by observing, that Dutch plate brass (which is made not by melting the copper, but by keeping it in a strong degree of heat whilst surrounded by *lapis calaminaris*) also possesses that property; at least, all the pieces of it, which I have tried, have that property.

"From these observations it follows, that when brass is to be used for the construction of instruments wherein a magnetic needle is concerned, as dipping needles, variation compasses, &c. the brass should be either left quite soft, or it should be chosen of such a sort as will not be made magnetic by hammering; which sort, however does not occur very frequently."

Importance of these in the construction of compasses, &c.

56. A remarkable instance of the magnetic power of brass is mentioned by Mr. Barlow, in page 17 of his *Essay on Magnetic Attractions*, and a similar effect was very recently noticed by Mr. Napier, a Master in the Navy. This gentleman having been requested to determine the variation of the needle, somewhat particularly, at one of our stations on the American coast, made use of one of the azimuth compasses belonging to his vessel, and found some part of the brass of it so strongly magnetic that he obtained a result differing by several degrees, according as the instrument was used direct or reversed. He found the same effect produced by several different azimuths, and which remained even after the pieces had been heated in a furnace. This shows the impropriety of using hammered brass bowls for compasses, and the great advantage of cast and turned bowls for this purpose; a practice that has been long adopted by Messrs. W. and T. Gilbert, instrument makers to the Hon. East India Company.

§ IX. *Description of magnetical instruments.*

Magnetical instruments. Fig. 28, 29, 30—40.

57. *Of the horizontal needle.* The first and most simple magnetical instrument is the horizontal needle; it consists of an uniform steel bar of almost any shape and any length, from an inch or less to a foot or more. In its centre it is made sometimes rather broader than in any other part, and is there tapped with an interior screw to receive the exterior screw turned upon a brass cap, furnished with an agate, on which the needle is balanced upon a steel point. We have shown a needle of this kind in fig. 28; the brass cap and agate, fig. 30; a section of the latter, fig. 29; and the needle balanced in fig. 31. Other forms of the needle are seen in fig. 32, 33, 34, 35, and 36.

Land compass.

58. *Of the land compass.* For the purpose of observation, the needle prepared as above, is placed upon a point in the centre of a brass or wooden box, furnished with a graduated limb, and sometimes the ends of the needle are made to carry a vernier scale, in order to bring down the reading to minutes or lower. The box, which may be square, round, or octangular, is moreover furnished with two straight edges of brass or index marks, to set to any proposed line, and sometimes with sights; the top being covered with glass to prevent the needle from being disturbed by the action of the air; see fig. 37 and 39. There are also two small pieces of brass, one of them turning on a fixed point, seen in the figure, but more particularly in fig. 38, which is used to check the oscillations of the needle. That is by pressing upon the upper end, the ring at the other end of the lever is raised till it touches the needle, which latter is thereby rendered steady: the lever is then let down and the needle left to find its proper direction. Other forms of the horizontal compass are shown in fig. 39 and 40. In the latter the needle is mounted with a card, divided into points and quarter points of the compass; the north and south points being made to correspond very exactly with the axis of the needle; in this form the general direction of an object will be known by

observing its bearing with the card, which will always arrange itself according to the magnetic meridian of the place of observation.

Schmalcalder's surveying compass.

59. *Schmalcalder's horizontal or surveying compass.* This is a very ingenious and useful instrument, it consists of a small round brass box, with two sights; that one towards the eye being furnished with a prismatic lens, and the other an open sight with a horse-hair line in its centre. The pupil of the eye being bisected by the upper edge of the prism, the object and its bearing are seen at the same time, the former by direct vision, and the latter by reflection in the inclined face of the prism from the divisions of the card beneath; which divisions, passing to the eye from a single reflection, are inverted, in order that they may arrive at the eye in their proper position. As in this instrument it is necessary to preserve the box horizontal, it would be difficult to observe the bearing of an object having any considerable degree of elevation; a defect which is removed by placing a dark reflector behind the horse-hair sight, so that by inclining this at a certain angle, the object reflected from it may be carried horizontally to the eye, but at the same time the eye-piece is furnished with shades or screens of coloured glass, in order to enable the observer to take the bearing of the sun; so that it becomes, in this case, an azimuth compass; but it is only calculated for land observations. The above description will be better understood when we speak of Gilbert's patent azimuth compass, the two instruments being founded on precisely the same principles, and parts of the same patent right.

Gilbert's steering compass.

60. *Of the sea compass or steering compass, by Gilbert.* This instrument, except the box or bowl in which the card is suspended, resembles that shown in fig. 40; but to fit it for its particular purpose, the card is supported upon a pin fixed in a heavy brass box or bowl; the bowl itself being hung in gimbals in a wooden square box, or, when at sea, fitted to the ship's binnacle. On that part of the compass bowl which is fixed directly forward in the vessel, on the inside, is a clear black line, (the inside of the bowl itself being white:) this is called the *lubber line*, and is that which the steersman uses to keep his required course; that is, he must always keep the point of the card which indicates his course, directly coinciding with the lubber line. A compass of this kind, in its gimbals, is shown in fig. 41.*

Fig. 41.

Considering the importance of this instrument in navigation, it is astonishing to see how very imperfectly it is generally constructed, and what little value appears to be set upon it by the generality of seamen. The following directions for its construction, however, are deserving of their consideration, as well as that of the makers of such instruments.

In the first place, the needle ought to be of some uniform figure, (but it does not appear to be of much consequence what that figure is,) and mounted with a brass cap, furnished with an agate, turned or ground down to a decided centre, not too sharp at the bottom. The needle should balance itself very accurately on a point before the card is applied, and the card itself should be made with equal uniformity, so that the balance may be still preserved when it is fixed upon the needle. For the better fixing of the needle to the card, the hole in the middle of the latter should have a thin rim of brass fixed round it, and the brass cap should be made sufficiently broad at bottom but very thin, whereby the latter may lie a little over the former when this is screwed into the needle, by which alone the card and needle may be held fast together. But to render every thing secure, the needle is tapped with a small screw hole, at about half an inch from each of its ends; and then the card being brought so that its meridian coincides with the axis of the needle, two holes are made in the former, and a small screw, with rather a broad and thin head, is passed through each and screwed into the needle; by this means the card is permanently attached, and then the centre screw or brass cap is introduced as above described. This method of fixing the needle to the card is practised by Messrs. W. and T. Gilbert, and is decidedly preferable to that commonly adopted, of rivetting the needle and card together by leaden rivets. The card being thus prepared, the next thing is to attend to its suspension and to the construction of the bowl. This latter ought, for the reasons assigned in the last section, to be made of new brass, first cast into its proper bowl form as nearly as possible, and then reduced to its proper hemispherical figure by turning. The advantage of which figure is, that at every inclination of the bowl itself, the edge of the card will always be at the same distance from the lubber line inside, and whereby this distance may be reduced to as small a quantity as possible, it being first observed to place the point of suspension in the common centre of curvature of the bowl. This latter is now to be placed in its gimbals, observing to adjust them in such a way that the point of suspension, on which the card is supported, may fall exactly in a line with the axis of motion given to the gimbals; by this means a very considerable motion may be given to the bowl without in any way displacing the point; that is, without giving motion to the point on which the card is suspended, and thereby the unsteadiness of the latter at sea, so much complained of by navigators, is in a great measure prevented. In the above description we have exactly indicated the practice adopted by the makers above mentioned, who have long been deservedly distinguished for the accuracy they have carried into the construction of their nautical instruments.

Other forms of sea compasses.

61. There is nothing peculiar to the above construction, except the care and accuracy employed in the manufactory: but there have not been wanted various contrivances for producing those effects which, in the above instance, are due only to the delicacy of the workmanship. These, however, have been too often rather fanciful than useful, or if they have answered the intended purpose on shore they have failed altogether on shipboard. There are however some exceptions, and of these we shall describe the following.

M'Culloch's patent compass. Fig. 42,

62. *M'Culloch's patent compass.* This is represented in our fig. 42 and 43; the former being a section, and the latter a perspective view of the instrument. In this compass, both the compass box and the needle, with the card, are suspended upon points, the extremities of which come very near to each other; which construction is intended to keep them horizontal without any gimbals, as will be easily manifested by the following description in fig. 42, *a, a, a, a,* is the common wooden box, with its lid; *c c* the brass compass box, with the glass cover to it; *d d* the hollow

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* By mistake the points on the card are omitted.

conical bottom; *e* the prop upon which the compass is supported, instead of gimbals, the spherical top of which is finely polished, and the apex of the hollow cone is fitted to receive it; *f f* is a quantity of lead run round the bottom and the cone of the compass box, to balance and to keep it steadily horizontal; *g g* is the card and the magnet needle bent in such a manner as to bring the point of the conical pivot, on which it moves and is supported, very near to the centre of gravity, as well as to the centre of motion; *h h* are two guards, which, by means of two pins, affixed to the compass box, prevent its turning round, and deceiving the steersman. In fig. 43, both the lid and the front of the box are removed; *h h* are the guards; *b* the brass compass box; and *e* is the brass pin which supports it. This compass however has never been a favourite with seamen.

63. A different kind of sea compass is shown in fig. 44; here the needle itself, fig. 46, is suspended by gimbals, fig. 47; being attached to the latter by four screws, the holes for which are seen in the needle, and two others at the extremities, whereby the latter is still more intimately attached to the card. From the centre of the upper arc of the gimbals proceeds a descending point, seen in fig. 47, and also in the section, fig. 44. This point rests on an agate let into the brass ball *a*, which fits a socket of the same form *b b*, from the bottom of this ball proceeds a stem *c c*, at the lower part of which a weight is attached, and *d d b b* is a hollow conical support within which the weight has room to vibrate. This weight having a constant tendency to preserve the verticality of the wire stem *c c*, will of course keep the lower face of the agate horizontal, and thus preserve a steady position of the card. Lastly the bowl itself is hung in gimbals in the same manner as in fig. 41. This is the construction of Mr. Preston; and Alexander's compass, which was of a prior date, is on nearly the same principles, except that in the latter the agate in its cap is supported by gimbals instead of the ball and socket joint above described. One great objection to these compasses is, that if by any accident they meet with damage, which they are of course more subject to than a simple needle, they are not readily repaired on board, and consequently become useless for the remainder of the voyage.

It would be as endless as it would be useless to attempt a description of all the varieties of form that has been given to sea compasses, we shall therefore only slightly mention the following.

64. *Pope's Compass.* In this the needle has a motion in the direction of the dip, and was thought not only to be competent to point out both the magnetic meridian and the dip, but that it possessed a greater directive power; this, however, was an erroneous assumption, for whether the needle be horizontal or inclined, its directive powers to the meridian is the same; and as to its compound motion we cannot but consider it as a disadvantage rather than otherwise.

65. *Jenning's patent insulating compass.* This differs very little in its general character from the common compass except that the brass bowl is made double, so as to leave a space between the outside and inside surfaces of about a quarter or half an inch, which is filled with pure iron dust or filings; the needle is therefore surrounded by iron, and was thought thus to be insulated from the action of the guns and other iron of the ship. The idea however was erroneous, and the instrument is very little used, although it received a good report from Captain Ross, in his voyage to the north.

Hanging compass.

66. *Hanging or cabin compass.* In this compass the bowl is hung up with its glass downwards, across which passes a light brass bar carrying a point upwards, on which the card is suspended. The brass cap carrying the agate is, in this case, screwed into the needle, the contrary way to the common compass; the upper part of the agate being on the plain or back part of the card instead of being in front, as in the usual construction. Instruments of this kind are commonly used by Captains of vessels in their cabins, where they are hung up over head, and whereby at any moment a mere glance of the eye will show the course of the vessel; and be the means of detecting either error or treachery in the steersman.

Common azimuth compass.

Fig. 47, plate IV.

67. *Of the azimuth compass.* The common form of this instrument differs very little in its general character from the steering compass, the only difference being that it is fitted with sights, and is hung in detached gimbals, which turn on a stout vertical pin in the box, or the pin itself is fixed below the box, and being inserted in a proper support, the instrument, as in the other cases, may be directed to any object to determine its bearing or azimuth, and hence its denomination. See fig. 47, plate IV. where E is the vertical pin, B A C D the gimbals, and F G the sights, which in some instruments shut down over the glass when the compass is not in use, but in others they are connected together by a brass bar, and are then altogether removed when the instrument is set aside. The card differs also in some respects from the common steering cards, the latter being divided only to points and quarter points, which are brought as nearly as possible to the outside edge of the card, whereas in the azimuth the points are merely shown as a matter of ornament, or for general reference, but the limb is divided into degrees, &c. sometimes on the card itself, but more commonly upon a light metal rim properly graduated. The sight G is that to which the eye is applied, and the other is directed towards the object, so that the latter may be seen upon the fine silk or horse-hair line in the sight F. A fixed line or sight, inside the box, placed vertically below the horse-hair, serves then to indicate the bearing of the object, by pointing out the particular division or degree corresponding with it on the card.

In consequence of the motion of the vessel, it is seldom that the card is sufficiently steady to allow the bearing of the object to be taken, and then for that bearing to be read by the same person; therefore the sight in the box, to which we have alluded, is generally made moveable, either on a joint below, or it is fixed to a spring, whereby, in either case, the observer as soon as he has got the object in view, and finds the card steady, may, by a contrivance for the purpose, bring the sight in contact with the card, and thereby prevent its future motion, and he can then read off the bearing at his leisure. It is seldom, however, on shipboard, that the card is sufficiently steady at the moment the object is properly seen; and moreover the very act of bringing the sight suddenly up to the card will frequently displace it considerably, and give a false reading; this defect is very completely avoided in the following instrument.

Gilbert's patent azimuth compass.

Fig. 48.

68. *Gilbert's patent azimuth compass.* We have given a perspective view of this ingenious instrument in fig. 48, plate IV., removed out of its box, and placed upon a tripod for observation; the brass box A B, differs but little from that of the common azimuth box described in the last article, except being made more accurately cylindrical, and of greater thickness of metal; E F shows the gimbal ring, and C D a support of brass to which the whole is appended. In the horizontal part G H, is a circular hole, which fits a brass pin proceeding upwards from the plate, at the upper part of the tripod, and about which the compass may be turned in any direction; but in case it should in any instance be necessary to leave the instrument fixed, this may be effected by a set screw I, seen just below G H; K L, at the back of which the eye is applied for taking an observation, is furnished with a triangular prism of glass in a brass case *e*, its lower face being brought to the figure of a lens by which the degrees, &c. on the card beneath are magnified. The eye then being applied to the slit behind K L, sees, just over the upper edge of the prism, the object whose bearing is required, in a line with the hair sight; and at the same time the reflected image of the degrees, &c. on the graduated edge of the card beneath, by which means the object and its bearing are taken at the same time and by the same person, a condition of the greatest importance in observations at sea, and which is unattainable in any other manner; *f*, *g*, are screens of coloured glass for making observations on the sun, and which, turning on a joint, may be brought down or set up at pleasure; the whole frame K L turns also on an axis *a b*, in order to adjust the prism in such a way that the object and its degree of bearing, shown from the card beneath, may be seen in complete contact.

Fig. 49.

The optical action of the prism will be better seen in fig. 49, where B C is the vertical plane side, A B the lens, and A C the reflecting face. Here it is obvious that *a b*, being an object placed horizontally, will be reflected from *a′ b′*, and is thence seen magnified and in a reversed position, by an eye placed at D, and which, at the same time, sees the object *s* directly over the upper edge of the prism. On account of this inversion of the object, the figures on the rim of the card are engraved reversed. As in this instrument it is necessary, on account of the card, to keep the whole horizontal, it would be difficult to obtain the bearing of an object that had any considerable altitude; to obviate which difficulty a plane reflector *p n*, fig. 48, turns on a joint at the bottom of the perpendicular sight N O, and which must be placed at such an angle as to bring the reflected rays in nearly a horizontal direction; in this case the eye sees the reflected instead of the direct object. The pin seen at M is attached to a moveable piece of brass inside the box, for the purpose of stopping the card when its vibrations are too great, and the brass pin *q p*, is connected with a lever resting on the bottom of the box inside, and which is pressed down when the sight N O is turned on its joint, in order to bring it flat on the top of the box; and by this means the other end of the lever is raised, and the card thus lifted off its pivot when the compass is not in use. The whole of the upper part of the instrument may then be removed from the tripod and placed in a box made to receive it.

This very beautiful instrument is one of the most important acquisitions to nautical science which it has received for many years.

Variati
compa

69. *Variation compass.* This being only intended for ascertaining small changes in the declination of the magnetic needle, either annual or diurnal, is commonly more limited in its arc of vibration, and is generally longer than the needles used for more common purposes; it is also furnished with conveniencies for reading closer than the common needle; otherwise any needle so mounted and furnished as to be very sensible and capable of being read off to minute divisions might be used as a variation compass. In general, however, needles designed for this purpose are constructed in a particular manner, having only a very small part of the circle graduated; and the means of distinguishing small changes better suited to the eye than they generally are in common horizontal needles. The place in which they are fixed requires also to be particularly firm and steady, and in short, every precaution is necessary to be had recourse to, in order to avoid small irregularities; because the changes to be observed are in themselves so small, that without the greatest accuracy, they may be either overlooked, or the irregularities of the instrument so combined them, that every attempt to deduce laws from them would be unavailing.

The following is the variation needle employed by Colonel Beaufoy, in making a valuable series of magnetic observations from the year 1813 to 1821; during which period the needle passed through its maximum of westerly variation in London. These results were published by Colonel Beaufoy, in successive numbers of the *Annals of Philosophy*; and we are indebted to the liberal scientific views of this gentleman for his permission to use his drawing and description given in the number of the above work for August, 1813.

Colonel Beaufoy's variation compass.

Plate V.

70. *Colonel Beaufoy's variation compass.* The principal part of this instrument, viz. its needle, with the box and divided arc, to measure the angle of variation, is very similar to the instrument in the possession of the Royal Society; but it is greatly improved by the addition of a small transit telescope, which readily and accurately determines the true meridian, or the zero from which the degrees of the variation is to be measured. This addition renders it particularly well adapted for travellers who wish to make accurate observations in different parts of the globe, as it contains every thing necessary for the observation within itself; and as every part admits of adjustment, it may be depended upon for accuracy. Plate V. contains a representation of the whole instrument in perspective, explaining all its parts at once. The box containing the needle is not fixed, as in many compasses, but turns horizontally on the centre, and has an index fastened to it, pointing to a divided arc on the brass frame on which it turns. The method of observing is to move the box until a line drawn on it points exactly to the end of the needle, which being done, the angle of variation is shown by the divisions on the arc; F F is a mahogany board, which is the support of the whole instrument; it stands on the points of three screws G, H, I, by which it can be levelled. Above this is a flat plate of brass E E, attached to the board by the centre pin, and resting upon three studs projecting from the board, to insure its having a perfect bearing, whilst it admits of a small horizontal motion round the centre by means of the

screw W X. The plate E E has the divided arc *m*, fixed upon one end of it; and a vernier D, traverses against the divisions; it is divided upon a projecting part of a brass plate C C, which moves upon the common centre pin of the plate E, and also the needle A A. The plate C C has two segments of brass at each end of the needle, and these have the centre lines drawn upon them, which are brought to the points of the needle when the observation is made. A light brass box *d d*, with a glass cover, is fitted over the plate C C, to preserve the needle from disturbance by the wind; it also supports a small double microscope M, intended to assist in examining when the index line points exactly to the end of the needle; the stem of the microscope is fitted to a dovetail groove and can be removed to the opposite end of the box at pleasure.

The centre pin of the plates E and C terminate in an extremely fine point, on which the needle is suspended, having an agate cap B, to diminish the friction as much as possible. The needle is of a cylindrical figure, about the five-hundredth of an inch in diameter, and ten inches long; weighs sixty-five grains and a half, and is terminated by two conical points; the circular enlargement of the centre has a hole through it, for the reception of the brass socket B, which has the agate cap fixed in it; the needle is, of course, provided with the usual apparatus for lifting it off the point when not in use, to avoid wearing the point of suspension; *a, a,* are two arms, screwing down upon the plate, carrying the vernier D; they support a brass frame, which has the usual clamp screw L, and tangent screw R, the former to fasten it to the arc *m*, and the latter to give a slow motion, and adjust the box for the observation.

The transit telescope O P, is supported over the instrument by two pillars N, N, fixed on the brass plate E E, and having small frames or boxes *f*, at the top, for the reception of the *y*'s, in which the pivots of the axis Q, of the telescope are supported; this axis is conical, and is fixed exactly at right angles to the tube O P, in the same manner as other transit instruments.

On the extremity of one of the pivots of the axis, a small divided circle R, is fixed, and has an arm at the opposite side turning about at the centre, and provided with verniers to read the divisions of the circle; to this arm a small level S, is attached; the whole forming a vertical index to set the telescope at any required altitude, by setting the index at the proper division, and then moving the telescope till the bubble of the level S, shews the index to be horizontal; the eye-piece *h*, of the telescope has a small dovetail in it to admit a dark glass for observation of the sun; *g g*, are the screws for adjusting the wires in the eye-piece as usual; there is also a detached level to adjust the instrument.

71. *Manner of adjusting the instrument previous to observation.* First place the feet, of the detached level, upon the brass plate E E, in different directions, and bring it level by the screws H, I, G; then apply the level upon the two pivots of the transit, the covers of the boxes *f* opening with hinges for this purpose, and the bottom of the feet having notches to rest upon the pivots; if the axis does not prove level, one of the *y*'s must be elevated or depressed by a screw in the farther pillar N, until the level stands horizontally, and, reversing it end for end, proves every thing to be correct.

Now set the vernier D, at zero, and put an additional object glass over the glass P, of the telescope, and in this state the marks A A, against which the needle reads, can be seen through it when directed to them; this proves the zero of the division to be exactly in the plane of the telescope's motion; a small screw near Q will rectify it, if necessary, by moving the *y* a small quantity in the box *f*; the transit may be reversed in its *y*'s to rectify every thing.

The telescope is now to be adjusted to the meridian by the transit of the stars in the usual manner, the screws W, X, being used to turn the whole instrument round when requisite; but after these observations have been once made in a satisfactory manner, distant marks should be set up, both north and south, and these will give the means of adjusting it at once, and in the day time.

When the instrument is properly placed, the taking of the observation is extremely simple. The needle being suffered to settle, the box is turned about on its centre till its mark comes very near the point of the needle; the clamp screw L, is then fastened, and the screw R, is employed to make the coincidence exact, in which the microscope M, greatly assists the eye. The vernier D, now shows the angle of variation. It is proper, after the needle has been once observed, to attract it with a piece of iron, and cause it to make a slight vibration. When it settles again, it will not in all cases come to the same point, because of the friction of the suspending point; it is therefore advisable to make this trial three or four times, and take the mean of the whole.

The instrument is placed on a pillar in the open air, free from any iron, and is adjusted before every observation; the vernier is divided into half minutes, and if the meridian be not at a considerable distance, an advantage will be gained by diminishing the aperture of the object glass by a pasteboard cap.

It may be worth while to remark, that this instrument will serve as a transit instrument for naval officers to examine the rate of their chronometers while on shore.

Dipping needle.

72. *Dipping needle.* We have already stated (art. 7.) that when an uniform steel bar is first correctly suspended at its centre of gravity, and then magnetised, it will take a certain inclination; one extremity dipping below the horizon, and the other rising above it. We have seen also in our historical chapter, that this curious fact was first noticed in London, by Norman, in 1576, and that at this time the inclination in London was more than 71°. It was many years after this date, however, before the inclination was known to have a variation, but the fact has since been well ascertained, and various instruments denominated *dipping needles*, have been constructed with a view of ascertaining the amount of the dip, in different places, at the same time, or in the same place, at different periods; but it was only towards the latter end of the last century that sufficient accuracy could be carried into their construction, to render them fully competent to the delicate task they were intended to perform; indeed it is questionable, whether, even at present, they are not susceptible of farther improvements. We shall, however, furnish our readers with a description of one of the most recent and improved construction.

Gilberts's dipping needle.

Fig. 51.

73. *Dipping needle by Messrs. W. and T. Gilbert.* A perspective view of this instrument is shown in fig. 51. It consists of a lower brass plate, supported on a plane board or stand by the three adjusting screws A, B, C, which answer as feet; on this plate, and concentric with it, is another brass plate, moveable on a centre pin, like the upper plate of a theodolite. This carries two levels shown in the figure, and by means of these the instrument is adjusted to horizontality, when employed in experiments. To the upper plate are firmly attached the four principal supports E, F, G, H, which bear the circular box, or principal case for the needle; this being furnished with two glass faces, which are easily removed and replaced at pleasure. Across this case are firmly fixed two brass bars, one of which I K, is seen at large, and another equal bar just appears behind it, in the figure, the upper edge of each being a little below the horizontal diameter of the circular case: to the centre of each of these cross bars is attached, by screws, admitting of adjustment, two other brass pieces, furnished on the upper side with two finely polished planes of agate, upon which the axis of the needle N S, rests, and on which it turns freely. The brass boxes P, Q, attached to the side of the principal case, are for the following purpose. Inside the fixed cross bars I K, are two thin moveable pieces of brass, which turn on an axis in the case Q, and are raised and lowered by means of the milled head R, in the box P; this pin carrying an eccentric piece of brass inside for the purpose. The middle of each of these inside pieces is cut in the form of a *y*, so that when the pin R is turned, the axis of the needle is caught into the *y*'s, and lifted off the agate planes; and when the pin is turned back again, the *y*'s leave the axis on the agate planes, exactly coinciding with the centre of the graduated circle; the *y*'s being carefully adjusted for the purpose. Without this contrivance it would be impossible to prevent the axis slipping on one side, and thus rendering the circular arc and the needle eccentric, and consequently destroying the accuracy of the instrument. In the particular needle in question, the ends N, S, are graduated so as to answer the purpose of a vernier scale to 6′; viz. the arc is divided to half degrees, and the needle is exactly such as to admit of five degrees of the arc being set off upon it in its whole breadth; and this being divided into twelve equal parts, obviously admits of reading down to every 6′; this division however is not attempted in the drawing. The instrument is also furnished with a small microscope for reading off; it is so attached to the rim of the glass face as to be easily put round to any place where the needle settles. The screw M, seen on the circular plate, is for the purpose of clamping the two plates when the instrument has been brought into the meridian, or into any other proposed direction, which is sometimes effected by a tangent screw applied at this place. Every dipping needle is, or ought to be, provided with a horizontal needle, which turns on a point attached to some part of the instrument, and furnished with indexes or lines parallel to the face of the principal brass box or case, for the purpose of bringing the latter into the plane of the meridian. In the instrument we are describing, the horizontal needle turns on a point, fixed in a brass plate, which latter is nicely fitted, so as to be easily placed just above the agate planes, the dipping needle being removed; and when this adjustment is made the horizontal needle is taken away, and the former rested on its agate planes as above described.

Formerly dipping needles were made upon a large scale, viz. the needle was commonly twelve or thirteen inches in length; latterly they have been made smaller, not being more than eight or nine inches, and the particular instrument above described has its needle only six inches in length; and as far as our practice goes in these instruments, we have invariably found, the smaller the instrument the more accurate its action. This however, of course, has its limits, and perhaps six inches is the least dimensions that can safely be depended upon; although we should much like to see the attempt made on a needle of only four inches. It is, of course, of great importance to reduce the size of dipping needles, not only on account of rendering them less expensive, but because they thus become more portable, and consequently more useful.

Of the use of the dipping needle, and method of observation.

Method o observing with a dipping needle.

74. The instrument being placed on a steady foundation, and at a distance from any iron, it is first to be very correctly levelled, by adjusting with the screws which serve for its support; this being effected in one position, the clamp screw is loosened, the upper plate turned on the one below, and any further adjustment made that may be necessary; this part of the operation being precisely the same as that with the common theodolite.

One of the glass faces is now to be removed, and the horizontal needle and its plate applied to its proper place above the agates, or to any other place intended for it. The instrument is now turned till, by means of this needle, the face of it is found to be exactly in the plane of the magnetic meridian, when the clamp screw is turned, and the whole rendered perfectly firm and secure.

The horizontal needle is now to be removed, and the dipping needle, resting by its axis on the horizontal agate planes, is left to vibrate; the glass face being replaced, when the vibrations have nearly subsided, turn the screw head R, thereby lifting the needle, and supporting it in the *y*'s; and which now, when again let down, will have its axis exactly coinciding with the centre of the graduated arc.

Every thing being thus prepared, apply a small magnet near one extremity of the needle, thereby drawing it out of its present state of rest; then removing the magnet leave the needle to vibrate till it settles, and if it be properly adjusted it will now point out the dip sought, which at this time in London is about 70°.

It is, however, very rarely that a needle can be so nicely balanced as to give the exact dip at a single observation; the usual mode of proceeding is therefore as follows: Having taken eight or ten observations by vibrating the needle so many times, as above described, turn it completely about, viz. if, in the first instance the face was to the east, turn it now to the west; or to the east, if its face was in the first instance turned to the west; which is very easily effected by the graduations on the lower limb of the instrument. Here again the same number of observations are to be taken in the same manner as above; which, from slight defects of construction, may not agree precisely with the former: let, however, the mean in both cases be preserved, by dividing the sum of each set of observations by the number of them. The needle is now to be removed from the box, and its poles inverted by any of the methods of magnetising described in section II.; so that when replaced on its axis, that end which was before below the horizon will be now above it, and if the needle be correctly balanced by exactly the same quantity; but if not, as is most likely to happen, we must get two other means as above, and the general mean of the four will be very nearly, or exactly, the true dip.

The following set of observations will serve to illustrate the above directions:

Face of instrument to the east.			Face of instrument to the west.		
Obser.	°	′	Obser.	°	′
1	70	12	1	70	0
2	70	0	2	70	0
3	69	50	3	70	0
4	69	54	4	70	0
5	69	54	5	70	0
6	69	54	6	70	0
7	70	0	7	70	0
8	70	0	8	70	0
9	70	0	9	70	0
10	70	0	10	70	0
10)	699	44	10)	700	0
Mean	69	58·4	Mean	70	0

Poles inverted.

Face to the east.			Face to the west.		
Obser.	°	′	Obser.	°	′
1	69	54	1	69	54
2	69	54	2	70	0
3	69	52	3	70	0
4	69	50	4	70	0
5	69	54	5	70	0
6	69	54	6	70	6
7	69	54	7	70	6
8	69	54	8	70	6
9	69	50	9	70	6
10	69	54	10	70	0
10)	698	50	10)	700	18
Mean	69	53	Mean	70	1·8

	°	′
First mean	69	58·4
Second mean........	70	0
Third mean	69	53
Fourth mean........	70	1·8
4)	279	53·2
True dip	69	58·3

It appears, by examining the preceding results, that there is an uncertainty in the action of the needle amounting to 15′ or 20′; the dip in the different observations in the same set of results varying by this quantity, and we never before used a needle in which the agreement fell within such narrow limits; this therefore is not to be considered as a defect in the particular instrument in question, but as a fault common to all dipping needles acting on these principles. Indeed, if we examine the nature of the force by which the needle is brought into its direction, we shall see that it varies as the sine of the angle, between its true position and that in which it is actually found at any time; and therefore when the needle is within 15′ of its natural position, the lever by which the power is communicated to the needle is only equal to the sine of 15′; or the power of the terrestrial magnetism on the needle, in this position, is to the same when the latter is at right angles to the dip, as radius to sine 15′ or as 1 : ·0043. It is obvious therefore that the least possible defect in the axis, or the interposition of the smallest particle of dust, will be sufficient to stop the motion of the needle, and lead to that inaccuracy in the results that has been noticed above.

Another method of construction.

Mayer's dipping needle.

75. The above considerations have led to another method of construction, or principle of observation, which will, we have no doubt, be attended with considerable advantages. It consists in tapping the needle with a fine interior screw, to receive a fine steel screw which projects some distance from the needle, and on which traverses a small brass ball. By this means the needle is deflected any quantity, at pleasure, from the actual dip, and by this means the terrestrial action upon it increased in almost any proportion; this, as we have seen, always varying as the sine of the angle between the natural line of the dip and the position of the needle.

With an instrument thus constructed we never get, by observation, the correct dip, the position of the needle being partly due to magnetism, and partly to gravity; but by reserving the instrument, as in the preceding experiments, the effect of gravity may be separated from that of the terrestrial magnetism, and the correct dip computed to the greatest accuracy.

Let us conceive the needle to be truly constructed, and the screw correctly inserted as supposed above. Let the true dip of the needle be denoted by δ, and the observed angle with the face of the instrument in one direction by F, and in the opposite one by f; the former exceeding, and the latter being, of course, less than δ. Call the weight of the ball and screw W, and the force of magnetism M; then as the needle is by the supposition in equilibrio in these two positions, we shall have

$$\sin f \,.\, W = \sin(\delta - f)\, M$$

$$\sin F \,.\, W = \sin(F - \delta)\, M$$

Hence

$$\frac{\sin(\delta - f)}{\sin f} = \frac{\sin(F - \delta)}{\sin F}$$

or,

$$\frac{\sin\delta\cos f - \sin f\cos\delta}{\sin f} = \frac{\sin F\cos\delta - \sin\delta\cos F}{\sin F}$$

or,

$$\sin\delta\cot f - \cos\delta = \cos\delta - \cot F\sin\delta$$

Whence dividing by $\sin\delta$, we have

$$\cot\delta = \tfrac{1}{2}(\cot F + \cot f)$$

that is, the cotangent of the dip is always equal to half the sum of the cotangents of the two observed angles, with the face of the instrument to the east and to the west. This, however, supposes that the needle is constructed with every possible degree of accuracy, so that prior to its being magnetised, it will balance itself on its axis like the beam of an assay balance, having the brass weight, or sphere, vertically below the line of suspension; a nicety not easily attained: it will therefore in this case, as in the former, be better to invert the poles of the needle, and to obtain the four angles due to the four positions of the instrument: then calling the first two arcs F and f, and the two latter G and g, take

$$\tan F + \tan f = A$$
$$\tan F - \tan f = B$$
$$\tan G + \tan g = C$$
$$\tan G - \tan g = D$$

then $$2 \cot \delta = \frac{A.D}{B + D} + \frac{B.C}{B + D}$$

See *Transactions of the Royal Society of Sciences at Gottingen* for 1814.

To determine the dip of the needle in any place, by comparing the times of its performing a given number of vibrations in different positions.

Dip deduced from the times of vibration.

76. When a dipping needle of the usual construction, is arranged in the plane of the meridian, it will, as we have seen, take up the line of the dip either exactly or approximatively; but as there will always be some doubt of the accuracy of the results, in consequence of the extreme delicacy of these instruments, every means of detecting error and correcting the observations should be had recourse to, and the two following are perhaps the best that can be employed; the first being best adapted for those parts of the earth where the dip is small, and the second where it is considerable.

First method.

First method.

77. When the needle is in the meridian, and nearly in the line of the dip, the whole force M, of the terrestrial magnetism is employed in producing its vibrations; but when it is placed with its face to the north or south, in which case the needle becomes vertical, the force which produces the vibrations will be, by the principle of the resolution of forces, only $M \sin \delta$, δ denoting the dip; therefore the time in which a given number of oscillations are performed in the two cases, will be as $\sqrt{M \sin \delta} : \sqrt{M}$, or as $\sqrt{\sin \delta} : 1$. Let therefore t be the time of performing any number (as, for example, 100) vibrations with the needle in the plane of the meridian, and T the time of performing the same with the face to the north or south, that is in a direction perpendicular to the meridian; then it is obvious that we shall have,

$$t^2 : T^2 :: \sin \delta : 1$$

whence, $$\sin \delta = \frac{t^2}{T^2}$$

the sine of the dip sought.

Second method.

Second method.

78. The force which gives direction to the horizontal needle, is to that which gives direction to the dipping needle, as $M \cos \delta$, to M, as is obvious from the simple resolution of the forces. If therefore the same needle can be hung so as to vibrate first in the dip, and then horizontally, and the times employed in making a given number of vibrations in each case be denoted as before by t and T, we shall have as above,

$$t^2 : T^2 :: M \cos \delta : M$$

or $$\cos \delta = \frac{t^2}{T^2}$$

Note. The best way of obtaining the vibration in the horizontal direction, is to suspend the needle by attaching a piece of untwisted silk to its axis, and to balance it, if necessary, by a small piece of leaf copper: the vibrations should be performed under cover, and the piece of untwisted silk of as great a length as possible. Captain Sabine took the dip of the needle in London, in September 1821, by each of the three latter methods, and his results agree so nearly with each other, as to confirm in a remarkable manner the accuracy of each. They are as below:

	Dip.
By ten experiments with Mayer's needle..	70° 02·9
By the times of oscillation in the magnetic meridian, and in a plane perpendicular to it	70 04
By the times of vertical and horizontal oscillation..........................	70 02·6
Mean	70 03

§ X. *Experiments made with a view of acertaining the best form and other conditions of a magnetised needle, so that its directive force may be the greatest.*

Form, temper, and de- scriptio of steel examin

79. Having, in the preceding section, given a pretty general view of the most approved magnetic instruments, it will be proper to conclude this part of our treatise with an examination of the circumstances under which the needle, which is common to them all, may be rendered the most efficacious as far as this depends on its form, description of steel, temper, &c.

In this examination we shall avail ourselves of the experiments of Captain Kater, which were wholly directed to this inquiry, and which seem to leave little farther to be desired on this subject.

Captain Kater's ex- perimen

80. Two needles were prepared of blister steel, and two of spur steel, the weight of each was sixty-six grains. They were of the form of a long ellipse, in length five inches, and in width half an inch. One of each sort was pierced in the middle, the other an uniform plate; but the former made thicker than the other, that the weights might be the same in both, although there was considerable difference in the extent of surface.

Captain Kater had also two needles made of shear steel, in the shape of a rhombus; they were, when complete, of the form shown in fig. 36, the cross piece of one being of brass, and in the other of steel. The weight of each of these needles was forty-five grains. To ascertain the directive force, the author employed the balance of torsion. This instrument, as is well known, consists of a fine wire attached to an index moveable round a circle divided into degrees; to the other end of the needle is fixed a cradle, to receive the needle which is the subject of experiment. The needle being in the magnetic meridian when the wire has no torsion, is afterwards forced to deviate from it to a mark distant about 60°, by turning the index, and consequently twisting the wire. The number of degrees passed over by the index will be as the directive force of the needle.

The needles above described were first made soft, and then hardened merely at their ends; they were not polished, and were magnetised to saturation.

Experiment I.

[Needles soft and then hardened at the ends.]

	Weight of needle.	Directive force.
Blister steel, solid ellipse..........	66	500
Blister steel, open ellipse	66	520
Spur steel, solid ellipse	66	540
Spur steel, open ellipse	66	500
Shear steel, rhombus	45	235
Shear steel, rhombus with cross piece of brass......................	45	435

By the experiments on magnetism, by M. Coulomb, it appears that the directive forces of needles of similar form are to each other as their masses; the directive force, therefore, of a needle of the form of a pierced rhombus of sixty-six grains would be expressed, according to the preceding experiment, by 638. From many other experiments made by Captain Kater, but not registered at the time, it appeared that shear steel was capable of receiving the greater magnetic force, and that the pierced rhombus was the best form for a compass needle. Needles of cast steel were tried, but found so very inferior as to be at once rejected.

81. The next object was to determine the effect of polish, and of various modes of hardening and tempering the needles; with this view, in addition to the former needles, two were made of *clock spring* of the pierced rhombus form, five inches long, two inches wide, and weighing sixty-six grains. One of these was first softened, then hardened at the ends, and left unpolished: the other, as well as the solid elliptical needle of spur steel, was hardened throughout, and polished. The needles were then magnetised to saturation.

Experiment II.

	Directive force.
Unpolished rhombus, hard at the ends......	800
Polished rhombus, hard throughout........	367
Polished elliptical needle, hard throughout ..	380
Polished elliptical needle, softened in the middle by laying it on a red hot poker....	620
Polished rhombus, softened in the middle in the same manner......................	590

The needles were now laid aside till the following day, when the directive force was again examined.

Unpolished rhombus, hard at the ends......	805
Polished elliptical needle, softened in the middle	625
Polished rhombus, softened in the middle ..	580

The polished rhombus was now softened throughout, and the extremities being hardened at a red heat, the directive force was found to be 800. It is scarcely necessary to say, that the needles were remagnetised to saturation previous to each experiment.

From these experiments the author draws the following conclusion.

That of the steel employed, shear steel is the best kind for compass needles.

That the best form for a compass needle is that of a pierced rhombus.

That polish has no influence on the directive force.

That hardening the needle throughout, considerably diminishes its capacity for magnetism.

That a needle soft in the middle, and its extremities hardened at a red heat, appears to be susceptible of the greatest directive force. That the directive force does not depend on the extent of surface, but on the mass.

82. The next experiments were made with three needles, two of which were rectangular parallelograms of equal length and weight, but the one only half the width of the other. The third needle was a pierced rhombus; the whole were made of clock spring. These needles were made perfectly hard, and magnetised, as was always the case, to saturation.

Experiment III.

[Needles perfectly hard.]

	Directive force.
Wide parallelogram	490
Narrow parallelogram	480
Pierced rhombus..........................	532

An accident happened to these needles which rendered them unfit for further experiment. It however appears from that above stated, that the directive force is nearly as the mass, and not as the surface; and that the pierced rhombus is superior to the parallelogram.

M. Coulomb having found that a needle of the rhombus form that was pierced, and which he called "*une lame taillée en flèche,*" was susceptible of a greater directive power than a parallelogram. Captain Kater was therefore desirous of repeating this experiment, as well as comparing this form with the pierced rhombus. For this purpose four needles were made, four inches and a half long, each weighing sixty-three grains; one was a parallelogram, 0·44 inch wide; another a rhombus, which we may call the large rhombus, 0·8 wide; the third, a pierced rhombus, 1·4 wide in the middle, having its sides 0·2 wide; these were made of clock spring. The fourth needle or rhombus, which may be called the small rhombus, 0·4 wide, was made of that kind of steel which is used for saw blades, or shear steel. This last needle was much thicker than the other. The steel of which these needles were made had been exposed to a sufficient degree of heat to render it soft enough to be worked, and in this state the needles were magnetised.

Experiment IV.

[Steel soft as worked.]

	Directive force.
Parallelogram	720
Small rhombus	530
Large rhombus	765
Pierced rhombus..........................	813

Experiment V.

[The ends of the needles at an obscure red heat.]

	Directive force.
Parallelogram	715
Small rhombus	577
Large rhombus	790
Pierced rhombus..........................	840

Experiment VI.

[The ends hardened at a red heat.]

	Directive force.
Parallelogram	742
Small rhombus	583
Large rhombus	745
Pierced rhombus..........................	844

Experiment VII.

[Hardened with a bright red heat, and then softened by a red heat from the middle towards the ends, the extremities for about an inch remaining hard.]

	Directive force.
Parallelogram	611
Small rhombus	710
Large rhombus	660
Pierced rhombus	685

Experiment VIII.

[Softened at a red heat between two plates of steel, the whole being allowed to cool gradually, and then the extremities of the needle hardened at a red heat.]

	Directive force.
Parallelogram	520
Small rhombus	585
Large rhombus	554
Pierced rhombus	590

Effect of repeated heating.

As it appeared from the above experiment that the needles had suffered a gradual deterioration, it was imagined that this might have occurred in consequence of their having been exposed to the heat of a coal fire, by which some portion of the carbon of the steel might have been destroyed; they were therefore recarbonized, by surrounding them with shreds of leather, and exposing them for several hours, in a close vessel, to a considerable heat. After they had gradually cooled, the ends were hardened at a red heat.

Experiment IX.

[Needles soft and then the ends hardened at a red heat.]

	Directive force.
Small rhombus	477
Large rhombus	415
Pierced rhombus	450

Here we may remark, that though the needles were apparently in the same state, except being recarbonized, as in the last experiment, they had suffered considerable deterioration.

The needles were now covered with a mixture well known to workmen to prevent decarbonization; this had been before neglected, but was used in all the subsequent experiments. They were hardened at a bright red heat, and afterwards tempered throughout rather beyond a blue colour. The large rhombus and parallelogram were accidentally broken.

Experiment X.

[Hardened at a bright red, and then tempered beyond a blue.]

	Directive force.
Small rhombus	660
Pierced rhombus	577

From these last experiments little can be gathered, except that the needles became less susceptible of directive force from repeated exposure to heat, and that this effect was not occasioned by a decarbonization of steel. The small rhombus of saw blade, perhaps from being the thickest, suffered less than those made of clock spring. The springs of clocks are made by passing the steel between rollers; and it thus undergoes great compression. May not this state be favourable to magnetism; and the repeated expansion of steel by heat, destroying this state, have occasioned the deterioration?

Experiments on other needles

83. The needles which were made of saw blade having suffered less than the others in the preceding experiment, three other needles were procured of this material; they were cut out of the same plate; the weight of each was 120 grains, and their length four inches and a half. One was a parallelogram, 0·46 inch wide; and another a rhombus as before, 0·87 inch wide; and the third, or pierced rhombus having the middle 1·5 inch, and its sides 0·25 wide.

These needles were made without its being found necessary to soften the steel plate; they consequently were all as nearly as possible of the same degree of temper. In this state they were magnetised.

Experiment XI.

[Steel the same as worked.]

	Directive force.
Parallelogram	1148
Rhombus	1020
Pierced rhombus	1065

Wishing to try whether the needles were magnetised to saturation, these were carefully remagnetised.

Experiment XII.

[Needles remagnetised.]

	Directive force.
Parallelogram	1140
Rhombus	955
Pierced rhombus	1069

It was now found that the directive force, instead of increasing, had lessened in each of the needles, and the author became anxious to discover the cause of so unexpected a phenomenon. It has been observed by M. Coulomb, and more fully entered upon by Biot, that if a needle be magnetised to saturation by strong magnets, and afterwards weaker magnets be applied, the needle will lose some part of the force it had before acquired. Now, if in using the same set of magnets, a certain degree of force be communicated to a needle, and the magnets be afterwards arranged in a manner less favourable for imparting the magnetic force, it should seem that this second operation would produce the same effect as would follow the use of magnets of less force, and that the magnetism of the needle would suffer a diminution.

The eff of mag- netising

84. The method the author had hitherto employed in magnetising the needles was that of Du Hamel, by joining the opposite poles of the magnets, and placing them on the centre of the needle, so inclined that each formed an angle, as was afterwards ascertained, of about thirty degrees with the horizon; the magnets were then slid from the centre to the extremities of the needle, and their poles being again joined at the distance from the needle, the operation was repeated. As the author could in no way account for the diminution of directive force which has been remarked above, except by supposing that he had inadvertently changed the inclination at which the magnets were held, he resolved to try whether a variation of this angle produced any considerable difference in the degree of magnetism communicated. For this purpose he remagnetised the needles, by laying the magnets, with their opposite poles joined, flat upon the needle, the junction of the magnets being upon the centre. They were then separated and drawn to the extremities of the needle, the surface of the needle and that of the magnets being in contact the whole time. The poles were then joined and the operation repeated, using but little pressure.

Experiment XIII.

[Magnet moved flat upon the needle with little pressure.]

	Directive force.
Parallelogram	1265
Rhombus	1048
Pierced rhombus	1130

This manner of magnetising, therefore, appears much superior to that before employed. The needles were again magnetised in the same manner as in the last experiment, except that the ends of the magnets were pressed pretty strongly against the needle.

Experiment XIV.

[Magnets moved as before but with strong pressure.]

	Directive force.
Parallelogram	1263
Rhombus	1005
Pierced rhombus	1131

No advantage appears to have resulted from the increased pressure, but the arrow shaped needle suffered a diminution of power.

The magnets were now slid from the middle to the extremities of the needle, at an inclination of only two or three degrees, and the following were the results.

Experiment XV.

[Magnets inclined in an angle of two or three degrees.]

	Directive force.
Parallelogram	1275
Rhombus	1051
Pierced rhombus	1150

This method appears to be preferable to any the author had yet tried, and was therefore employed in the subsequent experiments of the present series. The rhombus of sixty-three grains, which in Experiment X. was left with a directive force of 577, on being magnetised in this manner had its power increased to 600.

The ends of the needles were now hardened at a red heat, the middle remaining soft as before.

Experiment XVI.

[Needles hardened at the ends at a red heat.]

	Directive force.
Parallelogram	1315
Rhombus	1020
Pierced rhombus	1185

Experiment XVII.

[Ends hardened at a bright red.]

	Directive force.
Parallelogram	1258
Rhombus	970
Pierced rhombus	1085

The ends hardened at a red heat as near to that employed in Experiment XVI. as possible.

Experiment XVIII.

[Ends hardened at a red heat.]

	Directive force.
Parallelogram	1350
Rhombus	1121
Pierced rhombus	1205

Here it should seem that an increase of power was obtained by the ends of the needles having been first hardened at a higher temperature, and then at a lower.

The needles were now hardened throughout at a bright red heat.

Experiment XIX.

[Hardened throughout at a bright red.]

	Directive force.
Parallelogram	1120
Rhombus	1205
Pierced rhombus	1080

The needles softened by laying them on a red hot poker till they passed beyond the blue or greyish white. This was carried to within an inch of their extremities, which remained hard.

Experiment XX.

[Softened from the middle to a greyish white, ends hardened.]

	Directive force.
Parallelogram	1360
Rhombus	1140
Pierced rhombus	1210

The tempering was carried throughout the needles, the parallelogram was reversed for another purpose.

Experiment XXI.

[Softened throughout to a greyish white.]

	Directive force.
Rhombus	1075
Pierced rhombus	1145

Experiment XXII.

[Softened throughout to a greyish white, the ends hardened at a red heat.]

	Directive force.
Rhombus	1025
Pierced rhombus	1185

Experiment XXIII.

[Hardened throughout, and then softened to a greyish white, as in Experiment XXI.]

	Directive force.
Rhombus	1065
Pierced rhombus	1180

This last series of experiments presents a curious circumstance. From the experiments made by Coulomb, as well as from the general tenour of the above, the rhombus was found capable of receiving a greater directive energy than the parallelogram; yet here we perceive that the parallelogram, though formed of the very same plate of steel as the other needles, is not only, under every circumstance, superior to the rhombus, but also to the pierced rhombus. It is difficult to form any plausible conjecture as to the cause of this difference.

The weight of the rhombus, in Experiment X. made of clock spring, was sixty-three grains; that made of saw blade weighed 120 grains or very nearly double. The directive energy of the former, after having suffered great deterioration, and when not tempered in the most favourable manner, compared with the greatest directive energy of the latter, was as 600 to 1210; but if we refer to Experiment VI. it may be seen that the greatest directive energy of the clock spring rhombus was 844; which gives it an advantage of about one-third in directive energy over a needle of equal weight made of saw blade.

From Experiment XX. it should seem that a needle is susceptible of the greatest directive power, other circumstances being similar, when it is hardened throughout at a red heat, and then softened from the middle to within an inch of the extremities, till the blue colour which arises has again disappeared. See *Philosophical Transactions* for 1821; see also other experiments by this author in our section II.

Remarks. 85. The importance of obtaining the greatest possible directive power in a compass needle will be a sufficient apology for the length at which we have detailed Captain Kater's experiments, as it will also for the few further remarks we have to make on this subject. It is the opinion of many seamen, that in rough weather it is necessary to give considerable weight to the card to preserve it steady, and it is too often done by loading it with sealing wax; the consequence of which is, that the weight being thus increased while the directive force of the needle remains the same, the latter has not power to overcome the friction on the point; and thus to avoid one defect, another is introduced which is of still greater consequence; for the card remaining steady, the steersman, unsuspicious of danger, pursues his course, till too frequently he is first warned of the error of his compass by finding himself on a shore supposed at many miles distance. We are inclined to think that this defect, if weight be actually necessary, might be avoided by using three or five needles parallel to each other; for by this means the weight and magnetic power increasing in nearly the same ratio, the directive power will be the same as in the simple needle, and every necessary weight may thus be obtained without effecting the due action of the instrument.

Investigation of the circumstances attending the directive force of a compass card carrying five needles parallel to each other, and placed at equal intervals, as in figure 52, plate V.

Advantage of a card being mounted with several needles. Fig. 52.

86. Let N E S W represent the card in question; N S, N′ S′, N″ S″, &c. being five parallel needles crossing the diameter E W at the equal intervals W a', a' b', b' O, &c. The radius being 1, W $a = \frac{1}{3}$, W $b = \frac{2}{3}$, and the arcs E N″ = 48° 12′, E N′ = 70° 32′, N N′ = 19° 28′, N N″ = 41° 48′.

Let l denote the length of the middle needle, then

$l \sin 70° 32' = \cdot 942\, l =$ length of N′ S′, N‴ S‴

$l \sin 48\ \ 12 = \cdot 745\, l =$ length of N″ S″, N^{iv} S^{iv}

And since the directive forces of needles are as their masses, or as their lengths when their other dimensions are the same (art. 30, exper. 14.) if we denote the intensity of the centre needle by i,

the intensity of N′ S′, N‴ S‴ will be $\cdot 942\, i$

the intensity of N″ S″, N^{iv} S^{iv} will be $\cdot 745\, i$

Or, denoting ·942 by m, and ·745 by n, we shall have

length of centre needle $= l$, its intensity $= i$

length of N′ S′, N‴ S‴ $= m\, l$, intensity $= m\, i$

length of N″ S″, N^{iv} S^{iv} $= n\, l$, intensity $= n\, i$

Let now the card be freely suspended and drawn out of its natural direction, so that N comes to some point a, and N′ to b; S to d, and S′ to c, and let the angle N O a, be denoted by δ; then N O b will be 19° 28′ + δ, and S O c = 19° 28′ − δ; or denoting the arc N N′ = 19° 28′, by a; these will become

$$\text{N O } a = a + \delta,\ \text{S O } c = a - \delta$$

and in the same way representing the arc N N″ = 41° 48′ by b, the angles formed by the radii connecting the extremities of the other needle N″ S″, will be $b + \delta$, and $b - \delta$.

Now the power of the centre needle to return to its proper direction by means of the directive power at a, will be $i \sin \delta$, or taking into the account the force at each extremity it will be $2\, i \sin \delta$. In the needle N′S the power at b to cause the needle to return will be $m\, i \sin (a + \delta)$ and at c, it will be $m\, i \sin (a - \delta)$ which acting in opposition to each other will give a resultant $= m\, i \left\{\sin (a + \delta) - \sin (a - \delta)\right\}$; and in the same way the power which tends to bring back the needle N″ S″ to its proper bearing, will be $n\, i \left\{\sin (b + \delta) - \sin (b - \delta)\right\}$ or, taking into account all the several needles, we shall have for the sum of all the directive forces.

$$\text{Needle N S} = i \sin \delta$$

$$\text{Needles N}'\text{S}', \text{N}'''\text{S}''' = 2\, m\, i \left\{\sin (a + \delta) - \sin (a - \delta)\right\}$$

$$\text{Needles N}''\text{S}'', \text{N}^{iv}\text{S}^{iv} = 2\, n\, i \left\{\sin (b + \delta) - \sin b - \delta)\right\}$$

or, by reduction;

$$\text{Needle N S} = i \sin \delta$$

$$\text{Needles N}'\text{S}', \text{N}'''\text{S}''' = 4\, m\, i \cos a \sin \delta$$

$$\text{Needles N}''\text{S}'', \text{N}^{iv}\text{S}^{iv}, = 4\, n\, i \cos b \sin \delta$$

Or, since $\cos a = m$, $\cos b = n$, these may be expressed in one sum, viz.

$$\left.\begin{array}{l}\text{directive force of card or}\\ \text{combination of needles}\end{array}\right\} \begin{array}{l}= i \sin \delta \left\{2 + 4\, (m^2 + n^2)\right\}\\ = 7 \cdot 768\, i \sin \delta\end{array}$$

The directive force therefore of such a combination of needles will be nearly four times that of a single needle. It may be observed, however, that the weight and friction on the point being also increased in nearly the same proportion, no advantage would be gained by the construction but for this circumstance, viz. that in sea compasses, and particularly in those for boats, it is always necessary to make the *cards heavy*, in order to render them steady: now, in the present construction, we have a sufficient weight for this purpose, and it is all effective in giving direction to the needle; whereas when the card is incumbered with a dead weight, the power of a single needle is insufficient to bring it to its proper bearing; and hence the only essential quality of the compass is, in a great measure counteracted or destroyed; whereas, by employing three or five needles, the weight and directive power being nearly proportional to each other, we thus obtain the former quality without sacrificing the latter.

PART III.

THEORETICAL INVESTIGATIONS CONNECTED WITH THE LAWS OF MAGNETISM.

§ XI. *On the magnetic properties of simple iron.*

Introduction.

87. ALTHOUGH it is our intention in this part of our treatise to investigate, on mechanical and theoretical principles, the laws of magnetic action, yet it will be advantageous to the reader to be informed of the nature of the experiments which led the way to, and laid the foundation of the theory in question; and if, by this means, we in some measure blend experiment with theory, and thus far interrupt the scientific concatenation that some readers might expect from the title to this division of the work, yet we are convinced that it will so much simplify the ideas we are desirous of conveying, that we do not hesitate to adopt this arrangement, in preference to that of a rigid adherence to the common mode of treating physical subjects.

Description of Mr. Barlow's apparatus for making magnetical experiments on simple iron bodies.

Mr. Barlow's apparatus. Fig. 54, plate VI.

88. This apparatus is shown in fig. 54, plate VI. It consists of a large and substantial round table, rendered perfectly steady, and its plane horizontal. Its upper face is divided into the points of the compass, degrees, &c. In its centre is a hole eighteen inches and a quarter in diameter, and above the latter is suspended, by a system of pulleys, an eighteen inch iron shell, or any smaller ball or shell, which by means

of the pulleys, (the power of which is twenty-four to one,) may be raised and lowered at the pleasure of the experimenter. The table is so adjusted that one of its diameters is brought very exactly into the magnetic meridian, and then the whole is rendered perfectly firm, steady, and immoveable.

Every thing being thus prepared, it is found, that in whatever part of the table a compass needle is placed, except in the meridian, the south end of the needle is drawn towards the ball, when the latter is wholly above the plane of the table; and supposing it to be considerably above, the first attraction is very small; but as the ball descends the attraction increases to a certain point, which is the maximum; it then, as the ball is lowered still more, diminishes, and at length vanishes: continuing still to lower the ball, the north end of the needle is attracted by it, and this attraction now increases to a certain point where it is a maximum, and thence decreases again towards zero as the ball is made to descend farther. This, as we have stated above, is found to be the case in whatever part of the table (except in the meridian) the compass is placed; that is, in all cases, we pass through a point where the attraction changes, and where the action of the iron on the compass is zero.

89. If we examine the situation of the ball at the moment when this action ceases, we shall find it different, according to the situation of the compass and its distance from the centre. In some positions of the compass, that is, when it is due east or west of the ball, the action is zero when the horizontal great circle of the ball is coincident with the pivot of the needle, or when the ball is just half above and half below the plane of the table, or rather the plane parallel to it passing through the needle; but if the compass be between the east or west and the north, then, when the action ceases, the ball is found below the plane of the table, and it is so much the lower as the compass is nearer to the north, and more distant from the centre. But when the compass is towards the south of the east or west, then the centre of the ball is above the plane of the table when the action ceases, and so much the more as we are nearer the south, and more distant from the centre of the table.

These facts clearly show that the several points of *no attraction* are, at least, not situated in the same horizontal plane; but, if in a plane at all, in one forming an oblique angle with the horizon descending from the north to the south.

After several experiments of the kind above indicated, (in which the compass was kept at the same constant distance from the centre of the table, and the height or depth of the centre of the ball, above or below the plane of the table when the attraction vanished, carefully measured,) it was found, that those points of no attraction were actually all situated in one plane, and that this latter formed an angle with the horizon equal to the complement of the dip of the needle, and hence this fundamental principle was established, viz.

In every ball or shell of plane unmagnetised iron there exists a plane of no attraction, or a plane in which the iron produces no disturbance in the needle, and which plane inclines from north to south, (magnetic) forming with the horizon an angle equal to the complement of the dip.

90. *Divison of the magnetic sphere.* 1. The plane above specified obviously cuts the surface of the ball in a great circle, which may be called the magnetic equator of the sphere: it is represented in fig. 54, by the circle *e q*.

Division of the magnetic sphere.

Fig. 54

2. This preliminary step naturally suggested to the author, the idea of giving two poles to this equator, which would necessarily have a direction corresponding with that of the dipping needle: these are indicated in the figure by *n*, *s*.

3. And hence again by a natural step, succeeded the idea of defining any point on the sphere by its magnetic latitude and longitude; by meridian and parallels, as indicated by *s n*, *s m* and *l l*, &c. The hemisphere below the equator is considered as the northern, and that above as the southern, the latitude in the two cases having the same designation

4. Any meridian, at pleasure, might be assumed as the principal meridian: the author has taken that which passes through the east and west points, because in this meridian the attraction was supposed to be greater than in any other.

5. We have hitherto spoken of these circles as applied immediately to the surface of the ball; but it is obvious that by merely conceiving these planes to be produced, to cut any other sphere circumscribing and concentric with the ball, the same mode may be adopted for defining the situation of any point in space with reference to the centre of the sphere. This is illustrated in fig. 55.

Fig. 55.

6. Moreover, instead of conceiving the imaginary sphere to surround the ball, we may imagine a similar sphere to be concentric with the pivot of the needle: then it is obvious that the centre of the ball will have the same relative position on the latter sphere as the pivot of the compass has with respect to the former, except the name of the latitude and longitude being changed, so that the reference may be made to either at pleasure.

Experiments made with a view of deducing the laws of magnetic action on different points of the sphere.

Mr. Barlow's experiments.

91. Having established the above fundamental fact relative to the plane of no attraction, or the magnetic equator, and having found the attraction to be of different characters on its opposite sides; and of the same character, but of different magnitudes in different situations on the same side; the next object was to ascertain, whether the quantity of deviation, which had obviously a relation to the particular point in which the compass was placed, could be expressed by any function of the latitude and longitude, the distance from the centre of the ball and the mass of the latter remaining constant. But as it was desirable to separate in this investigation the effect of latitude and longitude, the first experiments were made by positing the compass in such a situation on the table, and the ball in such a point in the vertical line, that the former might always be found in that circle, or principal meridian, in which the longitude was zero, but in different latitudes, varying from nothing to 90°, both north and south. The next object therefore was to compute at what azimuth the compass must be placed on the table, and at what distance from its centre, and how much the ball must be raised above, or depressed below the plane of the table, that the centre of the needle might be situated in the plane of the principal meridian at a given distance from the centre of the ball and in a given latitude. To effect this the following calculations were made, in which, however, instead

of supposing the latitude given, and thence computing the arc of azimuth on the table, the latter is assumed and the former computed; but this is merely a matter of convenience, which enabled the experimenter to place his compass on some one of his fixed divisions on the table, and thereby throwing the fractional results upon the latitude, instead of having them attached to his azimuths. The calculations necessary in this case may be thus stated.

Relative to the law of the latitudes.

Fig. 56.

92. Let H Z R (fig. 56) represent the sphere concentric with the shell whose centre is at C; N, S its north and south poles with reference to the equator Q Q′; Z the zenith of the sphere; H R a circle parallel to the horizon; and S E another circle or principal meridian passing through the poles, and through the magnetic east and west points of the horizon, where it also meets Q Q′. Imagine also the quadrant Z L V to be drawn to any point V in the horizon, cutting S E in the point L; from L let fall the perpendicular L M, which will meet the plane H R in the line joining V with the centre C.

Now the arc E V being supposed given, we may readily compute the point L, where the arc S E is intersected by the quadrant Z V, and the arc L E will be the latitude of that point, with reference to Q Q′ as an equator, and the lines L M and C M, (the sine and cosine of the arc V L to any assigned radius,) will shew how much the compass ought to be elevated above the centre of the ball, and at what distance it ought to be placed from the centre of the table to correspond with the point L.

Formula for the computation.

93. The formulas for these computations are as follow, viz. in the right-angled spherical triangle V E L, we have given the right angle at V, the angle L E V equal to the dip and the arc E V, to find the hypothenuse L E, or the latitude, and arc or perpendicular L V. For the former we have

$$\tan LE = \tan VE \,.\, \sec VES$$

and for the latter,

$$\tan VL = \sin VE \,.\, \tan VES$$

and hence again,

$$LM = LC \,.\, \sin VL$$
$$MC = LC \,.\, \cos VL$$

V L being the given distance.

By means of this computation it was easy to compute the latitude of a variety of points in the arc N E S, and to observe the corresponding deviations of the needle; and the results, when compared with each other, showed very distinctly that there was such a relation between the deviations and the several latitudes as was sought; for it was found that the rectangle of the sine and cosine of the latitude, divided by the tangent of the deviation, gave always a constant quantity. The law therefore in this case might be stated as follows:

Law in reference to the latitude.

94. *The tangent of deviation is proportional to the rectangle of the sine and cosine of the latitude; or to the sine of the double latitude.* The mass and distance being constant.

Law as depending on the latitude and longitude.

95. The next step was to find the law of the deviation as it depended on the longitude, and here three different methods presented themselves, viz. first, the compass might be passed over a great circle in which the longitude was constant; secondly, over a small circle, in which the latitude was constant; or thirdly, over a circle in which both the latitude and the longitude were variable. The first and third presenting the least difficult cases for computation were those selected by the author. The nature of the operations in the former case may be thus illustrated.

Fig. 57.

96. Let H Z R (fig. 57) represent a sphere concentric with the ball, whose centre is at C; Q Q′ the equator; H R the horizon; N S the north and south poles; Z the zenith of the sphere, and S L any meridian, which let be produced to meet H R in B; the longitude E P of the point P being given, which will also be the longitude of the point L, situated in that meridian; or we may otherwise, as in the former case, consider the point B as given, and compute the longitude of the point P, or of the meridian S L P. Imagine also a quadrant Z V to be drawn from Z, perpendicular to H R, and cutting the circle S P in L; also let fall the perpendicular L M, which will fall in the line joining V and the centre C; so will C M (the cosine of the angle L C V to any proposed radius) represent the distance of the point L, from the vertical axis of the ball, or from the centre of the table, and L M (the sine of the same angle) the altitude of L, above the plane H R. In order therefore to have the centre of the compass placed so as to coincide with the point L, the centre of the ball must be depressed below the plane of the table by a quantity equal to L M, at the same time that the compass must be approximated towards the centre of the table in the line V C, till its distance is equal to C M. The latitude and longitude may then be computed as follow:

Formula for comp- tation.

Data { the arcs E B, the distance C L, and the angles S C H, and H C Q, or P E B.

$$\text{Here } \tan EP = \tan \text{long.} = \tan EB \cos PEB$$
$$\tan EBP = \cos PEB \,.\, \sec EB$$
$$\sin PB = \sin EB \,.\, \sin PEB$$

Now, asuming different arcs for B V, we may from the preceding data find the corresponding arcs B L, V L; from B L deducting B P, we shall have the latitude P L of the point L; and the sine and cosine of L V to any proposed radius, will give the position of the ball and compass as explained above. The following are the formulas for these determinations, viz.

$$\tan LB = \tan VB \,.\, \sec VBL$$
$$\tan LV = \sin VB \,.\, \tan VBL$$
$$LM = LC \sin LCV \,;\; MC = LC \,.\, \cos LCV$$

These computations and the corresponding observations being made, it was found that the product of the sine of the double latitude and cosine of the longitude, divided by the tangent of the deviation, gave again the same constant numbers as in the preceding case, but with rather stronger aberrations. Whence was deduced this second law, viz.

Law in ference latitud and lo tude.

97. *The mass and distance remaining constant, the tangent of the deviation is nearly proportional to the sine of the double latitude multiplied by the cosine of the longitude.*

These two laws being established, the next object was to find the law for the distance; and with this view having compared together the constant numbers obtained on the principles above indicated at different distances, it was found that,

Law i refer to dis

98. *All other things being the same, the tangent of the deviation is inversely proportional to the cube of the distance.*

It remained now only to examine the law which was observed with respect to the mass, and here a most unexpected result was obtained, viz.

Law i refere to ma surfac

99. *That the tangent of the deviation is directly proportional to the cube of the diameter of the ball or shell; but is still wholly independent of the mass, being the same*

tism. *in quantity whatever be the thickness of the metal, provided only that it exceed about $\frac{1}{30}$th of an inch.*

We shall as we proceed exhibit the numerical results by which these several laws have been experimentally established; the heads are merely stated in this place as useful preliminary steps to the general investigation on which we are now about to proceed.

§ XII. *Investigation of the analytical formula exhibiting the magnetic action of iron spheres.*

s of ga-

100. Although we have thought it advantageous to enumerate the several experimental laws above detailed, as a means of indicating to the reader the objects of our investigation, yet there is only one of them that it is essential to employ as an established experimental fact, which is, that last described; viz. *that the entire magnetic power of an iron sphere resides on the surface and is independent of the solidity*; this it is necessary to introduce to enable us to arrive at specific numerical formulas; the other laws are then deducible from the investigation, and thus serve to confirm reciprocally the accuracy both of the experimental and theoretical deductions.

hesis.

101. The hypothesis on which we shall proceed may be thus enumerated.

1. Magnetic phenomena are due to the existence of two fluids in a greater or less degree of combination, and such that the particles of the same fluid repel, and those of an opposite nature attract each other.

2. These fluids in iron bodies exist naturally in a state of combination and equilibrium, till that state is disturbed by some exciting cause.

3. But if a body already magnetic, i. e. one in which the fluids are held in a state of separation, be brought within the vicinity of a mass of iron, such as is supposed above, the concentrated action of each fluid in the magnetised body will act upon the latent fluids in the quiescent body, by repelling those of the same, and attracting those of the contrary kind, and thus impress upon the latter a temporary state of magnetic action, which will remain only while the two bodies maintain their respective situations.

4. The quantity of action thus impressed upon the iron body will depend, *first*, upon the intensity of the exciting magnet; *secondly*, upon the capacity of the quiescent body for magnetism, or the quantity of those fluids contained in it; and *thirdly*, upon the cohesive power of the iron; which latter quality determines the depth to which the exciting magnet is able to disengage the two fluids.

The above embraces every case, viz. of any magnet, natural or artificial, developing the magnetism in any given iron body; but that to which our attention will be principally directed, namely the displacement occasioned by the magnetic action of the earth on spheres of iron, we shall find more limited in its results, and more susceptible of correct mathematical investigation.

5. In this case, for instance, we may suppose the action to take place on every particle of the mass in lines parallel to each other, and corresponding in direction with the dipping needle; also that every particle is at the same distance from the centre of the disturbing force, and consequently that the displacement in each particle is equal also; conditions which throw great facilities into the analytical investigation of the laws of action.

Part III.

Hypothesis illustrated.

Fig. 58.

6. For the sake of illustration, let A B E D, fig. 58, represent a sphere of iron in its non-magnetic or quiescent state, and let C M be the line in which the terrestrial magnetism is exerted from a centre of action M, which is at such a distance that the diameter of the sphere is inconsiderable in comparison with it; then every particle on its surface, and to a certain distance within it, will be acted upon by equal powers, and in directions parallel to each other; whereby the fluids in the quiescent body, before in a state of combination, will be separated in each particle; and the two fluids may now therefore be conceived to form two spherical shells A e B d, A e' B d', whose centres of action will be in $c\,c'$, their distance from each other being greater or less according to the circumstances stated in No. 4.

7. Therefore in computing the action of such a mass of iron, in its temporary state of magnetism, upon a distant particle of magnetic fluid, we may refer it to those centres; we shall also asume that the law of action in this, as in all other cases of emanating forces, is inversely as the square of the distance.

102. Such is the hypothesis upon which is founded the following investigation, and which it will be seen, will enable us to deduce all those laws previously drawn from experiment, as likewise to infer several curious consequences arising out of our analytical formulas; the probable accuracy of which will be estimated by the coincidence in those cases between theory and experiment where the comparison can be made.

It should be here observed that this hypothesis differs from that advanced by Mr. Bonnycastle, in *Philosophical Magazine*, vol. 55, p. 132, only in this, i. e. he imagined the fluids to be separated from each other, and to be accumulated in distinct poles or centres of action; whereas, according to the supposition made above, the displacement takes place in each particle, and the centres of action are therefore indefinitely near to each other in the common centre of attraction of the surface of the body. It differs also from the theory advanced by Coulomb in this, that he conceives the displacement to take place in every particle of the mass, whereas we suppose it to be confined to the surface, a fact which has been already established by experiment, (art. 97.) His centres of action are therefore in the centre of attraction of the mass, and ours in the centre of attraction of the surface. These are coincident in spheres but in no other bodies.

General investigation.

103. Agreeably to the hypothesis which has been advanced, let A B (fig. 59) represent a sphere of soft iron, which has acquired a magnetic action in the direction S N, from the effect of the terrestrial magnetism; S N denoting the line of the dip; and let P be an indefinitely small magnetic particle. Then this, by the supposition, will be acted upon, first by the terrestrial magnetism in a direction parallel to S N, and also by the sphere A B from the two centres c, c indefinitely near to each other, and to the geometrical centre of the sphere O; being repelled from one of those centres and attracted towards the other, by forces varying inversely as $\mathrm{P}\,c^2$ and $\mathrm{P}\,c'^2$; and in consequence of these forces the particles at P will assume a certain direction, which it is our object to compute.

104. Let c' be the centre towards which the particle is attracted, and c that from which it is repelled;

and let the effect due to the former be denoted by $\frac{f}{\mathrm{P}\,c'^2}$, then that due to the latter will be $\frac{f}{\mathrm{P}\,c^2}$. In the same way if we consider the action on the other pole of the particle at P, we shall find the repulsion upon it denoted by $\frac{f}{\mathrm{P}\,c'^2}$ and the attraction by $\frac{f}{\mathrm{P}\,c^2}$, which will produce precisely the same results as the former forces; we may therefore consider each of these forces as being doubled and to become $\frac{2f}{\mathrm{P}\,c'^2}$ and $\frac{2f}{\mathrm{P}\,e^2}$.

Let the distance P O be denoted by d, the angle S O P by ϕ, and the indefinitely short distance c O$=$ c' O by e.

Conceive the force $\frac{2f}{\mathrm{P}\,c^2}$ to be resolved into two forces in the direction O P, c O, and the force $\frac{2f}{\mathrm{P}\,c'^2}$ into two, in the directions P O, O c'.

That is, resolve

$$\frac{2f}{\mathrm{P}\,c^2} \text{ into } \frac{2fd}{\mathrm{P}\,c^3} \text{ and } \frac{2fe}{\mathrm{P}\,c^3}$$

$$\frac{2f}{\mathrm{P}\,c'^2} \text{ into } \frac{2fd}{\mathrm{P}\,c'^3} \text{ and } \frac{2fe}{\mathrm{P}\,c'^3}.$$

105. The first of each of these pairs of forces are opposed to each others action, and their resultant is therefore equal to their difference; but the others acting in the same direction $c\,c'$ must be added; our forces thus become

$$\frac{2fd}{\mathrm{P}\,c'^3} - \frac{2fd}{\mathrm{P}\,c^3} \text{ in the direction P O,}$$

and $$\frac{2fe}{\mathrm{P}\,c'^3} + \frac{2fe}{\mathrm{P}\,c^3} \text{ in the line } c\,c' \text{ or N S.}$$

The latter being opposed to that of the terrestrial action. Now by trigonometry,

$$\mathrm{P}\,c = \sqrt{(d^2 + e^2 + 2\,d\,e\cos\phi)}$$
$$\mathrm{P}\,c' = \sqrt{(d^2 + e^2 - 2\,d\,e\cos\phi)}$$

The above expressions therefore become

$$\frac{2fd}{(d^2+e^2-2\,d\,e\cos\phi)^{\frac{3}{2}}} - \frac{2fd}{(d^2+e^2+2\,d\,e\cos\phi)^{\frac{3}{2}}} \text{ in P O}$$

$$\frac{2fe}{(d^2+e^2-2\,d\,e\cos\phi)^{\frac{3}{2}}} + \frac{2fe}{(d^2+e^2+2\,d\,e\cos\phi)^{\frac{3}{2}}} \text{ in N S.}$$

But since e is indefinitely small in respect to d, we may omit in the developement of these formulas, all those terms in which e enters in any power higher than the first, and which thus reduce to

$$\frac{12\,f\,e\cos\phi}{d^3} = \text{force in P O} \qquad (1)$$

$$\frac{4fe}{d^3} = \text{force in N S} \qquad (2)$$

The forces therefore arising from the action of the sphere upon the particle P are resolved into two, which may be represented by the lines P m, P n; and consequently their resultant will fall somewhere within the angle m P n. Let us however before we attempt this composition introduce the natural directive power of the earth upon the needle, and ascertain the proper analytical value of the force denoted by f.

106. With respect to the former, since it is constant in the line S N, it will be sufficient to denote it by any constant quantity M; and for the latter, we may observe that agreeably to our experiments (art. 97.) it follows that in iron of the same species, and of a thickness greater than the depth of metal at which the developement is effective, the quantity of displaced magnetism will be directly as the surface, or as the square of the radius; but its central action upon a particle at the surface, will be inversely as the square of the distance, that is of the radius; consequently the action of those centres on a particle at the surface will be the same for spheres and shells of all diameters and dimensions: hence our force (2) at the surface, which there becomes $\frac{4fe}{r^3}$ is a constant quantity, whatever may be the value of r; make it equal to C and we shall have

$$\frac{4fe}{r^3} = \mathrm{C}, \text{ or,}$$

$$f = \frac{\mathrm{C}\,r^3}{4\,e}$$

substituting this value of f into our forces (1) and (2), and combining with the latter the constant directive force M, we shall have for the forces acting on the particle P

$$\frac{3\ \mathrm{C}\ r^3\cos\phi}{r^3} = \text{force in P O} \qquad (3)$$

$$\mathrm{M} - \frac{\mathrm{C}\,r^3}{d^3} = \text{force in S N} \qquad (4)$$

For the more convenient estimation of these forces, let the former be resolved into two, the one perpendicular to S N, and the other parallel to it; by which means our forces will become

$$\frac{3\,\mathrm{C}\,r^3\cos\phi\sin\phi}{d^3} = \text{force perp. to S N} \qquad (5)$$

$$\mathrm{M} + \frac{3\,\mathrm{C}\,r^3\cos^2\phi}{d^3} - \frac{\mathrm{C}\,r^3}{d^3} = \text{force in S N} \qquad (6)$$

Let Δ represent the angle which the resultant of these forces form with N′ S′, or P N′; then, by the principle of forces, we shall have

$$\tan\Delta = \frac{3\,\mathrm{C}\,r^3\cos\phi\sin\phi}{\mathrm{M}\,d^3 + 3\ \mathrm{C}\ r^3\cos^2\phi - \mathrm{C}\,r^3}$$

$$\text{or,} \quad \tan\Delta = \frac{3\cos\phi\sin\phi}{\frac{\mathrm{M}\,d^3}{\mathrm{C}\,r^3} + 3\cos^2\phi - 1} \qquad (7)$$

107. In this expression, Δ obviously denotes the deviation of the magnetic particle, from its natural direction S N, or of an indefinitely short needle freely suspended at that point. This deviation being therefore that due to a needle freely suspended, will necessarily take place in the plane which passes through the centre of the needle and the attracting body. To estimate the effect of this force in deflecting a horizontal needle, the two forces (5) and (6) may be resolved into two, perpendicular and parallel to the meridian; and to estimate the same on a dipping needle suspended in the plane of the meridian, they may in like manner be resolved into two, the one perpendicular and the other parallel to the horizon. Or which is perhaps more simple, we may project the arc Δ (7) in the plane above alluded to, upon the horizontal plane and on the plane of the meridian, and thus determine the deviations required.

Of the horizontal needle.

108. First, for the horizontal needle; let S N (fig.

60.) represent the needle in its natural direction, about which, and concentric with it, conceive a sphere to be described; let P denote the centre of the attracting body, and S S′ the arc of deviation in the plane S S′ O P, (which has been denoted by Δ) H H′ the horizon, Q Q′ the equator, or plane of no attraction; and let the angle which the plane S O P makes with the meridian be denoted by i; that is, make $\angle$ Z S S′ $= i$ (Z being the zenith) the angle S O P $= \phi$ as before, and the arc H S $= \delta$ the dip of the needle.

If now from the zenith Z, we draw the arc Z S V, H V will be the projection of the arc S S′ upon the horizontal plane, and will exhibit the angle of deviation in a needle limited in its motion to that plane.

Make this last angle $= \Delta'$, then by a trigonometrical formula readily deduced, we have

$$\tan \Delta' = \frac{\tan \Delta' \sin i}{\cos \delta - \dfrac{\sin \delta \cos i}{\cot \Delta}}$$

Into which introducing the values of $\tan \Delta$, already determined, viz.

$$\tan \Delta' = \frac{3 \cos \phi \sin \phi}{\dfrac{\mathrm{M}\, d^3}{\mathrm{C}\, r^3} + 3 \cos^2 \phi - 1}$$

we find

$$\tan \Delta' = \qquad (8)$$

$$\frac{3 \cos \phi \sin \phi \sin i}{\dfrac{\mathrm{M}}{\mathrm{C}} \cdot \dfrac{d^3}{r^3} \cos \delta + (3 \cos^2 \phi - 1) \cos \delta - 3 \sin \phi \cos \phi \cos i \sin \delta}$$

or which is the same

$$\tan \Delta' =$$

$$\frac{\frac{3}{2} \sin 2\lambda \cos l}{\dfrac{\mathrm{M}}{\mathrm{C}} \cdot \dfrac{d^3}{r^3} \cos \delta + (3 \sin^2 \lambda - 1) \cos \delta \mp 3 \sin 2\lambda \sin \lambda \sin l}$$

or making

$$(3 \sin^2 \lambda - 1) \cos \delta \mp 3 \sin 2\lambda \sin \delta \sin l = \mathrm{N}$$

The lower sign applying to the case of $i \angle 90^\circ$, we have more concisely

$$\tan \Delta' = \frac{\sin 2\lambda \cos l}{\dfrac{\mathrm{M}}{\mathrm{C}} \cdot \dfrac{2 d^3}{3 r^3} \cos \delta + \frac{2}{3} \mathrm{N}} \qquad (9)$$

where λ denotes the complement of the angle ϕ, or the latitude, and l the complement of the angle i, or the longitude, conformably with the ideas advanced in (art. 90.)

109. The above is a rigorous formula for all distances, and for a ball or shell of any radius, and of any description of iron; but the value of the coefficient $\frac{\mathrm{M}}{\mathrm{C}}$ is different in iron of different kinds. In the case of cast iron, it will be seen in a subsequent article that it is nearly equal to unity, and consequently in this case, when d exceeds two three times r, as in the following experiments, the quantity denoted by N, which is made up with the sums and differences of products, of which each of the trigonometrical factors are less than unity, is small in comparison with the other term $\frac{\mathrm{M}}{\mathrm{C}} \cdot \frac{2 d^3}{3 r^3} \cos \delta$, and we shall therefore, by dropping the former, have as an approximate formula

$$\tan \Delta' = \frac{\mathrm{C}}{\mathrm{M}} \cdot \frac{3 r^3}{2 d^3 \cos \delta} \,(2\lambda \cdot \cos l) \qquad (10)$$

which exhibits in one term all the approximative laws to which we have alluded, (art. 94, *et seq.*) and which were, as we have seen, deduced wholly from experiment independent of any theory whatever.

It follows, for instance, from this formula;

1. That although by the hypothesis the developement of the magnetism of the ball or shell takes place only at the surface, yet the effect, as shown by the tangent of the deviation, is proportional to the cube of the diameter.

2. That the tangent of deviation is inversely as the cube of the distance.

3. That the tangent of deviation is proportional to the sine of the double latitude and cosine of the longitude; the latter being estimated from the east or west points.

Which are precisely the laws we have stated in the article above referred to as being deduced from experiments only.

We shall now exhibit the detail of these experiments, that the reader may be able to judge of the approximation between the results obtained from actual observation, and those derived from calculation.

§ XIII. *Comparison of the preceding theoretical formulas, with the results obtained from experiments on the horizontal needle.*

Comparison of experiment with theory.

110. We have already explained the apparatus and the method of performing the experiments on the iron ball, and also the formulas and the principles of calculation by which the compass was brought into any proposed situation, with respect to its latitude and longitude on the imaginary sphere; without therefore retracing our steps we may proceed at once to exhibit the results of the experiments, observing only that the last column in each of the following tables is obtained by simply dividing the product of the sine of double the latitude into the cosine of the longitude, by the tangent of the observed deviation; the quotients arising from which, according to our approximate formula (10), ought (at the same distance, and with the same ball) to give a constant number, very nearly; that is to say, we ought to find

$$\frac{\sin 2\lambda \,.\, \cos l}{\tan \Delta'} = \frac{\mathrm{M}}{\mathrm{C}} \cdot \frac{2 d^3 \cos \delta}{3 r^3}$$

a constant quantity.

The ball with which the following experiments were made was a solid thirteen inch mortar *proof ball*, weighing 288 lbs.; it was suspended as described in fig. 54, plate VI. and the observations were made in the manner explained in article 91, *et seq.*

Law in reference to the latitude.

111. *First set of experiments to compare the law of deviations as it depended on the latitude only. The compass having been carried round the ball in the principal meridian at the several distances, twelve, fifteen, eighteen and twenty inches.*

EXPERIMENTS.

In the circle S E, of which the longitude is 0°. Ball 288 lbs.

RADIUS OF CIRCLE TWELVE INCHES.

Position of compass.		Latitude.		Longitude.		Height of centre.	Distance from the centre of the table.	Deviation of compass, east.		Deviation of compass, west.		Mean deviation.		Ratio of $\frac{\sin . 2\lambda}{\tan . \Delta}$[*]
°	′	°	′	°	′	inches.	inches.	°	′	°	′	°	′	
2	30 E. or W.	7	27	0	0	1·465	11·91	10	0	10	15	10	7½	1·439
5	0	14	41	ditto		2·87	11·65	19	30	19	45	19	37½	1·375
7	30	21	31	ditto		4·15	11·26	26	30	26	30	26	30	1·369
10	0	27	51	ditto		5·28	10·77	32	0	31	30	31	45	1·335
12	30	33	35	ditto		6·26	10·28	34	15	34	0	34	7½	1·365
15	0	38	45	ditto		7·08	9·68	35	45	35	30	35	37½	1·363
17	30	43	22	ditto		7·76	9·15	36	15	36	15	36	15	1·362
20	0	47	28	ditto		8·33	8·63	35	30	36	0	36	15	1·328
25	0	54	24	ditto		9·20	7·76	34	0	35	0	34	30	1·378
30	0	59	58	ditto		9·79	6·93	32	15	32	30	32	22½	1·367
35	0	64	30	ditto		10·21	6·30	30	0	29	45	29	52½	1·353
40	0	68	18	ditto		10·51	5·79	26	45	26	45	26	45	1·363
50	0	74	21	ditto		10·88	5·03	21	0	21	0	21	0	1·354
60	0	79	6	ditto		11·10	4·54	15	15	15	15	15	15	1·362
70	0	83	4	ditto		11·22	4·23	10	0	10	0	10	0	1·359
80	0	86	38	ditto		11·30	4·05	5	0	5	45	4	52½	1·375
												Mean		1·365

In the circle S E, of which the longitude is 0°. Ball 288 lbs.

RADIUS OF CIRCLE FIFTEEN INCHES.

Position of compass.		Latitude.		Longitude.		Height of centre.	Distance from the centre of the table.	Deviation of compass, east.		Deviation of compass, west.		Mean deviation.		Ratio of $\frac{\sin . 2\lambda}{\tan . \Delta}$
°	′	°	′	°	′	inches.	inches.	°	′	°	′	°	′	
2	30 E. or W.	7	27	0	0	1·83	14·88	5	15	5	0	5	7½	2·867
5	0	14	41	ditto		3·58	14·56	10	15	10	0	10	7½	2·746
7	30	21	31	ditto		5·19	14·07	14	0	14	15	14	7½	2·712
10	0	27	51	ditto		6·60	13·47	16	45	17	0	16	52½	2·724
12	30	33	35	ditto		7·82	12·79	18	30	18	45	18	37½	2·736
15	0	38	45	ditto		8·85	12·11	19	30	20	0	19	45	2·719
17	30	43	22	ditto		9·71	11·43	20	0	20	30	20	15	2·706
20	0	47	28	ditto		10·42	10·79	20	0	20	15	20	7½	2·719
25	0	54	24	ditto		11·50	9·65	19	0	19	30	19	15	2·711
30	0	59	58	ditto		12·24	8·67	17	30	17	45	17	37½	2·728
35	0	64	30	ditto		12·76	7·93	16	0	16	0	16	0	2·710
40	0	68	18	ditto		13·14	7·24	14	0	14	10	14	5	2·732
50	0	74	21	ditto		13·60	6·30	10	30	11	0	10	45	2·736
60	0	79	6	ditto		13·88	5·67	7	30	7	30	7	30	2·821
70	0	83	4	ditto		14·03	5·29	5	0	5	0	5	0	2·739
80	0	86	38	ditto		14·12	5·07	2	30	2	30	2	30	2·685
												Mean		2·737

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[*] λ here denotes the latitude, and Δ the angle of deviation from the magnetic north.

In the circle S E, of which the longitude is 0°. Ball 288 lbs.

RADIUS OF CIRCLE EIGHTEEN INCHES.

Position of compass.	Latitude.	Longitude.	Height of centre.	Distance from the centre of the table.	Deviation of compass, east.	Deviation of compass, west.	Mean deviation.	Ratio of $\frac{\sin . 2\lambda}{\tan . \Delta}$
° ′	° ′	° ′	inches.	inches.	° ′	°	° ′	° ′
2 30 E. or W.	7 27	0 0	2·32	17·87	3 0	3 0	3 0	4·907
5 0	14 41	ditto	4·31	17·48	5 45	5 45	5 45	4·869
7 30	21 31	ditto	6·23	16·90	8 15	8 0	8 7½	4·780
10 0	27 51	ditto	7·92	16·17	10 0	10 0	10 0	4·686
12 30	33 35	ditto	9·39	15·36	11 0	11 0	11 0	4·741
15 0	38 45	ditto	10·62	14·53	11 40	12 5	11 52½	4·642
17 30	43 22	ditto	11·65	13·73	12 0	12 30	12 15	4·590
20 0	37 28	ditto	12·51	12·96	11 50	12 10	12 0	4·687
25 0	54 24	ditto	13·80	11·58	11 0	11 30	11 15	4·759
30 0	59 58	ditto	14·69	10·41	10 15	10 15	10 15	4·793
35 0	64 30	ditto	15·32	9·46	9 30	9 15	9 22½	4·707
40 0	68 18	ditto	16·77	8·69	8 30	8 30	8 30	4·595
50 0	74 21	ditto	16·32	7·56	6 20	6 20	6 20	4·681
60 0	79 6	ditto	16·66	6·82	4 30	4 30	4 30	4·719
70 0	83 4	ditto	16·84	6·35	3 0	2 45	2 57½	4·638
80 0	86 38	ditto	16·95	6·09	1 30	1 20	1 25	4·741
							Mean	4·709

In the circle S E, of which the longitude is 0°. Ball 288 lbs.

RADIUS OF CIRCLE TWENTY INCHES.

Position of compass.	Latitude.	Longitude.	Height of centre.	Distance from the centre of the table.	Deviation of compass, east.	Deviation of compass, west.	Mean deviation.	Ratio of $\frac{\sin . 2\lambda}{\tan . \Delta}$
° ′	° ′	° ′	inches.	inches.	° ′	° ′	° ′	
2 30 E. or W.	7 27	0 0	2·44	19·85	2 15	2 15	2 15	6·545
5 0	14 41	ditto	4·78	19·42	4 30	4 30	4 30	6·231
7 30	21 31	ditto	6·92	18·77	6 0	6 10	6 5	6·403
10 0	27 51	ditto	8·80	17·96	7 20	7 15	7 17½	6·456
12 30	33 35	ditto	10·43	17·06	8 10	8 15	8 12½	6·389
15 0	38 45	ditto	11·80	16·14	8 40	8 30	8 35	6·465
17 30	43 22	ditto	12·94	15·25	8 45	9 0	8 52½	6·394
20 0	47 28	ditto	13·89	14·39	8 40	8 45	8 42½	6·460
25 0	54 24	ditto	15·33	12·86	8 20	8 20	8 20	6·463
30 0	59 58	ditto	16·32	11·56	7 45	7 30	7 37½	6·473
35 0	65 30	ditto	17·02	10·51	6 45	7 0	6 52½	6·446
40 0	68 18	ditto	17·52	9·65	6 10	6 0	6 5	6·447
50 0	74 21	ditto	18·13	8·39	4 40	4 45	4 42½	6·308
60 0	79 6	ditto	18·51	7·57	3 20	3 20	3 20	6·376
70 0	83 4	ditto	18·71	7·05	2 10	2 10	2 10	6·338
80 0	86 38	ditto	18·83	6·76	1 0	1 0	1 0	6·717
							Mean	6·432

The very near approximation of the numbers or ratios in the last column in each of the preceding tables towards an equality, cannot allow us for a moment to doubt that the law which has been deduced from our theoretical investigation, is actually the law of action between the iron and the compass, viz. *that the tangent of the angle of deviation is proportional to the rectangle of the sine and cosine of the latitude, or to the sine of the double latitude, the longitude being zero.*

112. *Second set of experiments to compare the law as it depends on latitude and longitude combined. The compass having been carried round the ball in a meridian, whose longitude was* 58° 31′; *at the distances of eighteen and twenty inches.*

EXPERIMENTS.

In the circle S B, of which the longitude is 58° 31′. *Ball* 288 *lbs.*

RADIUS OF CIRCLE EIGHTEEN INCHES.

Position of compass from B.	Latitude.	Longitude.	Height of centre.	Distance from the centre of the table.	Deviation of compass, east.	Deviation of compass, west.	Mean deviation.	Ratio of $\frac{\sin.2\lambda\cos l}{\tan.\Delta}$
° ′	° ′	° ′	inches.	inches.	° ′	° ′	° ′	
2 30 E. or W.	2 45S.	58 31	4·30	17·48	0 30	0 30	0 30	....*
5 0	9 50N.	ditto	7·94	16·16	2 20	2 20	2 20	4·314
7 30	20 14	ditto	10·68	14·49	4 15	4 25	4 20	4·473
10 0	28 30	ditto	12·60	12·86	5 20	5 30	5 25	4·615
12 30	34 19	ditto	13·93	11·41	6 0	6 0	6 0	4·628
15 0	40 8	ditto	14·85	10·71	6 0	6 0	6 0	4·786
17 30	44 16	ditto	15·51	2·14	6 30	6 30	6 30	4·582
20 0	47 35	ditto	15·99	8·28	6 30	6 30	6 30	4·565
25 0	52 41	ditto	16·60	6·96	6 30	6 0	6 15	4·598
30 0	56 23	ditto	16·97	6·02	6 0	6 0	6 0	4·582
35 0	59 12	ditto	17·20	5·31	6 0	6 0	6 0	4·271
40 0	61 27	ditto	17·36	4·78	5 30	5 30	5 30	4·554
50 0	64 52	ditto	17·54	4·06	5 0	5 0	5 0	4·590
60 0	67 27	ditto	17·64	3·60	4 30	4 30	4 30	4·699

* This number is omitted in consequence of the latitude and deviation being of a contrary name to all that follow; the compass in this experiment being situate between P and B fig. 57.

In the circle S B, of which the longitude is 58° 31′. *Ball* 288 *lbs.*

RADIUS OF CIRCLE TWENTY INCHES.

Position of compass from B.	Latitude.	Longitude.	Height of centre.	Distance from the centre of the table.	Deviation of compass, east.	Deviation of compass, west.	Mean deviation.	Ratio of $\frac{\sin.2\lambda\cos l}{\tan.\Delta}$
° ′	° ′	° ′	inches.	inches.	° ′	° ′	° ′	
2 30 E. or W.	2 45S.	58 31	4.77	19·42	0 30	0 45	0 37½	....
5 0	9 50N.	ditto	8·82	17·95	1 30	1 30	1 30	6·712
7 30	20 14	ditto	11·86	16·10	3 10	3 10	3 10	6·135
10 0	28 30	ditto	14·00	14·28	4 15	4 0	4 7½	6·085
12 30	34 19	ditto	15·47	12·67	4 0	4 0	4 0	6·954
15 0	40 8	ditto	16·50	11·30	4 30	4 15	4 22½	6·741
17 30	44 16	ditto	17·23	10·15	4 30	4 45	4 37½	6·465
20 0	47 35	ditto	17·76	9·20	4 45	5 0	4 52½	6·109
25 0	52 41	ditto	18·44	7·73	5 0	4 45	4 52½	5·913
30 0	56 23	ditto	18·85	6·68	4 30	4 30	4 30	6·119
35 0	59 12	ditto	19·11	5·90	4 0	4 0	4 0	6·879
40 0	61 27	ditto	19·28	5·31	4 0	4 0	4 0	6·271
50 0	64 52	ditto	19·48	4·51	3 45	3 45	3 45	6·127
60 0	67 27	ditto	19·59	4·00	3 30	3 30	3 30	6·048

The numbers in the last columns of the above tables are not so approximative as in the four preceding, in which the latitude only enters; but, on the other hand, it is to be observed that the deviations are here considerably smaller than in that case; and consequently any trifling error of observation produces a more sensible effect; while the compass itself is brought much nearer the centre of the table, where a small error in the adjustment is more easily made, and gives rise to a greater discrepancy. Moreover the formulas we have taken are not perfectly correct but approximative, and ought therefore to be expected to give aberrations in the results fully to the amount of those in the table. We shall in a subsequent article submit a series of experiments to a test by the correct formulas, and in these we shall find a nearer coincidence. At present we may be permitted to assert, that all other things being the same the tangents of the deviations are *very nearly* proportional to the sine of the double latitude multiplied by the cosine of the longitude, as exhibited in our approximate formula (art. 110.)

Comparison of the preceding results with the laws in reference to distance.

113. We have seen that according to the theoretical deductions (art. 109) we ought to find, when all other things are the same, that the tangents of deviation are inversely proportional to the cubes of the distances; and we might therefore content ourselves with examining how nearly the constant numbers obtained in the preceding tables correspond with this law; but it will perhaps be more satisfactory to ascertain what law we shall be able to deduce from these results independent of any theory whatever, on the principles by which the author arrived at this important deduction. On this point he observes,

"Having, in the experiments for a comparison of the latitudes, obtained four mean results, corresponding to the four distances 12, 15, 18, and 20 inches, let us examine whether any law exists between those mean ratios and any power or powers of the distances. These ratios and distances are as below:

Distance.	Ratio.
12 inches	1·365
15	2·737
18	4·709
20	6·432

"In order to investigate this question, let m denote any indeterminate index, and let us assume that

$$12^m : 15^m :: 1365 : 2737$$
$$12^m : 18^m :: 1365 : 4709$$
$$12^m : 20^m :: 1365 : 6432$$
$$15^m : 18^m :: 2737 : 4709$$
$$15^m : 20^m :: 2737 : 6432$$
$$18^m : 20^m :: 4709 : 6432$$

"If, in all these proportions, we find m equal, or nearly equal, to one determinate number, we may conclude that such value of m will express the power of the distance, which is proportional to the inverse ratio of the tangents of the deviations, the position of the ball and compass remaining the same in respect to latitude and longitude.

"Now the preceding proportions give

$$1 \ldots\ldots m = \frac{\log 2737 - \log 1365}{\log 5 - \log 4} = 3{\cdot}118$$

$$2 \ldots\ldots m = \frac{\log 4709 - \log 1365}{\log 3 - \log 2} = 3{\cdot}054$$

$$3 \ldots\ldots m = \frac{\log 6432 - \log 1365}{\log 5 - \log 3} = 3{\cdot}035$$

$$4 \ldots\ldots m = \frac{\log 4709 - \log 2737}{\log 6 - \log 5} = 2{\cdot}976$$

$$5 \ldots\ldots m = \frac{\log 6432 - \log 2737}{\log 4 - \log 3} = 2{\cdot}970$$

$$6 \ldots\ldots m = \frac{\log 6432 - \log 4709}{\log 10 - \log 9} = 2{\cdot}959$$

Mean .. 3·019

"The approximation of all these ratios towards the integer 3, seems to indicate that the tangents of the deviations vary reciprocally as the cubes of the distances, the position, with regard to latitude and longitude, remaining the same; but as this law is not exactly uniform in the above results, let us examine and determine the least change that can be made in the numbers, as drawn from our experiments, in order that a complete coincidence may be obtained.

"This will be best done by a particular application of the method of *minimum squares*, as follows: let $1{\cdot}365 = a$, $2{\cdot}737 = a'$, $4{\cdot}709 = a''$, and $6{\cdot}432 = a'''$.

"Now, if these numbers had the exact ratio of the cubes of the respective distances 12, 15, 18, 20, they would, by the supposition, require no correction; but this not being the case, let us endeavour to find the four quantities x, x', x'', x''', which added respectively to a, a', a'', a''', shall make,

$$a + x,\ a' + x',\ a'' + x'',\ a''' + x''',$$

proportional to 12^3, 15^3, 18^3, 20^3,
and such also that

$$x^2 + x'^2 + x''^2 + x'''^2 = a \textit{ minimum}.$$

"That is, we must have

$$\frac{a + x}{a' + x'} = \left(\frac{12}{15}\right)^3 = \left(\frac{4}{5}\right)^3 = b$$

$$\frac{a + x}{a'' + x''} = \left(\frac{12}{18}\right)^3 = \left(\frac{2}{3}\right)^3 = b'$$

$$\frac{a + x}{a''' + x'''} = \left(\frac{12}{20}\right)^3 = \left(\frac{3}{5}\right)^3 = b''$$

and $x^2 + x'^2 + x''^2 + x'''^2 = a \textit{ minimum}.$

'The first three equations give

$$a + x = b\,a' + b\,x'$$
$$a + x = b'\,a'' + b'\,x''$$
$$a + x = b''a''' + b''\,x'''$$

$$\text{or, } x' = \frac{a - b\,a' + x}{b} = \frac{x + c}{b}$$

$$x'' = \frac{a - b'\,a'' + x}{b'} = \frac{x + c'}{b'}$$

$$x''' = \frac{a - b''\,a''' + x}{b''} = \frac{x + c''}{b''};$$

consequently,

$$\left(\frac{x + c}{b}\right)^2 + \left(\frac{x + c'}{b'}\right)^2 + \left(\frac{x + c''}{b''}\right)^2 + x^2 = a \textit{ min.}$$

"This expression, when reduced, gives

$$x = -\left(\frac{c\,b'^2\,b''^2 + c'\,b^2\,b''^2 + c''\,b^2\,b'^2}{b^2\,b''^2 + b^2\,b'^2 + b'^2\,b''^2 + b^2\,b'^2\,b''^2}\right)$$

Whence we find in numbers $x = {\cdot}024$; and thus the preceding ratios, when corrected, become

$$\left.\begin{array}{l} x\ \ + a\ \ = 1{\cdot}389 \\ x' + a' = 2{\cdot}712 \\ x'' + a'' = 4{\cdot}687 \\ x''' + a''' = 6{\cdot}430 \end{array}\right\}\text{the corrections being}\left\{\begin{array}{l} {\cdot}024 \\ {\cdot}025 \\ {\cdot}022 \\ {\cdot}002 \end{array}\right.$$

"These numbers have exactly the inverse ratio of the cubes of the distances, and either of them will give for the number corresponding to the unit of distance, A = ·00080382. We have therefore the following formula for computing the deviation for any distance, the latitude and longitude being given, and the mass remaining the same, viz.

$$\tan\Delta = \frac{\sin 2\,\lambda.\ \cos l.}{{\cdot}00080382\ d^3}$$

where λ is the latitude, l the longitude, and d the distance in inches."

Experiments relative to distance.

114. As the above constant coefficient is obtained from the corrected numbers, and as it might appear doubtful how far these corrections would affect the uniformity of the law as determined from the preceding experiments, the author undertook to compute and observe the deviations in four different positions of the compass, for the several circles whose radii are 12, 14, 16, 18, 20, 22, and 24 inches, selecting for those positions of the compass the divisions 15°, 17½°, 20°, 25°, from the east or west, these being the points where the deviations are the greatest, in order that any error might render itself the more obvious. The following are the results of these observations with the corresponding computed deviations:

Experiments in the circle S E to several radii. Ball 288 lbs.

Position of compass from E. or W.	Latitude.	Longitude.	Distance.	Computed deviation, $\tan\Delta = \frac{\sin 2\lambda.\ \cos l.}{{\cdot}00080382\ d^3}$	Mean of observed deviations.	Error.
°	° ′	° ′	inches	° ′	° ′	
15	38 45	0 0	12	35 18	35 37	+19
17½	43 22	ditto	12	35 54	36 15	+21
20	47 28	ditto	12	35 50	35 50	0
25	54 24	ditto	12	34 28	34 30	+ 2
15	38 45	ditto	14	24 2	24 0	− 2
17½	43 22	ditto	14	24 31	24 30	− 1
20	47 28	ditto	14	24 28	24 30	+ 2
25	54 24	ditto	14	23 23	23 30	+ 7
15	38 45	ditto	16	16 37	16 30	− 7
17½	43 22	ditto	16	16 59	17 0	+ 1
20	47 28	ditto	16	16 56	17 0	+ 4
25	54 24	ditto	16	16 9	16 0	− 9
15	38 45	ditto	18	11 51	11 52	+ 1
17½	43 22	ditto	18	12 7	12 15	+ 8
20	47 28	ditto	18	12 5	12 0	− 5
25	54 24	ditto	18	11 30	11 15	−15
15	38 45	ditto	20	8 42	8 35	− 7
17½	43 22	ditto	20	8 53	8 52	− 1
20	47 28	ditto	20	8 52	8 42	−10
25	54 24	ditto	20	8 26	8 20	− 6
15	48 45	ditto	22	6 34	6 30	− 4
17½	43 22	ditto	22	6 42	6 40	− 2
20	47 28	ditto	22	6 41	6 30	−11
25	54 24	ditto	22	6 22	6 25	+ 3
15	38 45	ditto	24	5 4	5 0	− 4
17½	43 22	ditto	24	5 11	5 15	+ 4
20	47 28	ditto	24	5 10	5 10	0
25	54 24	ditto	24	4 55	5 0	+ 5

The remarkable coincidence, or at least the close approximation, between the observed and computed results in these experiments, can leave no doubt that the law of attraction as respects the distance is, *that the tangents of the angles of deviation are reciprocally proportional to the cubes of the distances.*

Since the magnetic force varies inversely as the square of the distance, and the tangent of deviation inversely as the cube of the same; it follows that the square of the tangent of deviation is directly as the cube of the force; or that *the tangent of deviation varies directly as the $\frac{3}{2}$ power of the force.*

On the law of attraction, as dependent on the mass and surface.

115. This is one of the most important laws we have hitherto had to examine, because it is this which has enabled us to reduce our theoretical investigation to numerical and determinate results. Prior to the experiments we have been detailing, no doubt was ever entertained that the power of attraction in iron bodies of similar forms, was directly proportional to their masses, or to the cubes of their like linear dimensions; and it was under this impression that Mr. Barlow, for the sake of verification, procured a solid ten inch ball, such as is employed in proving ten inch mortars, and which weighs 128 pounds, or just four-ninths of the thirteen inch ball, with which the preceding experiments were performed, and with the former he repeated all the experiments detailed in art. 111. The following are the results as given by each ball.

Experiments in the circle S E, with balls of 288 lbs. and 128 lbs. Radii 15 inches and 12 inches.

Position of compass.	Distance 12 inches.		Ratio of tangents.	Distance 15 inches.		Ratio of tangents.
	Deviation, ball 288 lbs.	Deviation, ball 128 lbs.		Deviation, ball 288 lbs.	Deviation, ball 128 lbs.	
2° 30′	10° 7½′	4° 0′	2·552	5° 7½′	2° 15′	2·279
5 0	19 37½	8 30	2·385	10 7½	4 30	2·267
7 30	26 30	11 45	2·397	14 7½	6 30	2·207
10 0	31 45	15 15	2·269	16 52½	7 45	2·227
12 30	34 7½	17 0	2·216	18 37½	8 30	2·254
15 0	35 37½	17 30	2·272	19 45	9 0	2·267
17 30	36 15	18 15	2·353	20 15	9 15	2·265
20 0	36 15	18 15	2·353	20 7½	9 15	2·250
25 0	34 30	17 0	2·248	19 15	8 45	2·269
30 0	32 22½	15 45	2·246	17 37½	8 0	2·259
35 0	29 52½	14 15	2·261	16 0	7 15	2·254
40 0	26 45	12 0	2·386	14 5	6 15	2·319
50 0	21 0	9 45	2·238	10 45	4 45	2·285
60 0	15 15	7 0	2·222	7 30	3 20	2·261
70 0	10 0	4 0	2·521	5 0	2 15	2·227
80 0	4 52½	2 15	2·172	2 30	.. ..	.. ..

The mean of the first ratios is 2·318, and of the second ratios 2·259; and the mean of the two 2·288. Now the ratio of the masses, or of the cubes of the diameter, is 2·25, whence then we may conclude, agreeably to our theoretical deduction, *that the tangents of the deviations are proportional to the cubes of the diameters,* all other things being the same.

116. The cubes of the diameters being proportional to the masses, the obvious conclusion seemed to be, that the tangents of the deviation were also proportional to the masses; and such, in fact, was the conclusion the author had drawn, when he fortunately made trial of a ten inch shell, whose weight was ninety-six pounds, or just three-fourths of that of the last solid ball of the same dimensions; and he was not a little surprised to find, that he could observe no difference whatever between these results and the former. He then determined on a regular course of experiments with the shell, at the same distances, &c. as had been adopted with the ball; and having completed them, he found, on a comparison of the results, that they agreed with each other through-

out. In fact, it appeared *that the power of attraction resided wholly on the surface, and was independent of the mass.*

Being, however, unwilling to leave any thing doubtful respecting a result which appeared so extremely novel and unexpected, he tried two other ten inch shells, lest there should have been any thing peculiar in the one above referred to: he then employed other shells of different diameters and thicknesses, the whole of which still indicated the same law; viz. that *the tangents of the deviations are proportional to the cubes of the diameters, or to the $\frac{3}{2}$ power of the surfaces, whatever may be the weight and thickness.*

The author however found afterwards that this law only obtains within certain limits, for having procured a ten inch shell of tin, and another of iron plate, the former weighing two pounds eleven ounces or forty-three ounces, and the latter two pounds thirteen ounces or forty-five ounces. He found that the power of neither of these was so great as that of the solid ball of the same diameter, but approached to the former in about the ratio of two to three. Now the thickness of the iron being here at a medium about $\frac{1}{30}$th of an inch, the conclusion which the author drew from it was that the magnetic fluid appears to require a certain thickness of metal exceeding $\frac{1}{30}$th of an inch, in order effectually to develope itself and to act with its maximum power: a deduction which he afterwards verified by other experiments on plates of iron of various thicknesses.

Captain Kater's verification of the above law.

117. Considering the novelty and importance of the deduction, that the magnetic power of iron bodies resides wholly on their surfaces, it was to be expected that philosophers would verify it by distinct experiments before they would feel disposed to admit it as an established philosophical fact, and this task was first undertaken by Captain Kater, who published the result of his experiments in the *Philosophical Transactions* for 1821. For these experiments, he had three cylinders made of soft iron, about two and a half inches in diameter, and nearly the same in height. One of the cylinders was of sheet iron, less than $\frac{1}{10}$th of an inch in thickness; the second, of that sort called chest plate 0·185 inch thick; and the third was of solid wrought iron: the first weighed 2760 grains, the second 9376 grains, and the solid cylinder 22929 grains. Previous to the experiments they were all made red hot, to destroy any accidental magnetism. The compass employed was of a very delicate construction, and the cylinder was so placed that its centre was in the direction of a tangent to the zero of the compass, and at the distance of 4·85 inches from the southern extremity of the needle. The position of the cylinder was varied six times, and the following were the deviations of the needle.

Sheet iron cylinder.			Chest plate cylinder.		Solid cylinder.	
	°	′	°	′	°	′
	2	1·5	2	50	2	55
	2	1·5	3	4	3	15
	2	4·5	3	20	2	57
	2	5·	3	45	2	50
	2	5·	3	10	2	55
	2	16	3	30	2	30
Mean	2	16	3	16	2	54

Suspecting an error in the experiment with the solid cylinder, from an accident which occurred, Captain Kater repeated the whole with the utmost attention. The position of each cylinder was now varied eight times, and the results were as follows.

Sheet iron cylinder.			Chest plate cylinder.		Solid cylinder.	
	°	′	°	′	°	′
	2	3	2	55	3	15
	2	22	2	50	3	12
	2	32	3	20	3	15
	2	20	3	40	3	0
	1	50	3	40	3	15
	2	45	3	28	2	50
	2	45	3	10	2	45
	1	55	3	5	2	58
Mean	2	19	3	16	3	4

"The surfaces of the cylinders, determined by very careful measurement, were, the sheet iron 28·54 inches; the chest plate 30·77 inches; and the solid cylinder 28·94 inches.

"Reducing the deviations to the same extent of surface; viz. to that of the solid cylinder, they become respectively 141, 184, and 184 minutes." Which may be considered as a highly satisfactory verification of the law in question. See *Philosophical Transactions*, part i. for 1821.

Determination of the numerical value of the constant coefficient $\frac{C}{M} \cdot \frac{3\,r^3}{2\,d^3 \cos\delta}$.

Value of numerical coefficient determined

118. The value of this constant coefficient may be deduced from our experimental results given in the four tables, art. 111, or rather from the two latter; because the distance in these are greater than in the two former, and it is obvious from our formula, that the approximation will be nearer as the distance is greater, the rejected term N (art. 109.) being in that case less considerable in respect to the constant part.

Hence, since

$$\tan \Delta' = \frac{C}{M} \cdot \frac{3\,r^3}{2d^3 \cos\delta}(\sin 2\,\lambda \cos l)$$

we have, when $\cos l = 1$, as in the experiments in question,

$$\frac{\sin 2\,\lambda}{\tan \Delta'} = \frac{M}{C} \cdot \frac{2\,d^3 \cos\delta}{3\,r^3}.$$

Now the exact diameter of the ball on which the experiments were made, is 12·8 inches, or radius 6·4 inches, and in one case the distance is eighteen inches, and in the other twenty inches.

Hence our tabular numbers give,

$$1\ldots\ldots\ldots\frac{M}{C} \cdot \frac{2\cdot18^3.\cos 70\tfrac{1}{2}^\circ}{3.\overline{6.4}^3} = 4\cdot717$$

$$2.\ldots\ldots\ldots\frac{M}{C} \cdot \frac{2\cdot20^3.\cos 70\tfrac{1}{2}^\circ}{3.\overline{6.4}^3} = 6\cdot418$$

From the first of these we have $\frac{C}{M} = 1\cdot0496$,

and from the second $\frac{C}{M} = 1\cdot0539$

Mean $\frac{C}{M} = 1\cdot0539$

This number is constant for cast iron balls and shells of every diameter, and for all distances and positions; our correct formula (9) therefore becomes

$$\tan \Delta' = \frac{\sin 2\lambda \cos l}{\frac{2}{3}\left\{\frac{d^3}{1{\cdot}0539\, r^3} + 3 \sin^2 \lambda - 1\right\}\cos \delta \mp \sin 2\lambda \sin \delta \sin l}$$

or,

$$\cot \Delta' = \qquad (11)$$

$$\frac{2}{3 \sin 2\lambda}\left\{\frac{d^3}{1{\cdot}0539\, r^3} + 3\sin^2\lambda - 1\right\}\cos \delta . \sec l \mp \sin \delta \tan l$$

which is the most convenient form for computation.

Mr. Christie's experiments.

119. As this formula is wholly theoretical, with the exception of the numerical coefficient $\frac{1}{1{\cdot}0539}$, it will be satisfactory to compare the results deduced from it with those obtained from actual experiment, and we have an excellent opportunity of doing this by means of the table of experiments published by Mr. Christie, in the first part of the *Transactions of the Cambridge Philosophical Society*. These were made by that gentleman on the same apparatus which Mr. Barlow employed, but in different circles; namely, in parallels of latitude corresponding to 30°, 45°, and 60°, and at every tenth degree of longitude. They were made with a new and accurate compass constructed for the purpose, and every precaution was used to assure the greatest possible accuracy in the results.

The distance in all these experiments was eighteen inches, or $d = 18$; the radius of the ball was 6·4 inches, or $r = 6{\cdot}4$; and dip $\delta = 70° \, 30'$; whence the constant part of the formula (11) for each different parallel of latitude are, for

lat. 30°.... $\cot \Delta' = 5{\cdot}3601 \sec l \mp \tan l \sin 70\tfrac{1}{2}°$
lat. 45°.... $\cot \Delta' = 4{\cdot}8090 \sec l \mp \tan l \sin 70\tfrac{1}{2}°$
lat. 60°.... $\cot \Delta' = 5{\cdot}7455 \sec l \mp \tan l \sin 70\tfrac{1}{2}°$

TABLE.

120. *Containing a comparison between Mr. Christie's experiments, and the theoretical results computed by the preceding formula.*

Longitude.	Latitude 30°.			Longitude.	Latitude 45°.			Longitude.	Latitude 60°.		
	Computed.	Observed.	Error.		Computed.	Observed.	Error.		Computed.	Observed.	Error.
°	° ′	′ ′		°	° ′	° ′		°	° ′	° ′	
80 N.	2 14	2 20	− 6	80 N.	2 34	2 33	+ 1	80 N.	2 4	2 5	− 1
70	4 22	4 44	− 22	70	5 2	5 0	+ 2	70	4 1	3 58	+ 3
60	6 17	6 52	− 35	60	7 9	7 12	− 3	60	5 48	5 43	+ 5
50	7 53	8 16	− 23	50	8 57	8 53	+ 4	50	7 18	7 29	− 11
40	9 9	9 21	− 12	40	10 20	10 1	+ 19	40	8 29	8 26	+ 3
30	10 2	9 58	+ 4	30	11 18	10 54	+ 24	30	9 19	9 5	+ 14
20	10 34	10 23	+ 11	20	11 50	11 29	+ 21	20	9 50	10 1	− 11
10	10 44	10 38	+ 6	10	11 58	11 45	+ 13	10	10 1	9 46	+ 15
0	10 34	10 17	+ 17	0	11 45	11 32	+ 13	0	9 52	9 43	+ 9
10 S.	10 7	9 56	+ 11	10 S.	11 13	11 8	+ 5	10 S.	9 27	9 21	+ 6
20	9 23	9 8	+ 15	20	10 22	10 16	+ 6	20	8 48	8 38	+ 10
30	8 27	8 16	+ 11	30	9 19	9 17	+ 2	30	7 56	7 49	+ 7
40	7 19	7 3	+ 16	40	8 3	7 58	+ 5	40	6 53	6 45	+ 8
50	6 2	6 3	− 1	50	6 38	6 37	+ 1	50	5 41	5 35	+ 6
60	4 38	4 38	0	60	5 5	5 0	+ 5	60	4 21	5 25	− 4
70	3 8	3 9	− 1	70	3 27	3 20	+ 7	70	2 58	2 54	+ 4
80	1 35	1 38	- 3	80	1 44	1 40	+ 4	80	1 29	1 30	1

It would be useless to expect a closer approximation between theory and practice in experiments of such a nature, which, notwithstanding all the care that can be used, are subject to errors which it is impossible to estimate; the daily variation alone, for example, may cause an error of 15′, and very few of our errors in the preceding table exceed this amount; besides which we have to allow for any trifling deviation in placing the meridian line on the table in the plane of the actual meridian, in the adjustment of the compass at each place of observation, and in measuring its distance from the centre of the table, and its depth below the centre of the ball. When these sources of error are properly considered, it will be found that the agreements between the theory and experiment is as close as can be reasonably expected. It is to be further observed, that we have assumed, for the sake of simplifying the computation, that the needle is indefinitely small, whereas it must necessarily be of a determinate length; but while this is less than one-third of the distance, the errors thence arising, although of some amount, are very inconsiderable.

§ XIV. *Comparison of the preceding theoretical formulas with the results obtained by observations on the dipping needle.*

121. Having completed our investigation relative to the horizontal needle, let us next inquire what the effect will be upon a dipping needle limited in its motion to the meridian. For this purpose we have only to project the arc Δ, determined in (art. 106.) upon the plane of the meridian, by drawing the arc S′E (fig. 60) perpendicular to the circle H Z H′, so will S E be the deviation sought.

Since S′E S is a right-angled triangle, we have immediately,

$$\tan \mathrm{S\,E} = \tan \mathrm{S\,S'} \cos i$$

But $\mathrm{S\,S'} = \Delta$, and by formula (7)

$$\tan \Delta = \frac{3 \sin \phi \cos \phi}{\frac{M}{C} \cdot \frac{d^3}{r^3} + 3 \cos^2 \phi - 1}$$

Whence denoting the arc S E by Δ'', we have

$$\tan \Delta'' = \frac{3 \sin \phi \cos \phi \cos i}{\frac{M}{C} \cdot \frac{d^3}{r^3} + 3 \cos^2 \phi - 1} \qquad (12)$$

From which the angle Δ'' may in any case be computed.

But for the purpose of computation, it will be best to convert the above expression into

$$\cot \Delta'' = \frac{\frac{M}{C} \cdot \frac{d^3}{r^3} + 3 \cos^2 \phi - 1}{3 \sin \phi \cos \phi \cos i}, \text{ o}$$

$$\cot \Delta'' = \left\{ \left(\frac{M}{C} \cdot \frac{2 d^3}{3 r^3} - \frac{2}{3} \right) \operatorname{cosec} 2 \lambda + \tan \lambda \right\} \operatorname{cosec} l$$

In which it is only requisite to introduce the proper value of the several quantities as determined in (art. 118.)

ared expe- it. 122. The experiments of which the results are given in the following table were made on the same apparatus as that employed in the preceding cases.

The needle was placed only in the plane of the meridian, at such distances from the centre of the table, and the ball elevated so much above the centre of the needle, as to bring it into the several latitudes, $7\frac{1}{2}°$, 15°, $22\frac{1}{2}°$, &c. keeping, in all the experiments, the centre of the ball and centre of the needle at the constant distance of twenty inches from each other; we have therefore $\frac{M}{C} = \frac{1}{1 \cdot 5039}$ (art. 118.), $r = 6 \cdot 4$ inches, $d = 20$ inches, and $\cos l = 1$. Hence the above formula in numbers becomes

$$\cot \Delta'' = 18 \cdot 639 \cos 2 \lambda + \tan \lambda$$

From which the column of computed deviations in the first of the annexed tables has been calculated.

If we take $r = 8 \cdot 85$, the radius of an eighteen inch shell, then the above becomes

$$\cot \Delta'' = 6 \cdot 6341 \operatorname{cosec} 2 \lambda + \tan \lambda$$

which is the formula employed in Table 2.

123. *Table showing the deviations produced in a dipping needle by an iron ball thirteen inches in diameter, and the computed deviations from the preceding formula. Distance twenty inches, mean detached dip 70° 40′.*

Latitude.		Observed dip.		Observed deviation.		Computed deviation.	
°	′	°	′	°	′	°	′
0	0	70	40	0	0	0	0
7	30	69	40	1	0	0	48
15	0	68	55	1	45	1	31
22	30	68	20	2	20	2	9
30	0	68	10	2	30	2	36
37	30	67	40	3	0	2	51
45	0	67	40	3	0	3	4
52	30	67	55	2	45	2	41
60	0	68	10	2	30	2	24
67	30	68	40	2	0	1	58
75	0	69	10	1	30	1	23
82	30	69	40	1	0	0	43
90	0	70	55	0	15	0	0

The above mean dip 70° 40′, was with one end of the needle only, without inverting the poles; the error of the instrument is not included.

124. *Table showing the deviations produced in a dipping needle by an eighteen inch shell, and the computed deviations from the preceding formula. Distance twenty inches, mean detached dip 70° 11′.*

Latitude.		Observed dip.		Observed deviation.		Computed deviation.		Time of making 40 vibrations.
°	′	°	′	°	′	°	′	″
0	0	70	11	0	0	0	0	125·3
15	0	66	7	4	4	4	14	124·44
30	0	63	49	6	22	6	56	121·3
33	30	……		……		……		120·3
45	0	62	43	7	28	7	28	116·5
60	0	64	22	5	49	6	5	113·58
75	0	65	44	3	27	3	21	112·3
90	0	70	11	0	0	0	0	110·2

The above experiments were made with a different instrument to the former; the needle was on Captain Kater's construction, viz. diamond formed; seven inches in length: the action of the needle remarkably correct. The above is the dip with one end only.

125. The last column shows the number of seconds in which the needle made forty vibrations in each position; the object was to compare the computed with the observed intensity, as will be explained in a subsequent article. These observations were made with great care, the time being counted by a machine which would register to 40ths of seconds. The face of the instrument was first turned to the east and then to the west. The dip and vibrations being registered four times in each position, and the mean of the eight results taken as they are given in the foregoing table.

Confining ourselves here only to notice the deviations, we think it must be admitted that the agreement is as near as can be reasonably expected; both the instruments employed were perhaps amongst the most perfect of their kind; but every one who is acquainted with the dipping needle, will be aware that it is not so constant in its action as we might desire; the difference between any two dips, however, in the same position, never exceeded half a degree; and as to the number of vibrations they seldom differed from each other by half a second.

General results.

126. If we now collect under one point of view our principal formula, (7), (9) and (10), and substitute in them the proper values in λ and l, instead of their complements ϕ and i, we shall have, after the requisite reductions, Summary of the preceding results.

$$\tan \Delta = \frac{\sin 2 \lambda}{\frac{M}{C} \cdot \frac{2 d^3}{3 r^3} + 3 \sin^2 \lambda - 1} \qquad (13)$$

$$\tan \Delta' = \frac{\sin 2 \lambda \cos l}{\frac{M}{C} \cdot \frac{2 d^3}{3 r^3} \cos \delta + \frac{2}{3} N} \qquad (14)$$

$$\tan \Delta'' = \frac{\sin 2\lambda \sin l}{\frac{M}{C} \cdot \frac{2 d^3}{3 r^3} + 3 \sin^2 \lambda - 1} \qquad (15)$$

127. These are all rigorous formulas; the first shows the deviation of a needle freely suspended, or free to move in every direction; the second, the deviation of a needle limited in its motion to the plane of the horizon; and the third, the deviation caused in the dipping needle placed in the plane of the meridian. And if, in these formulas, we reject all the terms in their respective denominators beyond the first, for the reasons assigned (art. 109), we shall have the following simple approximate formulas answering to these respective cases, viz.

$$\tan \Delta = \frac{C}{M} \cdot \frac{3 r^3}{2 d^3} \sin 2\lambda \qquad (16)$$

$$\tan \Delta' = \frac{C}{M} \cdot \frac{3 r^3}{2 d^3} \sin 2\lambda \cos l \sec \delta \qquad (17)$$

$$\tan \Delta'' = \frac{C}{M} \cdot \frac{3 r^3}{2 d^3} \sin 2\lambda \sin l \qquad (18)$$

Theoretical deductions. 128. From these formulas, or from the three preceding ones, we may draw several curious results, viz.

1. It is obvious, that in all the above formulas when $\lambda = o$, the deviation will be zero, which answers to the case of the body being placed in the plane of no attraction.

2. In the second formula we shall also have $\Delta' = o$, when $\cos l = o$, or when l, the longitude, is 90°, which answers to the plane of the meridian.

3. Again, in the third, $\Delta'' = o$, when $\sin l = o$, or when the longitude is zero, which answers to the plane cutting the equator or plane of no attraction, at right angles, and passing through the east and west points of the horizon.

There are therefore three planes of no attraction, of which, however, one only is general for all cases; the two others are partial, one belonging to the horizontal, and the other to the dipping needle, as above stated.

129. We may draw similar conclusions in reference to the maximum of deviation, but they would be somewhat complicated if we deduced them from our correct formulas, (13), (14), (15); they are, however, equally as simple as the above, if we confine ourselves to the approximate formulas, (16), (17), (18), which, although they are not strictly correct, will give results very nearly true.

From these, it appears,

1. That every thing being supposed constant but the latitude, the deviation will be the greatest when $\sin 2\lambda$ is the greatest, that is, when λ or the latitude $= 45°$. This is common to all the three cases.

2. Every thing being constant but the longitude, the deviation will be the greatest in the horizontal needle, when $\cos l$ is the greatest, that is, when $\cos l = 1$, or $l = o$.

3. The same being supposed as above, the deviation will be the greatest in the dipping needle, when $\sin l$ is the greatest, or when $\sin l = 1$, or $l = 90°$.

Hence the deviation in the dipping needle will be the greatest where the horizontal deviation is nothing, and the deviation in the horizontal needle the greatest where that of the dipping needle is nothing. Every thing but the longitude being supposed constant.

4. We may also compare the actual quantity of the deviations in the two needles under the same circumstances of latitude, longitude, &c. for it is obvious that

$$\tan \Delta' : \tan \Delta'' :: \cos l \sec \delta : \sin l, \text{ or}$$
$$\tan \Delta' : \tan \Delta'' :: 1 : \tan l \cos \delta$$

When $l = 45°$, the last analogy becomes

$$\tan \Delta' : \tan \Delta'' :: 1 : \cos \delta$$

Which is also the ratio of the respective maximum values of those angles; for from what is stated above, these will be

$$\tan \Delta' : \tan \Delta'' : \sec \delta : 1 :: 1 : \cos \delta$$

5. As the $\cos \delta$ in these latitudes is nearly equal to one-third, it follows, that in London the greatest deviation caused in a dipping needle by the action of a mass of iron, will be but one-third of the greatest effect which the same mass will produce in a horizontal needle; the distance in both cases being the same.

6. But when $\cos \delta = 1$; that is, when the dip is zero; then the deviations of both needles at their maximum, and in the longitude 45°, will be equal to each other.

130. Another remarkable deduction from our formula (17) is, that while the compass is placed in the horizontal circle passing through the centre of the ball, the amount of deviation is independent of the dip; or is the same in all parts of the world. For let m (fig. 62, plate VII.) be the plane of the compass in the horizontal circle H H′, then its latitude λ will be denoted by the arc $m n$, and its longitude l by W n, also the angle m W n will be the complement of the dip. Fig. 62.

Hence in the right-angled triangle m W n we have

$$\sin W m = \sin m n \operatorname{cosec} m W n = \sin \lambda \sec \delta$$
$$\cos W m = \cos m n \cos W n = \cos \lambda \cos l$$

therefore

$$\sin W m \cos W m = \sin \lambda \cos \lambda \cos l \sec \delta$$
$$\text{or,} \qquad \sin 2 W m = \sin 2\lambda \cos l \sec \delta$$

Hence in the particular case in question, our formula (17) becomes, (by making W $m = h$)

$$\tan \Delta' = \frac{C\, 3 r^3}{M\, 2 d^3} \sin 2 h$$

Where h is simply the arc subtended between the compass and the east or west points of the horizon.

When the compass therefore is thus situated, the deviation is the same at the same points in all parts of the world; the radius r, and distance d being constant.

131. This, however, is on the supposition that the ratio of C to M is a constant quantity, which may not perhaps be strictly true: according to our hypothesis C, which denotes the power developed at the surface of the sphere, is greater or less according to the power of the exciting magnet, and therefore ought to vary as M varies, the latter representing the natural magnetic intensity at any place; and that this change actually takes place there can be no doubt; but we are not so certain that the value of C is exactly proportional to M, because the resisting power of the iron is the same in all cases, and therefore opposes in all an equal resistance; and the coefficient C having reference to the resistance to developement as well as to the exciting power, may not be exactly proportional to M. This however is a question which can be satisfactorily answered only by experiment in different parts of the earth.

Should this equality of ratio between C and M be ultimately established, then it will follow from our formula (17), that under like circumstances of mass, distance, and magnetic position, the deviation in a horizontal needle will vary as the secant of the dip; but that in the dipping needle the deviation will be constant; the formula (18) expressing this deviation, having reference only to mass, distance, and position.

§ XV. *Comparison of the preceding formulas with the result of experiments on regular formed iron bodies of different figures.*

132. Hitherto we have only submitted to the test of experiment the action of iron spheres or shells, which bodies possess the peculiar property of having the centre of attraction in the centre of the mass; and it would be questionable, perhaps, if we confined ourselves wholly to the result thus deduced, how far the laws we have hitherto found to obtain were general in their application; it will be proper therefore, before we conclude this part of our treatise, to examine the case of iron bars and plates, in which the peculiarity above noticed has not place.

On the magnetic action of iron bars.

133. In order to compute the action of a bar (considered as a line) on a magnetic needle, we must first determine the position and distance of the centre of attraction, or at least the amount of that attraction on a given point. This part of the computation may be effected as follows:

Let A B (fig. 63, plate VII.) represent the given bar, C the place of the compass, or the point to which the attraction is referred. Draw the perpendicular D C and join A C, B C. Make A D $= b$, D C $= a$, A C $= c$, and m D, any variable distance, $= x$. Then the attraction being inversely as the square of the distance, the attraction between m and C will vary as $\frac{1}{\mathrm{C}\,m^2} = \frac{1}{a^2 + x^2}$. Resolve this into the two directions D C, D m, and we shall have

$$\frac{a}{(a^2 + x^2)^{\frac{3}{2}}} = \text{attraction in the line D C}$$

$$\frac{x}{(a^2 + x^2)^{\frac{3}{2}}} = \text{attraction in the line D } m$$

consequently,

$$\int \frac{a\,dx}{(a^2 + x^2)^{\frac{3}{2}}} = \frac{x}{a\,(a^2 + x^2)^{\frac{1}{2}}} = \text{attraction in D C}$$

$$\int \frac{x\,dx}{(a^2 + x^2)^{\frac{3}{2}}} = \frac{-1}{(a^2 + x^2)^{\frac{1}{2}}} + \frac{1}{a} = \text{attraction in D } m$$

$\frac{1}{a}$ being the correction.

These results, when x becomes equal to b, that is, the whole attraction of the line D A, will be

$$\frac{b}{a\sqrt{(a^2 + b^2)}} = \frac{b}{a c} = \text{force in D C}$$

$$\frac{1}{a} = \frac{1}{c} = \text{force in D A}$$

In the same manner, denoting the lines on the other side of D, by the corresponding letters a, b', c', we shall have for the attraction of the part D B

$$\frac{b'}{a c'} = \text{force in D C}$$

$$\frac{1}{a} - \frac{1}{c'} = \text{force in D B}$$

The two rectangular forces therefore due to the whole line will be

$$\frac{b}{a c} + \frac{b'}{a c'} = \text{force in D C}$$

$$\frac{1}{c'} - \frac{1}{c} = \text{force in D A}$$

Let the former of these be denoted by m, and the latter by n; then the resultant R, by the known principle of forces, will be

$$\mathrm{R} = \sqrt{(m^2 + n^2)}$$

If now we denote by ϕ the angle which this resultant makes with the line A B, we shall have

$$\tan\phi = \frac{n}{m}$$

Whence the attraction of the line A B, upon the point C becomes determined, both in quantity and direction.

134. But to apply this result to the case of magnetic attractions we must refer to our approximate law (art. 114.); from which it appears that the tangent of deviation of a needle varies as the cube of the distance inversely (or as the $\frac{3}{2}$ power of the force directly) multiplied by the product of the sine and cosine of the latitude (or of the colatitude) and the cosine of the longitude.

In the present case if we suppose the bar to be placed in the direction of the dipping needle, the angle ϕ will be the colatitude, and we have obviously

$$\sin\phi = \frac{m}{\sqrt{(m^2 + n^2)}}$$

$$\cos\phi = \frac{n}{\sqrt{(m^2 + n^2)}}$$

The tangents of deviation ought to vary therefore as

$$\mathrm{R}^{\frac{3}{2}} \sin\phi \cos\phi \cos l$$

(l being the longitude), or substituting the above values of R, cos ϕ and sin ϕ, the above becomes

$$\frac{m\,n \cos l}{(m^2 + n^2)^{\frac{1}{4}}}$$

that is $\tan\Delta = \mathrm{A}\,\frac{m\,n\cos l}{(m^2 + n^2)^{\frac{1}{4}}}$ (19)

Where Δ denotes the angle of deviation, and A, a constant coefficient.

In the experiments with which it is proposed to compare this formula, the compass was placed due east and west of the bar, consequently the longitude of its position was zero, and therefore cos $l = 1$. Hence in this particular case the formula is farther reduced to

$$\tan\Delta = \mathrm{A}\,\frac{m\,n}{(m^2 + n^2)^{\frac{1}{4}}}$$

or, $\tan\Delta \,.\, \frac{(m^2 + n^2)^{\frac{1}{4}}}{m\,n} = \mathrm{A}$ *a constant quantity*, (20)

whatever may be the distance of the compass or its position, provided the longitude be zero.

Magnetism.

The experiments we shall have recourse to were performed by Mr. Charles Bonnycastle, on a bar twenty-four inches in length, and one inch and a quarter square; inclined in the direction of the dipping needle, and the compass was placed to the east and west of the bar; first opposite to its centre, and then at every three inches from the centre towards the extremities, at the distances of twelve and sixteen inches from the axis of the bar.

The following are the observed results, and the computed value of the constant coefficient.

135. *Table of observed deviations, and computed value of the constant quantity* A.

Dist. of compass from bar.	Dist. below centre.	Observed deviation.	Value of m.	Value of n.	Value of $\frac{m\,n}{(m^2+n^2)^{\frac{3}{2}}}$	Value of A.
inches.	inches.	° ′				
16	3	2 20	·07338	·00887	·002396	17·03
ditto	6	4 45	·06860	·01699	·004384	17·62
ditto	9	5 45	·06123	·02355	·005631	17·88
ditto	12	6 0	·05201	·02783	·005960	17·63
12	3	5 20	·11500	·01400	·004840	19·28
ditto	6	10 0	·10610	·02830	·008989	19·62
ditto	9	12 0	·09251	·03949	·011521	18·45
ditto	12	11 30	07450	·04606	·011600	17·54
					Mean	18·13

In examining the computed value of A in the last column of the above table, it should be remembered that we have only employed our approximate formula, which at small distances gives rise to certain inequalities sufficient to account for all the discrepances in the coefficients in the fifth and sixth experiments; but it may be proper to observe, that the excess of these above the mean, amounts to but a small fraction of a degree; and therefore we may be allowed, it is presumed, to cite these experiments, as well as those in the foregoing sections, as a proof of the accuracy of the hypothesis upon which all our investigations hitherto have been founded. We shall likewise see in the following article, that the same accurate agreement has place when we make use of iron plates, instead of the balls and bars; and that we have therefore every reason to conclude the laws we have deduced are general and applicable to bodies of every form.

On the magnetic action of iron plates.

Magnetic action of iron plates.

136. The experiments and investigation we shall here refer to, were also as the former made by Mr. Charles Bonnycastle, and communicated by him to Mr. Barlow, who has published the series in his *Essay on Magnetic Attractions.* The following is a copy of the former gentleman's letter.

Fig. 64.

"I placed the plate with its edge A B, (fig. 64) parallel to the horizon, and at right angles to the meridian $a\,b$; and having caused it to revolve about A B until the angle D A b was 20°; I then observed its attraction upon a compass at O, which is in the line B A produced; this attraction was zero, as it ought to be, very nearly, which renders it probable that there is but little permanent magnetism in the plate. Having ascertained this point, I next revolved the plate about A B, and through the various angles D′ A b, D″ A b', &c., taking at each the deviation of the compass at O; the results of these experiments are recorded in the table which is given below, and it only remained for me to calculate theoretically the corresponding deviations in order to decide upon the point in question.

Part I

"For the attraction to the plane A B E D in the directions O A, A D, is the same as that of the plane A B E′ D′ in the directions O A, A D′; or of A B D″ E″ in the directions A O, A D″, &c.; and, therefore, the deviation of the needle will vary as the same function of the angle D A b, whatever be the form of the plate in question, or the quantity of its attraction, provided that the depth to which the magnetic action penetrates is always the same; but if this depth depends on the position of the surface, with respect to the dip, it is manifest that in the various positions of the plate which are represented above, the law of attraction will vary considerably from that obtained on the hypothesis of a constant penetration.

"That this hypothesis is correct, will, however, readily appear from the following calculation of the deviations according to theory, and the comparison of the results with those actually observed.

Fig. 65

"Let C (fig. 65) be the centre of the compass, C O the line about which the plate revolves, E the centre of attraction, S N the dip, A D B F the equator, and $a\,b$ the meridian. Join C E; draw E d perpendicular to A B, and join C d. Then the supplement of the colatitude is, evidently, equal to C E d, and the longitude is equal to d C O: but by what has been observed above, the centre of attraction E, is always found in the circle B E A, and therefore as C O is invariable, C E will be also invariable; and (putting colat $= \phi$,) $\cos\phi$ will vary as E D, that is as the cosine of S E. For the same reason $\sin\phi$ will vary as C d; and $\sin\phi$, $\cos l$ will vary as C d, $\cos d\,C\,o = C\,o$, which is constant; and consequently $\sin\phi$, $\cos l$ are also constant.

"But the approximate formula for the deviation is

$$\tan\delta = A\,.\,\frac{\cos^2\,.\,\sin\phi\,\cos\phi}{d^3}$$

and in this case d^3, and the rectangle $\cos l\,\sin\phi$ are constant, and $\cos\phi$ varies as $\cos$ S E, or as $\cos\epsilon$, (putting S E $= \epsilon$); wherefore

$$\frac{\tan\delta}{\cos\epsilon} = M,$$

where M is a constant quantity.

Angle ϵ.	Observed deflection.	Value of M.
110	5° 0′ W.	·2558
90	0 0	....
50	9 20 E.	·25807
20	14 35	·27687
0	15 45	·282029
− 30	12 20	·28542
− 70	5 40	·29012

"*Note.* The distance of the compass from the plate was six inches, and the deflections were the means of two sets of observations made with opposite ends of the plate, the results of which, however, differed in no case more than twenty minutes from each other.

"As the first and last values of M ought, evidently, to be the same, it appears that the difference between the numbers in the last column arises from a permanent magnetism in the plate."

When it is remembered that the above numbers are obtained from an approximate formula, and the aberrations are such as would be compensated by a change of only a few minutes in the observed angles; there can, we presume, be no doubt that the developement of magnetism takes place according to the hypothesis that has been advanced, viz. to only a certain depth, and to the same depth whatever be the position of the body or the thickness of the metal, provided it exceed a certain quantity.

§ XVI. *On the change of magnetic intensity of a needle as effected by iron spheres.*

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netism.

137. We have hitherto limited our investigations to the determination of the quantity of deviation produced on a freely suspended horizontal or dipping needle, by the magnetic action of iron bodies of different figures; that is, we have had only to compute the angle which the resultant of our forces (5) and (6) makes with the natural direction of the magnetic force; we propose now to find the value of that resultant, or of the magnetic intensity of the needle when under this influence.

This from the known principles of mechanics will be, from our formulas (5) (6).

$$\mathrm{I} = \sqrt{\left\{\left(\frac{3\,\mathrm{C}\,r^3}{d^3}\cos\phi\sin\phi\right)^2 + \left(\mathrm{M} - \frac{\mathrm{C}\,r^3}{d^3} + \frac{3\,\mathrm{C}\,r^3}{d^3}\cos^2\phi\right)^2\right\}}$$

or dividing all the terms by $\frac{3\,\mathrm{C}\,r^3}{d^3}\cos\phi$

$$\mathrm{I} = \frac{3\,\mathrm{C}\,r^3}{d^3}\cos\phi\sqrt{\left\{\sin^2\phi + \left(\frac{\mathrm{M}\,d^5 - \mathrm{C}\,r^3}{3\,\mathrm{C}\,r^3}\sec\phi + \cos\phi\right)^2\right\}}$$

and making $\frac{\mathrm{M}\,d^3 - \mathrm{C}\,r^3}{3\,\mathrm{C}\,r^3} = \mathrm{A}$

this reduces to

$$\mathrm{I} = \frac{3\,\mathrm{C}\,r^3}{d^3}\cos\phi\sqrt{\left\{\sin^2\phi + (\mathrm{A}\sec\phi + \cos\phi)^2\right\}}$$

$$\text{or}\quad \mathrm{I} = \frac{3\,\mathrm{C}\,r^3}{d^3}\cos\phi\sqrt{(1 + 2\,\mathrm{A} + \mathrm{A}^2\sec^2\phi)}$$

$$\text{or}\quad \mathrm{I} = \frac{3\,\mathrm{C}\,r^3\,\mathrm{A}}{d^3}\sqrt{\left(\frac{1 + 2\,\mathrm{A}}{\mathrm{A}^2}\cos^2\phi + 1\right)}$$

Or reestablishing the value of A, and calling the natural magnetic intensity M = 1, in which case C = 1·0539 (art. 118.) we have for the resultant I.

$$\mathrm{I} = \sqrt{\left\{\frac{2\,d^3 + \mathrm{C}\,r^3}{d^6}\,3\,\mathrm{C}\,r^3\cos^2\phi + \frac{(d^3 - \mathrm{C}\,r^3)^2}{d^6}\right\}} \qquad (21)$$

from which I may be computed.

As an example, if $d = 20$, or distance = 20 inches; the radius of the ball $r = 6{\cdot}4$ inches, and C = 1·0539, as determined above for cast iron, then we shall find

$$\mathrm{I} = \sqrt{}\,({\cdot}16743\cos^2\phi + {\cdot}93213)$$

Or, if all the rest remain the same except r, and we make this = 8·85, the radius of what is called an eighteen inch shell, we shall have

$$\mathrm{I} = \sqrt{}\,({\cdot}57288\cos^2\phi + {\cdot}82582)$$

$$\text{or}\ \mathrm{I} = \sqrt{}\,({\cdot}57288\sin^2\lambda + {\cdot}82582)$$

138. In order to compare these results with experiment, we must consider the vibrations of a magnetised needle under the same point of view as those of a simple pendulum; that is, we must estimate the force as proportional to the square of the number of vibrations made in a given time; or the number of vibrations as proportional to the square root of the force; we have therefore only to give the proper values to our angle ϕ, or λ, and compute the value of I. Then since the natural magnetic intensity has been called unity, we shall have

$$1 : \sqrt{\mathrm{I}} :: \mathrm{N} : n$$

where N is the number of vibrations made by a dipping needle in any given time, when removed from every extraneous magnetic action; and n the number, made in the same time, when the instrument is placed in any given latitude on either of the iron balls, and at twenty inches distance from their centre. If the distance was different from this, the number in the above numerical formula would be different also. This distance is merely chosen for convenience, viz. that we may be able to compare the formulas with the results obtained in our experiments in the last column of the table, (art. 124) where we have given the number of seconds employed by the needle (seven inches in length) in making forty vibrations. The time was taken by a machine which gave the intervals to the 40ths of seconds, but it was not attempted to register nearer than to quarter seconds.

139. *Table showing the computed and observed intensities of a dipping needle in different positions in the vicinity of an eighteen inch shell.*

Value of $\mathrm{I}^2 = ({\cdot}57288\sin^2\lambda + {\cdot}82582)$			Time of 40 vibrations detached from iron.	Computed time of 40 vibrations in different latitudes.	Observed time of 40 vibrations in different latitudes.
Latitude.		Value of R²			
0°	0′	·82582	120″	125″78	125·3
15	0	·86419	ditto	124·46	124·44
30	0	·96940	ditto	120·95	121·3
33*	30	1·00000	ditto	120·00	120·3
45	0	1·11226	ditto	116·85	116·5
60	0	1·25548	ditto	113·36	113·58
75	0	1·36033	ditto	111·12	112·3
90	0	1·39870	ditto	110·35	110·25

140. The remarkable approximation to equality between the computed and observed times of vibration in the last two columns of the above table, is one of the best tests we have yet met with for deciding respecting the accuracy of our hypothesis, by which the action of the ball is referred to two centres indefinitely near to each other, and to the geometrical centre of the ball itself, and by which it is distinguished from the theory of Coulomb, or at least the interpretation usually given to his theory, where the two centres are

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[*] The proper value of λ when I = 1; that is, when $\cdot57288\,\lambda^2 = \cdot17418$ is $\lambda = 33°\,28'$.

assumed to be at the extremity of that axis which coincides with the magnetic direction of the earth. Whereas according to our hypothesis, the ball has merely polarity given to it, but it is supposed to be distributed over the surface, and to have its centres of action indefinitely near to each other as above stated. This latter condition is one of the distinguishing characteristics of soft unmagnetised iron, and in which its action differs the most from bodies actually magnetic or possessing fixed poles; for in the latter case we must no longer refer our centres of action to the centre of attraction of the mass or surface, but to two fixed and determinate points; the situation of which will depend upon the accumulation of the magnetic fluids developed in the act of magnetising. This distinction will form the subject for investigation in the following section.

§ XVII. *On the deviations produced in horizontal needles by the action of natural and artificial magnets, and on the general phenomena exhibited by such bodies.*

On the laws of natural and artificial magnets.

141. We have seen that in order to compute the effect which simple iron bodies produce in deflecting a magnetised needle from its natural direction, we are under the necessity of referring the centres of action to two points indefinitely near to each other in the common centre of attraction of the surface of that body; but this assumption will not enable us to compute the effect produced by the action of a body permanently magnetic; we must in this latter case refer our centres of attraction and repulsion to two points, whose situations are determined, and which in magnetised steel bars are commonly very nearly at the two extremities. This fact may be in some measure exhibited by a very easy experiment, as follows:—take a bar magnet of any length, let us say eighteen inches, and place it in the magnetic meridian; take also a small and very delicate needle on a pivot, or suspended by a thread of fine silk, and carry it very near the bar from one extremity to the other; and it will be seen that, when the needle is in the centre of the bar, it will stand parallel to the latter; its direction being inverted, or at least the north end of the needle will point towards that part in which the south end of the bar is placed, and its south end towards the north. As we now carry the needle towards either extremity, it will advance more and more towards a perpendicular position with respect to the bar, and which it will completely attain just before we arrive at the actual extremity, and to this point in the bar the end of the needle will continue to be directed while we pass the latter round the end of the former; thus exhibiting an ocular demonstration of a strong centre of attraction at this point; and precisely the same phenomena occur at the other extremity of the bar, but with the contrary end of the needle, as ought obviously to be the case. Another method of exhibiting this effect is to place an artificial bar magnet on a table in any direction, and arranging about it a number of small needles, as shown in fig. 66; when they will take the several positions indicated in the figure; and although, in this case, the needles naturally disturb each other, and are under the influence also of the natural magnetism of the earth, yet their directions are still such as obviously to indicate the existence of two centres of action not in the centre of the bar, as in the case of simple iron balls or shells, but at the extremities as above stated.

Exhibited by experiment.

Second perimen

Fig. 66.

142. These experiments, however, after all, are not sufficiently conclusive for mathematical purposes; it will therefore be proper to explain a much more decisive and indisputable method of demonstrating the truth of the doctrine we have advanced above. This is as follows: place a small compass, nicely suspended on a pivot, in the centre of a large table, and neutralise the terrestrial action upon it, by opposing two magnets to its extremities, so that its directive force may be altogether imperceptible. Describe about the compass, taking its pivot as a centre, a circle of any diameter from two to three feet; and divide this circle into points and quarter points. At each of these divisions, in succession, place a bar magnet, exactly in the magnetic meridian, and with its centre coinciding with the division in the circumference. Then compute the actual deviation which ought to arise from the action of the two poles of the bar according to the above hypothesis, on each of these points of division; and afterwards observe the same experimentally, and it will be seen that the agreement between the theory and practice is so complete, as to leave no doubt as to the truth of the principles on which the former is founded. The computation alluded to will stand thus.

Examin mathema- tically.

143. Let N S (fig. 67) represent a bar magnet, and $e\ e'$ its two centres of attraction and repulsion, which may at present be considered as indeterminate, but equally distant from the centre o. Let P be a magnetic particle or very short needle, which by having the terrestrial action neutralised as above described, will be free to obey any force impressed upon it. Make $o\,P = d$, and $o\,e = o\,e' = e$. Then adopting the same principle of resolution as in art. 105, we shall have

Compute deviatio

Fig. 67.

$$\frac{2fd}{(d^2+e^2-2de\cos\phi)^{\frac{3}{2}}} - \frac{2fd}{(d^2+e^2+2de\cos\phi)^{\frac{3}{2}}} = \text{force in P}\,o$$

$$\frac{2fe}{(d^2+e^2-2de\cos\phi)^{\frac{3}{2}}} + \frac{2fe}{(d^2+e^2+2de\cos\phi)^{\frac{3}{2}}} = \text{force in N S}$$

If now instead of assuming e to be indefinitely small, as in the article referred to, we consider it as representing half the length of the magnet, and the distance $d = m\,e$ (m being an indeterminate multiplier) the above become

$$\frac{2mef}{(d^2+e^2-2de\cos\phi)^{\frac{3}{2}}} - \frac{2mef}{(d^2+e^2+2de\cos\phi)^{\frac{3}{2}}} = \text{force in P}\,o \qquad (1)$$

$$\frac{2ef}{(d^2+c^2-2de\cos\phi)^{\frac{3}{2}}} + \frac{2ef}{(d^2+e^2+2de\cos\phi)^{\frac{3}{2}}} = \text{force in N S} \qquad (2)$$

or,
$$\frac{2\,m\,e\,f}{\{(m^2+1)\,e^2-2\,m\,e^2\cos\phi\}^{\frac{3}{2}}}-\frac{2\,m\,e\,f}{\{(m^2+1)\,e^2+2\,m\,e^2\cos\phi\}^{\frac{3}{2}}}\text{ - - - - force in P }o \qquad (3)$$

$$\frac{2\,e\,f}{\{(m^2+1)\,e^2-2\,m\,e^2\cos\phi\}^{\frac{3}{2}}}+\frac{2\,e\,f}{\{(m^2+1)\,e^2+2\,m\,e^2\cos\phi\}^{\frac{3}{2}}}\text{ - - - - force in N S} \qquad (4)$$

Let $\frac{m^2+1}{2\,m}=n$, then we have

$$\frac{2\,m\,f}{e^2(2\,m)^{\frac{3}{2}}}\left\{\frac{1}{(n-\cos\phi)^{\frac{3}{2}}}-\frac{1}{(n+\cos\phi)^{\frac{3}{2}}}\right\}\text{ - - - - force in P }o \qquad (5)$$

$$\frac{2\,f}{e^2(2\,m)^{\frac{3}{2}}}\left\{\frac{1}{(n-\cos\phi)^{\frac{3}{2}}}+\frac{1}{(n+\cos\phi)^{\frac{3}{2}}}\right\}\text{ - - - - force in N S} \qquad (6)$$

Resolve the former into two forces, perpendicular and parallel to N S; viz.

$$\frac{2\,m\,f\sin\phi}{e^2\,(2\,m)^{\frac{3}{2}}}\left\{\frac{1}{(n-\cos\phi)^{\frac{3}{2}}}-\frac{1}{(n+\cos\phi^{\frac{3}{2}}}\right\}\text{ - - - - perpendicular to N S} \qquad (7)$$

$$\frac{2\,m\,f\cos\phi}{e^2\,(2\,m)^{\frac{3}{2}}}\left\{\frac{1}{n-\cos\phi)^{\frac{3}{2}}}-\frac{1}{(n+\cos\phi)^{\frac{3}{2}}}\right\}\text{ - - - - parallel to N S} \qquad (8)$$

Deducting from the latter, equation (6) and dividing by equation (7), we have

$$\cot\Delta=\cot\phi-\frac{1}{m}\operatorname{cosec}\phi\left\{\frac{(n+\cos\phi)^{\frac{3}{2}}+(n-\cos\phi)^{\frac{3}{2}}}{(n+\cos\phi)^{\frac{3}{2}}-(n-\cos\phi)^{\frac{3}{2}}}\right\}$$

or reestablishing *m*

$$\cot\Delta=\cot\phi-\frac{1}{m}\operatorname{cosec}\phi\left\{\frac{\left(\frac{m^2+1}{2\,m}+\cos\phi\right)^{\frac{3}{2}}+\left(\frac{m^2+1}{2\,m}-\cos\phi\right)^{\frac{3}{2}}}{\left(\frac{m^2+1}{2\,m}+\cos\phi\right)^{\frac{3}{2}}-\left(\frac{m^2+1}{2\,m}-\cos\phi\right)^{\frac{3}{2}}}\right\}$$

An expression from which we may compute the deviation for any situation of the magnet or any angle ϕ and at any distance $d = m\,e$.

If we take the distance of the needle equal to the length of the magnet, or $m = 2$, then the above becomes

$$\cot\Delta=\cot\phi-\tfrac{1}{2}\operatorname{cosec}\phi\left\{\frac{\overline{\tfrac{5}{4}+\cos\phi}^{\frac{3}{2}}+\overline{\tfrac{5}{4}-\cos\phi}^{\frac{3}{2}}}{\overline{\tfrac{5}{4}+\cos\phi}^{\frac{3}{2}}-\overline{\tfrac{5}{4}-\cos\phi}^{\frac{3}{2}}}\right\}$$

From which the numerical values of Δ are easily determined, and may then be compared with observation; we may in like manner compute the same for any other value of *m*, or any distance $d = m\,e$, and by a comparison of the numbers thus obtained with experiment, it will be seen that in the case of fixed poles, these latter are not, as in bodies passively magnetic, in the common centre of attraction of the surface, but in the extremities; as we have assumed in the beginning of this section.

We must acknowledge that we are unable to give any satisfactory explanation of this difference in the action of bodies permanently and temporarily magnetic; but that such is actually the case is unquestionable, and our business is rather with facts than hypotheses; we have little doubt however that the energy with which magnetical investigations are pursued at present, not only in England but throughout all Europe, will soon lead to some facts which will serve to reduce to one general principle these two systems of phenomena, which are at present apparently so distinct from each other.

§ XVIII. *Illustration of certain popular magnetic experiments, properties of the magnetic curve, &c.*

n of ets on filings.

144. If we cover a strong bar magnet with a piece of white paper, or with a piece of thin smooth glass, and spread over the paper or glass a quantity of fine iron or steel filings, using for this purpose a sieve, or a box similar to that which is common for dusting sand over writing; we shall find that these filings will arrange themselves in very elegant curves about the poles of the magnet; varying in their figure as we employ one, two, or more magnets, and as we place towards each other poles of the same or of a contrary name. See fig. 68, and 69. These curves have given rise to the most fanciful theories of magnetism, having been assumed as the traces of an invisible fluid perpetually circulating between the poles of the magnet; and this circulation has been afterwards employed for illustrating every variety of magnetic phenomena, but in such a way as to leave the subject involved in greater mystery than at first. The fact is that these curves are the necessary consequences of the forces we have been just examining in the preceding articles, and the arrangement of the needles exhibited in fig. 66, may be considered as the elementary principle of the curves in question. We have only indeed to conceive the needles in this figure to be reduced in length to that of the particles of iron dust, and the number of them to be indefinitely increased, and we shall see immediately the principle to which we ought to refer the arrangement exhibited in the diagrams above referred to.

Fig. 68, 69.

The curve, which any line of these particles will assume, is called a magnetic curve; it has certain very curious properties belonging to it, and gives rise to some pleasing geometrical deductions; which although not necessarily connected with our present inquiry, may, it is presumed, be introduced without impropriety in conclusion of this part of our treatise. In stating these properties we shall avail ourselves of the article *Magnetic Curves*, as given by Professor Leslie in his *Geometrical Analysis*.

Properties of magnetic curves.

Geometrical properties of magnetic curves.

145. Def. I. The lines drawn from the two poles of any magnet to any point in the curve are called the radiants; as A C, B C, fig. 71, 72.

Def. II. The line which joins the two poles is called the *base* of the curve, and its extension the *axis*.

Fig. 71, 72.

Def. III. The angles B A C and A B C, which the radiants make with the base are called the *polar angles*.

Generation of the curve.—Let A C and B C be two radiants which turn about the poles A and B, and let them in the same instant move into the proximate positions A C, B C; if the angle C A *c* be to C B *c* as A C to B C, the points C and *c* will occupy a magnetic curve.

It is obvious that the magnetic curve will assume two distinct forms, according as its radiants turn in the same or in opposite directions. In the latter case both the polar angles augment at the same time, and the curve will consist of a single concave arc; but in the former case, the angles from one pole increasing, while those from the other are diminishing, the curve will divide and spread itself into two branches. The one species of curve is hence called a converging curve, or a *convergent*, fig. 71, and the other a *divergent*, as in fig. 72.

Propositions.

146. Prop. I. The sines of the angles, which a tangent forms in the magnetic curve, with the radiants drawn to the point of contact, are proportional to the square of those lines. Let T C, (fig. 71, 72) touch the curve at the point C, and make the angles T C A and T C B, with the corresponding radiants, then will

$$\sin TCA : \sin TCB :: AC^2 : BC^2$$

For assume the approximate point *c*, to which inflect the radiants A *c*, B *c*, make B D equal to δ A C, and from A and B describe the minute arcs *c a*, C β, D δ.

The arcs *c a* and D δ are obviously the measures of the elementary angles C A *c* and C B *c*, and consequently from the genesis of the curves; $ca : D\delta :: AC : BC$; but since the sectors C B β and D B δ are similar, $D\delta : C\beta :: BD$ or $AC : BC$; wherefore by composition of ratios $ca : C\beta :: AC^2 : BC^2$. Again the right angled elementary triangle *c a a* and C β *a* having a common verticle angle, are similar and $ca : ac :: C\beta : Ca$, or alternately $ca : C\beta :: ac : Ca$; whence by identity of ratios $ac : Ca :: AC^2 : BC^2$. Now in the triangle C *a c*, $ca : Ca :: \sin cCa$ or $\sin TCA : \sin Cca$ or $\sin TCB$; wherefore

$$\sin TCA : \sin TCB :: AC^2 : BC^2$$

The same analytically.

147. The same may be demonstrated analytically by a simple resolution of forces on the principles employed in the preceding articles and sections of this treatise.

Fig 70.

Let A B (fig. 70) be the two poles of a magnet, and P any point or particle, make $AP = d$, $BP = e$, and let the force at the distance 1 be 1, then the force at the distance d will be $\frac{1}{d^2}$ and at the distance e, $\frac{1}{e^2}$; let the latter represent attraction and the former repulsion.

Let the force $\frac{1}{d^2}$ be represented by P *a*, and resolve it into two forces, perpendicular and parallel to P B; viz. $\frac{\sin\phi}{d^2}$, and $\frac{\cos\phi}{d^2}$, (the angle A P B being denoted by ϕ); then the rectangular forces at P will be $\frac{\sin\phi}{d^2}$ and $\left(\frac{1}{e^2}+\frac{\cos\phi}{d^2}\right)$; and the resultant from the known principle of mechanics will be

$$\text{result}^2 = \frac{\sin^2\phi}{d^4} + \frac{\cos^2\phi}{d^4} + \frac{2\cos\phi}{e^2 d^2} + \frac{1}{e^4}, \text{ or}$$

since $\sin^2\phi + \cos^2\phi = 1$, it becomes

$$\text{result}^2 = \frac{1}{d^4} + \frac{1}{e^4} + \frac{2\cos\phi}{d^2 e^2}$$

$$\text{and } \sin^2\Delta = \frac{\frac{\sin^2\phi}{d^4}}{\frac{1}{d^4}+\frac{1}{e^4}+\frac{2\cos\phi}{d^2 e^2}}$$

Δ being the angle which the resultant makes with P B.

Or if, instead of this resolution, we resolve the force $\frac{1}{e^2}$ upon the line A P, and perpendicular to the same, we shall find precisely in the same way that

$$\sin^2\Delta' = \frac{\frac{\sin^2\phi}{e^4}}{\frac{1}{e^2}+\frac{1}{d^4}+\frac{2\cos\phi}{e^2 d^2}}$$

where Δ' is the angle which the resultant makes with P A,

therefore,

$$\sin^2\Delta : \sin^2\Delta' : \frac{\sin^2\phi}{e^4} : \frac{\sin^2\phi}{d^4} :: d^4 : e^4$$

Whence again

$$\sin\Delta : \sin\Delta' :: d^2 : e^2$$

the same result as before.

Cor. I. Referring to our geometrical investigation it follows, that a tangent cuts the extended base into segments, which are as the third powers of the corresponding radiants. For draw A E parallel to B C, and the triangle E A C (fig. 71, 72) being similar to *c a* C,

$AE : AC :: ca : Ca :: AC^2 : BC^2$; wherefore $AE : BC :: AC^3 : BC^3$; but from the property of parallel and diverging lines, $AE : BC :: AT : BT$ and consequently $AT : BT :: AC^3 : BC^3$. In fig. 71 and 72 the radiants A C and B C are as 2 : 3; whence $TA : TB :: 8 : 27$. Wherefore A T in fig. 71, is $\frac{8}{19}$ths of A B; but in fig. 72, it is equal only to $\frac{8}{35}$ths of the same distance.

Cor. II. Hence if tangents be drawn from any point in the axis to different magnetic curves constituted upon the same base, the points of contact will lie in the circumference of a given circle. For T being a given point, the ratio of A T to B T or that of A C to B C is given, and consequently that of A C to B C is given, and the locus of C (Leslie's *Anal.* iii. 13) is a given circle. In the illustrative fig. 74 and 75, where the radiants A C and B C are as two to three, the distance A X of the circle of contact is two-fifths of

A B, and its radius six-fifths. The correspondence with theory in these delineations is sufficiently striking.

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The method of applying a tangent to the magnetic curve is hence easily perceived. In the convergent curve, the tangent C T lies beyond the nearer pole, but in the divergent curve, C T falls within the poles. The former species of curve is therefore always concave towards its base, while the latter is convex.

Since $Ca : ca :: BC^3 : AC^3$, the little spaces C a and c a, in the direction of the poles A and B, are inversely as the squares of the distances C A and C B. These spaces may therefore represent the forces of magnetic attraction or repulsion emanating from A and B. An attraction C a, combined with a repulsion a c, would produce the oblique direction C c, fig. 71; while the same attractive force conjoined with another attraction ca', would give the result C c fig. 72. Hence very short needles or iron filings, subject to those attractive or repulsive powers, but allowed to arrange themselves freely, will coalesce into the elementary portions of the curve. If A and B be dissimilar poles, the curve, being composed of projecting portions, C c will exhibit the convergent form; but if these poles be similar, the curve consisting of minute reclined portions will become divergent.

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148. Prop. II. The sum of the cosines of the polar angles in the convergent magnetic curve, but their difference, in the divergent form of the curve remains constant.

The cosines of the angles B A C and A B C fig. 71, compose by addition a certain invariable amount at every point of the curve; but in fig. 72, the cosines of the polar angles have always the same difference throughout each branch of the curve. For let fall upon the axis the perpendiculars $c\phi F$ and DdG, and draw the parallels $a\phi$ and δd. The elementary triangle $c\phi a$ is evidently similar to the right angled triangle AFc, and $ac : a\phi :: Ac$ or $AC : cF$; but the triangles $Dd\delta$ and BFc, being likewise similar, $D\delta : \delta d :: BC$ or $Bc : cF$. By alternating both analogies $ac : AC :: a\phi : cF$, and $D\delta : BC :: \delta d : CF$; but ac and $D\delta$, being the measure of the variation of the polar angles B A C and A B C, are as the radiant A C to B C, consequently $a\phi : cF :: \delta d : cF$, and $a\phi = \delta d$ or $fF = Gg$. Now fF (fig. 71) is the diminution which the cosine A F suffers in the movement of the radiant A C, from the point C to c; and the equal portion δd is the equal increase which cosine B G receives in the corresponding advance of the radiant B C. Wherefore the cosines A F and B G must maintain the same amount throughout the convergent curve.

In the divergent curve (fig. 72) $a\phi'$ lies on the other side of the perpendicular cF, the cosines of the polar angles B A C and A B C both receive equal increments, $f'F'$ and $G'g'$ as the radiants expand from c to C, and consequently those cosines AF' and BG', must have always the same difference.

race a netic e.

It is hence easy to trace through any point C, a magnetic curve whose poles are given. From A and B with a radius A O, (fig. 71 and 72) equal to half their distance, describe two semicircles; draw A C and B C to cut the circumference in the points H and I; from which let fall the perpendiculars H K and I L; take any segment K Q and make L R equal to it, either on the same or on the opposite side, according as the curve is convergent or divergent; erect the perpendiculars Q M and R N to meet the circles, and inflect the radiants A M P and B R N; whose intersection P will be a point in the curve. But if L r (fig. 72) be made equal to O Q, and the perpendicular r n erected, the radiant B n will by its intersection with the radiant A M, assign the point p in the other branch of the curve.

When the radiant B C coincides with the axis, the radiant A C will become a tangent to the curve, and will consequently mark the direction of its origin. To find this, make K S equal to L O, either on the same side, as in fig. 71, or on the opposite side, as in fig. 72; erect the perpendiculars S V to meet the circle, and S V will be the radiating tangent.

Construction of a system of magnetic curves.

But a system of magnetic curves may be constructed more expeditiously. Divide the radius of the circle into any even number of equal parts, suppose twelve, erect ordinates from all the points of extended section, and draw radiants through their summits. Suppose the radiants of the pole A (fig. 71) to be reckoned from X to Y, and those of B from X to Z. For the convergent curve, the intersection of the radiant 1, from the pole A, with the radiant 1, from the pole B, will assign the limit of the curve next the axis; the intersection of the radiants 2 and 1 from A, with the radiants 1 and 2 from B, will mark two points in the next curve; the intersection of the radiants 3, 2, and 1 from A, with the radiants 1, 2, and 3 from B, will give three points in the curve beyond it; and the intersection of the radiants 4, 3, 2, and 1 from A, with the radiants 1, 2, 3, and 4 from B, will trace another exterior curve. In this way the process may be pursued through the whole series of curves, the corresponding initial and final tangents being the succeeding radiants 2, 3, 4, 5, &c. from the poles A and B.

For the divergent curve the intersections of the radiants 1, 2, 3, 4, 5, &c. from the pole B, with the radiants 2, 3, 4, 5, 6, &c. from the pole A, will mark the first branch from O towards Y; the intersections of the same radiants, with the radiant 3, 4, 5, 6, 7, &c. will trace the next branch, and the intersection of still the same radiants with the radiants 4, 5, 6, 7, 8, &c. will assign the third branch; this procedure is easily carried on, and a similar mode will assign the opposite branches. The radiants 1, 2, 3, 4, 5, &c. from A, will by their intersection with the radiants 2, 3, 4, 5, &c. from B with the radiants 3, 4, 5, 6, 7, &c.; and with the radiants 4, 5, 6, 7, 8, &c. will trace out the successive diverging curves from O to Z.

The tangents to all these curves at A and B are the preceding radiants 0, 1, 2, 3, &c.

The figures 74 and 75 were reduced from curves actually delineated in this way, omitting only for the sake of distinctness some of the initial and intermediate curves. Leslie's *Geometrical Analysis.*

§ XIX. *Application of the laws of induced magnetism, to the correction of the local attraction of vessels.*

On the local attraction of vessels.

149. After what has been illustrated in reference to the action of masses of iron, in deflecting a magnetised needle from its natural direction; it will seem astonishing that navigators were not sooner aware of the effect which the iron, entering into the construction and equipment of vessels, particularly in ships

of war, must necessarily have in deflecting the compass from its proper bearing; and that even after the irregularities of the needle under different directions of the ship's head had been remarked, it was still not known to what cause the error ought to be attributed. It appears that the first notice of such an effect occurs in the voyages of Captain Cook; having been first observed by Mr. Wales his astronomer; but the cause of the deviation, as stated above, was not then suspected, and the subject at that time attracted but little attention. The next reference to the local attraction of vessels, that is, to the deflection of the needle from its natural bearing, in consequence of the influence of the ship's iron, and in which the cause is clearly pointed out, is found in Walker's *Treatise on Magnetism*, published in 1794; it is contained in a report from Mr. Downie, master of his Majesty's ship Glory, in which he says; "I am convinced that the quantity and vicinity of iron in most ships have an effect in attracting the needle; for it is found by experiment that the compass will not always point in the same direction, when placed in different parts of the ship; also it is rarely found that two ships steering the same course, by their respective compasses, will go exactly parallel to each other; yet these compasses when compared on board the same ship, will agree exactly with each other."

A few years after this, the action of the iron of the vessel was more minutely noticed by Captain Flinders, who was the first to trace its connection with the dip of the needle, and through whose perseverance some attention was paid by Government to the subject, and several experiments were made by order of the Admiralty, on various ships at the Nore and Spithead, by which the general fact was fully established. The subject, however, seems to have been again lost sight of, till Mr. Bain published his valuable *Treatise on the Variation of the Compass*, where the fatal consequences attending this source of error are put in so clear a point of view, as to strike the most indifferent reader. And as at this time our first Arctic expeditions were in contemplation, the local attraction of the vessels was one of the objects to which the attention of the officers was particularly directed. The results of the experiments made in these instances are given by Captain Ross in the account of his voyage; by Mr. Fisher, who accompanied Captain Buchan, in the *Journal of the Royal Institution*; and by Captain Sabine in the *Philosophical Transactions*, part i. for 1819; from which latter paper we make the following abstract, in order to apprise our readers of the amount of the error that may thus be produced in a ship's reckoning, and the necessity there must be of adopting some method of correction. In order, however, to understand Captain Sabine's remarks, it may be proper to state that Captain Flinders had not merely made barren observations on the local attraction of his ship, but had furnished seamen with a rule for correcting the same. He assumed that on the magnetic equator of the earth, the iron of the ship produced no disturbance of the needle, but as the dip increased the power of the iron increased; also that the attraction was greatest with the ship's head east or west, nothing at north and south; and that the deviation at any other point was to that at east or west, as the sine of the angle between the ship's head and the north or south point to radius. He also gave a rule for determining a multiplier for different dips of the needle, so as to suit his formula to every part of the globe, provided the dip of the needle was known.

Captain Sabine's observations on the local attraction of the Isabella.

This being premised, the following remarks and observations of Captain Sabine will be readily understood; we shall give them nearly in his own words.

150. The first opportunity Captain Ross found of ascertaining the local attraction of the Isabella, was while that vessel was at anchor in Brassa Sound, Shetland. Here her head was placed, by means of warps, on each point of the compass successively, and the bearing of a pile of stones on the summit of a distant hill, noted by her compass at each point. At the same time that these observations were made on board, her bearing from the hill was also observed by a compass placed on the pile of stones; the agreement in bearing showed the points of no error, and the difference, the error at each point.

The following table exhibits, at one view, the results of these observations; the amount of the errors towards the east and west, it will be seen, is half a point nearly in opposite directions, and consequently is such as to produce a total error of nearly a point in the ship's course.

Table of the errors in the Isabella's compass. Shetland.
Dip 74° 21′.

Direction of ship's head.	Deviations.	Direction of ship's head.	Deviations.
North.	+1°26′	North.	+1°26′
N. by E.	+0 26	N. by W.	+2 26
N. N. E.	−0 19	N. N. W.	+3 26
N. E. by N.	−1 19	N. W. by N.	+4 26
N. E.	−2 09	N. W.	+5 11
N. E. by E.	−3 04	N. W. by W.	+5 46
E. N. E.	−3 34	W. N. W.	+5 46
E. by N.	−4 04	W. by N.	+5 41
East.	−4 34	West.	+5 11
E. by S.	−5 34	W. by S.	+4 11
E. S. E.	−5 34	W. S. W.	+3 56
S. E. by E.	−5 34	S. W. by W.	+2 56
S. E.	−4 59	S. W.	+1 11
S. E. by S.	−4 24	S. W. by W.	+0 26
S. S. E.	−3 34	S. S. W.	−0 0
S. by E.	−3 04	S. by W.	−1 34
South.	−2 04	South.	−2 04

Local attraction of the Alexander.

Of the Alexan

The Alexander being alongside a float of ice in Baffin's Bay, the true magnetic bearing from the ship of a very distant and well defined object on the main land was found, by carrying a compass on the ice in an opposite direction, to a distance which ensured its being perfectly free from local influence. The ship's head being then warped round to each point of the compass successively, the errors in each were determined by the differences in bearings, as in the last instance.

Table of the errors in the Alexander's compass. Baffin's Bay. Lat. 75° 25′; *long.* 56° 26′ *W.; dip* 84° 30′; *variation* 87° 30′ *W.*

Direction of ship's head.	Deviations.	Direction of ship's head.	Deviations.
North.	−4°39′	South.	+6°46′
N. N. E.	−4 09	S. S. W.	+4 46
N. E.	−3 44	S. W.	+3 20
E. N. E.	−2 24	W. S. W.	+3 11
E. by N.	−2 14	W. by S.	......
East.	−1 44	West.	+1 41
E. by S.	+0 06	W. by N.	......
E. S. E.	+0 56	W. N. W.	−0 19
S. E.	+2 51	N. W.	−1 44
S. S. E.	+4 01	N. N. W.	−3 09

“The regularity in the above results is the best testimony that the method is a satisfactory one, but certain precautions must be attended to: thus the object must be sufficiently distant, that the space occupied in warping the ship round may not subtend any sensible parallax. The direction of the ship's head should be noted by the compass by which the bearings are taken. A short time must be allowed to elapse after the ship is steady on any point, to ensure the traversing of the cards: this is particularly necessary in high latitudes, where the compasses move very sluggishly. And, lastly, the observations should be repeated.

“It will be observed by the above results, in the Isabella and Alexander, that the points of no error were not coincident in either ship with those of the magnetic meridian; in the Alexander especially, they were more nearly at right angles to it. That this ship should have differed so materially from all the instances on record, may be attributed to her compass being so near the level of the deck, and therefore being more affected by the influence of a considerable quantity of iron articles, (such as ice anchors, ice saws, &c.) which were carried on the after-part of the deck for convenience in use, than it would have been had it been raised higher. This was proved by placing a compass on a plank, elevated for experiment in front of the companion, to the same height as in the Isabella, namely, nine or ten feet. The points of no error were found, in this position, to be about north and south, and the amount of error at eight points, nearly twenty degrees; the same as in the Isabella: the greatest error at the same time, by the Alexander's standard compass, viz. the one nearer the decks being 8° 20′ at N. N. E. The dip was 84° 09′.

“The propriety of Captain Flinders's recommendation to determine the points of no error in a fixed compass, by actual observation in every ship as soon as the distribution of iron is completed, may therefore be considered as confirmed by the observations in the Isabella and Alexander: whilst his rule of proportion may receive a verbal alteration to render it more suitable for general application: so corrected, it would be as follows; the expressions substituted being marked in italics, and the original words entered in the margin.

“The error produced in any direction of the ship's head, will be to the error at *the point of the greatest irregularity*, as the sine of the angle between the ship's head and *the points of no error* to the sine of eight points or radius.

East or west.
Magnetic meridian.

“Thirdly, Captain Flinders's experience in the Investigator, showed that the maximum of error in the same compass would be different in different parts of the world, although the use of the compass was confined to one particular spot in the ship, and every precaution taken to avoid an interference with the distribution of the ship's iron.

“It is worthy of remark, that by multiplying observations, and by comparing the series one with another, he was thus *practically* led to trace a connection between the amount of the errors and the dip of the needle; a knowledge of the fact preceding, in his mind, any theoretical suggestion that such might be the case.

“It does not appear indeed that the principal cause of this connection was subsequently known to him; he perceived that the influence of local attraction on the compass needle increased as the dip became greater. He endeavoured to account for this circumstance, on a supposition that all iron might receive an *absolute* increase in the intensity of its attractive power by approaching the magnetic pole.

“The increase, however, which was the subject of his observation, was a *relative* one, being in comparison to the directive power of magnetism. A diminution of the latter would therefore produce the effect equally with an absolute augmentation in the former; and that such a diminution does take place, and in a degree which is sufficient to account for all the effects observed, will be evident to every person, who reflects that although the magnetic force is greatest at the pole, its directive power must then have wholly ceased, having become less on the horizontal traversing of the needle in proportion as the point of attraction has been brought beneath the compass; indicated by the angle which the dipping needle makes with the horizon. This is doubtless the principal cause of the connection which Captain Flinders was the first to trace.

“It is not designed to say that this cause may not be aided by the increased magnetism of portions of the ship's iron, such as bars and stanchions; which, being fixed in an upright position, may receive an addition to their attractive power where the position of the dipping needle is always coincident with theirs; but merely to observe, that a cause is known to exist for the connection, independently of supposition; which cause, conjointly with experience, shows the inadequacy of the rule proposed by Captain Flinders, whereby the amount of error, under any known dip, being ascertained, the amount may be calculated for any other dip, by using as a multiplier the decimal expression of the proportion which the error, in the one ascertained instance, may have borne to the dip.

“In the observations made in the Isabella at Shetland, where the dip is 74° 21½′, the maximum of error was 5° 34′ easterly of the true variation, with the ship's head at E. S. E., and 5° 46′ westerly at W. N. W., making an extreme difference of 11° 20.′

“By Captain Flinders's rule, the common multiplier for this compass would have been about one-twelfth, or .083, which at a dip of 86° 09′, which was the greatest observed during the late voyage, would have given an error of between 7° and 8°, making the extreme difference 15°; whereas repeated observation showed it to be at that time more nearly 50°, if not exceeding that amount.

"The inadequacy of the rule will also appear by reference to the observations made by the Alexander in Baffin's Bay. The error at eight points being 6° 46′, at a dip of 84° 30′; it ought scarcely to have exceeded 7° at the greatest possible dip, making an extreme difference of less than 15°. No opportunity occurred indeed of making accurate observations at a greater dip than the above; but the difference in the bearing of objects before and after tacking, indicated with sufficient certainty that the error had increased to an amount far beyond 15°; frequent instances of an extreme difference of from three to four points being remarked as the ship approached the farthest western longitude to which she attained in a high latitude: this was in Lancaster's Sound of Baffin, into which inlet the expedition sailed beyond the 18° of west longitude in the parallel of 74° and a few minutes.

"It is much to be regretted that the service did not admit an opportunity to be afforded of making observations on the various magnetic phenomena, with the excellent instruments supplied to the expedition, at this very interesting place, where a nearer approach was made to one of the magnetic poles than had ever been known before.

"But in the absence of any actual observation on the dip of the needle, this fact of the error of the compasses having increased from local attraction so greatly beyond the amount which had been before observed, is worthy of notice, as affording an indication that the dip had also increased, and not inconsiderably. The greatest which was observed was 86° 09′; and, after this observation, the ship continued to sail for six days in the direction in which the dip had hitherto been found to increase.

"In concluding this paper, it may be permitted to remark, that it is to the voyages of discovery undertaken during the reign of his late Majesty that a knowledge of the extent and causes of the errors to which a compass is subject in ships, is to be principally attributed; as well as the steps that have been taken towards the investigation and remedy of the inconvenience they occasion, to practical navigation.

"The care and exertions of Captain Flinders in collecting observations for this purpose, give his opinions and rules a peculiar claim to attentive consideration. No one could have been more fully persuaded than he was, that a rule, founded on the effects experienced in a few ships, would require a far more extensive trial before it could be depended on for general application.

"To carry this on, therefore, is to follow his useful example, and to effect what he was desirous to have done himself, had his life been spared." *Philosophical Transactions,* part i. 1819.

151. The foregoing observations, which are given rather more at length by Captain Ross, are the first that were published after the revival of this inquiry, but these having shown the great amount of this source of error, particularly in high latitudes, much attention has been since drawn to the subject by many eminent philosophers and navigators, not only in England but in Europe and America. Admiral L. Löwenörn of the Danish navy; Admiral Krusenstern, of the Russian navy; and several officers in the American naval service are at this time engaged in investigating and experimenting upon this source of error, which, if suffered to remain, would be a disgrace to the science and philosophy of the nineteenth century. It is scarcely credible that to this day the course of a vessel should be in error or uncertainty as much as a point, or 11° 15′, without that error, in nine cases out of ten, being even suspected and much less corrected. We are aware that in the present state of navigation much less attention is paid to the compass than formerly, because the situation of a vessel may be much better determined from celestial observations than from the mere reckoning by the log. But when we consider that it frequently happens that those observations, in consequence of the state of weather, cannot be made for several days together, and that then the compass is the only guide the navigator can have recourse to; it certainly becomes an important question to rid this instrument of every possible source of error, and with this view those experiments, to which we have above alluded, are now proceeding with all that earnestness the importance of the subject demands.

Mr. Barlow's proposition for correcting the local attraction of vessels.

Mr. Barlow's method correcti local attractio

152. A moment's consideration will be sufficient to show that the laws of magnetic action discovered by Mr. Barlow, and which are described in sections XI. and XII. are strictly applicable to the correction in question. For from these it appears that the two centres of action in any mass of iron are indefinitely near to each other in the common centre of attraction of the mass, or rather of the surface of the mass; so that whatever may be the dip of the needle, the compass and the common centre of attraction of the iron will remain at the same constant distance in all parts of the world, the situation of the iron and compass not being altered during the voyage. The only change of magnetic power will therefore proceed from the change of the latitude and longitude of the centre of attraction, as referred to the ideal sphere circumscribing the compass, agreeably to the idea advanced in art. 90. Hence having determined the situation of the centre of attraction, by experiment, in any part of the world, the same will become known in every other part where the dip of the needle had been previously ascertained, and the local attraction being observed in the former case, may then be computed in the latter.

This, method, however, although it possesses all the accuracy that can be desired, involves rather a troublesome course of calculation, and unfortunately like every other formula yet proposed, it requires, as a first datum, the dip of the needle, which generally is not known, and cannot be determined on ship board; so that, notwithstanding the correctness of the principles upon which it is founded, it is actually impracticable at sea.

By expe ment.

153. The author was aware of this defect, and therefore sought an experimental method of proceeding, which should give the attraction of the vessel at all times and in all places, independent of the

dip, and the method he has suggested is certainly extremely simple and practicable; for since the deflection produced by a mass of iron depends upon its distance, and a certain function of the latitude and longitude, as referred to the ideal sphere to which we have alluded, it is obvious that whatever effect a large mass of iron may produce on the compass at a great distance, a small mass placed with its centre of attraction in the line which joins the centre of attraction of the whole mass and the pivot of the needle, but at a less distance, may be made to produce the same effect; and since the power of the iron is resident on the surface, and is independent of its mass, it is very easy to introduce a plate or ball of iron near the compass, so that under all circumstances its effect on the needle shall be the same as the iron of the vessel, that is so as to double the original effect.*

154. Nothing can be more obvious than this principle of correction, nor more simple in its application, but still it would necessarily be considered speculative, till submitted to actual practice; and with a view of obtaining this test, the author addressed a memorial to the Lords of the Admiralty, requesting permission to have the experiment tried on any vessel or vessels their lordships might be pleased to direct; and having received a favourable reply to this request, experiments have since been made, and the method of correction practised with the best success in several of his Majesty's ships, of which we have some accounts in the author's *Essay on Magnetic Attractions,* from which the following is extracted:

An account of some experiments made on board his Majesty's ships Leven, Conway, and Barracouta.

peri- nts on local action H. M. S. en,Con- r, and ra- ta.

155. The first opportunity which presented itself of putting the orders of the Lords Commissioners of the Admiralty into effect, was on board the Leven, which was fitting at Woolwich, for a survey of some part of the western coast of Africa.

The first object, in course, in all cases is to determine the quantity of the local attraction of the vessel; and the method adopted for this determination in the present instance, will be easily understood by the following extract from Mr. Barlow's report to the Admiralty.

"The Leven having dropped down to Northfleet on the fifteenth of April, 1820, I went down on the seventeenth, for the purpose of making a series of experiments before the guns should be brought on board, these observations were conducted as below.

"'First, Finding that there would be great difficulty in warping the vessel round in the tide way of this place, I proposed, and it was agreed to proceed in the following manner.

"'I took on shore an excellent azimuth compass, by Messrs. W. and T. Gilbert, which I had procured for the purpose, as also a theodolite, by Schmalcalder. With the azimuth, the bearing of a distant object was taken, and found to be N. 35° 50′ E., and the theodolite was then adjusted to the same reading, viz. 35° 50′ from zero; by means of which the zero of the theodolite was brought to the true magnetic north, and consequently the bearing of an object might now be determined without any further reference to the needle. It will of course be understood, that the theodolite was fixed immediately over the spot where the azimuth compass was first erected. The latter instrument was now taken on board, for the purpose of the experiments, while Lieutenant Mudge remained on shore to take the bearings of the pedestal,* or pillar, on board with the theodolite.

"'The ship now beginning to swing to the tide, the word was given "look out," at which signal Lieutenant Vidal, at the azimuth compass on board, kept Lieutenant Mudge on shore, in the line of the sights, while the latter gentleman kept, in the same way, Lieutenant Vidal in the field of the telescope. Being thus prepared, the word "stop" was given, at which each registered the bearing of the other at the same instant. These bearings, independent of the local attraction of the vessel, ought to have been diametrically opposite, and consequently the difference between the two readings, was the error due to the attraction of the iron on board.

"'The first observation being registered, the word "look out" was again given, and then the word "stop," and the same was repeated as often as possible while the vessel was swinging; Lieutenant (now Captain) Baldey taking every time the bearing of the ship's head, by the ship's azimuth compass at the capstan.

"'The advantages of this method are, that both bearings, viz., on board and on shore, are made to depend on the same compass, and thus the errors arising from the use of different needles are avoided, as are also those arising from the parallax of a distant object while the vessel is swinging; a source of error which must have attended all former observations of this kind.

"'The only thing actually necessary in this case, is a fine free azimuth compass, those commonly served out to the Navy are so sluggish, that it is impossible (while there is no motion in the vessel) to depend upon their settling within two or three degrees of the true magnetic north.

"'The experiments above referred to, were made before the guns were got on board, but the same were again repeated on the nineteenth of April, after they had been all shipped. The following are the results of both series of observations:'"

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* It should be observed that, strictly, it is not the angle of deviation that is doubled by the experiment, but the tangent of the angle; but as, in small angles, the tangents have nearly the same ratio as the arcs, it will be sufficiently correct to consider it as above stated. If greater accuracy should be thought desirable, let x be the angle of deviation produced by the guns, &c. and a the angle produced beyond the former by the plate; then we shall have

$$\tan(x + a) = \frac{\tan x + \tan a}{1 - \tan x \,.\, \tan a} = 2 \tan x$$

whence,

$$\tan x = \frac{1 + \sqrt{(1 - 8 \tan^2 a)}}{4 \tan a}$$

This formula may, however, in general be dispensed with.

* Captain Bartholomew had ordered a pedestal to be erected just before the mizen mast, as a fixed situation for taking his azimuths during the voyage.

Experiments on the Leven, first voyage.

156. *Experiments on board H. M. S. Leven, at Northfleet, April 17 and 19, 1820, by Mr. Barlow and the Officers of the above vessel. Dip 70° 30′.*

Guns not on board.			Guns on board.			
No. of experiments.	Bearing of ship's head.	Difference in bearing or local attraction.	No. of experiments.	Bearing of ship's head.	Difference of bearing or local attraction.	Local attraction shown by the plate.
1	N. 77° 0′ W.	+ 2° 22′	1	N. 71° 0′ W.	+ 2° 51′	
2	N. 68 30 W.	+ 2 25	2	N. 64 0 W.	+ 2 27	2 20
3	N. 57 0 W.	+ 1 37	3	N. 57 0 W.	+ 1 39	
4	N. 47 0 W.	+ 1 54	4	N. 47 0 W.	+ 1 45	
5	N. 32 0 W.	+ 1 12	5	N. 31 0 W.	+ 1 39	1 30
6	N. 20 0 W.	+ 1 20	6	N. 24 0 W.	+ 1 10	1 0
7	N. 14 30 W.	+ 0 12	7	N. 15 0 W.	+ 1 19	
8	North.	− 0 15	8	N. 6 0 W.	+ 0 17	0 40
9	N. 5 0 E.	− 0 54	9	N. 4 0 W.	− 0 8	
10	N. 16 0 E.	− 1 32	10	North.	− 0 24	0 0
11	N. 32 0 E.	− 1 43	11	N. 5 0 E.	− 0 11	
12	N. 45 0 E.	− 2 25	12	N. 13 0 E.	− 0 29	0 40
13	N. 52 0 E.	− 2 36	13	N. 23 0 E.	− 0 46	1 0
14	N. 67 0 E.	− 3 15	14	N. 57 0 E.	− 1 27	1 30
15	N. 74 0 E.	− 3 6	15	N. 59 0 E.	− 2 32	
16	N. 83 0 E.	− 2 13	16	N. 72 0 E.	− 2 23	2 10
17	East.		17	N. 80 0 E.	− 2 51	
18	S. 81 15 E.	− 2 34	18	N. 86 0 E.	− 2 11	2 30
19	S. 74 30 E.	− 2 30	19	N. 85 0 E.	− 2 34	2 30

The rapidity and force of the tide at Northfleet, at the time of making the experiments on the Leven, would not admit of warping the vessel about point by point, which is doubtless the best way. This, however, is easily done in Portsmouth harbour, and was the method adopted by Captain Hall, in his experiments on the Conway: reported in the following table.

Where the apparent or observed westerly bearing exceeds the true westerly bearing, the error or local attraction is marked + (plus); and where the former is *less* than the latter, the error is marked — (minus). With the ship's head at west, the object on shore could not be seen.

Experiments on the Conway

157. *Experiments on the local attraction of H. M. S. Conway, Portsmouth harbour. The numbers marked *, were the mean of several observations. July 24, 1820. By Captain Basil Hall and Mr. Barlow, Royal Military Academy.*

Dip, 70° 30′ 3.									
No. of mean observations.	Direction of ship's head.	Observed bearing of the station on shore from the ship, by Captain Hall on board.	Bearing of compass on board by Mr. Foster on shore.	Local attraction.	No. of mean observations.	Direction of ship's head.	Observed bearing of the station on shore by Captain Hall.	Bearing of compass on board by Mr. Foster on shore.	Local attraction.
1	S. by S.	N. 97° 0′ E.	S. 95° 40′ W.	−1° 20′	17	S. S. E.	N. 97° 0′ E.	S. 97° 15′ W.	+0° 15′
2	South.	96 0	94 3	−1 57	18	S. E. by S.	95 50	96 22	+0 32
3*	S. by W.	95 20	92 57	−2 23	19	S. E.	94 10	95 16	+1 6
4*	S. S. W.	95 10	92 19	−2 51	20	S. E. by E.	93 20	94 24	+1 4
5*	S. W. by S.	94 8	91 0	−3 8	21	E. S. E.	91 0	92 30	+1 30
6*	S. W.	94 2	90 47	−3 15	22	E. by S.	89 30	91 52	+2 22
7*	S. W. by W.	93 35	90 15	−3 20	23	East.	87 50	91 15	+2 25
8	W. S. W.	93 30	88 32	−4 58	24	E. by N.	85 0	89 5	+4 5
9*	W. by S.	92 10	87 32	−4 38	25	E. N. E.	83 20	86 34	+3 14
10	West.				26	N. E. by E.	82 10	85 31	+3 21
11	W. by N.	88 0	84 25	−3 35	27	N. E.	82 15	84 58	+2 43
12	W. N. W.	86 35	83 12	−3 23	28	N. E. by N.	83 0	85 13	+2 13
13	N. W. by W.	85 20	82 27	−2 53	29	N. N. E.	85 50	88 4	+2 14
14	N. W.	83 25	81 46	−1 39	30	N. by E.	84 40	85 47	+1 7
15*	N. W. by N.	84 17	82 7	−2 10	31*	North.	83 0	83 7	+0 7
16*	N. N. W.	83 35	82 3	−1 32	32*	N. by W.	82 28	81 38	−0 50

Experiments on board his Majesty's ship Leven, (Captain Owen,) and on his Majesty's brig Barracouta, (Captain Cutfield,) previous to the departure of these vessels for the eastern coast of Africa.

158. The Leven on her return from the western coast of Africa, having been ordered to survey the eastern coast of that continent, and the command of her given to Captain Owen; who deservedly enjoys the reputation of one of the most able scientific officers of the Navy; the author had another excellent opportunity of making his experiments both on this vessel and on the Barracouta.

He proceeded here nearly as in the cases above, except that the land station being at a greater distance, signals by flags were employed; warps were also used in this instance, and the observers thus enabled to steady the vessel on any proposed point, and to take the observation on board with more precision.

A second situation was also taken for a compass, a little forward of the foremast, where a pillar was fixed, and the two compasses placed under like circumstances, except their different situations in the vessel. Another compass was also placed on the gun-room table, with which the ship's head was registered every time it was taken with the other two.

The following are the results of these experiments.

159. *Experiments made to ascertain the quantity of local attraction in H. M. S. Leven, on two compasses, one forward and one aft, under the direction of Captain Owen, January 15, 1822, at Northfleet.*

After compass.				Foremast compass.				Ship's head by compass on gun-room table.
Bearing of ship's head by after compass.	Bearing of Lieut. Owen at shore station from after compass.	Bearing of Lieut. Vidal at after compass from shore station.	Difference or local attraction.	Bearing of ship's head by foremast compass.	Bearing of Mr. Daniel's at shore station from foremast compass.	Bearing of Lieut. Boteler at foremast compass from shore station.	Difference or local attraction.	
N. 4° 00′ E.	S. 34° 30′ W.	N. 34° 23′ E.	+0° 7′	N. 5° 00′ E.	S. 33° 30′ W.	N. 32° 20′ E.	− 1° 10′	—° —
10 00	30 30	34 6	−3 36	18 30	Intercepted by the foremast.	33 30	— —	N. 17 30 E.
16 00	30 30	33 30	−3 0	21 30		32 40	— —	— —
20 00	29 30	34 3	−4 33	28 30		33 30	— —	25 30
22 00	30 00	34 23	−4 23	29 30		34 00	— —	26 30
28 00	33 30	35 52	−2 32	36 30		34 12	— —	33 00
40 30	29 30	34 43	−5 13	51 00		34 47	— —	— —
50 00	29 00	34 40	−5 40	— —		35 10	— —	— —
57 00	28 50	34 35	−5 45	— —		35 17	— —	— —
67 00	29 00	34 19	−5 19	78 0	43 15	35 54	+ 7 21	72 00
82 00	26 00	33 47	−7 47	84 30	43 15	35 45	+ 7 30	N. 89 00 E.
89 00	26 30	33 25	−6 55	83 00	43 0	35 26	+ 7 34	S. 88 00 E.
East.	27 20	34 11	−6 51	— —	— —	— —	— —	— —
S. 80 00 E.	27 00	33 01	−6 01	S. 67 30 E.	42 30	34 47	+ 6 43	73 00
70 00	27 40	32 45	−5 45	56 15	41 0	34 20	+ 6 40	65 00
63 00	27 10	32 28	−5 18	50 30	40 30	34 27	+ 6 13	58 30
56 00	27 0	32 25	−5 25	47 15	39 30	34 12	+ 4 48	52 30
49 00	26 20	31 57	−5 37	39 30	38 45	33 45	+ 5 00	45 30
44 00	28 00	31 56	−3 56	38 30	37 30	33 40	+ 3 50	41 30
40 00	28 00	31 45	−3 45	33 15	37 00	33 23	+ 3 37	37 30
27 00	27 00	30 57	−3 57	20 00	— —	27 00	— —	— —
South.	32 00	31 14	+0 46	S. 5 00 E.	31 30	30 00	+ 1 30	S. 6 00 W.
S. 13 00 W.	— —	— —	— —	S. 7 00 W.	25 30	31 23	− 5 57	9 00
17 00	34 20	31 05	+3 15	12 30	25 30	30 50	− 5 20	13 00
26 00	34 00	31 06	+2 54	18 00	24 30	30 50	− 6 20	21 00
33 00	34 30	31 02	+3 28	26 00	23 00	30 31	− 7 31	28 30
39 00	35 00	31 09	+3 51	31 30	22 30	30 38	− 8 08	34 00
50 00	35 40	31 36	+4 4	37 30	22 30	30 49	− 8 19	43 30
57 00	34 00	— —	— —	45 00	21 30	30 58	− 9 28	50 30
60 00	35 00	31 23	+3 47	— —	22 30	29 30	− 7 00	— —
70 00	38 50	33 24	+5 26	57 30	21 30	32 00	−10 30	65 35
80 00	40 20	34 05	+6 15	68 30	22 00	32 54	−10 54	75 30
West.	40 00	34 36	+5 24	77 30	22 30	32 36	−10 06	S. 87 30 W.
N. 74 00 W.	41 10	35 00	+6 10	N. 87 00 W.	21 30	32 56	−11 26	N. 76 00 W.
63 00	39 50	35 22	+4 28	77 00	22 00	33 02	−11 02	64 30
53 00	39 40	35 29	+4 11	58 30	24 15	33 15	− 9 00	50 00
43 00	40 30	35 32	+4 58	50 00	25 30	33 08	− 7 38	48 00
33 00	— —	— —	— —	— —	— —	— —	— —	— —
22 00	39 10	— —	— —	27 30	30 45	36 00	− 5 15	— —
16 00	38 30	37 30	+1 0	19 00	30 45	35 22	− 4 37	— —
12 00	38 00	37 57	+0 3	15 00	31 30	35 43	− 4 13	10 30

(Signed) A. H. Vidal, First Lieutenant of H. M. S. Leven

160. *Experiments made to ascertain the local attraction of H. M. S. Barracouta, at Northfleet, January* 15, 1822, *by the Officers of that vessel.*

Experiments on the Barracouta.

Direction of ship's head	Bearing of shore station from compass on board, Lieut. Mudge.	Bearing of compass on board from shore station, Mr. Durnford.	Local attraction.	Direction of ship's head.	Bearing of shore station from compass on board, Lieut. Mudge.	Bearing of compass on board from shore station, Mr. Durnford.	Local attraction.
N. by E.	S. 37° 00′ W.	N. 37° 00′ E.	0° 0′	North.	S. —° —′ W.	N. —° —′ W.	
N. N. E.	34 40	38 40	− 4 0	N. by W.	— —		
N. E. by N.	35 40	39 10	− 3 30	N. N. W.	48 10	41 30	+ 6 40
N. E.	34 40	41 20	− 6 40	N. W. by N.	51 00	42 00	+ 9 00
N. E. by E.	33 00	42 20	− 9 20	N. W.	— —	44 47	— —
E. N. E.	— —	— —	— —	N. W. by W.	52 30	44 20	+ 8 10
E. by N.	— —	— —	— —	W. N. W.	56 40	43 40	+13 00
East.	26 20	42 40	−16 20	W. by N.	55 40	43 30	+12 10
E. by S.	27 20	40 40	−13 20	West.	54 00	40 59	+13 01
E. S. E.	25 00	39 30	−14 30	W. by S.	54 40	43 00	+11 40
S. E. by E.	32 00	38 00	− 6 00	W. S. W.	49 18	41 30	+ 7 48
S. E.	33 00	36 00	− 3 00	S. W. by W.	— —	— —	— —
S. E. by S.	32 00	35 30	− 3 00	S. W.	— —	— —	— —
S. S. E.	32 50	35 00	− 2 40	S. W. by S.	— —	— —	— —
S. by E.	— —	— —	— —	S. S. W.	— —	— —	— —
South.	36 00	34 58	+ 1 2	S. by W.	— —	— —	— —

(Signed) Wm. Mudge, First Lieutenant of H. M. B. Barracouta.

The points omitted, with the exception of the N. W., were those on which the vessel could not be brought, in consequence of the wind and tide; the observation (on board) at the N. W. is omitted, there being obviously an error in registering the result.

The great difference in the amount of the local attraction of the Leven, this time and the former, is attributed by the author to her being fitted with a new patent capstan, the spindle of which is about eleven feet long, and its mean diameter not less than five inches; it commences from the capstan head, passes through the upper deck and gun deck, and steps into a carling below the beam. Being vertical, the power on the compass is very considerable, although the station for observation is taken as far aft as possible.

In the Barracouta, the spindle is of the same length, and the place for observation is necessarily, from the small dimensions of the vessel, proportionally nearer to it.

Description of the correcting plate, method of adjusting it, &c.

Correcting plate described.

161. The plate employed in all the preceding experiments, as well as those which have been sent with Captain Parry, in the Fury, and with Captain Sabine, in the Iphigene frigate, have been double; viz., each consists of two thin plates of iron screwed together, in such a manner as to combine any strong irregular power of one plate, with a corresponding weak part of another, by which means a more uniform attraction is obtained; it does not, however, appear that these precautions are necessary, if iron weighing about six pounds to the square foot is made use of; but with thinner plate iron, viz. of about three pounds to the foot it is requisite, not only for the purpose above stated, but to prevent any accidental bending by a fall or otherwise.

The plates are twelve or thirteen inches in diameter, having a hole in their centre, through which is passed a brass socket with an exterior screw; a brass nut about an inch and a half in diameter, screws on the exterior of each end of the socket, compressing thereby the plates together, with an interposed thin circular piece of board, which is intended to increase somewhat the thickness of the plate, without increasing its weight. It appears also that the compound plate is more powerful when the two, of which it is formed, are thus separated from each other.

In order to render the union the more permanent, the plates are also fixed to each other by two or three small iron screws near their edges.

Method fixing th plate. Fig. 76.

162. The plate being thus prepared, the next object is to ascertain its proper situation in the ship; for which purpose we may proceed as follows: let a box or log of wood, having no iron about it, as A B (fig. 76.) be taken on shore, and let holes be bored in it, at eight, nine, ten, eleven, &c. inches from its upper part, to receive the brass or copper horizontal rod R, which is to carry the plate. This pin being inserted in one of the holes, and the compass placed securely on the top of the box or log, turn the latter with the pin successively to the several points of the horizon, and by attaching and removing the plate, observe its power of attraction. If the results thus obtained agree with those observed on ship board, we have the right position of the plate; but if this should not be the case (as is most likely to happen) shift the height and distance of the plate, and repeat the experiments again, and after a few trials we shall be able to obtain the same attraction with the plate, as was observed in the vessel.

163. This being done, measure carefully the distance of the plate from the vertical, passing through the pivot of the needle, and its depth below the plane of the card, and cause a hole to be bored, and a socket

to be introduced into one of the legs of the tripod, used for the azimuth compass on board, so that when the brass pin is inserted as shown in fig. 77, the centre of the plate may be at the same depth below, and distance from the vertical passing through the pivot of the compass as was determined on shore, and it will be the fixed situation required. If a fixed pedestal be used on board, as a stand for the azimuth compass, which is to be preferred, the same direction of course will apply to this as to the tripod. The plate and pin are both moveable, and are laid aside except at the time of observation.

It may be proper to observe, that as, from the unavoidable errors in observations, it is probably impossible to get the plate to give the same attraction as the vessel at every point, the object ought to be to take a mean between the deviations at (S.E., S.W.), (N.E., N.W.), (E.W.), (N.S.), and if the mean at those points in the ship, and with the plate on shore agree, the other errors will be inconsiderable.

164. After all, however, this is by far the most difficult part of the experiment, and the best way is undoubtedly to procure a plate already corrected, that is, a plate whose attraction has been found experimentally for several distances and positions; the results of which are given in a table with it, so that having obtained the local attraction of the vessel, the same attraction may be selected out of the table above-mentioned, and opposite to it will be found the proper corresponding height and distance. Plates, with the requisite tables are sold by Messrs. W. and T. Gilbert, 148, Leadenhall-street.

It is to be observed, however, that these *tables* only apply to experiments made in these latitudes; viz., in any British harbour, but they will not be applicable if the experiments are made in a port where the dip of the needle is very different from what it is in London. That is, the experiments on the plate, and on the vessel ought always to be made in the same place.

Method of using the plate on ship board.

165. In the first instance Mr. Barlow proposed to apply the plate to the binnacle compasses, but on the suggestion of the officers of the Leven, he changed the application of it to the azimuth compass, as being more susceptible of nice observation, besides possessing advantages not to be obtained in the other case.

The following are the directions left with the officers of the above ship for using the plate:

166. "When an azimuth, or amplitude of the sun, or other heavenly body, is taken for the purpose of determining the variation, the observations are to be first made in the usual way, and then immediately repeated with the plate attached, and the difference between the two bearings will be the local attraction.

"For example: suppose the mean of the first series of observations to give the bearing 67°, and the second with the plate attached 70° 30′;

Then, 70° 30′ 2d. mean from 67° 0′
67 00 1st. mean take 3 30

3 30 local attraction. 63 30 {correct azimuth.

"Again, let the amplitude by common observation be 13° 30′, but with the plate only 10° 30′, then

13° 30′ 1st. mean to 13° 30′
30 10 2d. mean add 3 0

3 0 local attraction. 16 30 {true compass amplitude.

"*Note.*—In all cases, when the first observed bearing of an object is diminished by the plate, the difference or local attraction is to be added to the first bearing; and when the first angle is increased by the plate, the difference is to be subtracted."

167. It is to be observed, that this supposes the plate to be applied immediately after the mean of the first series of observations has been obtained, and before any considerable change has taken place in the azimuth of the celestial body. To avoid every possible chance of error from this cause, the officers of the above ship, computed their variation for both series of observation; viz. with, and without the plate, and the difference gave them the local attraction. This is some additional trouble, but the result is proportionately more correct.

The above are all the directions necessary for using the plate, and they are such as cannot, it is presumed, be misunderstood by any seaman entitled to the character of a practical navigator.

We had intended to follow up this subject, by reporting the results of the experiments made in the Conway by Captain Basil Hall, through more than 110° of latitude, viz. from 50° N., to 60° S. We find, however, it would carry us too far; we can therefore only inform our readers that the experiments were attended with the best success, and that they answered in every particular to the entire satisfaction, both of the experimenters and the author. Similar results were obtained by the officers of the Leven, but this voyage was more limited in its extent, and consequently the deductions made in it less decisive.

§ XX. *On the effects produced in the rates of chronometers, by the magnetic action of a ship's guns, &c. on the balance.*

Effect of iron on chronometers.

168. It having been ascertained during the voyage made by Captain Buchan to the Arctic regions, in the year 1818, (see *Phil. Trans.* 1820,) that the rates of the chronometers were considerably different on board and on shore, and this change having been attributed to the iron of the vessel, Mr. Barlow undertook the task of ascertaining whether the proximity of a mass of iron had actually any effect in changing the rate; and, secondly, supposing this to be the case, to determine, if possible, the laws and principles by which that action was governed. See *Phil. Trans.* for 1820, part ii.

He accordingly procured six excellent chronometers, and having made the requisite preparations, he began his series of observations on them on the 11th of March, 1821, and continued them daily till the 25th of May; when, having obtained a considerable number of results, they were discontinued. It will however be proper, before we proceed to the detail of particulars, to explain the views the author had formed on the subject, and the principles upon which he conducted the experiments.

It was conceived, that if such an effect as that described by Mr. Fisher, were generally produced on the rates of watches and chronometers, it must arise from the spring, or some part of the balance having become magnetic, and the consequent attraction of the iron upon it; for the experiments and observations which Mr. Fisher describes as having been made with a strong bar magnet, brought within two inches of the balance, are perfectly distinct in their nature from those which were made by him on board and on shore at Spitzbergen; for a magnet of such power, brought within the distance of two inches of any small piece of steel, will, whether the latter be previously magnetic or not, impress upon it a strong temporary derangement of its latent magnetism, and give to the part nearest the magnet, a contrary pole to that by which it is opposed; and consequently, there will exist between the balance and the magnet a strong power of attraction sufficient to cause that acceleration so strongly indicated in Mr. Fisher's experiments; and this will be the case whichever end of the magnet is opposed to the balance, and to whatever part of the latter the application is made; because, in this instance, the effect does not depend upon the previous magnetic state of the balance, but upon that temporary state excited by the proximity of the magnetic bar, and which ceases when the bar is removed.

This explanation will not, however, apply to the action of plain unmagnetised iron; for notwithstanding, according to the theory of magnetism that has been advanced, every mass of soft iron becomes a temporary magnet by induction from the earth; yet we are not aware that ever any particular action has been discovered between two pieces of iron, whether hard or soft, which had not previously acquired a polar quality; the largest mass of iron, for instance, will not attract and give direction to the lightest and most freely suspended needle of soft iron, or of unmagnetised steel.

Now, if this be admitted, it necessarily follows, that plain unmagnetised iron can only be supposed to act on the balance of a chronometer, when that balance has acquired a polar or directive quality; and then, as we have already stated, it will have a tendency to produce an acceleration, or retardation, according to its position with respect to the balance, and the previous adjustment of the machine.

If this be actually the case, it may probably appear singular, that all Mr. Fisher's chronometers were accelerated; but it is not much less so, that five out of the six used in these experiments were as decidedly retarded. It will likewise, after examining these experiments, be difficult to account for that high degree of acceleration noticed by Mr. Fisher; for it will be seen that, although some of the chronometers were brought to within two or three inches of the surface of an iron ball thirteen inches in diameter, the utmost effect which could be produced did not exceed 4″ per day; whereas Mr. Fisher makes *his* amount to 8″ or 9″ per day; and yet we can scarcely imagine that he brought his chronometers so closely within the immediate sphere of action of any mass of iron, more powerful than that described in these experiments; indeed we are led strongly to suspect, that the remarkable change in the rates of the nine chronometers of the Dorothea and Trent, reported by Mr. Fisher, must have been produced by some extraordinary cause, not commonly operating on ship board.

We have already observed, that, according to the idea we entertain of the action of iron on the balance of a chronometer, it is actually necessary to conceive, that part of the machine, or at least its spring, to have acquired a certain polar or directive quality, whereby, independent of any other power, the balance would have a tendency to assume a certain direction, when brought within the sphere of action of a given mass of iron; and the amount of that tendency might, it was conceived, be estimated, by counting the number of vibrations which a small magnetised needle would make in a given time, in any assigned situation, near the iron, and comparing the result with the number it would make under like circumstances, and in the same time, when wholly removed from any attracting mass.

169. In order to illustrate this view of the subject a little more particularly, let A B C D (fig. 78.) represent the balance of a chronometer, *s s′* its spring, and let D be that part of the rim which is attracted by the centre *o*, of an iron ball or shell. If now we conceive the spring to be detached from the fixed part of the machine, it will be free, with the balance itself, to take any position. The point D will therefore be attracted towards *o*; and if it be displaced from this position, it will have a tendency to oscillate on each side of the point D; and the number of vibrations which it would make in a given time would serve, if we could obtain such results, to estimate the intensity of action of the attracting body. Fig. 78.

But although we cannot detach the balance for such an experiment, we may still form some idea of the intensity of action, by causing a small magnetised needle to oscillate in the place of the balance, and by counting the number of its vibrations as above described. Indeed there is not much difficulty in estimating, theoretically, the change of intensity due to a certain change in the position and distance of the attracting body; but experiment was in this case preferred as more satisfactory to those who may not be able to follow out completely the mathematical investigation on which such a computation must depend. With this previous view of the subject, the author began with first ascertaining the time in which forty vibrations were made with a small magnetic needle in different situations with respect to an iron shell eighteen inches in diameter, and at eighteen inches distance from its centre; the weight of the shell being 496 lbs.

But as the degree of intensity, as well as the quantity of deviation, occasioned by the iron ball, has reference, not to the plane of the horizon, but to *the plane of no attraction,* he proceeded with these experiments as follows:

170. Let S Q N Q′ (fig. 79.) represent the iron shell, or a sphere concentric with it; Q Q′ its magnetic equator, or plane of no attraction, and *a b*, *c d*, *e f*, &c. parallels of latitude answering to 60°, 45°, 30°, &c. H H′ the horizon, and S N the natural direction of the magnetic action in this place; the circle S Q N Q′ denoting the plane of the magnetic meridian, agreeably to the division of the magnetic sphere, as described in (art. 90.) Fig. 79.

He now first placed the compass at Q, eighteen inches from the centre of the shell, and observed the number of seconds which the needle employed to make forty vibrations; then, still keeping the needle in the circle Q Q′, he placed it 30° from Q towards E, or in longitude 60°. He then brought it 30° nearer to E, or into longitude 30°; then to E, or longitude

0°; and so on at every 30° through the whole circle. The same was then repeated in the circles *a b*, *c d*, *e f*, &c.; and by taking the mean of the results for each corresponding situation on each side of the meridian, he obtained the numbers given in the following table:

171. *Table showing the time of making ten vibrations with a fine magnetic needle in different situations, eighteen inches from the centre of an iron shell, eighteen inches in diameter, weighing 496 lbs.*

Latitude.	Mean time of making ten vibrations.						
	Long. 90° N.	Long. 60° N.	Long. 30° N.	Long. 0°	Long. 30° S.	Long. 60° S.	Long. 90° S.
90° N.	30″·25						
60 N.	26·00	27″·25	28″·00	29″·75	34″·50	35″·75	38″·25
45 N.	26·25	27·50	28·25	30·00	35·25	39·25	43·50
30 N.	27·25	27·50	29·00	31·25	34·00	41·25	46·50
0	34·50	34·75	34·75	35·50	35·00	35·00	35·00
30 S.	46·50	41·25	34·00	31·25	29·00	27·50	27·25
45 S.	43·50	39·25	35·25	30·00	28·25	27·50	26·25
60 S.	38·25	35·75	34·50	29·75	28·00	27·25	26·00
90 S.							30·25
Mean time of making ten vibrations detached from the iron ball 32″·50ᵒ.							

These experiments were made with a small steel bar or magnetic needle, finely suspended with untwisted silk in a glass vessel, and some care was taken to get the time as accurately as seemed desirable for the purpose; but as the only intention of the experiments was to have some general ideas of those situations near the ball, where a compass needle would be the most affected in its vibrations, and where also, according to the ideas above advanced, the chronometer would be most affected in its rate, it was not considered necessary to carry these observations to the utmost degree of precision.

172. It does not appear requisite for our present purpose, to give the detail of these experiments at full length; it will be sufficient here to state the mean results, and to refer those readers who are desirous of more particular information to the volume of the Transactions above quoted.

In the following table, the first column describes the situation of the chronometer; the second the mean daily rate; the third the gain or loss per day in each position; the fourth the number of days the watch was in each situation; and the fifth the proportional magnetic intensity of the iron on the balance, estimated by the number of vibrations which a freely suspended horizontal needle made in the place where the chronometer stood; the intensity of the needle when wholly detached from iron being called 100.

TABLE.

173. *Of experiments on the rates of chronometers by the proximity of masses of iron.*

Situation, &c. of the chronometers.	Daily rate.	Gain or loss per day.	No. of days in each position.	Prop. mag. intensity.
Chronometer denoted by No. I.				
Rate of chronometer before it was applied to the ball - - - - - - -	−3″·2	—	14	100
Chronometer to the south of the ball 2·1 inches from the floor, and distant from vertical passing through the centre of the ball 17·31 inches, corresponding to lat. 0° long. 90°, distance from centre of ball 18 inches. 12 o'clock, south - - - - - - - - - - - - - - - - - -	−5·6	−2·4	6	91
Chronometer No. II.				
Rate of chronometer before it was applied to the ball - - - - - - -	+6	—	10	100
Ditto ditto -	+5	—	4	100
Chronometer placed above the ball, height from floor 23 inches, distance from vertical, through the centre six inches to the south, corresponding to lat. 90° S., and distance 18 inches. 12 o'clock, north - - - - - -	+6·5	+1·5	6	117
Detached from ball - - - - - - - - - - - - - - - - - - -	+5·8	—	3	100
Ditto ditto -	+5·2	—	6	100
Chronometer placed to the north of ball, height 10½ inches; distance from vertical 11·31 inches, corresponding to lat. 0° long. 90°, distance 12 inches. 12 o'clock, south - - - - - - - - - - - - - - - - -	+6·1	+0·9	3	94
Detached from ball -	+4·7	—	3	100
Placed above the ball; height from floor 17·3 inches, distance from vertical four inches to the south of the ball; or lat. 90° S.; distance 12 inches. 12 o'clock, south - - - - - - - - - - - - - - - - - -	+5·0	−0·2	5	162
Same latitude; distance 18 inches. 12 o'clock, south - - - - - - -	+4·0	−1·2	2	117
Detached from the ball - - - - - - - - - - - - - - - - - - -	+4·3	—	7	100

the top of the box a convenience for placing the chronometer. Then, having taken its rate in the usual way, let it be taken again while the chronometer is placed on the pedestal, keeping the plate, generally, at the distance of about twelve inches from the vertical through the centre of the dial, and its centre about the same depth below the plane of the balance, and the rate thus obtained, will be a very close approximation to the ship rate of the instrument, provided care be taken, when it is removed on board, to keep it out of the immediate action of any partial mass of iron. The plate for this purpose should be a double one, such as have been described in the preceding section.

It should be observed, that the plate is meant as a substitute for the iron forward; and therefore the chronometer when on board, should be placed in the same direction in reference to the ship's head, as it had with respect to the iron plate when its rate was determined; that is, if the 12 o'clock mark of the dial be turned towards the iron plate on shore, then must the same be turned towards the ship's head when taken on board.

177. The following table of the land and sea rates of the four chronometers, whose numbers are specified in it, is taken from the *Edinburgh Philosophical Journal*, for October 1, 1821. It is contained in a communication to the Editor, from Lieut. William Mudge, and is copied from the journal of H. M. S. Leven, the rates having been ascertained conjointly by the above gentleman, and by Lieut. (now Captain) Vidal.

Table of the land and sea rates of the chronometer supplied to H. M. S, Leven, during her voyage to the Cape de Verd Island, in the year 1819.

Particulars of time and place.	No. 1970, Arnold rate.	No. 498, Arnold rate.	No. 240, Harris and Hutton rate.	No. 505, Arnold rate.
	″	″	″	″
S. R. Lisbon, Jan. 2, to Jan. 28, 1819 - -	—	l. 3·70	g. 2·86	g. 7·74
S. R. St. Jago, Feb. 8, to Feb. 14 - - -	l. 17·30	l. 1·98	g. 5·53	g. 5·68
S. R. Isle of Sal. Feb. 28, to March 28 -	l. 16·27	l. 1·25	g. 6·83	g. 7·29
S. R. Ditto, March 28, to April 20 - - -	l. 16·99	l. 0·99	g. 6·82	g. 9·80
S. R. Quarl. Island, April 27, to May 4 -	l. 17·90	g. 0·26	g. 6·34	g. 10·39
S. R. Ditto, May 4, to May 12 - - - -	l. 17·66	g. 0·66	g. 6·55	g. 9·68
Mean S. R. from the above - - - - -	l. 16·95	l. 1·47	g. 6·52	g. 8·47
L. R. Madeira, June 20, to July 7 - - -	l. 14·88	g. 1·27	g. 2·00	g. 13·68
L. R. Ditto, July 7, to July 17 - - -	l. 13·90	g. 3·85	g. 1·85	g. 14·85
L. R. Ditto, July 17, to July 28 - - -	l. 13·72	g. 2·83	g. 3·64	g. 13·82
L. R. Ditto, July 28, to Aug. 6 - - -	l. 14·40	g. 2·73	g. 2·84	g. 13·51
L. R. Ditto, Aug. 6, to Aug. 24 - - -	l. 13·85	g. 2·60	g. 2·87	g. 13·15
L. R. Ditto, Aug. 24, to Sep. 1 - - -	l. 14·23	g. 2·76	g. 2·26	g. 12·90
L. R. Ditto, Sep. 1, to Sep. 13 - - -	l. 14·10	g. 3·20	g 3·50	g. 14·60
L. R. Ditto, Sep. 13, to Sep. 19 - - -	l. 14·10	g. 3·30	g. 3·50	g. 14·60
Mean L. R. from the above - - - - -	l. 14·17	g. 2·69	g. 2·75	g. 13·90
Difference between mean land and sea rates	2·78	3·22	3·77	5·33

The land rates were taken with the astronomical quadrant at the house of the British Consul.

Captain Scoresby's method of correction.

178. Captain Scoresby, whose name is so well known in the scientific world, has considered this subject under a different point of view. He argues, that if the magnetic action of the iron of a ship be sufficient to affect the balance of a chronometer, much more must the terrestrial magnetism affect it, according to the direction of the ship's head; this force far exceeding that proceeding from the iron. He proposes therefore to place the chronometer when on board, suspended in gimbals, and resting on a base to which is attached, at a distance below a needle sufficient to bring the chronometer always to the same bearing, and therefore always in the same direction with respect to the meridian, (fig. 80.) There seems, however, some danger in this case, of the magnet itself affecting the rate, not merely by its own action, but by communicating some portion of fixed magnetism to the balance, more than it originally possessed. See the last volume of the *Transactions of the Royal Society of Edinburgh*.

PART IV.

ON TERRESTRIAL MAGNETISM.

§ XXI. *Laws of terrestrial magnetism, expressing the relation between the dip and intensity of the needle in different parts of the earth.*

179. HITHERTO our investigations have been confined to the examination of the laws of magnetism exercised by temporary and permanent magnets; and in order to reduce these to computation, we have been under the necessity of mixing with the effect produced by the iron or magnet, the directive power of the earth itself; and afterwards, by means very commonly had recourse to in analytical investigations equating

The same otherwise.

183. We may arrive at this conclusion in a more simple way, by referring to our two forces (3) and (4), only rejecting M, as above, out of the latter, which thus become

$$\begin{array}{ll} A\,3\cos\phi & \text{in the direction } PO, \\ \left\{\begin{array}{l} -A \\ \text{or } A \end{array}\right. & \begin{array}{l} \text{in the direction } SN, \\ \text{in the direction } NS, \end{array} \end{array}$$

taking A to denote the coefficient $\frac{C r^3}{d^3}$.

Draw P H (fig. 61) perpendicular to P O, which will thus represent the horizon, or a line parallel to it; consequently S′ P H will be the latitude = λ, and H P *n* = the dip = δ. Resolve the latter force S N into two; one in the direction P H = A cos λ, and the other into O P = A sin λ; by this means our rectangular forces will be,

$$\begin{array}{l} A\cos\lambda \text{ in the direction } PH, \\ A\,(3\cos\phi - \sin\lambda) \text{ in the direction } PO. \end{array}$$

But cos φ = sin λ, they therefore reduce to

$$A\cos\lambda \text{ in the direction } PH \qquad (18)$$

$$A\,.\,2\sin\lambda \text{ in the direction } PO \qquad (19)$$

and, consequently, the angle of their resultant, or the angle it forms with the horizon will be

$$\tan\delta = \frac{A\,2\sin\lambda}{A\cos\lambda} = 2\tan\lambda \qquad (20)$$

the same result as above. This law of terrestrial magnetism was first deduced in a complex form by M. Biot, and afterwards in the above simple shape by Mr. Kraft, of St. Petersburg. See our historical part, (art. 16.)

Of the magnetic intensity in different latitudes.

184. With respect to the magnetic intensity in any place, this may be deduced from our formula (17), by making M = *o*, but it is more readily drawn from the above forces (18) and (19).

For here we have immediately, for the resultant of those forces, or for the intensity in the natural direction, (which we may denote by I.)

$$I = A\sqrt{(4\sin^2\lambda + \cos^2\lambda)} = A\sqrt{(3\sin^2\lambda + 1)} \quad (21)$$

while that for the horizontal needle I′, is expressed in the first force itself, viz.

$$I' = A\cos\lambda \qquad (22)$$

Law of intensity in the dipping needle.

The intensity therefore of a dipping needle, in any part of the earth, varies as the square root of three times the square of the sine of the magnetic latitude *plus* unity; and that of the horizontal needle varies as the cosine of the magnetic latitude.

In terms of the dip.

In order to express these values in terms of the dip, we have only to employ the relation already established between the latitude and dip, namely,

$$\tan\delta = 2\tan\lambda$$

or,
$$\frac{\sin^2\delta}{\cos^2\delta} = \frac{\sin^2\delta}{1-\sin^2\delta} = \frac{4\sin^2\lambda}{1-\sin^2\lambda}$$

whence,
$$\sin^2\lambda = \frac{\sin^2\delta}{4-3\sin^2\delta}.$$

In the same manner

$$\cos^2\lambda = \frac{4}{3+\sec^2\delta}$$

and substituting these values in the preceding expressions for I and I′, we have

$$I = 2A\sqrt{\frac{1}{4-3\sin^2\delta}} \qquad (23)$$

$$I = 2A\sqrt{\frac{1}{3+\sec^2\delta}} \qquad (24)$$

Second restrial magne law.

that is, the natural magnetic intensity varies inversely as the square root of four *minus* three times the square of the sine of the dip; and that of the horizontal needle, inversely as the square root of three increased by the square of the secant of the dip, which are both known laws of terrestrial magnetism, depending upon the investigation of M. Biot, above referred to.

Captai Sabine' experiments a compari of thos laws.

185. These formulas have been compared by Captain Sabine, with the results of observation made during the interesting voyage of Captain Parry through Barrow's Straits; that is, having taken the times of a dipping needle, making one hundred vibrations in London, Sheerness, Baffin's Bay, and Melville Island; and the times of making ten vibrations on two horizontal needles at the same stations, he was enabled to compute the value of the intensity in these cases, and afterwards to compare them with the values of the same as drawn from the preceding theory; and the agreement is certainly much closer than from the nature of the subject there was any good ground to expect.

Before we repeat this comparison, it may not be amiss to furnish our readers with Captain Sabine's actual observations.

186. *Observations on the intensity of the magnetic force, with Mr. Browne's dipping needle,* 1819, *March. In the Regent's Park, London, latitude* 51° 31′ 40″ N., *longitude* 8′ W. *Dip* 70° 30′·3 N.

In the magnetic meridian.					
No. of vibrations.	Interval.		No. of vibrations.	Arcs.	Interval.
	m s			°	s
100	8 01·5			71	
100	8 08		10	55	50
100	7 56·5		10	40	49
100	8 02	Mean	10	30	48
			10	23	49
			10	17	48
			10	13	47
			10	9	48
			10	6	48·5
			10	3	47
			10	1	47
Perpendicular to the meridian.					
No of vibrations.	Interval.		No. of vibrations.	Arcs.	Interval.
	m s			°	s
				90	
			10	55	52
100	8 18·5		10	40	52
			10	30	49·5
			10	22	49·5
			10	16	49
			10	11	48·5
			10	7	49·5
			10	4	48·5
			10	2	50·5
			10	1	49·5

1820, *July.* *In the observatory, Winter Harbour, latitude* 74° 47′ 19″ N., *longitude* 110° 48′ 30″ W. *Dip* 88° 43′·5 N.

In the magnetic meridian.

No. of vibrations.	Interval.		No. of vibrations.	Arcs.	Interval.
	m s			° 88	s
100	7 27·2		10	67·5	48·8
100	7 26·5		10	53·5	46·5
100	7 25·5		10	41·5	45
100	7 25·8		10	33	44·5
100	7 26·25	Mean	10	26.5	44
			10	21	43·3
			10	15·5	43·5
			10	12·5	43·2
			10	9·5	43·8
			10	6·5	43·2

Perpendicular to the meridian.

No. of vibrations.	Interval.		No. of vibrations.	Arcs.	Interval.
	m s			° 90	s
100	7 27		10	78	49·7
100	7 25·8		10	54	46·8
100	7 26·4	Mean	10	43	44·5
			10	34	44·5
			10	27	44
			10	22	44
			10	17	43·8
			10	13	43
			10	10	43·2
			10	8	43·5

1820, *December.* *In the Regent's Park, London, latitude* 51° 31′ N., *longitude* 0° 08′ W. *Dip* 70° 33′·3 N.

In the magnetic meridian.

No. of vibrations.	Interval.		No. of vibrations.	Arcs.	Interval.
	m s			° 70	s
100	8 02		10	46	51
100	7 58		10	36	50
100	8 00		10	28	50
100	8 00	Mean	10	21	48·5
			10	16	48
			10	12	47·5
			10	9	46·5
			10	7	46·5
			10	5	46
			10	3	46

With needles suspended horizontally.

187. The needles used in these experiments were bars of steel of seven inches in length, ·25 in breadth, and ·15 in thickness, magnetised to saturation previous to the voyage, and their magnetism not interfered with subsequently.

Three such bars had been provided, numbered from I to III, whereof Nos. II and III were kept together when not in use, with their opposite poles connected; No. I was kept by itself.

Each of the needles successively was placed for use in an instrument made for the occasion; consisting of a circular plate of ten inches diameter, centred as usual, and adjusted horizontally by foot screws, as shewn by spirit levels fixed at right angles to each other; a graduated ring, eight inches in diameter, was attached to the plate, and rose half an inch above it; on the outside of the ring were three pillars nine inches in height, bearing a tripod, from the centre of which, a stirrup was suspended by a silk line. The needle being slid into the stirrup until it balanced horizontally, and being left to itself, and the instrument protected from the air by a bell-glass, it indicated, on settling, the magnetic meridian. The brass plate was then moved in azimuth until the zeros of the ring coincided with it, when the needle being removed, and a brass bar substituted in the stirrup, all twist was taken out of the silk, and the point of suspension adjusted (it having a motion for that purpose) until the brass bar rested in the same direction as the needle had done; the needle being then replaced, was drawn by a magnet 60° from its natural direction, released, and suffered to vibrate until the arcs became too small to be discerned; the time and arc being noted at each tenth vibration.

The whole instrument was of brass; the silk line consisted of as few fibres of raw silk as were sufficient to support the weight of the needle, the length of the suspension being about nine inches.

1819, *May.* *On the Sea Beach, at Sheerness, latitude* 51° 26′ N., *longitude* 0° 44′ W. *Dip of the needle* 69° 55′ N.

No. of vibrations.	*Needle No. I.*		*Needle No. II.*		*Needle No. III.*			
	Arcs.	Interval.	Arcs.	Interval.	Arcs.	Interval.	Arcs.	Interval.
	° 90	m s	° 60	m s	° 60	m s	° 60	m s
10	48·5	1 39	47	1 34·5	42	1 29·5	44	1 30
10	39	1 37	37·5	1 32·5	33·5	1 27·5	35	1 28
10	32·5	1 37	30·5	1 31·5	26·5	1 26	28·5	1 27
10	27	1 35 5	25	1 30 5	21	1 26	23	1 26
10	23·5	1 35	22	1 31	17	1 25	19	1 26
10	20·5	1 34·5	17·5	1 30	14	1 24·5	16	1 25·2
10	16	1 34 5	12·5	1 30	10	1 24·5	12	1 25·3
10	14·5	1 34·5	12	1 30	7·5	1 25	10·5	1 25

1819, *June 26th. On Ice in Davis' Strait, latitude* 64° N., *longitude* 61° 50′ W. *distant* 222 *yards from the ships. Dip* 83° 4′ N., *by an indifferent observation.*

No. of vibrations.	*Needle No. II.*		*Needle No. III.*			
	Arcs.	Interval.	Arcs.	Interval.	Arcs.	Interval.
	°	m s	°	m s	°	m s
10	60	2 40	60	2 32	60	2 33
10	52	2 36	42	2 27	45	2 29
10	44	2 35	34	2 27	37·5	2 26
10	36	2 33	26	2 24	32	2 24
10	32	2 32	24	2 24	23	2 24
10	28	2 32	19	2 24	20	2 24
10	25	2 32	14·5	2 24	16	2 24
10	23	2 31·5	10·5	2 24	14	2 23·5
	21		9		10·5	

1819, *July 23rd and 24th. On Ice in Baffin's Bay, latitude* 73° N., *longitude* 61° 30′ N., *at a considerable distance from the ships; the dip was not observed on the spot, but from other observations, it must have been about* 84° 30′.

No. of vibrations.	*Needle No. II.*		*Needle No. III.*			
	Arcs.	Interval.	Arcs.	Interval.	Arcs.	Interval.
	°	m s	°	m s	°	m s
10	60	2 54	60	2 40	60	2 40
10	43	2 48	31	2 34·5	38	2 34
10	38	2 47·5	25	2 32	27	2 32·5
10	25	2 45·5	16	2 32	16	2 31·5
10	22	2 44	12	2 33	8	2 31·5
10	20	2 43·5	6	2 33	7	2 32
10	18	2 43·5	4		6	
10	16	2 43·5				
10	10	2 43				
10	9	2 43				
	8					

1819, 1820. *At the observatory, Winter Harbour, Melville Island, latitude* 74° 47′ N., *longitude* 110° 48′ W. *Dip* 88° 43′·5.

Number of vibrations.	*Needle No. I.*											
	July 10, 5 A. M.		July 10, 6 A. M.		July 10, 7 A. M.		July 10, 3 P. M.		July 12, 3 A. M.		July 12, 4 A. M.	
	Arcs.	Interval.	Arcs.	Interval.	Arcs.	Interval.	Arcs.	Interval.	Arcs.	Interval.	Arcs.	Interval.
	°	m s	°	m s	°	m s	°	m s	°	m s	°	m s
10	60	5 50·5	60	5 56	60	5 58	60	5 58	60	5 59·5	60	5 58
10	34	5 43·5	42	5 50	44	5 51	44	5 50	41	5 50	42	5 50
10	22	5 40·5	32	5 43·5	31	5 44	33	5 45	28	5 47	29	5 45
10	14	5 40·5	21	5 40·5	22	5 42	25	5 42·5	20	5 42·5	21	5 40
10	11	5 39·5	16	5 39·5	16	5 41·5	21	5 41	14	5 40·5	16	5 40
10	7		13	5 40	11	5 40	15	5 39·5	8		11	5 39
			9		8		11				8	

Number of vibrations.	*Needle No. II.*																	
	July 7, 6 P. M.		July 7, 7 P. M.		July 7, 10½ P. M.		July 7, 11½ P. M.		July 7, 12½ P. M.		July 9, 7 P. M.		July 9, 9 P. M.		July 9, 10 P. M.		July 10, 4 A. M.	
	Arcs.	Interval	Arcs.	Interval	Arcs	Interval	Arcs.	Interval	Arcs.	Interval	Arcs.	Interval	Arcs.	Interval	Arcs.	Interval	Arcs.	Interval
	°	m s	°	m s	°	m s	°	m s	°	m s	°	m s	°	m s	°	m s	°	m s
10	60	5 46	60	5 44	60	5 49	60	5 46	60	5 45	60	5 45·5	60	5 43·5	60	5 47·5	60	5 44·5
10	33	5 31·5	39	5 33·2	39	5 30	41	5 41	44	5 38	43	5 39	41	5 37·5	42	5 38·5	43	5 37·5
10	26	5 28	30	5 26·4	28	5 26	31	5 32	28	5 26·5	31	5 34	31	5 34·5	29	5 34·5	33	5 33·5
10	20	5 26	15	5 26·4	18	5 26	20	5 30	13	5 25	23	5 31·5	23	5 33	22	5 32	25	5 31·5
10	11	—	11	—	11	—	10	—	9	—	17	5 29·5	17	5 31	17	5 29	18	5 29·5
10	—	—	—	—	—	—	—	—	—	—	13	5 28	13	5 28	13	5 28	13	5 29·5
10	—	—	—	—	—	—	—	—	—	—	8	—	8	—	8	—	10	5 29
	—		—		—		—		—		—		—		—		7	

Observations at Winter Harbour continued.

Needle No. III.

Number of vibrations.	Nov. 4, Ther. zero.		Nov. 5, Ther. —8.		April 26, 3 to 5 P. M. Ther. + 5°.				July 10, 9 A. M.		July 10, 10 A. M.		July 10, Noon.		July 11, 10 A. M.		July 11, Noon.	
	Arcs.	Interval	Arcs.	Interval	Arcs.	Interval	Arcs.	Interval	Arcs.	Interval	Arcs.	Interval	Arcs.	Interval	Arcs.	Interval	Arcs.	Interval
	°	m s	°	m s	°	m s	°	m s	°	m s	°	m s	°	m s	°	m s	°	m s
10	60	5 34·5	60	5 36·5	60	5 35·75	60	5 37	60	5 37	60	5 32	60	5 30	60	5 36·5	60	5 39
10	43	5 26	39	5 29	38	5 31·75	41	5 29	43	5 30·5	43	5 25	43	5 19	44	5 28	44	5 27·5
10	32	5 20	27	5 24·5	25	5 24·5	29	5 25·5	31	5 24	32	5 21·5	30	5 17·5	33	5 25	34	5 22·5
10	28	5 17	22	5 20	15·5	5 18·5	22	5 21·5	23	5 22	22	5 15	22	5 16·5	25	5 22	24	5 22
10	18	—	17·5	5 16·5	12	5 16·5	18	5 15	16	5 21	15	5 13·5	15	5 16·5	18	5 20	18	5 30
10	—	—	10	—	9	—	12	—	11	5 20	11	5 13	10	5 13·5	12	5 17	13	5 18
	—		—		—		—		8		7		8		7		9	

See Appendix to Parry's *Voyage*.

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188. Collecting from the foregoing tables the several mean results, they will stand thus:

Dipping needle.

London dip 70° 30′·3 mean time of making 100 vibrations - 482″ } general mean, 481″
On the return of the expedition - - - - - - - - 480 }
Winter Harbour dip 83° 43′ - - - - 446·25

Horizontal needle.

Needles.		
No. 2. Sheerness dip 69° 55′	time of 10 vib.	90″
No. 3. Ditto	ditto	85
No. 2. Ice, Davis' Strait dip 83° 4′	ditto	151·5
No. 3. Ditto	ditto	143 75
No. 2. Baffin's Bay dip 84° 3′	ditto	163
No. 3. Ditto	ditto	152·5
No. 2. Winter Harbour dip 88° 43′·5	ditto	329
No. 3. Ditto	ditto	318

Now first for the dipping needle, according to our formula, we should find

$$\{4-3\sin^2(70^\circ 30'\cdot3)\}^{\frac{1}{2}} : \{4-3\sin^2(88^\circ 43')\}^{\frac{1}{2}} :: 481 : 446\cdot25$$

and we have in actual numbers for the last term $\overline{447\cdot84}^2$, which is a near approximation, and thus far therefore confirms the truth of our theory.

Again for the horizontal needle, comparing the results at the greatest of the above dips with the least, we ought to have, taking the means of the two needles,

$$\{3+\sec^2(69^\circ\ 55')\}^{\frac{1}{2}} : \{3+\sec^2(88^\circ\ 43')\}^{\frac{1}{2}} :: \overline{87\cdot5}^2 : \overline{323\cdot5}^2$$

and we have in actual numbers for our last term $\overline{317\cdot76}^2$, which is also a very close approximation.

In like manner we should have

$$\{3+\sec^2(69^\circ\ 55')\}^{\frac{1}{2}} : \{3+\sec^2(84^\circ\ 30')\}^{\frac{1}{2}} :: \overline{87\cdot5}^2 : \overline{157\cdot7}^2$$

and the last term is $\overline{154\cdot59}^2$.

Also

$$\{3+\sec^2(69^\circ\ 55')\}^{\frac{1}{2}} : \{3+\sec^2(83^\circ\ 4')\} :: \overline{87\cdot5}^2 : \overline{147\cdot6}^2$$

but the computation gives the last term $\overline{138\cdot29}^2$. This is a greater aberration than in the preceding cases; but still a very slight error in taking the dip would be amply sufficient to account for it.

Upon the whole therefore, we conceive we have a right to conclude, that the laws we have deduced are those actually existing in the case of terrestrial magnetism; that is, the forces which give direction to a magnetised needle at the earth's surface, must be referred to two centres, indefinitely near to each other in the centre of the terrestrial sphere; precisely as we have found it in the case of simple iron balls and shells, and not to the extremity of any assumed axis, as would be the case in bodies permanently magnetic. It is indeed surprising, that philosophers should have so long looked for the situation of the terrestrial magnetic poles so near the surface; for a little consideration would have shown them that, in this case, the increase of intensity in approaching the poles must have been incomparably greater than there was any reason to expect to find it, as far as could be judged from experiments and observations made in those regions.

Method of determining the dip at sea, by observations on the intensity of the horizontal needle.

189. The relation between the dip and intensity, or time of vibration, suggested to Captain Sabine the idea of obtaining the former by means of the latter. For example, let D denote the dip in any place where it is known, and let n be the number of vibrations made by a finely suspended horizontal needle in the same place, in any given time. Let also N be the number of vibrations which the same needle makes in the same time, in any other place where the dip (d) is required; then

To find the dip on shipboard.

$$(3+\sec^2 D)^{\frac{1}{2}} : (3+\sec^2 d)^{\frac{1}{2}} :: N^2 : n^2$$

or, $$(3+\sec^2 d)^{\frac{1}{2}} = \frac{(3+\sec^2 D)^{\frac{1}{2}}\, n^2}{N^2}$$

or, $$3+\sec^2 d = \frac{n^4}{N^4}(3+\sec^2 D)$$

whence $$\sec d = \left\{\frac{n^4}{N^4}(3+\sec^2 D) - 3\right\}^{\frac{1}{2}} \qquad (25)$$

Where n, N, and D, being given, d also becomes known.

It is to be observed that n and D, when once determined for any given place; as for example, London, will be constant; let therefore $n^4(3+\sec^2 D) = A$, then the above becomes simply

$$\sec d = \sqrt{\frac{A}{N^4} - 3}$$

or, if instead of counting the number of vibrations made in a given time, we take the time that the needle is in performing a given number of vibrations, (which is the best where we have a good stop watch, or other instrument for marking shorter intervals of time ;) then denoting the times by T and t, the above formula is transformed into

$$\sec d = \left\{ \frac{T^4}{t^4} (3 + \sec^2 D) - 3 \right\}^{\frac{1}{2}} \qquad (26)$$

which, like formula (25) is reducible to the simple form

$$\sec d = \sqrt{T^4 A' - 3}$$

after the dip D, and time t, have been ascertained in the first place of observation.

Examples.

190. We may take, as examples, the numbers given in (art. 188,) where it appears that, with the dip 69° 55′, the mean time of making ten oscillations, was 87″·5 ; whence

$$A = \frac{3 + \sec^2 69^\circ\ 55'}{\overline{87{\cdot}5}^4} = {\cdot}00000019584$$

Now at Winter Harbour, Melville Island, the mean time of making ten vibrations was 323″·5. Whence we have for the dip at the latter place,

$$\sec d = \sqrt{(\overline{323{\cdot}5}^4 \times {\cdot}00000019584 - 3)} = 86^\circ\ 46'$$

In Baffin's Bay,

$$\sec d = \sqrt{(\overline{157{\cdot}7}^4 \times {\cdot}00000019584 - 3)} = 84^\circ\ 43'$$

In Davis' Strait,

$$\sec d = \sqrt{(\overline{147{\cdot}6}^4 \times {\cdot}00000019584 - 3)} = 83^\circ\ 57'$$

Now the actual observed dips in those places were 88° 43′, 84° 30′, and 83° 4′; consequently, the approximation by computations, at least with the exception of the last, is as close as we have any reason to expect in calculations of this kind.

Apparent objection answered.

191. One apparent objection to this mode of determining the dip is, that the iron of the vessel may so far affect the intensity of the needle as to leave the results doubtful as to the actual amount of the quantity sought; but this objection is merely apparent, and not real, at least the effect thus produced on the intensity is very inconsiderable. Mr. George Harvey, of Plymouth, an ingenious and able mathematician, undertook, in consequence of certain ideas he entertained relative to the nature of local attraction, a very extensive series of observations on the intensity of needles in various parts of different ships of war in the above Port ; and from these it appears, that in no instance while the observations were made far aft, viz. in the Captain's cabin, and even on the quarter deck, was the intensity increased or diminished by more than $\frac{1}{30}$th part ; consequently, although an error to this amount may enter, yet we may still expect to obtain the dip on shipboard, by this method, much nearer than by any other we are acquainted with.

Method submitted to experiment.

192. A trial, it seems, of this principle of determining the needle's inclination, is at this time making on board H. M. S. Leven. The needle is suspended by a fine fibre of silk from an arc, rising above a common compass bowl fitted up for the purpose ; but it is still kept to its centre, by means of a fine point and agate cap on which it is slightly rested ; the silk suspension having an adjustment by which this can be effected. Mr. Barlow, who had embodied the formulas for this experiment, made the first trial of the instrument at Northfleet, at a time when, in consequence of the wind being very high, there was considerable motion in the vessel, and from which alone the needle was in a continual state of oscillation ; in short, arcs of about 5° on each side the north.

By means of a watch which registered time to 12ths of seconds, the time of the needle making 100 vibrations, was counted and found, by a mean of several trials, to be 6^m 22^s·4, therefore in this case

$$A' = \frac{\sec^2 D + 3}{(382''{\cdot}4)^4} = {\cdot}00000000056$$

and, consequently, in any other case the dip will be found by the formula,

$$\sec d = \sqrt{({\cdot}00000000056\ T^4 - 3)}$$

T being the number of seconds in which the needle will perform 100 vibrations.

It may be necessary to observe, that in order to preserve the needle when not in use, in a constant state of saturation, it should be kept connecting the extremities of a magnet in the form shown in (fig. 73.)

§ XXII. *On the situation and direction of the terrestrial magnetic axis.*

Situatio of terre trial ma netic ax

193. It appears from the preceding investigation and comparison, that the two centres which give direction to magnetised needles at the earth's surface, are situated in two points indefinitely near to each other in the centre of the terrestrial sphere. If we conceive the indefinitely short lines which unite these centres to be produced both ways to the surface, the diameter thus formed is called the ***terrestrial magnetic axis***. The circle cutting this axis at right angles is the *terrestrial magnetic equator* ; and the extremities of this axis, or diameter, the poles of this equator, or the *terrestrial magnetic poles*, where, it will be observed, the term poles is used simply according to the usual principles of geometry, and not as centres of action, as is commonly understood by this expression in magnetical inquiries. The magnetic sphere therefore is founded on these bases ; the circle last defined is its equator, and the extremities of the terrestrial magnetic axis, its poles. The *magnetic latitude* of any place is its distance from this equator measured on the meridian passing through it ; and the magnetic longitude may be estimated from any fixed meridian at pleasure.

Definitio of term

If the magnetic poles coincided with the poles of the earth, the magnetic equator would in like manner fall upon the terrestrial equator ; and the magnetic and geographic meridians would also coincide, and the needle, in all places, would point duly north and south ; moreover, a needle upon the equator would in that case be equally distant, and under equal influence from both poles, or rather from both centres, and would therefore have no dip or inclination.

We know, however, that the needle does not everywhere point duly north and south, and that the circle on the globe, in which the needle has no inclination, is not coincident with the terrestrial equator. The magnetic and terrestrial poles, therefore, are not coincident, and our object is to determine the situation of the former in comparison with the latter.

Situatio of the magneti equator.

194. If we knew distinctly the direction of the circle on the terrestrial surface in which the dip was zero, this alone would be sufficient to determine the place of the poles ; but, unfortunately, the imperfection of magnetical instruments, and the disturbance which iron and other ferruginous matter produce on the needle, are such as not to admit of our getting

Computed situation of the magnetic north pole of the earth continued.

Place of observation.	Date.	Terrestrial latitude and longitude.		Dip.	Variation.	Computed place of magnetic pole.		Name of observer.
		Latitude.	Longitude.			North latitude.	West longitude.	
Different stations in North America ...	1821 and 1822	56° 41′ N.	109° 51′	85° 7′	25° 2′ E.	65° 11′	160° 5′	Captain Franklin.
		58 42	111 18	85 23	22 49	64 47	102 14	
		62 17	114 9	86 38	33 36	67 35	104 25	
		64 15	113 2	87 20	36 54	68 17	104 24	
		67 1	116 27	87 31	44 11	70 17	106 21	
		67 47	115 36	88 5	46 25	69 51	107 31	
		67 19	109 44	88 58	41 43	68 58	105 54	
		68 18	109 25	89 31	44 15	68 50	107 33	

Every place appears to have its particular axis.

198. On these results it may be observed that, although in determinations relative to the dip and variation of the needle, we cannot expect the utmost accuracy, yet it is very obvious from the preceding table, that the aberrations in the latitude and longitude of the magnetic pole are much greater than can be attributed to errors of observation. It will be seen, that the place assigned to it differs in longitude as much as 57°, between one set of observations and another, and as much as 14 in latitude. It will also be observed, that the more we approach the north and west, the more westerly we find the place of the pole; and the more easterly the place of observation, the greater is its latitude. In short, it is evident from the few examples we have taken, that every place has its particular polarizing axis, which, probably in all cases, fall within the arctic circle; but that this is the narrowest limits we are able to assign; that is, the local attraction or particular influence which the compass in every place is under, besides that of the general directive power of the globe, is such, as to displace the needle so much from its natural direction, as to give a different pole to almost every different set of observations; so that instead of the magnetism of the earth possessing that degree of uniformity which appertains to a perfectly formed iron ball, it may rather be said to resemble that species of action which we might expect to find in an irregularly formed mass of iron, approximating in its general character to that of a globe, but not perfectly such; and if the magnetism of the earth be imagined to be under the influence, or affected by the iron distributed in its interior, we ought, in fact, *a priori*, rather to expect this sort of action, than that which belongs to a perfectly formed iron sphere.

199. It is true that the observations we have used were not made simultaneously, and that a change is perpetually going on in the direction of the axis of polarization, which circumstance alone would give rise to some discrepancies; but not, as will be seen in the following section, to the amount shown in the preceding table. Every place, therefore, appears to have its proper poles; and the only limit we are enabled to assign to their situation is, that as far as observations have yet been carried, they appear to fall somewhere within the two frigid zones, but varying through all possible degrees of longitude and latitude within these limits. These aberrations being, however, attributed to local inequalities in the distribution of the ferruginous parts of the terrestrial sphere, we ought still to expect a certain degree of uniformity in the annual changes which take place in the situations of the poles of any particular place, supposing these changes to arise from some general cause acting equally on all. Let us then examine the circumstances attending the annual variation of the needle, and ascertain how far this phenomenon is reducible to determinate laws.

§ XXIII. *On the annual change in the variation of the magnetic needle.*

Annu chang the va tion.

200. In the preceding section the needle has been considered as permanently pointing to the same part of the globe, viz. we have supposed the observations employed as made simultaneously, or at least within a short period of time; we have now to examine the circumstances respecting the change in the variation of the needle, as it takes place from year to year, in any given place. It has been seen that the needle does not in every place point due north and south; that is, there is a certain angle contained between the natural and the magnetic meridian, which angle is the measure of what is termed the variation of the needle. Where the north end of the needle points towards the west of the true meridian, the variation is called *west*; and where to the eastward, it is called *east*, viz. the variation is always indicated with reference to the north end of the needle.

For many years after the directive power of the needle was known, it was supposed to point due north and south; at length the variation was discovered; and then, although it was ascertained that it did not point due north, yet it was not suspected that this variation was subject to any change. About the year 1622, however, it was discovered that the variation was less east than it had been; and continued observations proved that the needle was constantly approaching towards the true meridian, which it passed in London about the year 1658 or 1660; from which time, till within the last four or five years, its westerly motion uniformly continued; but at the present time it is unquestionably again on the return to the true meridian, its greatest deviation having been about 24° 30′ west, in London, in 1818.

Annual change computed.

201. It is not our intention to offer in this place any conjecture respecting the cause of this annual change; it is sufficient that we know the general fact, viz. that the polarizing axis of any particular place is perpetually changing its direction. Now this direction being known at certain times, and the rate of change supposed uniform, we shall be able to compute what it ought to be at others, and then by comparing these results with known observations,

be enabled to judge of the accuracy of our first assumptions.

It has been seen (art. 197) that the longitude of the polarizing axis which governs the needle in London, was 67° 41′ W. in 1818, and that its latitude was 75° 2′ N. Now in 1660 the variation in London was nothing; consequently we have a right to assume that in the latter year the longitude of the pole was zero: and if it be farther supposed that the motion during this time has been uniform, and made at the same distance from the terrestrial pole, we shall find it amount to about 4° 14′ in ten years; and hence it will be easy to compute the situation of the pole, and what ought to have been the dip and variation of the needle in London, from the year 1660 to the present time, agreeably to these suppositions.

g. 82.

For example, let N S E Q (fig. 82) represent the terrestrial sphere, N its north pole, π the magnetic pole which governs the direction of the needle in London; S L N the meridian of the latter place, and $p\,p$ the parallel of latitude in which the pole π revolves. Then in the triangle π N L, the arc N L, or colatitude of London, is known, and π N, the colatitude of the magnetic pole is supposed to be given, as also the angle π N L, the latter angle increasing at the rate of 4° 14′ for every ten years, beginning from the year 1658 or 1660, when the pole π corresponded with the meridian S L N.

So that in the spherical triangle π N L, we have always the two sides π N, L N, and the included angle π N L given, to find the angle N L π, or the variation. And as, in this case, the two sides π N, N L are constant, and only the angle π N L variable, we may give for this computation the following rule:

"To the cotangent of half the angle π N L add the constant log. 1·65642; find the angle of which the sum is the tangent, and call it arc (A). To the same cotangent add the log. 0·33987, and find the arc of which the sum is the tangent, and call it arc (B)."

Then B − A will be the variation, or angle π L N.

To find the dip we must compute the arc π L, and then $2 \cot \pi L = \tan$ of the dip.

By this rule the following variations have been computed.

202. *Table of computed and observed variation in London, from the year* 1660 *to* 1818.

By computation.					By observation.					
Year.	Variation.		Dip.		Variation.		Dip.		Year.	Authority.
	°	′	°	′	°	′	°	′		
1658 or 1660	0	0	—	—	0	0	—	—	1658 or 1660	Bond.
1670	2	44	—	—	2	30	—	—	1672	Halley.
1680	5	25	—	—	—	—	73	30	1676	Bond.
1690	7	59	—	—	6	0	—	—	1692	Halley.
1700	10	26	—	—	—	—	—	—	—	
1710	12	43	—	—	—	—	—	—	—	
1720	14	47	76	27	14	17	74	42	1723	Graham.
1730	16	41	—	—	—	—	—	—	—	
1740	18	20	—	—	17	0	—	—	1745	Graham.
1750	19	47	—	—	17	48	—	—	1748	Ditto.
1760	21	1	—	—	—	—	—	—	—	
1770	22	4	73	40	21	9	72	19	1773	Heberden.
1780	22	54	73	18	23	17	72	8	1786	Gilpin.
1790	23	33	72	39	23	39	71	53	1790	Ditto.
1800	24	1	71	58	24	3	70	35	1800	Ditto.
1810	24	18	71	15	24	11	—	—	1809	Ditto.
1818	24	30	70	34	24	30	70	34	—	Kater, the dip.

Unfortunately we are not able to make this comparison between computation and observation in many places. There are no observations recorded far enough back to make this comparison, except in London, Paris, and Copenhagen. It has been seen by the preceding table how near the results, as computed, agree with those recorded; and we shall find, that upon the whole, the agreement is still better in the two following cases.

Variation at Paris.

Annual change at Paris.

203. According to the table, art. 197, the longitude of the magnetic pole for Paris, reckoning from London, was 67° 4′ west, in 1814; to which adding 2° 20′, the longitude of Paris, east of London, makes the longitude of the magnetic pole from Paris, in 1814, 69° 24′ west. Now in 1664, or 1666, the needle pointed due north at Paris, or its longitude was then zero; therefore in 150 years the poles have moved through 69° 24′ of longitude, which is at the rate of about 4° 37′ in ten years, or about 2′ annually faster than we found it for London. If, with this number, and with the proper latitude, &c. for Paris, we proceed as above for London, there will be obtained the following computed variations, which are compared with the observed variations as recorded by the best observers, (see the table of variations for Paris at the end of this treatise.)

Table of computed and recorded variation at Paris, from 1664 *to* 1814.

By computation.			By record.		
Year.	Variation.		Year.	Variation.	
	°	′		°	′
1664 or 1666	0	0	1666	0	0
1676	2	36W.	1680	2	40
1686	5	11	1686	4	30
1696	7	38	—	7	8
1706	9	58	—	9	48
1716	12	6	—	12	15
1726	14	7	—	13	45
1736	15	53	—	15	40
1746	17	27	—	16	15
1756	18	46	1757	18	0
1766	19	54	1770	19	55
1776	20	48	1777	20	27
1786	21	30	—	21	37
1796	22	3	1798	22	17
1806	22	24	1804 1807	22	25
1814	22	34	—	22	34

The agreement here, as stated above, is still more satisfactory than in the table of results for London.

Variation at Copenhagen.

Annual change for Copenhagen.

204. According to the table, art. 197, the longitude of the magnetic pole for Copenhagen, in 1813, was 67° 48′ west, to which adding the east longitude of Copenhagen 12° 35′, we have for the longitude of the magnetic pole in the above year 80° 23′ as referred

to the meridian of that place. Unfortunately, the time of the needle being due north, is not registered, we must therefore assume the annual motion; supposing it to have been at the same rate as at Paris, or, for the sake of simplifying, calling it 4° 40′ in ten years, instead of 4° 37′, we may then make the computation as above, and the results will be as follow.

Table of computed and observed variation at Copenhagen, from 1673 *to* 1813.

By computation.		By observation.	
Year.	Variation.	Year.	Variation.
	° ′		° ′
1673	4 0	1672	3 35
1733	13 51	1731	11 15
1763	15 45	1765	15 5
1773	17 23	—	16 12
1783	17 53	—	17 49
1793	18 14	—	18 15
1803	18 26	1806	18 25
1813	18 8	—	18 8

Although there is not in the above tables that accurate coincidence between the computed and observed variations we might desire; yet the agreement is too close in all those cases (and they are the only comparisons we can make for want of recorded observations,) to allow us to suppose that the agreement is all accidental; on the contrary, there can be no doubt that a motion of rotation, very similar to that supposed, actually takes place, although some of its elements may have been erroneously assumed. At the same time, however, great allowance is to be made for the errors and uncertainties of magnetical observations; our authorities are, it is true, derived from the best sources, but this will not insure us against local inequalities. Of this there is a remarkable instance in the volume of the *Philosophical Transactions* for 1821, where the dip is stated to be 71° 36′; which is unquestionably more than a degree greater than it really is in the neighbourhood of London, although it may be correct for the Royal Society Rooms, viz. some iron in the building, or some other hidden cause, may increase the inclination at least a degree beyond what the same would be found in the open fields.

Present amount of the dip in London.

205. That the dip at this time does not exceed 70½° is proved by a number of independent observations. For example, Captain Kater found the dip in the Regent's Park, in 1818, to be 70° 34′. Captain Sabine, on his return from the Arctic voyage of that year, found it still the same, and still less prior and subsequent to his second voyage. Mr. Barlow has also taken the dip at Woolwich with four different instruments, and the results of his mean observations have always fallen within the limits of 70° 35′ and 70° 17′; there cannot therefore be the least doubt that the dip published in the volume of the *Philosophical Transactions* referred to, is in error at least a degree. And if in the present day, when magnetism is becoming a mathematical science, we are liable to uncertainties of this kind, we may easily reconcile ourselves to the discrepancies between the computed and observed variations and inclinations, which appear in the preceding tables, without suspecting the hypothesis on which the computations are founded to be, in its general principle, erroneous.

Of the ti of greate westerly variation.

206. We have indeed a very remarkable coincidence, highly favourable to the supposition of an uniform motion of rotation of the polarizing axis; which is, by computing the time when the needle ought to have its greatest westerly variation, and commence again its return towards the true meridian, and comparing the result with observations.

It is obvious, from the principles on which we have proceeded, in our computation for London, that the variation will be the greatest when $L \pi N$ (fig. 82) is a right angle. We have therefore in this case the side $\pi N = 14° 58'$, and the $\angle L \pi N = 90°$, to find the angle $\pi N L$, which is found to be 70° 23′; the variation therefore ought to be the greatest when the longitude of the magnetic pole is 70° 23′ W.

Now it has been seen that this longitude was 67° 41′ in 1818, and as the pole revolves at the rate of 4° 14′ in ten years, it follows that the longitude ought to be 70° 23′, some time in the present year 1823; and consequently, in this year, the variation ought to be at its maximum, according to the principles and data on which we have proceeded; and as there is every reason to believe that the needle has already attained its maximum of westerly variation, and is about to return again towards the north,—the agreement in this case between computation and observation, is perhaps more satisfactory than the uncertain nature of our data could have led us to expect.

Capt. Sabine's observations on the dip.

207. As another instance, shewing the uncertain nature of observed results on the dip of the needle, we may refer to Captain Sabine's experiments on this subject, with his new needle in 1821, and published in the *Phil. Trans.* for 1822, of which we have given some account in art. 75. Here the dip was first taken by Meyer's needle; secondly, it was computed by a formula proposed by La Place, and explained in art. 77, and again by a formula proposed by Captain Sabine (art. 78.) The several means deduced from all these different methods were as follow:

	°	′
With Meyer's needle	70	2·9
La Place's formula.........	70	4
Capt. Sabine's formula......	70	2·6
3)	210	9·5
General mean....	70	3·2

The dip it appears is, therefore, nearly half a degree less than we have reckoned it to be in 1818, and more than a degree and a half less than the amount stated in the volume of the *Philosophical Transactions* to which we have referred.

Annual change in the dip computed

208. It is to be observed, however, that the dip we have used (viz. 70° 34′) was taken in the year 1818; and that according to the principles of calculation adopted in this chapter, this angle ought to diminish as the longitude of the magnetic pole increases. Let us therefore compute what the dip ought to be in the present year, (1823) at which time we have seen that the variation will be at its maximum, and the longitude of the magnetic pole 70° 23′: that is, referring to (fig. 82) we shall have in the right angled triangle

π N L, the side N π given, $= 14° 58'$; the angle π N L $= 70° 23'$ to find L π the magnetic colatitude of London; and consequently the magnetic latitude becomes known. And hence again the dip from the formula

$$\tan \text{dip} = 2 \tan \text{mag. lat.}$$

Now $$\tan \pi L = \sin \pi N \tan \pi N L$$

or

$$\tan \pi L = \sin 14° 58' \times \tan 70° 23' = \tan 35° 56'$$

whence mag. lat. of London $= 54° 04'$ in 1823,

and $$2 \tan 54° 4' = \tan 70° 5'$$

the dip at that time.

The dip therefore, as computed on our hypothesis, ought to be 70° 5′, in the year 1823; and it has been found, from the most accurate observations ever yet made, to have been 70° 3′·2 in September 1821: a coincidence, or exceedingly close approximation which could scarcely have been expected; and which will, we are persuaded, be duly estimated by the candid philosophical inquirer.

Captain Sabine, by comparing the present dip with the dip 47 years back, finds the mean annual diminution to be about 3′. According to our hypothesis the dip has not an uniform decrease, but is changing now more rapidly than it has ever before done since magnetical observations have been made. Its decrease during the last five years, has been nearly half a degree; and, if our principles be correct, it ought to decrease nearly the same during the next five years; a short time therefore will either confirm or refute the hypothesis on which we have founded the preceding computations. Agreeably to which we ought to find in

1828, the variation 24° 29′ dip 69° 43′
1833, ... 24 26 .. 69 21

The dip, therefore, is at present changing more rapidly than the variation; and it will continue to decrease with the latter for about 260 years, when the longitude of the magnetic pole will be 180°; the variation will therefore then be nothing, and the dip only 56°, which will be its minimum; they will then both increase together for the next 260 years, when the needle will have its greatest easterly variation, and will then again return towards the north, the variation decreasing, but the dip still increasing, for 165 years longer; viz. till about the year 2510, when the magnetic pole will be again on the meridian of London; the variation will be zero, and the dip being then at its maximum will amount to 77° 43′.

Such, at least, are the results arising out of the hypothesis on which the preceding calculations are founded. They are unquestionably, in some measure, speculative, and are only given as such; but we may perhaps be permitted to say that as far as comparison could be made with well authenticated observation it has been done; and the approximations towards coincidence have been throughout more favourable than, from the nature of the inquiry, we could have had any reason to expect. Further comparisons may also still be made, and a few years will be sufficient to confirm or refute the hypothesis, of an uniform motion of rotation in the terrestrial polarizing axis.

It is however proper to state, that if we had carried our computations back to the sixteenth century, and compared them with observation, the agreement would not have been found so close as in the cases mentioned in the preceding table; but we think it may be questioned how far these observations are to be depended upon; one of which, out of the only three, recorded, having been made previous to the time of the variation in the variation being known. At the same time we are by no means disposed to assert that the elements of the motion here assumed are perfect. It is to be observed, that we have deduced them from two observations only; viz. the dip, as taken by Captain Kater in 1818, and the variation at that time; it is from these only we have determined the latitude and longitude of the magnetic pole, and thence, by assuming the longitude to have been zero in 1660, for London, we have determined the annual motion. There is little doubt that we should have found a nearer approximation by taking the polar distance of the magnetic pole greater or less than has been done; but our object has not been to find how nearly it was possible to approximate to observation, but how nearly the deductions, legitimately arising out of our first hypothesis, corresponded with the same. Churchman, in his *Magnetic Atlas*, has assumed a certain distance and movement of rotation, which give as near approximations as those found above, but they are dependent upon no previous principle, and are inconsistent with every other magnetic law; whereas our distance and motion are drawn immediately from an independent hypothesis, and are perfectly consistent with every known principle of terrestrial magnetism. See Barlow's *Essay on Magnetic Attractions*.

§ XXIV. *On the diurnal variation.*

Nature of the daily variation explained.

209. It has been stated in our historical chapter, that the horizontal needle, besides the annual change in direction, examined in the preceding section, is also subject to a daily change, amounting, at certain seasons of the year, to about 14′ or 15′. This extraordinary motion was first noticed by Mr. Graham in London, about the year 1722; and the circumstances attending his observations were communicated to the Royal Society in 1723; it has been since registered with great diligence and attention by Wargentin, Canton, Gilpin, Van Swinden, Col. Beaufoy, and others; and various theories have been advanced to account for the phenomenon; but no one, that we are aware of, has been considered perfectly satisfactory. There seems no question that the sun is the agent, by which this change in the direction of the needle is effected; but in what manner, whether simply by heat, or whether the solar beams, acting on the atmosphere, produce any magnetic or electro-magnetic action which is sufficient to cause the disturbance in question, is at present undetermined. When the daily variation was first noticed, it was supposed that the needle had only two changes of motion during the day; viz. that from about seven o'clock in the morning the north end began gradually to deviate towards the west, and that it attained its greatest daily westerly bearing about two o'clock in the afternoon; and that from two or three o'clock it again returned to the eastward, till it obtained its original bearing, and then remained stationary till the next morning.

More accurate observations, however, at length shewed that this was not strictly the daily course of the needle; it was discovered that the motion began much earlier than seven o'clock, but that it was to the eastward; and still more recent observations have shewn that it attains its maximum in that direction about seven o'clock or half past seven in the morning; that it continues moving westward as above stated till two o'clock in the afternoon; it then returns to the eastward till the evening; it has then again a slight westerly motion; and in the course of the night, or early in the morning, attains the bearing it had twenty-four hours before, or very nearly. It has also been admitted by all observers, that the daily motion during the summer months is the greatest, and during the winter months the least; but the particular month in the summer, when the daily change is the greatest, is a little uncertain; Canton and Wargentin make it about July; but Col. Beaufoy found it greater in June and August than in July; the difference, however, is inconsiderable, and may in the particular case in question be accidental; although it must be acknowledged, that it was the same in five successive years while this gentleman continued his observations.

The following are Col. Beaufoy's mean results for two years.

Colonel Beaufoy's observations.

210. *Table of the mean monthly diurnal variation of the compass from April* 1817 *to March* 1819. *By Colonel Beaufoy, at Stanmore Heath.*

From April 1817 to March 1819.		The years 1817 and 1818, wes terly variation.			The years 1818 and 1819, westerly variation.			Difference, morning, noon, evening.	Difference, morning, noon, evening, 1817, 1818.		Difference, morning, noon, evening, 1818, 1819.		Mean of the two.	
		°	′	″	°	′	″		′	″	′	″	′	″
April	Morning	24	31	52	24	34	6	n. — m.	12	51	10	44	11	48
	Noon	—	44	43	—	44	50	n. — e.	8	45	8	14	8	30
	Evening	—	35	58	—	36	36	e. — m.	4	6	2	30	3	18
May	Morning	—	32	20	—	36	18	n. — m.	10	15	9	31	9	53
	Noon	—	42	35	—	45	49	n. — e.	7	50	7	14	7	32
	Evening	—	34	45	—	38	35	e. — m.	2	25	2	17	2	21
June	Morning	—	31	9	—	33	47	n. — m.	11	5	11	24	11	15
	Noon	—	42	14	—	45	11	n. — e.	8	9	7	31	7	50
	Evening	—	34	5	—	37	40	e. — m.	2	56	3	53	3	25
July	Morning	—	31	14	—	34	24	n. — m.	10	52	10	35	10	43
	Noon	—	42	6	—	44	59	n. — e.	6	23	6	45	6	34
	Evening	—	35	43	—	38	14	e. — m.	4	29	3	50	4	9
Aug.	Morning	—	31	16	—	34	40	n. — m.	11	35	11	18	11	26
	Noon	—	42	51	—	45	58	n. — e.	9	6	8	8	8	34
	Evening	—	33	45	—	37	50	e. — m.	2	29	3	10	2	52
Sept.	Morning	—	33	2	—	34	29	n. — m.	8	34	10	53	9	44
	Noon	—	41	36	—	45	22	n. — e.	6	58	7	54	7	26
	Evening	—	34	38	—	37	28	e. — m.	1	36	2	59	2	18
Oct.	Morning	—	31	6	—	35	36	n. — m.	9	40	7	52	8	46
	Noon	—	40	46	—	43	28	n. — e.						
	Evening	—	—	—	—	—	—	e. — m.						
Nov.	Morning	—	31	49	—	33	24	n. — m.	6	6	8	17	7	10
	Noon	—	37	55	—	41	41							
	Evening	—	—	—	—	—	—							
Dec.	Morning	—	34	3	—	37	4	n. — m.	3	59	4	16	4	7
	Noon	—	38	2	—	41	20							
	Evening	—	—	—	—	—	—							
Jan.	Morning	—	34	2	—	35	42	n. — m.	5	55	4	12	5	3
	Noon	—	39	57	—	39	54							
	Evening	—	—	—	—	—	—							
Feb.	Morning	—	34	22	—	34	17	n. — m.	6	29	5	38	6	3
	Noon	—	40	51	—	39	55							
	Evening	—	—	—	—	—	—							
March	Morning	—	33	18	—	33	18	n. — m.	8	19	8	24.	8	22
	Noon	—	41	37	—	41	42	n. — e.	7	50	6	25	7	7
	Evening	—	33	47	—	35	17	e. — m.	0	29	1	59	1	15

Reference to recorded observations.

211. It was intended, and is stated in our historical chapter, that we would, in this place, give the several numerical results obtained by Wargentin, Canton, and Gilpin, relative to this daily change in the direction of the magnetised needle; but our treatise having already exceeded the limits intended, we shall content ourselves with referring the reader to those observations as they are recorded in the *Philosophical Transactions* for 1756, and 1806, and particularly to Hansteen's *Treatise on Magnetism*, to which we have already frequently referred; and the remaining part of this chapter we shall appropriate to the experiments on the daily variation recently made by Messrs. Barlow and Christie at Woolwich, which appear to have thrown a new light on this interesting subject.

ssrs. low and istie's eri- nts.

212. It occurred to the former gentleman about the beginning of the present year (1823,) that if he were to neutralize the terrestrial magnetic direction of the needle, by opposing to the latter two artificial magnets, so that while the needle should retain all its natural power, its direction should be reduced to zero, or to a very inconsiderable quantity; that, then, that power, whatever it may be which produces the daily change, would render itself more obvious, and increase the usual daily variation from about 14′ or 15′ to 3° or 4°, or, in short, to almost any amount at pleasure; and thereby render all the changes of the needle the more obvious, and point out with greater accuracy the time of maximum and minimum effect.

Suppose, for example, the time in which a finely suspended horizontal needle makes one vibration to be 2″, under the usual terrestrial influence, and that by reducing its power by magnets properly disposed, that time is increased to 8″, its original direction being preserved, then it would follow that the directive power was reduced to one-sixteenth of the former; and consequently that any lateral magnetic force, acting upon it, would produce an effect sixteen times greater than before; so that if the former were 12′, the new effect, or deviation, might be expected to amount to between three and four degrees, and therefore such as to admit of distinct and satisfactory observations.

A course of experiments carried on for a few days, convinced the author that his ideas were correct, and that he might, while the needle was kept in its natural direction, produce a daily variation to almost any amount at pleasure; he obtained, for instance, the first day, a maximum deviation of 3° 40′, the second day he increased it, by bringing up his magnets, to 7°; the third day it was reduced to 2°, and so on. He found also that he could exhibit a very sensible daily change with the needle in an inverted position, viz. with the north end held to the south, by a magnet placed either to the north or south, or one at each place; and so also with it at east and west; or, in short, in any position whatever. For this, it was only necessary, first, by means of one magnet, to draw or repel the needle into its desired direction; and then to modify its directive power by another magnet, in the same way as when it was in its natural meridian; or the same might be done by first neutralizing the power of the needle by two magnets placed in the meridian, with their opposite poles opposed to the corresponding poles of the needle, (the latter being in a line between them,) and then slightly changing their position till the desired direction was obtained.

Experiments on the needle in different observations.

213. With these ideas Mr. Barlow commenced a course of observations and experiments with the needle, directed first with the north end to the north, then two points from the north towards the east, four points to the east, and so on throughout the entire circle. The results of these several experiments are contained in the following table; by which it will appear that the north end being directed to any point from the south to the N. N. W., its motion, during the forenoon, is toward some point between the N. N. W. and N., passing, therefore, to the left hand of an observer, situated so as to be facing the north end of the needle; and that while it is directed to any point from the S. S. E. to N., it passes to his right hand, and is therefore still carried to some point between the N. and N. N. W., the south end of course at the same time passing, in both cases, to some point between the S. S. E. and S. So that it would appear that there ought to be some position of the needle between those limits in which the daily motion would become zero; which position, from the following table, appears to be somewhere very nearly at N. N. W., or rather at N. 16° W., and the line of maximum daily change at right angles to this line of the minimum. This will be fully understood by examining the following results, and comparing them with the diagram (fig. 83.) *Philosophical Transactions*, part ii. for 1823.

Table of observations.

214. *Table, containing the results of observations on the daily variation of the horizontal needle made in the open air, arranged according to the direction in which the needle pointed.*

The character of the deviations are distinguished as follow: The reader should conceive himself as facing the north end of the needle, then those deviations which are made towards his right hand (and which therefore increase the westerly variation with the north end to the north,) are marked *plus*, and those which pass to his left hand are marked *minus*. The direction of the needle at six o'clock every morning being marked *zero*.

North end, N.				North end, S.			
Hour.	First day.	Second day.	Third day.	Hour.	First day.	Second day.	Third day.
h m	° ′	° ′	° ′	h m	° ′	° ′	No observations on this day.
6 0	0 0	0 0	0 0	6 0	0 0	0 0	
7 0	− 0 10	+ 0 5	− 0 5	7 0	− 0 30	− 0 15	
8 0	+ 0 10	+ 0 15	+ 0 20	8 0	− 0 40	− 0 45	
9 0	+ 0 30	+ 0 50	+ 1 0	9 0	− 0 55	− 1 5	
10 0	— —	+ 1 0	— —	10 0	— —	− 1 25	
10 30	+ 1 5	+ 1 0	+ 1 50	10 30	− 1 10	− 1 50	
11 0	+ 1 45	+ 1 30	+ 2 10	11 0	− 1 10	− 1 50	
11 30	+ 2 5	+ 1 45	+ 2 20	11 30	− 1 10	− 2 0	
12 0	+ 2 5	+ 1 45	+ 3 0	12 0	— —	— —	
1 0	+ 2 50	+ 1 45	+ 3 10	1 0	− 1 15	− 2 20	
2 0	+ 2 40	+ 1 30	+ 2 45	2 0	− 1 5	− 2 20	
3 0	+ 2 30	+ 1 30	+ 2 45	3 0	− 1 0	− 2 20	
4 0	+ 2 0	+ 1 30	— —	4 0	— —	− 1 30	
5 0	+ 1 30	+ 1 0	+ 1 40	5 0	— —	− 1 10	
6 0	+ 1 30	— —	— —	6 0	− 0 45	− 1 10	
7 0	+ 1 10	— —	+ 1 0	7 0	− 0 35	— —	
8 0	+ 1 0	— —	+ 0 45	8 0	— —	− 0 45	
9 0	+ 0 45	— —	+ 0 30	9 0	− 0 15	− 0 30	
10 0	— —	+ 0 15	+ 0 30	10 0	— —	− 0 30	

North end, N. N. E.				North end, S. S. W.			
Hour.	First day.	Second day.	Third day.	Hour.	First day.	Second day.	Third day.
h m	° ′	° ′	No observations on this day.	h m	° ′	° ′	No observations on this day.
6 0	0 0	0 0		6 0	0 0	0 0	
7 0	− 0 5	0 0		7 0	− 0 10	− 0 20	
8 0	+ 0 15	+ 0 20		8 0	− 0 30	− 0 40	
9 0	+ 0 40	+ 0 45		9 0	− 0 45	− 1 0	
10 0	+ 1 0	— —		10 0	− 1 0	− 1 20	
10 30	+ 1 5	— —		10 30	− 1 10	— —	
11 0	+ 1 40	+ 1 45		11 0	1 30	— —	
11 30	+ 1 40	+ 2 0		11 30	− 1 30	− 2 0	
12 0	+ 2 0	— —		12 0	− 1 30	− 2 20	
1 0	+ 2 10	— —		1 0	− 2 0	− 2 30	
2 0	+ 2 10	+ 2 0		2 0	− 2 0	— —	
3 0	— —	+ 1 50		3 0	— —	— —	
4 0	— —	— —		4 0	— —	− 1 30	
5 0	+ 1 30	+ 1 45		5 0	— —	— —	
6 0	+ 1 35	— —		6 0	− 1 10	− 1 15	
7 0	— —	+ 0 55		7 0	− 1 0	− 0 55	
8 0	+ 1 10	+ 0 55		8 0	− 1 0	− 0 50	
9 0	+ 0 50	+ 0 45		9 0	− 1 0	— —	
10 0	+ 0 30	+ 0 0		10 0	− 1 30	0 40	
11 0	− 0 0	− 0 10		11 0	— —	— —	

Observations on the daily variation of the horizontal needle continued.

North end, N. E.				North end, S. W.			
Hour.	First day.	Second day.	Third day.	Hour.	First day.	Second day.	Third day.
h m	° ′	° ′		h m	° ′	° ′	
6 0	— —	— —	No observations on this day.	6 0	— —	— —	No observations on this day.
7 0	+ 0 10	+ 0 15		7 0	− 0 20	— —	
8 0	+ 0 30	+ 0 15		8 0	− 0 35	− 0 20	
9 0	+ 0 45	+ 0 50		9 0	− 0 45	− 0 30	
10 0	+ 1 0	— —		10 0	− 0 45	− 0 40	
10 30	+ 1 20	— —		10 30	− 1 10	— —	
11 0	+ 1 50	+ 1 30		11 0	− 1 10	— —	
11 30	+ 1 50	— —		11 30	− 1 10	− 1 30	
12 0	+ 2 0	+ 2 10		12 0	− 1 50	− 1 30	
1 0	+ 2 0	— —		1 0	− 1 50	− 1 30	
2 0	+ 1 45	+ 1 15		2 0	− 1 40	− 1 15	
3 0	— —	+ 1 15		3 0	— —	— —	
4 0	— —	+ 1 15		4 0	— —	− 1 0	
5 0	— —	+ 1 0		5 0	— —	− 1 0	
6 0	+ 0 45	+ 0 45		6 0	− 0 45	− 0 50	
7 0	+ 0 45	— —		7 0	− 0 30	— —	
8 0	+ 0 30	+ 0 25		8 0	− 0 30	— —	
9 0	+ 0 5	+ 0 25		9 0	— —	— —	
10 0	− 0 5	— —		10 0	+ 0 5	— —	
11 0	− 0 5	— —		11 0	+ 0 5	+ 0 10	

North end, E. N. E.				North end, W. S. W.			
Hour.	First day.	Second day.	Third day.	Hour.	First day.	Second day.	Third day.
h m	° ′	° ′		h m	° ′	° ′	
6 0	— —	— —	No observations on this day.	6 0	— —	— —	No observations on this day.
7 0	+ 0 10	+ 0 40		7 0	− 0 30	− 0 40	
8 0	+ 0 45	+ 0 55		8 0	− 1 0	— —	
9 0	+ 1 15	+ 1 30		9 0	− 1 0	— —	
10 0	+ 2 0	+ 2 10		10 0	− 1 45	− 2 0	
10 30	+ 2 10	+ 2 40		10 30	− 2 25	− 2 30	
11 0	— —	— —		11 0	− 2 50	− 2 45	
11 30	— —	— —		11 30	− 2 50	— —	
12 0	— —	— —		12 0	− 2 50	— —	
1 0	+ 2 0	+ 2 30		1 0	− 2 10	2 10	
2 0	+ 2 0	+ 2 10		2 0	— —	— —	
3 0	— —	— —		3 0	— —	2 15	
4 0	— —	+ 1 40		4 0	— —	— —	
5 0	— —	+ 1 0		5 0	− 1 30	2 0	
6 0	— —	— —		6 0	— —	— —	
7 0	+ 1 10	— —		7 0	— —	— —	
8 0	+ 0 40	+ 0 50		8 0	− 1 0	1 30	
9 0	+ 0 20	+ 0 20		9 0	− 1 0	1 15	
10 0	— —	− 0 5		10 0	− 0 25	0 50	
11 0	− 0 15	— —		11 0	+ 0 5	− 0 20	

Observations on the daily variation of the horizontal needle continued.

North end, E.				North end, W.			
Hour.	First day.	Second day.	Third day.	Hour.	First day.	Second day.	Third day.
h m	° ′	° ′	° ′	h m	° ′	° ′	° ′
6 0	— —	— —	— —	6 0	— —	— —	— —
7 0	+ 1 30	+ 0 30	+ 0 20	7 0	− 0 15	— —	− 0 10
8 0	+ 1 50	+ 1 0	+ 0 55	8 0	− 1 5	− 0 5	− 1 0
9 0	+ 2 0	+ 1 15	+ 1 30	9 0	− 1 35	− 0 35	− 1 20
10 0	+ 3 15	+ 1 35	+ 2 0	10 0	− 2 30	− 0 50	− 2 15
10 30	— —	+ 2 0	+ 2 15	10 30	− 2 55	− 1 25	— —
11 0	+ 3 15	+ 2 0	— —	11 0	− 3 5	− 1 40	− 2 45
11 30	— —	+ 2 0	— —	11 30	− 3 5	− 1 40	− 2 45
12 0	+ 3 15	+ 2 0	— —	12 0	− 2 55	− 1 20	− 2 0
1 0	+ 3 10	+ 2 0	— —	1 0	— —	− 1 0	— —
2 0	+ 2 30	+ 1 30	+ 1 15	2 0	− 2 30	− 0 40	− 2 0
3 0	+ 1 30	+ 1 30	+ 1 15	3 0	− 2 20	− 0 35	− 2 0
4 0	+ 1 20	+ 1 10	+ 1 10	4 0	− 2 10	− 0 30	− 1 45
5 0	+ 0 45	+ 1 0	— —	5 0	− 2 10	− 0 25	− 1 0
6 0	+ 0 45	+ 1 0	+ 1 15	6 0	− 2 0	− 0 20	— —
7 0	+ 0 15	+ 1 10	+ 1 0	7 0	− 1 55	− 0 10	− 0 30
8 0	— —	+ 0 50	+ 0 45	8 0	− 1 35	+ 0 5	− 0 30
9 0	− 0 5	+ 0 30	+ 0 40	9 0	— —	+ 0 20	— —
10 0	− 0 10	+ 0 5	+ 0 10	10 0	− 0 55	+ 0 25	— —
11 0	— —	— —	— —	11 0	— —	— —	— —

North end, E. S. E.				North end, W. N. W.			
Hour.	First day.	Second day.	Third day.	Hour.	First day.	Second day.	Third day.
h m	° ′	° ′	° ′	h m	° ′	° ′	
6 0	— —	— —	— —	6 0	— —	— —	No observations on this day
7 0	— —	+ 0 5	+ 0 10	7 0	— —	— —	
8 0	+ 0 10	+ 0 15	+ 0 30	8 0	− 0 20	0 15	
9 0	+ 0 20	+ 0 30	— —	9 0	− 0 40	— —	
10 0	+ 0 35	+ 0 35	— —	10 0	− 0 50	− 0 20	
10 30	— —	+ 0 45	— —	10 30	— —	— —	
11 0	+ 0 40	— —	— —	11 0	— —	— —	
11 30	+ 0 40	+ 0 55	— —	11 30	— —	— —	
12 0	+ 0 40	+ 0 55	+ 1 0	12 0	− 0 55	− 0 22	
1 0	+ 0 35	+ 0 50	+ 0 50	1 0	− 0 55	− 0 30	
2 0	+ 0 35	+ 0 55	+ 0 40	2 0	− 0 50	− 0 30	
3 0	+ 0 35	+ 0 45	— —	3 0	− 0 50	— —	
4 0	+ 0 35	+ 0 45	— —	4 0	− 0 50	− 0 30	
5 0	+ 0 30	+ 0 30	— —	5 0	— —	— —	
6 0	+ 0 25	+ 0 30	— —	6 0	— —	— —	
7 0	+ 0 15	+ 0 30	+ 0 20	7 0	− 0 30	— —	
8 0	+ 0 5	+ 0 10	+ 0 10	8 0	− 0 20	— —	
9 0	− 0 5	+ 0 5	— —	9 0	− 0 5	+ 0 10	
10 0	− 0 5	— —	− 0 5	10 0	+ 0 5	+ 0 15	
11 0	— —	— —	− 0 5	11 0	— —	— —	

Observations on the daily variation of the horizontal needle continued.

North end, S. E.						
Hour.		First day.		Second day.		Third day.
h	m	°	′	°	′	
6	0	—	—	—	—	No observations on this day.
7	0	+ 0	10	—	—	
8	0	+ 0	20	+ 0	25	
9	0	+ 0	35	+ 0	25	
10	0	+ 0	45	+ 0	40	
10	30	+ 1	0	+ 1	10	
11	0	+ 1	5	+ 1	10	
11	30	+ 1	5	+ 1	10	
12	0	+ 1	10	—	—	
1	0	+ 1	10	—	—	
2	0	—	—	+ 1	0	
3	0	—	—	+ 1	0	
4	0	—	—	—	—	
5	0	+ 0	40	+ 0	55	
6	0	+ 0	40	+ 0	45	
7	0	—	—	—	—	
8	0	—	—	—	—	
9	0	+ 0	20	+ 0	15	
10	0	+ 0	10	—	—	
11	0	—	—	+ 0	5	

North end, N. W.							
Hour.		First day.		Second day.*		Third day.	
h	m	°	′	°	′	°	′
6	0	—	—	—	—	—	—
7	0	— 0	10	— 0	10	— 0	5
8	0	— 0	10	— 0	16	— 0	20
9	0	— 0	15	— 0	30	— 0	30
10	0	— 0	25	— 0	45	—	—
10	30	—	—	— 0	50	— 1	0
11	0	— 0	20	— 0	55	— 1	0
11	30	—	—	—	—	—	—
12	0	— 0	15	— 0	55	— 1	0
1	0	— 0	10	— 0	50	—	—
2	0	— 0	10	— 0	50	— 0	45
3	0	0	10	—	—	—	—
4	0	—	—	— 0	50	— 0	45
5	0	—	—	—	—	—	—
6	0	—	—	— 0	45	— 0	40
7	0	— 0	10	— 0	30	—	—
8	0	—	—	—	—	— 0	30
9	0	—	—	— 0	5	—	—
10	0	+ 0	10	— 0	5	— 0	15
11	0	— 0	5	—	—	—	—

* The needle was readjusted this day, and the directive power reduced as much as possible.

S. S. E. ½ S.				
Hour.		First day.	Second day.	Third day.
h	m			
6	0	Very little daily motion.	No perceptible motion.	No motion till two o'clock this day; the needle then removed.
7	0			
8	0			
9	0			
10	0			
10	30			
11	0			
11	30			
12	0			
1	0			
2	0			
3	0			
4	0			
5	0			
6	0			
7	0			
8	0			
9	0			
10	0			
11	0			

North end, N. N. W.						
Hour.		First day.		Second day.		Third day.
h	m	°	′	°	′	
6	0	—	—	—	—	No observations on this day.
7	0	+ 0	6	—	—	
8	0	— 0	5	— 0	10	
9	0	— 0	25	— 0	30	
10	0	— 0	30	— 0	45	
10	30	— 0	40	— 0	45	
11	0	— 0	55	— 1	0	
11	30	— 1	10	— 1	10	
12	0	—	—	— 1	0	
1	0	— 1	10	—	—	
2	0	— 1	5	— 1	0	
3	0	—	—	— 1	0	
4	0	—	—	—	—	
5	0	— 0	45	—	—	
6	0	— 0	40	— 0	50	
7	0	— 0	40	— 0	45	
8	0	— 0	20	— 0	30	
9	0	— 0	20	— 0	10	
10	0	—	—	— 0	10	
11	0	—	—	—	—	

Time of maximum and minimum deviation.

215. While the author was pursuing this course of experiments, his friend Mr. Christie, also of the Royal Military Academy, undertook, at his suggestion, a totally independent course of similar but more extended experiments; from which it is shewn, (see *Phil. Trans.* part ii. for 1823 above referred to,) that while the needle is directed to the north, its first motion commences very early in the morning to the eastward; that it obtains its greatest deviation in this direction at about seven o'clock, or half past seven o'clock, in the morning; from this time the easterly deviation diminishes, (the motion of the needle being to the westward,) till about half past ten or eleven o'clock in the morning, when the deviation is zero; and when, it will be observed, the sun is upon the magnetic meridian; from this time the westerly deviation commences, and consequently the westerly motion of the needle continues till about half past one o'clock or two o'clock, when the westerly deviation is at its maximum, and then begins to diminish, the motion of the needle being now to the eastward; at about five o'clock the deviation again vanishes, and the needle points once more north and south, (magnetic,) but its easterly motion continues till very late in the evening. It appears, therefore, that three times during the day the needle points to the true magnetic north and south; that is to say, when the sun is upon the magnetic meridian, and about six hours before and after it attains this position; and that its maximum easterly and maximum westerly variation, happen about three hours before and after the transit of the sun over the same meridian.

General deductions.

216. These facts are highly important, as pointing to the first cause of the daily motion, which is thus very obviously shewn to depend upon the relative position of the sun with the magnetic meridian; but at present the results are too limited to admit of founding upon them a complete theory of the phenomena. The position of the sun, at the same hour of the day, but at different times of the year, changes very considerably, and it would therefore require us to be in possession of a complete set of observations for a year, before we could allow ourselves to venture upon a complete theory. The above observations were made during the months of April, May, and June; and, as far as they extend, it appears that while the sun is between the N. and E., (magnetic,) the north end of the needle deviates to the west; while it is between the east and south, the north end deviates to the east, pointing correctly, or very nearly, north and south, with the sun at the east or south, (magnetic.) Again, when the sun gets between the south and west, the deviation is to the west; and when between the west and north, the deviation is eastward: so that, generally, it may be said that the sun appears to repel that end of the needle to which it is the nearest, and that its action is greatest when the angle between the sun and the magnetic meridian is about 45°; modified, however, by its intensity of action and angle of elevation; and most probably by the position of the solar orb upon the imaginary magnetic sphere, which has been the foundation of all our analytical deductions in Part III.

Canton's theory of the daily variation.

217. Canton, who made a great number of observations and experiments on this subject, endeavoured to account for the daily variation by the operation of the solar heat; he ascertained, from experiment, that heat has a tendency to diminish the attractive powers of a magnet, and assuming that the direction of the needle was merely due to the resultant of all the magnetic forces of the terrestrial sphere upon it; he concluded, that while this body was to the eastward of the needle, the forces lying to the eastward suffered a diminution of power; in consequence of which the westerly force prevailed, and the north end deviated to the westward; and that when the sun was to the westward, the powers on that side diminishing, the needle returned again to the eastward: but not being in possession of some of the facts above stated, his explanation, although very ingenious, was certainly incomplete, particularly that part of it, in which he endeavours to explain what he denominates the irregular daily variation; that is, the small easterly motion of the needle in the early part of the morning. At the same time, the observations we have in some measure abridged, on the daily variation at Woolwich, seem to be strongly in favour of his hypothesis, after some slight modification rendered necessary by the new facts these experiments have elicited, of which the following is an example.

While carrying on his experiments, the author removed his apparatus from his house to the garden; and he was surprised to find that while the needle was pointing any where towards the east and west, according to the method of observation described in (art. 212.) the morning motion of the needle was in the house, the reverse of what it was in the garden, and that its maximum of variation happened at a different hour of the day. This circumstance caused him considerable perplexity, particularly as the very same effect was observed by Mr. Christie; but after some weeks observation, the latter gentleman began to suspect that it was caused by the difference of temperature within doors and in the garden; and the following experiment seems fully to confirm the accuracy of the supposition.

Let *n s* (fig. 84) represent a very delicate compass needle, and N S a strong magnet placed in the magnetic meridian, so near to the former as to deflect it into an east and west direction. If now only the hand be applied to the magnet N S for a few minutes, the warmth thus communicated to the steel will have an obvious effect upon the needle; the north end gently approaching towards its natural direction, and when the hand is removed, the needle will again return. If the needle is placed in a copper vessel in water, and a lamp be applied beneath it to raise the temperature gradually, the needle will as gradually approach towards the pole N of the magnet, moving in some cases about a degree for every increase of one degree of temperature. It appears, therefore, not only that temperature has an effect upon the power of magnetic bodies, but that the slightest change of temperature may be thus rendered perceptible. Consequently, if we assume, as there appears every reason to do, that the direction of the needle is due to the resultant of a system of forces acting on all sides of it, superficially distributed over the surface of the earth; we shall see that while the sun is between the magnetic east and south, those parts being the most heated, their power will be diminished, and the south end of the needle ought to incline to the west, or the north end to the east, as it is found to do by the experiments above reported; and we ought to expect that the greatest declination eastward should take place when the sun is found equally distant between those points; as the sun approaches nearer the south, the parts to the west of the magnetic meridian, as well as those to the east, become heated, and the eastern deviation ought to decrease and disappear entirely, as the sun transits the magnetic meridian; because then the effects on each side of that meridian are equal to each other, which is precisely consistent with the observations (art. 215.)

Beyond this period the south-western parts will receive the greatest power of the solar rays, become weakened in their action, and the south end of the needle will, in consequence, deviate to the eastward, or the north end to the westward, and continue increasing in its deviation till the sun becomes S. W. (magnetic,) which happens between one and two o'clock in the afternoon; and then, as in the morning, at S. E., its effect will be the most obvious; and will, in general, be greater than the morning easterly deviation; because it happens when the sun has a greater altitude, and consequently a more intense action, which is precisely what has been observed in the experiments referred to.

From this period the western deviation ought to diminish, till the sun becomes west (magnetic,) when it ought to cease entirely; because then the parts on the western side of the needle being equally heated, both to the north and to the south, there can be no tendency in either end to incline from the meridian; for although the power on the western side will be less than on the eastern, yet it will affect both ends the same; but beyond this point, the north-west parts will be more heated than the south-west, and the north end will deviate to the eastward, the motion of the needle continuing the same as before the sun passed the western point; only in the former, the needle was losing its western deviation, whereas it will now be increasing in its eastern; but as the sun approaches the north-west, its altitude and power will be small, and its effect upon the deviation will, in consequence, be but inconsiderable, and ought to vanish entirely, as the sun again crosses the magnetic meridian about eleven o'clock at night.

To illustrate this by experiment, let *n s* (fig. 85) be a finely suspended compass needle, which may be neutralised by two magnets, as described in (art. 212.) and let S N, S′ N′, &c. be four magnets so placed as to give to the needle its true direction. If now to represent the effect of the sun at the magnetic east, we apply an artificial heat to the magnets S N, S′ N′ we shall find no disturbance in the needle; but if we heat alone the magnet S′ N′, its power being diminished, the south end will approach towards the west, and its north end the east, as happens when the sun is at S E (magnetic,) about seven o'clock in the morning. If now again, we equalise the heat upon the magnets S′ N′, S″ N″, the defect of power being equal on both sides, the needle will return to its true bearing. This represents the effect of the sun at half past ten or eleven o'clock in the morning, when he is upon the magnetic meridian.

Let now the heat be applied principally to the magnet N″ S″, then the power of it being thus diminished, the south end of the needle will pass to the east, and, consequently, the north end to the west, as really happens when the sun is towards the south-west magnetic point, at about half past one or two o'clock in the afternoon. Again, let the magnets N″ S″ and N‴ S‴ be equally heated, and then, as before, the needle will attain its true bearing, as at half past four or five o'clock, when the sun is west of the magnetic meridian. Lastly, let the magnet N‴ S‴ be rendered the hottest, and it will be seen that the north end will again pass to the eastward, as it is found to do in the experiments above referred to. According to this view of the subject, there is no occasion to have recourse to the irregular attraction mentioned by Canton; for every known motion of the needle is thus in its general character explained; and with regard to quantity, although we are not enabled to ascertain its actual amount at different seasons of the year, yet attributing its effect to heat, it is obvious that it ought, according to the suppositions we have made, to be greater in summer than in winter, as it is shewn to be by all observations yet made; but at the same time it ought not to be very considerable in parts near the equator, where the heat is very great; because here the sun always rises nearly east and sets nearly in the opposite quarter, its daily course being almost wholly in the direction, east and west; so that in these cases, the heat being equal on each side of the east and west line, little or no motion will ensue; and this also is consistent with observation, see *Phil. Trans.* for 1796. This hypothesis is still farther confirmed by the observations on the daily variation made by Lieutenant Hood, during Captain Franklin's recent expedition to the mouth of the Copper-mine river, of which an abstract is given in the following table. It will be seen in this, that the variation being easterly, it diminishes while ours increases, that is, during the morning; but which, of course, still indicates a westerly motion of the north end: it is remarkable also, that the westerly motion commences and attains its maximum about three hours after the time it does in London. Now our variation being about 24½° west, and at Cumberland house 17° 17′ east, the magnetic meridian is about 42° more to the west than it is here; and therefore the transit of the sun over it, will happen very nearly three hours later, which is a singular confirmation of the hypothesis in question. The unfortunate death of this valuable young officer will be much regretted by every lover of science; all his records display talents which, had his life been spared, could not have failed of placing him in the first rank, not only in his profession but in science. The variations given by Captain Franklin in the following tables, cannot be used in the same manner as a test, because they are blended with the effect of the aurora borealis, and are not due to the daily variation only, as those are of Lieutenant Hood's to which we have referred. See pages 595 and 629 of Captain Franklin's *Journey to the Polar Sea.*

A table of daily variations, at different stations in North America.

Mean diurnal variation of the compass for four months, observed at Cumberland House, by Lieutenant Hood, R. N.; latitude 53° 56′ 40″ N., longitude 102° 16′ 41″ W. Dip 83° 12′ 50″.

Month.	8 A. M.	9 A. M.	1 P. M.	4 P. M.	8 P. M.	12 P. M.
1820.	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″
February	— — —	17 16 0E.	17 11 2	17 9 9	17 12 4	17 13 0
March	— — —	17 16 9	17 9 5	17 9 7	17 12 1	17 13 4
April	17 15 7E.	— — —	17 8 8	17 9 6	17 12 3	17 13 9
May	17 16 9	— — —	17 7 8	17 8 3	— — —	17 14 7

Monthly abstract of diurnal variation observed at Fort Enterprise; latitude 64° 28′ 24″ N., longitude 113° 6′ W. Dip 86° 58′ 42″.

Month.	Variation, East.	Variation, East.	Variation, East.	Variation, East.	Variation, East.	Variation, East.	Variation, East.
1821.	° ′ ″ 10 A. M.	° ′ ″ Noon.	° ′ ″ 2 P. M.	° ′ ″ 3 P. M.	° ′ ″ 6 and 5 P. M.	° ′ ″ 9 P. M.	° ′ ″ Midnight.
January	36 27 57	36 22 9	36 12 39	36 12 3	36 18 57	36 16 9	36 9 9
February	9 A. M. 36 31 45	36 21 57	36 17 27	36 15 51	5 and 4 P. M. 36 13 3	36 13 51	36 11 57
March	8 A. M. 36 29 36	36 17 21	36 21 9	36 10 45	4 P. M. 36 6 15	36 6 37	36 12 33
April	7 A. M. 36 7 45	36 6 3	— — —	36 2 51	5 P. M. 36 0 15	35 55 15	35 40 3
May	7 A. M. 36 42 3	36 29 45	— — —	36 21 9	36 14 57	36 12 9	36 11 33

Mean variation observed with the needle attached to the transit instrument, placed at a greater distance from the buildings at Fort Enterprise.

	8 and 9 A. M.	Noon.	4 and 5 P. M.	9 and 10 P. M.	Midnight.
Between May 12, and June 13, 1821.	° ′ ″ 37 25 5E.	° ′ ″ 37 24 1	° ′ ″ 37 26 1	° ′ ″ 37 22 8	° ′ ″ 37 23 0E.

Monthly abstract of diurnal variation observed at Mouse Deer Island, Slave Lake; latitude 61° 11′ 8″ N., longitude 113° 51′ 37″ W.

Month.	Variation, East.	Variation, East.	Variation, East.	Variation, East.	Variation. East.	Variation, East.
1822.	From 7 A. M. to 9 or 10 A. M.	Noon.	3 P. M.	5 P. M.	9 P. M.	Midnight.
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″
January	25 42 52	25 42 23	25 41 12	— — —	25 40 42	25 39 55
February	25 40 10	25 38 56	25 38 9	25 39 27	25 39 7	25 37 26
March	25 40 52	25 39 6	25 37 50	25 37 16	25 37 47	25 35 2
April	25 42 16	25 40 23	25 38 21	25 38 10	25 38 49	25 38 15
Mean of the three complete months, Feb. March, April . . .	25 41 6	25 39 28	25 38 7	25 38 18	25 38 34	25 37 14

Note.—In the last three tables the effect of the Aurora is included; in the first it is excluded.

§ XXV. *Of magnetic charts.*

218. It has been already stated in our historical chapter, that Dr. Halley was the first who undertook the construction of a magnetic chart, limited, however to the Atlantic and Indian seas; this was published in 1701, and was afterwards republished on an enlarged scale in Holland. In the year 1745, the same chart was again reprinted, with considerable alterations, by Messrs. Mountain and Dodson, making the necessary corrections for the time elapsed; and this again was also copied, on a reduced scale, by M. Bellin, hydrographer of France, in 1765; and in 1794, a corrected edition of Messrs. Mountain and Dodson's chart was printed in London for that period.

In 1807 a new magnetic chart was published in London by Mr. Yeats, in which the variation is given for all known seas, between lat. 60° N. and lat. 60° S., and which contains, besides the Halleyan lines, indicating the lines of equal variation, the curves, shewing the actual bearing of the needle in every part, geometrically as well as numerically. Professor Hansteen of Norway, has also published a very extensive Magnetic Atlas, containing charts both for the variation and dip. As this atlas, although not of so recent a construction as that above mentioned, has been formed on a series of most extensive observations, we have been induced in a few doubtful cases to adopt some of its lines, in preference to the former; see plate IX. Unfortunately, the perpetual, but inconstant change to which the compass is subject, renders such charts as those above described, of a very temporary use; yet they would be highly important and interesting, in pointing out the direction of the motion of the magnetic poles, if they could be securely depended upon. But they are constructed from observations made on shipboard, which we have seen, in our chapter treating on Local attraction, must necessarily be erroneous; because hitherto little attention has been paid to this source of inaccuracy; and where ship observations fail, recourse is had to calculations, founded on principles altogether imaginary. So that we must only consider variation charts as giving very rough approximations to the actual state of the needle; and probably many of the fanciful formed lines which they all exhibit, are wholly attributable to inaccuracies of observations.

In the Atlantic Ocean, as the part of greatest traffic, we have, perhaps, the most certain information relative to the state of the needle; and the charts of recent construction are not, perhaps, in these parts, very defective. Here, therefore, we may trace, with some certainty, the course of the line of no variation. If there were a decided north and south magnetic pole, diametrically opposite to each other, the line of no variation would necessarily fall on some one meridian, and would exhibit itself in the form of a right line on the chart, although all the lines of equal variation, on each side of this, would necessarily assume that particular form shewn in the chart, or one approaching towards it. But instead of the line of no variation falling on any one meridian, it appears, from the recent observations of Capt. Parry, (who found the place of the pole in 102° W. longitude, and in latitude 73° N.) to proceed from this point, to the extreme point where it is marked in Yeats's chart for 1807; viz. in longitude 17° W. and in latitude 60° S. Its direction, therefore, instead of being north and south, is nearly N. N. W. and S. S. E.; and a part of the other line of no variation in the Indian Ocean, runs for a considerable distance nearly east and west.

This view of the subject seems to confirm what we have advanced in section XXIII., that is to say, that every place has its particular polarizing axis; and that nothing can be more mistaken than the idea of searching for certain determinate poles common to the whole earth. At the same time, the near approximation of observation and theory, on those points in which the variation is not concerned, viz. as respects the dip and intensity of the needle, seems to shew, that our general principle of computation on these matters is founded on correct views of the subject; and the same appears from the calculation relative to the annual change in the direction of the needle for the last 150 years in London and Paris. The conclusion, therefore, to which we are led from all these considerations is, that the needle is subject to some general polarizing principle resident on the terrestrial surface, and is so far under the dominion of certain laws, probably, as determinate and fixed in their operation as those which govern the celestial motions; but that, beside this general directive power, the needle, in every place, is under a certain local influence from iron mines and other ferruginous matter, sufficient to change altogether its bearing, without, in any important degree, affecting the intensity of its action, or the rate and direction of its annual motion. It is obvious, for instance, that we may very considerably deflect a needle from its natural meridian, by the application of a ball of iron at some distance; and yet the needle may vibrate as many times in a given period, as when it was undisturbed, or very nearly. Moreover, the intensity of the needle in the natural direction of the dip, is greater than when brought into a horizontal plane, in the ratio of the secant of the dip to radius; and if, therefore, the whole disturbing power acted on the dipping needle, it would be deflected less than the horizontal needle, in the ratio of radius to secant, or in our latitudes nearly as 1 to 3; but in case of a lateral attraction towards the east or west, the ratio will be much greater; it follows, therefore, that such a disturbance may take place in the horizontal needle, as totally to destroy all uniformity in the conditions of variation, while all the other magnetic laws may remain almost wholly unaffected.

With respect to the cause of magnetic action and direction, it is perhaps impossible to form any conjecture; and the cause of the constant change in the direction of the needle in any given place, is not much less mysterious, although the recently discovered science of Electro-Magnetism seems calculated to throw some light upon this interesting subject; we shall reserve, however, what we have to say on this head, and on the effect of the aurora borealis, till we have to treat upon the latter science in the following volume; and shall here conclude our treatise on magnetism with a general table of the variation of the needle in different places, and at different periods, from the earliest authentic records to the present time; and a similar table of observed dips, if not to the same extent, at least to the greatest that has been yet given. We are indebted to Mr. Hansteen for the fundamental parts of these tables, which he has collected with great care and labour; and they will, we trust, be found highly useful as references, to every one interested with magnetical inquiries.

Table, shewing the variation of the needle in various parts of the earth from the earliest observations to the present time. All taken on shore.

Note.—Where more than one observation is made at any place, the name of the place is inserted in the last column. In other cases, only the latitude and longitude are given.

Denmark, Norway, Sweden.

Authority.	Date.	Variation.		Place, or Latitude.		Longitude.	
		°	′	°	′	°	′
Vibe	1796	21	20W	58	34N	8	25E
Ditto	1799	17	25	59	23	15	45
Ditto	1799	17	40	59	55	16	33
Holm	1768	19	20	60	24	5	21
Vibe	1791	24	45	Bergen.			
Ditto	1792	25	30				
Ditto	1792	24	52	59	34	4	19
Lievog	1780	34	30	64	16	337	56
Holm	1768	16	30	63	7	8	6
Aubert	1794	22	0	58	8	7	58
Holm	1761	15	15	59	55	10	42
Ditto	1769	16	45				
Hansteen	1816	20	15	Christiania.			
Ditto	1817	20	3				
Ditto	1799	17	5	Carlberg.			
	1716	11	15	56	10	15	31
Lowenorn	1786	42	41	65	52	341	11
Ditto	1799	18	45	60	36	15	35
Ditto	1783	19	29	58	6	6	57
Hansteen	1810	18	50	Fredricksberg.			
Gädda	1694	8	30	57	42	11	54
Elvius	1748	12	40	Goetheberg.			
Lowenorn	1786	35	21	64	4	338	9
Ditto	1786	43	9	64	9	338	9
Holm	1765	6	50	70	40	23	36
Ditto	1766	7	0	71	8	N. Cape.	
Ditto	1768	16	25	59	2	11	6
Bartholin	1672	2	35	55	54	12	42
Astr. Jahrb. 1779	1761	14	0	55	57	12	38
Hiorter	1748	9	10	Hedmora, Sweden.			
Hellant	1776	11	30	67	51	20	55
Holm	1766	5	30	71	0	26	1
Ditto	1768	6	50	69	28	25	15
Vibe	1779	17	30	60	12	12	5
Schulten	1803	21	0	56	18	12	28
Ditto	1799	17	15	59	30	35	56
Ditto	1799	17	25	59	52	14	57
Hiorter	1746	9	30	Kongsor, Sweden.			
Hellant	1776	5	30	65	51	29	5
Luchtemacher	1649	1	30E	55	41	12	35
Bartholin	1672	3	35W				
Lous (Senior)	1730	10	37				
Ditto	1731	11	15				
Lous (Junior)	1765	15	5				
Ditto	1765	15	3				
Ditto	1767	15	7				
Ditto	1768	15	13				
Ditto	1768	15	0				
Ditto	1768	15	2				
Ditto	1768	14	50				
Ditto	1768	14	56	Copenhagen.			
Ditto	1769	15	30				
Ditto	1769	15	29				
Ditto	1769	15	34				
Ditto	1769	15	22				
Ditto	1770	15	32				
Ditto	1770	15	32				
Ditto	1770	15	32				
Ditto	1770	15	32				
Ditto	1770	15	37				
Ditto	1771	15	32				
Ditto	1771	15	42				
Ditto	1771	16	2				

Authority.	Date.	Variation.		Place, or Latitude.		Longitude.	
		°	′	°	′	°	′
Lous (Junior)	1772	16	17W	55	41N.	12	35E.
Verdun	1772	16	0				
Lous	1773	16	12				
Ditto	1773	16	16				
Ditto	1773	16	17				
Ditto	1773	16	22				
Ditto	1773	16	9				
Ditto	1773	16	20				
Ditto	1774	16	17				
Ditto	1774	16	20				
Ditto	1774	16	32				
Ditto	1774	16	27				
Ditto	1775	16	37				
Ditto	1775	16	20				
Ditto	1775	16	33				
Ditto	1775	16	26				
Ditto	1775	16	27				
Ditto	1776	16	27				
Ditto	1776	16	30				
Ditto	1776	16	30				
Ditto	1776	16	28				
Ditto	1776	16	32				
Ditto	1777	16	20				
Ditto	1777	16	39				
Ditto	1779	17	5				
Bugge	1782	17	41				
Ditto	1783	17	49	Copenhagen.			
Lous	1784	17	42				
Bugge	1784	18	0				
Ditto	1785	18	7				
Ditto	1786	18	9				
Ditto	1792	18	23				
Ditto	1793	18	15				
Wleugel	1806	18	25				
Ditto	1807	18	21				
Ditto	1808	18	22				
Ditto	1809	18	22				
Ditto	1810	18	16				
Ditto	1812	18	17				
Ditto	1813	18	22				
Ditto	1813	18	11				
Bugge	1813	18	10				
Ditto	1813	18	14				
Ditto	1813	18	8				
Wleugel	1814	17	58½				
Ditto	1814	17	56				
Ditto	1814	17	56				
Ditto	1815	18	6				
Ditto	1815	18	3				
Ditto	1815	18	5½				
Ditto	1816	18	15				
Ditto	1817	18	5				
Ditto	1817	17	55½				
Hall	1605	7	10E.	57	50	7	14
Hudson	1608	0	0	68	46	9	27
Ditto	1609	0	0	Lofoeden Isles, Norway.			
Pihl	1785	19	30W	58	27	6	36
Holm	1768	19	20	60	22		
Hiorter	1746	9	36	Lindesberg.			
Bayley	1769	6	0	71	10	25	50
Ditto	1799	18	35	59	30	14	59
Ditto	1799	17	35	60	8	15	21
Verdun	1772	33	30	65	36	336	16

Variation of the needle in various parts of the earth, continued.

Authority.	Date.	Variation.	Place, or Latitude.	Longitude.
		° ′	° ′	° ′
Jahrb	1770	5 50W	47 13N.	39 50E.
Schubert	1805	6 6E.	56 55	74 5
Ditto	1805	5 37	56 30	90 10
Stralenberg	1716	0 0	58 12	68 25
Chappe	1761	3 46E.	Tobolsk.	
Schubert	1805	7 9		
Inochodzow	1784	5 45W	52 44	41 45
Pictet	1769	3 30	66 44	34 13
Astr. Jahrb.	1769	1 30E.	54 43	55 53
Islenief	1770	2 0	49 56	82 40
Inochodzow	1785	3 52W	59 13	40 10
Ditto	1783	8 0	51 40	39 15
Astr. Jahrb.	1770	4 50	48 42	44 27

Holland, Netherlands, Prussia, Switzerland.

Authority.	Date.	Variation.	Place, or Latitude.	Longitude.
	1600	9 0E.	51 13	4 24
Astr. Jahrb.	1767	17 30W	52 22	4 52
Ditto	1772	16 40	48 24	10 53
Humboldt	1798	18 26	Augsburg.	
Kirch	1717	10 42	52 32	13 21
Ditto	1717	10 52		
Ditto	1724	11 45		
Ditto	1725	11 56		
Ditto	1725	11 52		
	1751	14 16		
Reccard	1764	14 15		
Bernoulli	1770	16 9		
	1773	16 48		
	1774	16 54		
Schulze	1775	17 1½		
Ditto	1777	16 42		
Ditto	1777	16 45		
Ditto	1777	16 42	Berlin.	
Ditto	1778	16 45		
Ditto	1779	16 46½		
Ditto	1780	16 48		
	1782	17 47		
	1783	17 51		
	1784	17 57		
	1785	18 3		
	1786	18 20		
Bode	1787	17 44		
Ditto	1788	17 5		
Ditto	1805	18 5		
Ditto	1805	17 57		
Ditto	1805	18 2		
	1782	17 20	50 41	7 1
	1784	17 41	Bonn.	
	1787	18 1		
	1788	18 55		
	1628	1 0	54 21	18 38
	1642	3 15		
	1670	7 20	Dantzic.	
	1682	8 48		
	1760	11 0		
Koch	1811	13 48		
	1783	20 0	51 14	6 46
Köhler	1797	18 30	51 3	13 42
	1769	15 40	50 53	13 17
	1774	16 32	50 7	8 38
V. Swinden	1771	19 30	Franeker.	
	1773	18 30		
	1797	19 40	46 12	6 9
	1800	21 30		
	1801	21 26	Geneva.	
	1802	21 27		
	1803	21 18		
	1804	21 13		
	1770	15 50	47 4	15 25
Mayer	1777	16 48	51 32	9 44
Vander Heyde	1782	20 16	52 3	4 5
Tallinger	1787	22 40	47 15	11 20

Authority.	Date.	Variation.	Place, or Latitude.	Longitude.
		° ′	° ′	° ′
	1600	0 0	54 42N.	20 26E.
	1628	1 0W	Konigsberg.	
	1642	1 5		
Reccard	1774	13 30		
Heinsius	1749	13 0	51 19	12 21
	1776	19 48	49 29	8 27
	1785	19 44		
	1786	19 53	Manheim.	
	1787	20 2		
	1788	20 5		
Beitler	1783	10 52	56 39	23 43
	1786	21 14	51 30	3 27
	1788	21 56		
Eimmart	1685	5 5	49 27	11 7
	1774	15 45	50 5	14 25
	1775	16 15		
	1781	16 50		
	1782	17 44	Prague.	
Kampf	1786	18 9		
Ditto	1787	17 20		
	1767	19 0	51 55	4 29
	1784	17 49	49 0	12 3
	1785	19 1	Regensburg.	
	1786	19 11		
Krafft	1747	13 34	48 31	9 3
Ditto	1748	14 22		
Ditto	1749	14 45	Tubingen.	
Ditto	1750	14 30		
Ditto	1752	14 37		
Stopel	1814	19 0	52 32	
	1638	0 0	48 13	16 22
Chappe	1760	13 0	Vienna.	
	1781	18 40	49 46	9 55
	1782	18 40		
	1783	18 39		
	1784	18 30	Wurtzburg.	
	1785	18 33		
	1786	18 31		
	1787	18 35		
	1762	15 15	47 22	8 29

France.

Authority.	Date.	Variation.	Place, or Latitude.	Longitude.
De la Hire	1682	3 40W	43 35	7 8
Ditto	1680	1 20	43 29	8 30
	1767	17 26	50 44	1 37
De la Hire	1679	1 45	48 23	355 29
Verdun	1771	20 10	Brest.	
Rochon	1798	25 30		
Picard	1681	4 30	50 57	1 51
	1767	19 30	Calais.	
Beaulieu	1619	6 30E.	49 55	1 4
	1767	18 33W	51 2	2 22
	1767	19 15	49 29	0 6
Lowenorn	1782	22 15	Havre de grace.	
Niebuhr	1761	18 0	43 18	5 22
Humboldt	1798	20 55	Marseilles.	
Picard	1681	2 0	48 39	357 58
Cook	1776	23 1	48 28	354 41
Bellarmatus	1541	7 0E.	48 50	2 20
Oront	1550	8 0		
Sennertus	1580	11 30		
Nantonnier	1603	8 45		
	1610	8 0		
Petit	1630	4 30		
	1642	2 30		
	1659	2 0	Paris.	
	1660	1 0		
Picard	1664	0 40		
Ditto	1666	0 0		
Ditto	1667	0 15W		
Ditto	1670	1 30		
Ditto	1680	2 40		
Ditto	1681	2 30		
	1682	2 30		

Variation of the needle in various parts of the earth, continued.

Authority.	Date.	Variation.		Place, or Latitude.		Place, or Longitude.	
		°	′	°	′	°	′
La Hire	1683	3	50W	48	50N.	2	20E.
Ditto	1684	4	10				
Ditto	1685	4	10				
Ditto	1686	4	30				
Cassini	1687	5	12				
	1688	4	30				
Cassini	1689	6	0				
	1691	4	40				
La Hire	1692	5	50				
	1693	6	20				
	1695	6	48				
La Hire	1696	7	8				
	1697	7	40				
La Hire	1698	7	40				
Ditto		8	10				
Cassini	1699	7	50				
Ditto	1700	7	40				
La Hire		8	12				
Cassini	1701	8	25				
La Hire		8	48				
Ditto	1702	8	48				
Cassini		8	50				
Ditto	1703	9	0				
La Hire		9	6				
Cassini	1704	9	20				
La Hire		9	20				
Ditto	Mar. 1705	9	25				
Cassini	July.	9	36				
La Hire	Dec.	9	35				
Cassini	1706	9	48				
La Hire	1707	10	10				
Cassini	1708	10	15				
La Hire	1708	10	15				
Cassini	1709	10	40				
La Hire	1709	10	30				
Cassini	1710	10	50				
La Hire	1710	10	50	Paris.			
Cassini	1711	11	0				
La Hire	1711	10	50				
Cassini	1712	11	25				
La Hire	1712	11	15				
Cassini	1713	11	40				
La Hire	1713	11	12				
Cassini	1714	12	0				
La Hire	1714	11	30				
Ditto	1715	11	10				
Cassini	1716	12	15				
Ditto	1716	12	30				
La Hire	1716	12	30				
Cassini	1717	12	45				
La Hire	1717	12	40				
Cassini	1718	12	30				
La Hire	1718	12	30				
Maraldi	1719	12	30				
Ditto	1720	13	0				
Ditto	1721	13	0				
Ditto	1721	13	0				
Ditto	1722	13	0				
Ditto	1722	13	0				
Ditto	1723	13	0				
Ditto	1723	13	0				
Ditto	1724	13	0				
Ditto	1725	13	0				
Ditto	1725	13	15				
Ditto	1725	13	15				
Ditto	1726	13	45				
Ditto	1727	14	0				
Ditto	1728	13	50				
Ditto	1729	14	10				
Ditto	1730	14	25				
Ditto	1731	14	45				
Ditto	1732	15	15				
Ditto	1733	15	45				
Ditto	1734	15	35				
Buache	1734	15	40W	48	50N.	2	20E.
Maraldi	1735	15	45				
Ditto	1735	14	55				
Ditto	1736	15	40				
Ditto	1736	15	40				
Ditto	1737	14	45				
Cassini	1738	15	10				
Ditto	1738	15	20				
Maraldi	1739	15	30				
Ditto	1740	15	45				
Ditto	1740	15	30				
Ditto	1741	15	40				
Ditto	1742	15	40				
Ditto	1742	15	10				
Ditto	1743	15	10				
Fouchy	1744	16	15				
	1745	16	15				
	1746	16	15				
	1747	16	30				
	1748	16	15				
	1749	16	30				
	1750	17	15				
	1751	17	0				
	1752	17	15				
	1753	17	20				
	1754	17	15				
Maraldi	1755	17	30				
Ditto	1757	18	0				
Ditto	1758	18	0				
Ditto	1759	18	10				
Ditto	1760	18	30				
Ditto	1765	19	0				
Ditto	1770	19	55				
Le Monnier	1771	19	50				
Ditto	1772	20	12				
Maraldi	1772	20	2				
Le Monnier	1773	20	4				
Maraldi	1773	20	0	Paris.			
Le Monnier	1774	20	17				
Ditto	1774	20	12				
Ditto	1777	20	27				
Ditto	1778	20	37				
Ditto	1778	20	41				
Ditto	1779	20	31				
Ditto	1779	20	35				
Ditto	1779	20	33				
Ditto	1779	20	40				
Ditto	1779	20	31				
Ditto	1779	20	35				
Ditto	1780	20	35				
Ditto	1780	20	45				
Ditto	1780	20	40				
Ditto	1780	20	42				
Ditto	1780	20	49				
Ditto	May 18	20	42				
Ditto	19	21	4				
Ditto	20	20	49				
Ditto	21	20	45				
Ditto	22	20	42				
Ditto	23	20	57				
Ditto	24	20	44				
Ditto	25	20	48				
Ditto	26	20	46				
Ditto	27	20	47				
Ditto	28	20	43				
Ditto	29	20	47				
Ditto	31	20	42				
Ditto	June 1	20	45				
Ditto	2	20	45				
Ditto	3	20	48				
Ditto	4	20	39				
Ditto	5	20	49				
Ditto	6	20	39				
Ditto	9	20	39				
Ditto	13	20	48				

Variation of the needle in various parts of the earth, continued.

Authority.	Date.	Variation.	Place, or Latitude.	Longitude.
Le Monnier ...	June16	21° 4′W	48° 50′N. Paris.	2° 20′E.
Ditto	18	20 43		
Ditto	19	20 45		
Ditto	22	20 54		
Ditto	24	20 50		
Ditto	25	20 42		
Ditto	26	20 39		
Ditto	27	20 42		
Ditto	28	20 38		
Ditto	30	20 44		
Ditto	July 3	20 36		
Ditto	5	20 51		
Ditto	7	20 39		
Ditto	8	20 44		
Ditto	Aug.16	21 0		
Ditto	Sep. 19	20 45		
Ditto	21	20 50		
Ditto	22	20 42		
Ditto	23	20 46		
Ditto	Dec. 19	20 56		
Ditto	1781	20 47		
Ditto	Jan. 27	20 40		
Ditto	Mar. 7	20 47		
Ditto	Apr. 1	20 59		
Ditto	17	20 50		
Ditto	May 8	20 51		
Ditto	12	20 51		
Ditto	29	20 51		
Ditto	June23	20 57		
Ditto	Oct. 8	21 3		
Ditto	10	21 9		
Ditto	11	21 3		
Ditto	1782	21 1		
Ditto	June28	21 16		
Ditto	July 4	21 12		
Ditto	1783	21 12		
Ditto	June23	21 22		
Ditto	Aug. 5	21 27		
Ditto	1784	21 27		
Ditto	Feb. 29	21 24		
Ditto	1785	21 35		
Ditto	1786	21 37		
Ditto	June21	21 27		
Ditto	1789	21 56		
Ditto	1790	21 0		
Ditto	Aug. 7	22 52		
Ditto	1791	22 2		
Ditto	July 30	22 4		
Ditto	1792	22 45		
Ditto	1793	22 49		
Ditto	1798	22 17		
Ditto	1799	22 49		
Cotte	1799	22 0		
Ditto	1800	22 12		
Ditto	1801	22 1		
Ditto	1802	21 45		
Bouvard	1802	22 3		
Ditto	1802	21 45		
Cotte	1803	21 59		
Ditto	1804	22 5		
Bouvard	1807	22 34		
Ditto	1814	22 34		
La'Hire	1680	1 20	45 37	358 58
Ditto	1682	3 45	43 7	5 57
Marcorelle	1747	15 10	43 36 Toulouse.	1 25
Ditto	1748	15 0		
Ditto	1749	15 10		
Ditto	1750	16 5		
Ditto	1751	16 0		
Ditto	1752	15 45		
Ditto	1753	16 45		
Ditto	1754	16 40		
Ditto	1755	15 0		
Ditto	1756	15 45		

England, Ireland, &c.

Authority.	Date.	Variation.	Place, or Latitude.	Longitude.
Sturmy	1666	1° 27′W	51° 28′N. Bristol.	357° 10′E.
Ditto	1667	1 33		
Kendrick	1745	18 0	53 21 Dublin.	353 41
Gibson	1751	19 0		
Harding	1772	23 30		
Ditto	1786	26 21		
Ditto	1788	26 50		
Ditto	1790	27 15		
Ditto	1791	27 23		
Burrows	1580	11 15E.	51 31 London.	00 00
Gunter	1622	5 56½		
Gellibrand......	1634	4 6		
Bond	1657	0 0W		
Gellibrand......	1665	1 22½		
Halley	1672	2 30		
Ditto	1692	6 0		
Graham.........	1723	14 17		
Ditto	1745	17 0		
Ditto	1745	17 0		
Ditto	1746	17 10		
Ditto	Mar.21	17 10		
Ditto	Apr. 22	17 15		
Ditto	May 4	17 18		
Ditto	14	17 20		
Ditto	16	17 15		
Ditto	Dec. 18	17 25		
Ditto	1747	17 30		
Ditto	1747	17 40		
Ditto	1748	17 40		
Heberden	1773	21 9		
Caverdish	1774	21 16		
Ditto	1775	21 43		
Gilpin	1786	23 17		
Ditto	1787	23 19		
Ditto	1788	23 32		
Ditto	1789	23 19		
Ditto	1790	23 39		
Ditto	1791	23 36		
Ditto	1792	23 36		
Ditto	1793	23 49		
Ditto	1794	23 56		
Ditto	1795	23 57		
Ditto	1796	24 0		
Ditto	1797	24 1		
Ditto	1798	24 0·6		
Ditto	1799	24 1·8		
Ditto	1800	24 3·6		
Ditto	1801	24 4·2		
Ditto	1802	24 6·7		
Ditto	1803	24 8·8		
Ditto	1804	24 8·4		
Ditto	1805	24 8·8		
Ditto	1809	24 11·0		
Ditto	1814	24 16·7		
Ditto	July	24 17·9		
Ditto	Aug.	24 21·2		
Ditto	Sept.	24 20·5		
Hoxton	1733	13 27W	Plymouth.	
Hutchins	1774	24 0	58 59	356 23

Portugal, Spain, Italy.

Authority.	Date.	Variation.	Place, or Latitude.	Longitude.
Humboldt	1798	19 25W	Aranjuez.	
Butler	1733	14 12	Albcrne Island.	
	1761	16 15	41 33	351 40
Lana	1676	4 0E.	45 29	10 7
Feuillée........	1724	5 25W	36 32 Cadiz.	353 43
Fleurien	1769	17 15		
Ditto	1769	18 40		
Verdun	1771	18 0		
Ulloa...........	1776	19 42		
Lowenorn......	1791	21 56		

Variation of the needle in various parts of the earth, continued.

Authority.	Date.	Variation.	Place, or Latitude.	Longitude.
		° ′	° ′	° ′
Butler	1733	13 49W	Cape St. Vincent.	
Ditto	1733	13 56	Cape de Gata, Spain.	
Ditto	1734	14 20	Cape St. Mary, Portugal.	
Wright	1589	7 40E.	42 52	350 41
	1768	21 4W	Cape Finisterre.	
Marchand	1792	19 23	38 40	
Lana	1677	2 0	44 54	11 36
Butler	1733	13 38	36 5	355 1
Ross	1761	17 11	Gibraltar Bay.	
Marchand	1792	22 6		
Martinius	1638	7 39E.	38 42	350 51
	1668	0 50W		
Ditto	1683	3 0		
Couplet	1697	4 18		
Noël	1706	6 30	Lisbon.	
Ross	1762	17 32		
Borda	1776	19 0		
Lowenorn	1782	19 51		
Condamine	1756	15 35	43 27	13 45
Humboldt	1799	19 59	40 25	356 31
Chazelles	1694	9 15	35 54	14 28
Ditto	1694	9 45	Malta.	
Feuillée	1708	10 25		
Butler	1733	14 34	Minorca.	
Poleni	1725	13 0	45 24	11 52
Ditto	1730	13 0	45 24	11 52
Ditto	1770	16 20	Padua.	
Auzout	1670	2 15	41 54	12 28
Ditto	1681	5 0		
Cassini	1695	7 30		
Ditto	1730	11 0		
	1782	16 49		
	1783	16 49	Rome.	
	1784	16 54		
	1785	17 0		
	1786	17 4		
	1787	17 7		
	1788	17 12		

Hungary and European Turkey.

Authority.	Date.	Variation.	Place, or Latitude.	Longitude.
	1771	9 25W	46 12	30 44
Astr. Jahrb.	1772	9 45	46 51	29 36
Inochodzow	1772	11 36	44 27	26 8
	1600	0 0	41 1	28 56
Fournier	1625	2 0		
Chazelles	1694	9 0	Constantinople	
Ditto	1694	12 0		
Ditto	1781	16 45	47 30	19 0
Ditto	1782	15 58		
Ditto	1783	15 36		
Ditto	1784	15 40	Buda.	
Ditto	1785	15 48		
Ditto	1787	16 26		
Ditto	1788	16 36		

Asia.

Authority.	Date.	Variation.	Place, or Latitude.	Longitude.
Chazelles	1694	14 22W	36 35	36 20
Middleton	1612	13 40	13 0	47 22
Daunton	1612	12 40		
	1674	15 0		
Mathews	1723	13 50		
Ditto	1723	13 42		
Duchatz	1689	5 0	21 0	96 20
Beauchamp	1781	12 30	36 11	37 11
Mathews	1723	14 20	12 40	44 25
Davis	1610	19 50	6 0 S.	72 0
Niebuhr	1762	11 50	14 31	43 1
Paris	1612	4 48E	0 51	AmazonBay
	1676	12 0W	19 0	71 45
Mathews	1721	5 12		
Ditto	Sept.12	5 16	Bombay.	
Ditto	1722	5 7		

Authority.	Date.	Variation.	Place, or Latitude.	Longitude.
		° ′	° ′	° ′
Mathews	1723	5 10W	Bombay.	
Yeates	1817	0 0		
Marchand	1791	0 0	1 30 S.	106 2
	1680	8 20	21 22	86 0
Mathews	1722	3 33	20 50	86 10
Vermanus	1616	1 30	22 13	113 35
Fontenay	1690	2 25		
	1722	1 30	Canton.	
Yeates	1817	0 0		
Cook	1779	0 32	22 8	113 43
Mathews	1722	4 5	22 34	88 22
Ditto	Nov.15	4 9	Calcutta.	
Ditto	1722	5 40	15 0	73 50
Carteret	1767	1 16	5 30 S.	117 45
Mathews	1721	5 27	18 40N.	72 0
Davis	1601	16 0	7 34	78 14
Best	1612	14 0		
Boners	1612	14 30		
	1620	14 20		
	1680	8 45	Cape Comorin.	
	1688	7 30		
Mathews	1723	2 51		
Ditto	Oct. 18	2 48		
Ditto	19	2 50		
Ditto	20	3 9		
Ditto	1722	2 12	Ceylon.	
Ditto	1722	2 21		
Castleton	1613	13 24	6 0	80 37
Mathews	1723	2 46		
Ditto	1723	2 45		
Bondier	1731	3 0	22 51	88 29
Ditto	1735	2 0		
	1743	1 20	Chandernagor.	
	1745	1 0		
	1747	0 0		
	1750	0 0		
Pring	1614	15 0	9 58	76 3
Noël	1706	6 20		
Mathews	1722	3 34	Cochin.	
	1722	3 53		
	1724	3 26		
	1724	4 16		
Best	1612	16 30	Daman.	
Daunton	1610	15 34	17 32	72 45
Ditto	1611	16 30		
	1712	12 0	43 55	47 23
Saris	1612	15 2		
Ditto	1613	5 20E.	2 35	126 50
Ditto	1613	2 50E.	33 30	329 25
Aleuisius	1609	16 0W	15 31	73 45
Noël	1706	6 40		
Mathews	1722	4 57	Goa.	
Ditto	1723	5 13		
Ditto	1723	5 3		
Ditto	1724	5 41		
Payton	1613	17 15	24 30	63 42
Childe	1616	18 0		
Saris	1613	0 50	18 19	Haynam.
Ditto	1609	3 0	6 20 S.	105 21
Wallis	1767	1 25	6 9 S.	106 54
Carteret	1768	0 25	Batavia.	
Davis	1605	3 20	6 45 S.	104 0
Beauchamp	1787	7 30	32 25	52 50
	1797	8 14	41 12	39 53
Middleton	1607	4 13	5 30 S.	101 35
Cook	1776	27 44	48 41	69 11
Gonye	1685	4 45	14 44	101 0
Jesuiten	1685	0 30 ?	Louveau.	
Fontenoy	1686	4 45		
Richaud	1688	4 30		
Wallis	1767	0 0	3 11S.	106 17
Mathews	1722	2 52W	13 15	79 57
Ditto	1723	3 16	Madras.	
Yeates	1817	1 0E.		
Carteret	1768	0 30W	7 7S.	112 49

Variation of the needle in various parts of the earth, continued.

Authority.	Date.	Variation.		Place, or Latitude.		Longitude.	
		°	′	°	′	°	′
Mathews	1722	5	24W	13	0N.	74	34
Hippon	1610	12	22	16	11	80	54
Saris	1612	4	12E.	0	18	126	20
Ditto	1613	3	28E.				
Davis	1605	17	0W	6	30	73	50
Mathews	1722	4	16				
Payton	1613	20	10	20	30	56	50
Mathews	1723	13	34	13	16	43	10
Panton	1776	11	20	Mocha.			
Carteret	1767	1	45E.	6	15	127	12
Thomas	1685	0	0	32	5	118	47
Davis	1605	7	5W	6	30	93	20
Hippon	1611	13	15	13	35	80	20
Marlowe	1613	13	10				
Hippon	1611	12	47	16	0	80	20
Ditto	1611	12	22	Petapolli.			
Marlowe	1613	13	50				
Richand	1689	7	0	11	56S.	79	51
Wallis	1767	1	0	6	36S.	105	17
Cook	1780	0	54	Java.			
Dales	1620	1	0	8	40	106	19
Cook	1780	0	14				
Amiot	1755	2	0	39	54	116	25
	1677	23	30	38	0S.	77	20
Davis	1610	23	30	11	0S.	65	35
Mathews	1722	4	58	16	50	73	10
Payton	1613	19	20	22	34	60	21
Daunton	1610	16	40				
Middleton	1611	16	30	Sually.			
Boners	1612	17	0				
Millword	1615	3	30	Sunda Straits.			
Payton	1613	16	45	24	30		
	1797	10	18	42	14	35	12
Best	1611	16	23	21	10	72	22
Boners	1612	16	50				
Mathews	1722	5	50	Surate.			
Ditto	1723	5	59				
	1723	5	22				
Fontenay	1689	3	17	34	17	108	57
Davis	1610	6	25	5	22	95	34
Macdonald	1794	1	10E.	3	45S.	101	50
	1795	1	8E.	Sumatra.			
Marlowe	1612	4	10W	0	50S.	99	43
Castleton	1613	4	50	0	38S.	Priaman.	
Mathews	1722	4	21	Tellicherry.			
Boners	1612	4	40E.	Tecu Island.			
Werbiest	1682	0	0W	41	56	123	54
Ditto	1682	1	40E.	44	20	127	15

Africa.

Authority.	Date.	Variation.		Place, or Latitude.		Longitude.	
Gravius	1638	5	45W	31	12	29	55
Niebuhr	1761	11	4	Alexandria.			
Nouët	1798	13	6				
	1678	1	0E.	7	56S.	345	39
Caille	1754	8	6W				
Wallis	1768	9	53	Ascension Island.			
Cook	1775	10	52				
Bönsöe	1806	15	40				
Mathews	1726	11	25	5	28	1	57
Saris	1611	12	1	16	10S.	37	30
Ditto	1612	17	23	11	27		
Mathews	1723	12	43	12	30	51	33
Condamine	1731	14	0	36	49	2	13
	1589	3	5E.	38	32	331	27
Cook	1775	22	7W	Fayal Bay.			
	1600	0	0	39	40	330	2
Davis	1610	1	40E.	36	57	334	51
Mathews	1723	14	20W	12	40	44	25
Davis	1610	19	50	6	0S.	72	0
Castleton	1614	22	48	20	58S.	55	30
Mathews	1722	19	49	St. Paul's Bay.			
Childe	1616	13	12	16	34S.	Boobam.	
Mathews	1726	11	55	5	0	0	50
Ditto	1726	12	10	5	0	0	50

Authority.	Date.	Variation.		Place, or Latitude.		Longitude.	
		°	′	°	′	°	′
Feuillée	1724	5	0W	28	0N.	342	20
Fleurieu	1769	17	30	Ferro.			
Tranberg	1802	19	55				
Davis	1610	6	6E.	28	32	346	56
Gray	1727	6	58W	32	36	342	57
Wallis	1766	14	10				
Carteret	1766	16	0				
Verdun	1771	18	0	Madeira.			
Loweuorn	1783	18	22				
Tranberg	1802	20	21				
Mudge	1820	19	59				
Davis	1610	6	6E.	Great Canaria.			
Fleurieu	1769	15	43W	28	27	343	45
Verdun	1770	15	30				
Cook	1776	14	41				
Don Ulloa	1776	15	55	Teneriffe.			
La Perouse	1785	15	25				
Bligh	1788	20	1				
	1792	16	32				
Krusenstern	1803	16	1				
Saris	1611	13	0	15	40S.	37	30
	1721	19	12				
Mathews	1721	19	44				
Ditto	1721	20	33				
Ditto	1722	20	39	Anjouan.			
		20	33				
Mathews	1725	4	5	14	54	336	31
Carteret	1766	8	20				
Wallis	1766	8	20	St. Jago.			
Verdun	1772	10	45				
Marchand	1791	14	12				
Mudge	1820	15	55				
Mathews	1725	3	32	15	6	336	50
Cook	1776	9	32¼	St. Mayo.			
Davis	1610	3	30E.	16	38	337	4
Chazelles	1694	12	15W	30	3	31	7
Niebuhr	1761	12	25	Cairo.			
Ditto	1762	11	40				
Nouët	1798	12	0				
Chazelles	1694	12	30	31	43	31	23
Saris	1611	17	36	6	27	48	50
Ditto	1611	17	20	6	27	48	50
Cook	1776	26	15	47	15S.	40	48
Keeling	1609	21	0	20	10S.	57	28
Mathews	1722	18	46				
	1722	18	39	Isle of France.			
	1722	19	7				
	1722	19	45				
Yeates	1817	15	0				
Davis	1605	0	30E.	33	55S.	18	24
Keeling	1609	0	12W	Cape Aguglias.			
Pring	1614	1	30	Saldanha Bay.			
Daunton	1614	1	45				
	1667	7	15				
Leydeker	1675	8	28				
	1687	8	30				
	1699	11	0				
	1702	12	50				
	1706	13	40				
	1708	14	0				
Mathews	1721	16	25				
	1724	16	27	Table Bay.			
	1724	16	18				
Caille	1752	19	0				
Wallis	1768	19	30				
Cook	1772	20	26				
Ditto	1775	21	14				
Ditto	1780	22	16				
Bligh	1788	23	16				
Vancouver	1791	25	40				
Dentrecasteaux	1792	24	30				
Bönsöe	1804	25	4				
Varin	1682	0	0	14	40	342	35
Fleurieu	1769	12	15	Goree.			
Verdun	1772	10	30				

Variation of the needle in various parts of the earth, continued.

Authority.	Date.	Variation.		Place, or Latitude.		Longitude.	
		°	′	°	′	°	′
Middleton	1610	17	35W	11	45N.	52	32
Saris	1612	17	34				
Mathews	1721	11	11	Guardafui.			
Ditto	1723	12	34				
Middleton	1610	18	55	11	72		
Saris	1612	17	23				
	1600	8	0E.	15	55 S.	354	12
Kolthurst	1604	7	45				
Davis	1610	7	13				
	1623	6	0				
Halley	1677	0	40				
	1691	1	0W				
Mathews	1724	7	30				
Nicholson	1764	11	38	St. Helena.			
Wallis	1768	12	47				
Cook	1775	12	18				
Lodberg	1785	14	18				
Hunter	1789	15	30				
Macdonald	1796	15	48½				
Krusenstern	1806	17	18				
Keeling	1607	15	30	23	35 S.	43	9
Ditto	1607	15	26				
Davis	1610	14	50	Augustin's Bay			
Saris	1611	15	11	Madagascar.			
Payton	1613	15	40				
Mathews	1721	23	48				
Flacourt	1661	22	30	15	28 S.	49	56
Le Gentil	1761	18	0	Antongil Bay.			
Flacourt	1661	19	0	25	1 S.	46	49
Le Gentil	1761	22	7½	Fort Dauphin.			
Ditto	1762	16	45	17	40 S.	49	56
Lancaster	1600	16	0	12	42 S.	47	40
Davis	1610	16	40				
Mathews	1722	19	1	St. Sebastian.			
	1722	18	36				
Davis	1610	19	50	16	30 S.		
Mathews	1722	19	53				
	1722	19	52	Madagascar.			
	1722	19	25				
Mathews	1722	20	0	13	4 S.	47	5
	1722	19	45	Madagascar.			
	1722	19	5				
Butler	1735	12	19	33	43 S.	353	22
Keeling	1608	1	50E.	8	36	346	37
Mathews	1725	5	12W	Sierra Leone.			
Lancaster	1606	21	0	31	48 S.		
Middleton	1611	16	0	12	47	56	0
Saris	1612	17	22				
	1674	17	0				
Mathews	1723	11	25				
Ditto	1723	11	36				
Ditto	1723	11	38	Socotra Island.			
Ditto	1723	12	4				
Ditto	1723	11	57				
Ditto	1723	12	20				
Panton	1776	8	6				
Yeates	1817	6	0				
Mathews	1726	14	48				
Ditto	1726	14	32	St. Thomas.			
Butler	1733	13	22	32	54	13	5
Schouten	1615	12	0E.	20	45 S.	330	30

America and its Islands.

Authority.	Date.	Variation.		Place, or Latitude.		Longitude.	
Anson	1744	3	0E.	16	51	260	10
Middleton	1730	23	0W	52	22	277	22
Hutchins	1774	17	0	Albany Fort.			
Mathews	1727	4	28E.	17	4	298	5
Halley	1670	5	30E.	8	0 S.	324	22
Barentz	1596	13	0W	74	10	16	0
Poole	1610	13	30	Bear Island.			
Willard	1781	7	6	42	36		
	1781	7	0				
	July 30	7	6	Beverley.			
	31	7	0				
		°	′	°	′	°	′
	1781	7	0W	42	36N.		
	Aug. 6	7	1	Beverley.			
Mathews	1726	4	24E.	13	5	300	23
Ross	1760	4	30W	Barbadoes.			
Ditto	1761	3	47				
Mathews	1726	3	26E.				
Ditto	1726	7	48	Bestimentos Island.			
	1708	9	0W	42	25	289	23
	1741	7	30	Boston.			
Baffin	1615	24	0	60	40	292	22
Middleton	1730	39	0	Button's Islands.			
Feuillée	1708	15	32E.	34	35 S.	301	29
Noël	1708	11	30	13	10 S.		
Brattle	1708	9	0W	42	23½	289	18
Winthorp	1763	7	0				
Williams	1780	7	2	Cambridge.			
Ditto	1782	6	45				
Ditto	1783	6	52				
Feuillée	1705	7	12E.	10	26	284	30
Mathews	1726	6	50	Carthagena.			
Lowenorn	1783	3	20	17	45	295	11
Thomson	1817	2	50	St. Croix.			
Mathews	1726	4	10	17	19	297	7
Ditto	1726	4	0	St. Christopher.			
Ditto	1726	4	24	23	2	278	16
Harris	1732	4	30	23	8	277	28
Richer	1672	11	0	4	56	307	45
Hayes	1982	5	30	Cayenne.			
Feuillée	1709	10	20	36	43 S.	287	20
Ditto	1700	8	32	29	55 S.	288	44
Ditto	1704	6	40	Coquimbo.			
Humboldt	1799	4	14	10	28	295	50
Hoxton	1732	4	58W	Chesapeak Bay.			
	1789	6	45	Cod Bay.			
Hall	1605	12	15	62	38	308	44
Frezier	1712	12	0E.	27	19 S.	312	20
La Perouse	1785	12	0	St. Catherine's Island.			
Krusenstern	1804	7	51				
Dampier	1684	4	28	5	0	282	30
Vancouver	1792	21	30	48	2	237	22
Ditto	1792	19	16	50	11	235	21
Ditto	1792	11	0	32	39	242	43
Mathews	1726	3	27	16	25	298	49
Verdun	1772	5	20	19	46	287	42
Don Ulloa	1776	5	30	St. Domingo.			
Lowenorn	1783	5	32	Cape François.			
Gray	1728	6	2	17	18	288	21
	1670	12	10	22	57 S.	318	28
Cook	1768	6	40				
Davis	1610	8	10	3	56 S.	327	22
Mathews	1726	3	26	Cape Florida.			
Cook	1774	24	43	55	52 S.	289	57
Ditto	1769	24	9	54	50 S.	294	35
Ginge	1784	50	30W	64	10	308	12
Ditto	1787	51	21				
Mathews	1726	3	22E.	15	59½	298	15
Feuillée	1707	20	0	56	6 S.	Hermit. Isl.	
Carteret	1767	11	0	33	40 S.	281	0
Mathews	1726	6	2	17	6	283	1
Ditto	1726	4	31	Port		283	14½
Harris	1732	6	2	Royal,			
Ditto	1726	6	20	Jamaica.			
Yeates	1817	6	0				
Feuillée	1709	6	15	12	3 S.	282	52
Don Alzate	1769	5	20	19	26	260	54½
	1769	5	35	Mexico.			
Hayes	1682	4	10	14	36	298	51
Feuillée	1704	6	5				
Ditto	1704	6	10	Martinique.			
Ditto	1706	6	10				
Ross	1760	5	41				
Mathews	1726	3	40	15	32	300	44
Feuillée	1704	7	6	11	20	285	51
Byron	1765	9	36	33	45 S.	279	6
Carteret	1767	10	24				

Variation of the needle in various parts of the earth, continued.

Authority.	Date.	Variation.		Place, or Latitude.		Longitude	
		°	′	°	′	°	′
Vancouver......	1795	9	15E.	33	45N.	279.	6
Carreri	1693	2	0	40	29	235	31
Perouse........	1786	14	24				
Hutchins	1774	17	0W	51	20	277	21
Pickersgill	1776	50	36	64	55	307	4
Vancouver......	1795	12	22E.	36	36	238	9
Ross	1760	5	32	16	48	297	52
Verdun	1772	19	15W	46	46	303	42
Cook	1778	19	45E.	49	35	233	23
Vancouver......	1792	18	22	Nootka Sound.			
Cook	1778	25	45	64	31	197	5
	1770	3	8W	40	10	284	44
Feuillée........	1704	7	25E.	9	33	280	10
Ditto	1707	7	0	Pisco.			
Couplet........	1698	5	35	6	58S.		
Bouguer	1742	8	30	0	14S.	281	15
Bressan........	1649	16	0W	46	55	290	7
Des Hayes......	1686	15	30	Quebec.			
Cook	1768	7	34E.	22	54	317	15
Hunter	1787	6	12				
Baffin..........	1615	24	6W	61	45	292	25
Ditto	1615	27	30	62	30	287	22
Ditto	1616	57	0	78	0	293	0
Espinoza	1794	14	28E.	33	28S.	289	10
Dampier	1683	23	10	51	0S.	293	42
Baffin..........	1613	13	11W	79	44	Bell Sound.	
Barentz........	1596	16	0	79	44		
Poole	1610	16	0	76	55		
Baffin..........	1613	13	0	Horn Sound.			
Ditto	1613	12	14				
Fotherby	1614	25	0	79	34	Magdalen S.	
Baffin..........	1613	15	21	78	24	Poppy Bay.	
Barentz........	1596	16	0	79	42	Read Bay.	
Phipps	1773	20	38	79	53	12	1
Cook	1778	19	59E.	53	5	193	28
Don Ulloa......	1776	7	30	Verra Cruz.			
Feuillée........	1709	9	30	33	0S.	287	46
Vancouver......	1795	14	49	Valparaiso.			
Basil Hall	1821	14	43				
Narborough	1670	8	10	Valdivia.			
Middleton	1725	21	0W	58	48	265	48
Ditto	1738	18	0	Prince of Wales Fort.			
Ditto	1742	17	0				
Wales	1769	9	41				
Feuillée........	1710	6	38E.	17	36S.	288	47
Wells..........	1686	8	45W	40	43		
Burnet	1723	7	20				
	1750	6	22	New York.			
Evans..........	1755	5	0				
	1789	4	20				
Parry	1820	60	20	67	37	265	34
Ditto	1820	118	16	71	10	261	46
Ditto	1820	82	2	71	13	262	57
Ditto	1820	108	46	69	40	260	50
Ditto	1820	127	46E.	72	12	257	14

Australasia.

Authority.	Date.	Variation.		Place, or Latitude.		Longitude	
Dentrecasteaux..	1792	1	14W	3	42S.	128	29
Tasman	1643	6	20E.	20	15	185	11
Cook	1774	9	47	Anamoca.			
Ditto	1777	8	30				
Carteret........	1767	6	30	2	19	145	23
Cook	1773	14	3	40	55	174	1
Ditto	1770	8	0	34	0	151	23
Dentrecasteaux..	1792	0	54W	3	22	127	21
Albrechtsen	1799	0	18E.	Bourou Island.			
Ditto	1799	0	19				
Wallis	1767	9	0	15	50	184	42
Vancouver......	1795	7	45	5	35N.	273	5
Broughton	1795	5	14	5	57	209	35
Wallis	1767	7	0	19	18	219	18
Ditto	1767	5	20	19	18	221	48
Cook	1773	15	2	41	2	174	0
Ditto	1778	6	22	1	59N.	202	30

Authority.	Date.	Variation.		Place, or Latitude.		Longitude.	
		°	′	°	′	°	′
Bayley	1773	13	31	41	6S.	173	26
Wales	1773	14	15	Charlotte Sound.			
Ditto	1773	13	52				
Ditto	1774	14	9	41	6	174	25
Cook	1777	12	4	Ship Cove.			
Tasman	1642	0	0	42	25	142	8
Bayley	1793	10	34E.	43	22	147	29
Ditto	1773	9	6				
Cook	1777	5	15	Adventure Bay.			
Bligh..........	1788	8	33				
Dentrecasteaux..	1792	7	40	43	32	147	6
Wales	1773	13	49	45	47	166	18
Vancourer......	1793	14	56	45	45	166	16
Forest	1775	1	30	0	21	131	0
Cook	1774	10	6	18	43	169	28
Wallis	1767	6	0	19	20	221	22
Bayley	1773	9	3	21	21	185	26
Cook	1774	4	30	27	6	250	14
		3	32	Easter Island.			
La Perouse	1786	3	10				
Cook	1774	10	29	14	39	166	49
Bayley	1773	12	12	40	18	173	10
Vancouver......	1794	8	0	0	44	267	54
Carteret........	1767	5	20	5	0	152	11
Vancouver......	1791	5	20W	35	6	118	14
Byron	1765	1	20	21	8N.	118	6
Wallis	1767	1	3	21	4	120	52
Ditto	1767	7	10E.	19	11	219	46
Carteret........	1767	8	20	7	56	158	48
Wallis	1767	7	40	16	46	205	39
Tasman	1643	8	40	34	35		
Cook	1769	11	25	34	0	172	0
Bayley	1773	6	31	16	45	208	51
Cook	1777	5	13¼W	16	43N.	208	52
Grant..........	1800	2	50E.	38	29	144	40
Cook	1777	8	12	20	15	185	11
Wallis	1767	10	0	15	53	184	39
Cook	1777	10	12	20	15	185	11
Le Gentil	1766	0	15W	14	34N.	120	51
La Perouse	1787	0	33	14	29N.	121	11
Bligh..........	1788	5	36E.	17	53	211	49
Cook	1769	12	40	35	50	174	45
Tasman	1642	9	0	40	30	174	0
Cook	1770	13	5				
La Perouse	1786	8	42	20	50N.	203	55
Cook	1774	11	3	29	2	168	16
Dentrecasteaux..	1793	9	30	20	17	164	36
Tasman	1643	5	30	0	26	126	0
Wallis	1767	6	0	17	48	210	37
Ditto	1767	5	30	17	30	209	52
Cook	1769	4	45½	17	29	210	30
Ditto	1773	5	40	17	29¼	210	39
Ditto	1774	5	49	17	29¼	210	18
Ditto	1777	5	34	17	29¼	210	22
Vancouver......	1794	6	12	17	30⅓	210	24
Broughton......	1795	3	43				
Cook	1774	3	31	9	55	220	51
Marchand	1791	5	50	9	59	221	51
Wallis	1767	6	0	17	51	212	22
Broughton......	1796	10	54	21	51N.	199	37
Cook	1779	8	6	19	28N.	204	0
Vancouver......	1793	7	47	Owhyee.			
Broughton......	1796	8	15				
		9	12				
Cook	1774	10	24	20	18	164	41
Ditto	1774	10	1	18	0	196	0
Carteret........	1767	2	46	25	2	226	43
Cook	1774	8	33	20	18	164	41
Wallis	1767	10	0	10	10N.	167	47
Ditto	1767	1	0W	2	28N.	104	52
Cook	1770	8	0E.	33	50	151	23
Hunter	1788	8	30	33	50	151	25
Broughton......	1795	11	9	33	52	151	10
Tasman	1643	7	30	22	26	184	1
	1773	9	51				

Variation of the needle in various parts of the earth, continued.

Authority.	Date.	Variation.	Place, or Latitude.	Place, or Longitude.
Cook	1770	14° 0′E.	41° 37′S.	174° 30′
Bayley	1773	13 1		
Wallis	1767	8 0	16 28	204 28
Ditto	1767	6 30	17 28	208 48
Cook	1774	11 25	19 3	190 20
Bayley	1773	13 40	38 21	178 34
Tasman	1643	7 15	21 9	184 55
Cook	1773	10 11	Tongatabo.	
Ditto	1777	9 44		
Dentrecasteaux	1793	10 0	21 8	184 50
Wallis	1767	6 20	14 58N.	150 42
Cook	1774	12 28	19 48	182 0
Ditto	1774	7 14	19 32	169 49
Bayley	1773	6 10	16 45	208 35
Wallis	1774	6 4	16 46	208 23
Cook	1777	6 19	16 45½	208 25
Broughton	1796	9 41	21 18N.	202 1
Forest	1775	1 0	0 0	127 0
Dentrecasteaux	1793	1 14	0 1	131 13
Wallis	1767	6 0	19 26	221 56
Ditto	1767	7 0	19 0	218 46
Ditto	1767	10 0	13 18	182 52
Ditto	1767	6 0	17 28	209 36
Oxley, (In the interior of New Holland)	1817	7 47	33 40	148 21
	ditto	7 0	33 15	147 16
	ditto	7 18	34 8	146 3
Oxley, (In the interior of New Holland)	1817	7° 44′E.	34° 11′ S.	145° 56′
	ditto	8 0	34 13	146 0
	ditto	7 45	33 24	145 46
	ditto	7 30	33 22	145 24
	ditto	6 49	33 22	145 5
	ditto	7 25	33 53	144 33
	ditto	7 51	33 14	146 7
	ditto	7 23	33 4	146 31
	ditto	5 20	32 47	147 23
	ditto	7 18	32 44	147 46
	ditto	8 38	32 32	148 51
	ditto	8 40	33 27	149 28
	1818	8 22	31 49	147 52
	ditto	7 48	31 18	147 31
	ditto	8 14	31 14	148 18
	ditto	6 22	31 13	148 38
	ditto	7 0	31 13	148 41
	ditto	8 42	30 57	149 20
	ditto	9 58	31 4	151 5
	ditto	8 51	31 1	151 26
	ditto	10 39	31 19	151 49
	ditto	8 22	31 23	152 8
	ditto	10 5	31 24	152 51
	ditto	10 11	31 25	151 53
	ditto	9 33	31 38	152 48

Note. In the preceding table of variations in Australasia, the latitudes not marked by the letter N are South; in all the preceding part, the latitudes not marked with S are North. The longitude throughout is East.

Table, shewing the dip of the needle in various parts of the earth, from the earliest observations to the present time, taken on shore.

Place.	Authority.	Date.	Dip.		Latitude.		Longitude.	
			°	′	°	′	°	′
Ancona	Basil Hall	1821	6	15 S.	11	46 S.	282	52
Alexandria	Nouet	1799	47	30 N.	31	12 N.	29	55
Altorf	Humboldt	1805	66	53 N.				
Antonio	Ditto	1800	14	25 N.	0	0	282	8
Ascension	La Caille	1754	11	10 N.	7	56 S.	345	37
	Cook	1775	8	57 N.				
Agulhas	Ekeberg	1771	45	30 S.	35	15 S.		
Antongil Bay	Le Gentil	1761	46	9 S.	15	28 S.	50	33
Anjenga	Panton	1776	4	23 S.	8	41 N.	76	54
Albany Fort	Hutchins	1775	79	20 N.	52	22 N.	277	22
Amboina	Dentrecasteaux	1792	20	37 S.	3	42 S.	128	29
Brest	Ditto	1791	71	30 N.	48	23 N.	355	29
Berlin		1755	71	45 N.	52	32 N.	13	21
Ditto		1761	72	0 N.				
Ditto	Euler	1769	72	45 N.				
Ditto	Humboldt	1805	69	53 N.				
Bologna	Ditto	1805	64	48 N.	44	30 N.	11	20
Barcelona	Ditto	1800	64	37 N.	41	23 N.	2	12
Bareedy	Panton	1776	34	21 N.	24	16 N.	38	30
Bocca Tigris	Ekeberg	1770	26	0 N.	23	0 N.		
Birch Bay	Vancouver	1794	73	13 N.	48	53 N.	237	33
Bourbon Island	Manevillette	1751	52	52 S.	20	52 S.	55	30
	La Caille	1754	54	22 S.				
Boulogne	Barlow	1820	69	45 N.				
St. Paul's	Le Gentil	1766	52	55 S.	20	58 S.	55	30
Botany Bay	Cook	1770	67	1 S	34	0 S.	151	23
Bourou Island	Dentrecasteaux	1793	20	30 S.	3	22 S.	127	21
Carrichana	Humboldt	1800	30	24 N.	6	34 N.	292	2
Cumana	Ditto	1799	39	47 N.	10	28 N.	295	50
Cumanacoa	Ditto	1799	38	53 N.	10	16 N.	296	1
Cathagena	Ditto	1799	35	15 N.	10	26 N.	284	30
Cocollar	Ditto	1799	38	20 N.	10	10 N.	296	1
Cuenca	Ditto	1799	8	43 N.	2	55 N.	280	46
Como	Ditto	1805	66	12 N.	45	44 N.	8	53
Cambrilis	Ditto	1799	64	34 N.				
Cenis	Ditto	1805	66	9 N.				
Carlos, Saint	Ditto	1800	20	47 N.	1	52 N.	292	10
Chancay	Ditto	1799	10	35 N.	11	33 N.	282	36
Cherson	Inochodzof	1782	65	0 N.	46	38 N.	32	40
Cruz, Saint, Teneriffe	Cook	1776	61	52 N.	28	30 N.	343	29
Ditto	Dentrecasteaux	1791	62	25 N.				
Ditto	Humboldt	1799	62	25 N.				
Ditto	Mudge	1820	58	22 N.				
Cambridge	Williams	1780	69	51 N.	42	23 N.	289	18
	Ditto	1782	69	41 N.				
	Ditto	1783	69	41 N.				
Coquimbo	Feuillée	1710	47	25 S.	29	55 S.	288	44
Conception	Ditto	1710	55	30 S.	36	43 S.	287	20
Christmas Island	Cook	1778	11	54 S.	1	59 N.	202	30
Cracatoa Island	Ditto	1780	26	3 S.	6	8 S.	105	36
Chalmer's Haven	Vancouver	1794	77	8½ N.	60	16 N.	213	22
Cross Sound	Ditto	1794	78	58½ N.	58	12 N.	223	55
Cocos Island	Ditto	1795	19	45 N.	5	35 N.	273	5
Catherine, Saint, Island	La Perouse	1785	30	30 S.	27	21 S.	312	20
Croix, Saint, Island	Lowenorn	1783	49	15 N.	17	45 N.	295	11
Cochin	Le Valois	1786	10	15 N.	9	58 N.	76	3
Charkow	Huth	1809	66	15 N.	50	0 N.	36	26

Dip of the needle in various parts of the earth, continued.

Place.	Authority.	Date.	Dip.	Latitude.	Longitude.
Charlotte Sound, New Zealand	Cook	1770	54° 50′ S.	41° 5′ S.	175° 25′
	Bayley	1773	64 44½ S.		
	Wales	1773	64 39 S.		
	Ditto	1773	64 21 S.		
Cove, Ship	Ditto	1774	64 38½ S.		
Ditto	Cook	1777	63 42 S.		
Cavite, Manilla	La Perouse	1787	11 5 N.	14 29 N.	111 11
Copenhagen	Lous	1773	71 45 N.	55 41 N.	12 35
	Bugge	1791	71 20½ N.		
	Wlengel	1813	71 26 N.		
Diemen's Land:					
Adventure Bay	Cook	1777	70 15 S.	43 21 S.	147 29
North Port	Bertrand	1792	70 50 S.	43 32 S.	146 56
Cape Diemen	Dentrecasteaux	1792	70 30 S.	43 32 S.	147 6
South Port	Ditto	1793	72 22 S.	43 34½ S.	146 56
	De Rossee	1793	70 10 S.	43 34½ S.	146 56
Dusky Bay, New Zealand	Cook	1773	70 6 S.	45 47 S.	166 18
Ditto	Vancouver	1791	69 43 S.		
Dauphin, Fort, Madagas.	Le Gentil	1761	52 5 S.	25 1 S.	46 49
Discovery Haven	Vancouver	1792	74 30 N.	48 2½ N.	237 22
Diego, Saint	Ditto	1793	59 13 N.	34 42 N.	243 7
Dynefiord's Haven	Lowenorn	1786	79 30 N.	65 52 N.	341 10
Esmeralda	Humboldt	1799	25 58 N.	3 11½ N.	293 57
Endeavour River	Cook	1770	36 0 S.	15 26 S.	145 12
France, Isle of	Manneville	1751	52 52 S.	20 10 S.	57 28
	La Caille	1754	52 17 S.		
Florenze	Humboldt	1805	63 57 N.	43 46½ N.	11 15
Faënza	Ditto	1805	63 54 N.		
Ferrol	Ditto	1799	68 32 N.	43 29 N.	351 45
Franeker	Brugmanns	1765	72 12 N.	53 11 N.	
Fayal Bay	Cook	1775	71 1 N.	38 32 N.	331 27
Foulpoint, Madagascar	Le Gentil	1762	48 32 S.	17 41 S.	49 56
Fuego, Terra del					
Success Bay	Cook	1769	68 51 S.	54 50 S.	294 35
	Ditto	1769	65 0 S.		
Christmas Sound	Ditto	1774	66 54 S.	55 22 S.	289 57
Saint Vincent	La Perouse	1786	63 0 S.	54 35 S.	294 16
Guayaquil	Basil Hall	1821	12 11½ N.	2 13 S.	280 15
Gottingen	Humboldt	1805	69 29 N.	51 32 N.	9 54
	Mayer	1805	69 26 N.		
	Ditto	1814	69 9 N.		
Genoa	Humboldt	1805	64 45 N.	44 25 N.	8 58
Guadarama	Ditto	1799	66 9 N.		
Gironna	Ditto	1799	64 34 N.		
Saint Gothard					
Saint Ursern		1799	66 42 N.		
Saint Hospiz		1799	66 22 N.		
Saint Airolo		1799	65 55 N.		
Good Hope	La Caille	1751	43 0 S.	33 55 S.	18 24
	Bayley	1772	45 37 S.		
	Ditto	1775	45 19 S.		
False Bay	Abercrombie	1775	46 26 S.		
Simon's Bay	Cook	1780	46 47 S.	34 20 S.	18 29
	Bertrand	1792	47 25 S.	33 55 S.	18 24
Gallipagos	Colnet	1793	6 0 N.	0 37 S.	270 22
	Vancouver	1795	2 20 N.	0 44 S.	267 54
George III. Sound	Ditto	1791	64 54 S.	35 5 S.	118 14
Gebel Zeghir	Panton	1776	11 37 N.	14 24 N.	43 35
Gebel Teir	Ditto	1776	13 23 N.	15 29 N.	43 12

Dip of the needle in various parts of the earth, continued.

Place.	Authority.	Date.	Dip.		Latitude.		Longitude.	
			°	′	°	′	°	′
Havnefiod's Haven	Lowenorn	1786	80	0 N.	64	4 N.	338	8
Holmen's Haven	Ditto	1786	78	30 N.	64	9 N.	338	8
Helena, Saint	La Caille	1754	9	0 S.	15	55 S.	354	12
	Ekeberg	1771	13	0 S.				
	Cook	1775	11	25 S.				
Heidelberg	Humboldt	1805	68	39 N.	49	24 N.	44	0
Heilbronn	Ditto	1805	68	1 N.	49	10 N.	9	7
Huaheine, Owhyee	Cook	1777	28	28 N.	16	43 N.	208	52
Judda	Panton	1776	26	0 N.	20	59 N.	39	56
Irkutsk	Schubert	1805	67	0 N.	52	17 N.	104	11
Javita	Humboldt	1799	24	19 N.	2	48 N.	291	58
Impossibile	Ditto	1799	38	50 N.	10	26 N.	295	54
Kola	Rumowsky	1769	77	45 N.	68	52 N.	33	0
Kleve	Humboldt	1805	70	8 N.	51	48 N.	6	5
Landscrone	Wilcke	1770	72	30 N.	55	52 N.	12	49
Loxa	Humboldt	1799	5	24 N.	4	0 S.	280	36½
Lima	Feuillée	1710	10	30 S.	12	3 S.	282	52
	Humboldt	1799	9	59 S.				
Lucern	Ditto	1805	67	10 N.				
Lucie le Bois	Ditto	1805	68	10 N.				
Lyons	Ditto	1805	66	14 N.	45	46 N.	4	49
London	Norman	1576	71	50 N.	51	31 N.	0	0
	Gilbert	1600	72	0 N.				
	Ridley	1613	72	30 N.				
	Bond	1676	73	30 N.				
	Whiston	1720	73 75	45 10				
	Graham	1723	74 74	42 42				
	Nairne	1772	72	19				
	Cavendish	1775	72	31				
	Gilpin	1786	72	8·1				
	Ditto	1787	72	2·5				
	Ditto	1788	72	4·0				
	Ditto	1789	71	54·8				
	Ditto	1790	71	53·7				
	Ditto	1791	71	23·7				
	Ditto	1795	71	11·4				
	Ditto	1797	70	59·4				
	Ditto	1798	70	55·4				
	Ditto	1799	70	52·2				
	Ditto	1801	70	35·6				
	Ditto	1803	70	32·0				
	Ditto	1805	70	21·0				
	Sabine	1821	70	3·2				
Lefooga Island	Cook	1777	36	55 S.	20	15 S.	185	11
Maria Island, Madagasc.	Le Gentil	1762	48	31 S.	16	30 S.		
Manilla	Ditto	1766	11	41 N.	14	34 N.	120	51
Madras	Abercrombie	1775	5	15 N.	13	15 N.	79	57
Mocha	Panton	1776	8	23 N.	13	22 N.	44	10
Macao	Ekeberg	1770	23	7½ N.				
	Cook	1779	21	1 N.	22	9 N.	113	49
Musketto Cove	Pickersgill	1776	81	22 N.	64	53 N.	307	3
Madrid	Humboldt	1799	67	41 N.	40	25 N.	356	17
Montpellier	Ditto	1799	65	53 N.	43	36 N.	5	52
Marseilles	Ditto	1799	65	10 N.	43	18 N.	5	23
Medina de Campo	Ditto	1799	66	9 N.				
Magdeburg	Ditto	1805	69	35 N.	52	10 N.	11	45
Michael, Saint	Ditto	1805	66	12 N.	45	23 N.		
Modane	Ditto	1805	66	6 N.				
Milan	Ditto	1805	65	40 N.	45	28 N.	9	11

Dip of the needle in various parts of the earth, continued.

Place.	Authority.	Date.	Dip.		Latitude.		Longitude.	
			°	′	°	′	°	′
Modena	Humboldt	1805	64	55 N.	44	39 N.	10	50
Mietau	Beitler	1783	73	0 N.	56	39 N.	23	43
Mahé	Le Valois	1786	9	0 N.	4	38 S.	55	35
Mexico	Don Algate	1778	38	0	19	26 N.	260	54
	Humboldt	1799	42	10 N.				
Melville Island	Parry	1820	88	43 N.	74	47 N.	249	12
Ditto	Vancouver	1794	63	0 N.	36	36 N.	238	9
Monterrey	Parry	1820	88	43 N.	74	47 N.	249	12
Madeira	Mudge	1820	63	47 N.	32	38 N.	342	6
North Cape	Bayley	1760	79	0 N.	71	10 N.	25	50
Nismes	Humboldt	1799	65	23 N.	43	50 N.	4	19
Naples	Ditto	1805	61	35 N.	40	50 N.	14	16
Nootka Sound	Cook	1778	72	30 N.	49	36 N.	233	23
	Vancouver	1794	73	56 N.				
Norton Sound	Cook	1778	76	25 N.	64	31 N.	197	13
Orel	Inochodzof	1781	67	0 N.	52	57 N.	35	57
L'Orient	La Caille	1754	72	0 N.	47	45 N.	356	37
Otaheite, Venus Point	Cook	1769	30	43 S.	17	29 S.	210	25
	Ditto	1773	29	43 S.				
	Ditto	1774	29	59 S.				
——— Matavai Point	Ditto	1777	29	12 S.	17	29 S.	210	22
——— ——— Bay	Vancouver	1791	30	53 S.	17	30 S.	210	24
Owhyee	Cook	1777	40	51 S.	19	28 S.	204	0
	Vancouver	1793	41	24 S.				
Oneehow	Cook	1778	42	1 S.	21	12 S.	200	40
Observatory Inlet	Vancouver	1793	75	54½ N.	55	15½ N.	230	16
Ohittahoo	Cook	1774	18	7 S.	9	55 S.	220	51
Paris	Richer	1671	75	0 N.	48	50 N.	2	20
	La Caille	1754	72	15 N.				
	Le Monnier	1776	72	25 N.				
	Cassini	1780	71	48 N.				
	Ditto	1791	70	52 N.				
	Bouvard	1798	70	10 N.				
	Humboldt	1798	69	26 N.				
	Ditto	1806	69	12 N.				
	Connais. de tems	1814	68	36 N.				
Petersburgh	Euler	1755	73	30 N.	59	56 N.	30	19
	Mallet	1769	73	35 N.				
	Kraft	1778	72	36 N.				
	Ditto	1802	76	42 N.				
Ponor	Mallet	1769	76	30 N.	67	4 N.	41	9
Petropaulowska	Cook	1779	63	5 N.	53	1 N.	158	43
	Krusenstern	1804	63	32 N.				
Pulo Condore	Cook	1780	2	1 S.	8	40 N.	106	19
Prince's Island	Ditto	1780	28	15 S.	6	36 S.	105	17
Pulo Penang	Le Gentil	1768	6	22 N.	5	25 N.		
Popayan	Humboldt	1799	20	53 N.	2	24 N.	283	35
Possession Bay	Parry	1820	86	4 N.	73	31 N.	285	38
Portsmouth	Barlow	1820	70	30 N.				
Plymouth	Ditto	1820	70	21 N.				
Portland	Ditto	1820	70	24·8 N.				
Perpignan	Humboldt	1799	65	18 N.	42	42 N.	2	53
Pavia	Ditto	1805	65	26 N.	45	11 N.	9	9
Piacenza	Ditto	1805	65	0 N.	45	3 N.	9	42
Parma	Ditto	1805	65	7 N.	44	48 N.	10	20
Pesara	Ditto	1805	64	18 N.	43	55 N.	12	53
Prague	Kampf	1768	71	3 N.	50	5 N.	14	25
Palma	Ekeberg	1770	64	22 N.	28	38 N.	342	16
Port Français	La Perouse	1786	73	52 N.	58	39 N.	222	30
Quito	Bouguer	1742	10	0 N.	0	13 S.	281	15
	Humboldt	1799	13	22 N.				

Dip of the needle in various parts of the earth, continued.

Place.	Authority.	Date.	Dip.	Latitude.	Longitude.
			° ′	° ′	° ′
Rome	Kircher	1640	65 40 N.	41 54 N.	12 28
	Humboldt	1806	61 57 N.		
Rimini	Ditto	1806	63 48 N.	44 4 N.	12 33
Rio Janeiro	La Caille	1751	20 0 S.	22 54 S.	317 15
Regent's Inlet	Parry	1820	88 26 N.	72 45 N.	270 19
Spoleto	Humboldt	1806	62 51 N.	42 15 N.	12 45
Java	Dentrecasteaux	1793	25 40 S.	7 14 S.	112 39
Jago, Saint	Marryat	1820	48 0 N.	14 54 N.	336 28
Socotra	Panton	1776	4 37 N.	13 16 N.	52 55
Suez	Ditto	1777	44 33 N.	29 58 N.	32 15
Scheulpar	Ditto	1776	2 30 N.	9 59 N.	72 30
Sunda	Ekeberg	1770	27 30 S.	6 36 S.	
Tubingen	Humboldt	1805	68 4 N.	48 31 N.	9 3
Turin	Ditto	1805	66 3 N.	45 4 N.	7 40
Tompenda	Ditto	1799	3 11 N.	5 31½ N.	281 23
Saint Thomas de la Guayana	Ditto	1799	35 6 N.	8 8 N.	296 5
Tornea	Celsius	1737	78 5 N.	65 51 N.	24 12
Tolago Bay	Cook	1773	62 26 S.	38 21 S.	178 34
Trinquemala	Panton	1776	4 37 S.	8 32 N.	81 30
Tulcaguana Bay	La Perouse	1786	50 45 S.	36 43 S.	286 50
Tongataboo	Cook	1777	39 1 S.	21 8 S.	184 55
Tristan da Acunha	Marryat	1821	37 53 S.	37 0 S.	347 50
Trinidad	Ditto	1821	10 37 S.	20 30 S.	331 0
Ulietea	Cook	1777	29 5 S.	16 45 S.	208 25
Unalaska	Ditto	1778	69 23½ N.	53 5 N.	193 30
Umba	Pictet	1769	75 10 N.	66 4½ N.	34 13
Vampo Haven	Ekeberg	1770	26 7½ N.		
Valparaiso	Vancouver	1795	44 15 S.	33 1 S.	288 29
	Basil Hall	1821	38 46 S.		
Vesuvius Crater	Humboldt	1805	62 0 N.		
Wellendingen	Ditto	1805	67 57 N.	48 9 N.	8 42
Wittenberg	Langguth	1803	70 37 N.	51 51 N.	12 42
Woolwich	Barlow	1818	70 34 N.		
Zurich	Humboldt	1805	67 22 N.	47 22 N.	8 32

The following observations, giving both the dip and variation, are taken from the appendix to Captain Franklin's Journey to the shores of the Polar Sea; to which are added the computed latitude and longitude of the terrestrial magnetic pole, (art. 197.)

Date.	Latitude, North.	Longitude, West.	Amount of dip.	Difference produced by changing the face of the instrument.	Variation, West.	Computed place of the pole. Latitude, North.	Computed place of the pole. Longitude, West.	Place of Observation.
1819. Sept. 4.	57° 0′	92° 26′	79° 29′	11° 3′	6° 0′	77° 6′	83° 3′	York factory.
26.	54 57	94 56	80 37	3 45	11 50	73 14	82 38	Magnetic Island, Knee Lake.
Oct. 7.	53 41	98 0	83 40	7 39	14 21	69 37	89 31	Norway house.
Nov. 24.	53 56	102 16	83 9	13 3	17 17	66 29	92 15	In a room at Cumberl. house.
Dec. 1.	53 56	102 16	83 12	12 50	17 17			In the air at ditto.
1820. June 11.	53 56	102 16	84 45 84 24	17 48 19 17	17 17			Cumberland house.
29.	55 26	107 52	84 5 84 21	16 29 17 43	25 15	65 27	96 7	Isle à la Crosse.
July 9.	56 43	109 53	85 9 85 5	6 39 9 21	25 2	65 11	100 5	North side of Methye Portage.
14.	58 43	111 18	85 23	6 2	22 47	66 47	102 14	Fort Chipewyan.
Aug. 1.	62 17	114 9	86 38	6 49	33 35	67 35	104 25	Fort Providence, north side of Slave Lake.
11.	63 47	113 46	87 11	7 4	36 45	68 5	104 47	Upper Carp Lake.
16.	64 15	113 2	87 20	7 13	36 50	68 19	104 24	Portage at the north end of Grizzle Bear Lake.
30.	64 28	113 6	87 25	6 44	35 45 36 15	68 30	104 51	Fort Enterprise.
1821. June 9.	64 28	113 6.	87 3 86 54	15 56 19 45	36 24 37 24	68 17	104 24	Fort Enterprise.
10.	67 2	116 28	87 28 87 34	16 39 15 34	44 11	70 17	106 21	Below the Rocky Defile Rapid Copper-Mine River.
21.	67 47	115 36	88 3 88 6	6 28 9 21	46 25	69 51	107 31	Mouth of the Copper-Mine River.
31.	67 19	109 44	88 54 89 2	5 20 0 1	40 59	68 58	105 54	The first Cascade in Hood's River.
Aug. 1.	68 18	109 25 110 5	89 56 89 6	2 42 0 24	44 15	68 50	107 33	Point Turnagain.

Note. The column showing the difference of dip, with the face of the instrument in different directions, appears to indicate a want of adjustment in the dipping needle, and may lead to some doubt respecting the accuracy of the mean results. It is presumed, however, that the error is not very considerable.

TABLE

OF THE

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  in London, Paris, and Copenhagen, from 1660 to 1818, computed - - - - 817
  diurnal - - - - - - - - - - - 819

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ERRATA IN THE TREATISE OF MECHANICS.

Page.	*Col.*	*Line.*	*Error.*	*Correction.*
30,		3,	describing,	described by.
85,	1,	41,	it can ascend,	we can ascend.
87,	do.	16,	space,	the space.
105,	do.	3 from bott.	plate III.	plate IV.
114,	do.	in fig. 44, referred to,		the letter O is omitted.
145,	do.	18 from bott.	Segnor,	Segner.

HYDRODYNAMICS.

Page.	*Col.*	*Line.*	*Error.*	*Correction.*
190,	2,	15, 24, from bott.	O,	O′.
273,	1,	3 from bott.	*k*,	K.
280,	2,	18 from bott.	A E F D,	A F E B.

PLANE ASTRONOMY.

The Table of Constellations, p. 507, the following principal Stars are misplaced, being set one line too high, *i. e.*

Betelgense *should be placed opposite* Orion.
Canopus " Argo.
Cor Hydræ " Hydra.
Page 522, *for* meridian, *read* horizon.

NAUTICAL ASTRONOMY.

Page.	*Col.*	*Line.*	*Error.*	*Correction.*
608,	1,	16,	80° east,	80° 30′ east.
do.	do.	39,	longitude,	latitude.
do.	do.	43,	*after* top,	*insert* under 4.
do.	do.	49,	$3^{m}\ 48^{s}{\cdot}5$,	$3^{m}\ 45^{s}{\cdot}5$.
do.	2,	10, 11,	$24^{h}\ 3^{m}\ 56^{s}{\cdot}55$; $3^{m}\ 56^{s}{\cdot}55$; $24^{h}\ 3^{m}\ 55^{s}{\cdot}91$,	$24^{h}\ 3^{m}\ 56^{s}{\cdot}55 : 3^{m}\ 56^{s}{\cdot}55 :: 24^{h} : 3^{m}\ 55^{s}{\cdot}91$.
do.	do.	21, 25,	February,	April.
610,	1,	55,	$-\ 55^{s}{\cdot}2$,	$-\ 15^{s}{\cdot}2$.
612,		10,	10′ 6″	9′ 54″, and make the corresponding corrections.
613,	2,	19,		*dele* (= GE *m*,).
617,	1,	33,	18 feet,	16 feet.
619,	2,	21,	PA,	ZA.
622,	do.	24,	lay-to,	lie-to.
628,	do.	34,	*after* time,	*insert* it.
632,		14,	Barometer + 0′ 35″,	Barometer + 0′ 16″.
do.		15,	Thermometer + 0′ 56″,	Thermometer + 0′ 22″, and make the corresponding corrections throughout the example.
636,	1,	44,	*after* horizon,	*insert* + the refraction.
645,		Table VI.	sideral,	sidereal.
646,		Table VII.	sideral,	sidereal.

In plate I. fig. 2, the centre of the circle ☽ *m* Z should be C and not E.

PHYSICAL ASTRONOMY.

Page.	*Col.*	*Line.*	*Error.*	*Correction.*
647,	1,	21,	power,	powers.
do.	do.	47,	conveying,	converging.
do.	do.	53,	limits,	stretch.
648,	do.	47,	laws,	law.
654,	2,	14,	gives,	have.
659,	do.	18,	$: \sqrt{AR}$	$: \sqrt{AC}$
664,	1,	marg. line 11,	*after* let S be the sun,	*insert* (Fig. 8.)
670,	do.	9,	plane,	planet.
679,		3 and 5,	$\frac{a\,t}{2}$	$\frac{a\,t}{2\,n}$
684,		9 from bott.	$(-a\,c)$	$(1-a\,c)$
do.		7 from bott.	$(a^2 + 2\,a\,a')$	$(a^2 - 2\,a\,a')$
689,		30,	Ω	Ω, *r*, *r*,′ &c.
716,		9,	$\frac{1}{\mu}$	$\frac{1}{u}$
717,		11,	(250)	(241,1)
do.		13,	(241)	(241,2)
718,		7,	$-\frac{m}{r^2}$	$-\frac{m'}{r'^2}$
721,		10,	A. cos ſ θ	A. cos B θ
722,		1,	to,	of.

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